Cubic core-free symmetric m-Cayley graphs

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Abstract

An *m-Cayley graph* Γ over a group G is defined as a graph which admits G as a semi-regular group of automorphisms with m orbits. This generalises the notions of a Cayley graph (where $m = 1$) and a *bi-Cayley graph* (where $m = 2$). The m-Cayley graph Γ over G is said to be *normal* if G is normal in the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ, and *core-free* if the largest normal subgroup of $\mathrm{Aut}(\Gamma)$ contained in G is the identity subgroup.

In this paper, we investigate properties of symmetric m-Cayley graphs in the special case of valency 3, and use these properties to develop a computational method for classifying connected cubic core-free symmetric m-Cayley graphs. We also prove that there is no 3-arc-transitive normal Cayley graph or bi-Cayley graph (with valency 3 or more), which answers a question posed by Li (in *Proc. American Math. Soc.* 133 (2005)).

Using our classification method, we give a new proof of the fact that there are exactly 15 connected cubic core-free symmetric Cayley graphs, two of which are Cayley graphs over non-abelian simple groups. We also show that there are exactly 109 connected cubic core-free symmetric bi-Cayley graphs, 48 of which are bi-Cayley graphs over non-abelian simple groups, and that there are $1, 6, 81, 462$ and 3267 connected cubic core-free 1-arc-regular 3-, 4-, 5-, 6- and 7-Cayley graphs, respectively.

Keywords: Symmetric graph, double-coset graph, m-Cayley graph, simple group.

2010 Mathematics Subject Classification: 05C25, 20B25.

Acknowledgements: The authors were partially supported by the National Natural Science Foundation of China (11571035, 11731002), the 111 Project of China (B16002), and the N.Z. Marsden Fund (UOA1626).

1 Introduction

Cayley graphs form an important class of vertex-transitive graphs, which have been the object of study for many decades because of their symmetry and their usefulness in a variety of contexts — for example since they often make good models for interconnection networks (see [12, 13, 14]). Cayley graphs admit a group of automorphisms that acts regularly (sharply-transitively) on vertices.

On the other hand, there are many important vertex-, edge- or arc-transitive graphs that are not Cayley graphs, such as the Petersen graph, the Gray graph, and the Hoffman-Singleton graph. These three examples are *bi-Cayley,* which means they admit a group of automorphisms acting semi-regularly on the vertices with two regular orbits of equal length. Bi-Cayley graphs can be used to construct non-Cayley vertex-transitive graphs, such as the generalised Petersen graphs, and edge-transitive bi-Cayley graphs were investigated in detail in [8].

The notions of a Cayley graph and a bi-Cayley graph can be generalised by defining an *m-Cayley graph* to be a graph on which some group of automorphisms acts semi-regularly with m orbits (of equal length). More specifically, if the group G of automorphisms of the graph Γ acts semi-regularly on $V(\Gamma)$ with m orbits, then Γ is called an *m-Cayley graph over the group G.* And in case it is not obvious, such a graph is a Cayley graph when $m = 1$, and a bi-Cayley graph when $m = 2$. Other examples include tri-circulants (with $m = 3$) and tetra-circulants (with $m = 4$).

Also we say that the m-Cayley graph Γ over G is *normal* if G is a normal subgroup of the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ, and *core-free* if the largest normal subgroup of $\mathrm{Aut}(\Gamma)$ contained in G is the identity subgroup. Clearly both of these kinds of m-Cayley graphs can play a role in the construction of larger examples. In particular, if G is a simple group then every m-Cayley graph over G is either normal or core-free.

In this paper we investigate properties of m-Cayley graphs in general, but also in particular for m-Cayley graphs of valency 3. One general theorem we prove is Theorem 5.1 in Section 5, namely that if Γ is a connected (G, s)-arc-transitive graph of valency at least 3, and N is a normal subgroup of G that acts semi-regularly on $V(\Gamma)$ with one or two orbits, then $s = 1$ or 2. We then use properties of m-Cayley graphs to develop a computational method for classifying connected cubic core-free symmetric m-Cayley graphs, with the help of the Magma system [2].

The case $m = 1$ for this classification has already been achieved. It was shown by Li [16] in his 1996 PhD thesis that for a connected symmetric cubic Cayley graph Γ over a non-abelian simple group G, either Γ is normal, or $G = A_5$, $\mathrm{PSL}(2, 11)$, M_{11}, A_{11}, M_{23}, A_{23} or A_{47}. Then in 2005, Xu, Fang, Wang and Xu [26] proved that either Γ is normal or $G = A_{47}$, and two years later, they proved further in [27] that if $G = A_{47}$ and Γ is not normal, then Γ must be 5-arc-regular, and up to isomorphism there are exactly two such graphs. Also Conder [3] showed that there are 6 connected cubic core-free 5-arc-transitive Cayley graphs, and Li and Lu showed that there are precisely 15 connected cubic core-free symmetric Cayley graphs, in [19, Theorem 1.1].

Our computational approach gives a new proof of the latter classification of connected symmetric cubic core-free Cayley graphs (with 15 in total, and with two being Cayley graphs over non-abelian simple groups); see Theorem 6.1 in Section 6. But also we take this much further. In the case $m = 2$, we use it to show there are exactly 109 connected symmetric cubic core-free bi-Cayley graphs, 48 of which are over some non-abelian simple group (namely one of A_{23}, A_{47} or A_{95}); see Theorem 7.1 in Section 7. Also in the cases $m = 3$, 4, 5, 6 and 7 we find there are 1, 6, 81, 462 and 3267 connected cubic core-free 1-arc-regular m-Cayley graphs respectively; see Theorems 8.1 to 8.3 in Section 8.

Finally, for a symmetric m-Cayley graph Γ over G, let $A = \mathrm{Aut}(\Gamma)$. If the core G_A of G in A has at least three orbits, then we obtain a symmetric core-free m-Cayley graph over G/G_A. On the other hand, if G_A has just one or two orbits, then Γ is at most 2-arc-transitive by our Theorem 5.1. In particular, there is no 3-arc-transitive normal Cayley or bi-Cayley graph, and so this answers a question posed by Li in [18]. The same answer was given also in [8], but using a different approach.

In Section 2 we explain our notation and give some further background, and then we consider properties of cubic core-free symmetric m-Cayley graphs in Section 3, and outline our computational approach in Section 4. We consider degenerate cases (not only for valency 3) and give our answer to the question of Li [18] in Section 5, and then deal with cubic core-free symmetric Cayley graphs in Section 6, and symmetric cubic core-free bi-Cayley graphs in Section 7. Finally we deal with 1-arc-regular cubic core-free m-Cayley graphs for $3 \le m \le 7$ in Section 8.

2 Notation and further background

Throughout this paper, all graphs are undirected, simple (without loops or multiple edges), and connected, unless otherwise specified. We will denote by $\mathbb{Z}_n$, D_n, A_n and S_n the cyclic group of order n, the dihedral group of order $2n$, the alternating group of degree n, and the symmetric group of degree n, respectively. The *core* of a subgroup H of a group G is the largest normal subgroup of G contained in H, sometimes denoted by H_G, and if this core is trivial, then we say that H is *core-free* in G.

In a permutation group G on a set Ω, we denote by G_α the stabiliser in G of a point $\alpha \in \Omega$, that is, the subgroup of G fixing the point α. We say that G is *semi-regular* on Ω if G_α is trivial for every $\alpha \in \Omega$, and *regular* (or sharply-transitive) on Ω if G is transitive and semi-regular on Ω.

A *symmetry* (or *automorphism*) of a graph Γ is a permutation of its vertex-set $V = V(\Gamma)$ preserving the edge-set $E = E(\Gamma)$. Under composition, symmetries of the graph Γ form a group, called the *automorphism group* of Γ, and denoted by $\mathrm{Aut}(\Gamma)$. If $\mathrm{Aut}(\Gamma)$ is transitive on V, then Γ is said to be *vertex-transitive*, and if $\mathrm{Aut}(\Gamma)$ is transitive on E, then Γ is said to be *edge-transitive*. An *arc* in a graph Γ is an ordered pair of adjacent vertices (or equivalently, an ordered edge), and if $\mathrm{Aut}(\Gamma)$ is transitive on the set of all arcs in Γ, then Γ is said to be *arc-transitive*, or *symmetric*.

Next, an *s-arc* in Γ is an ordered $(s+1)$-tuple $(v_0, v_1, ..., v_s)$ of vertices of Γ such that any two consecutive v_i are adjacent, and any three consecutive v_i are distinct — that is, v_{i-1} is adjacent to v_i for $1 \le i \le s$, and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$. The graph Γ is said to be *s-arc-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the set of all s-arcs of Γ, and *s-arc-regular* if this action is regular. Also we sometimes say that Γ is *s-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the s-arcs but not on the $(s+1)$-arcs of Γ.

More generally, if G is any subgroup of $\mathrm{Aut}(\Gamma)$, we can say that G is s-arc-transitive, s-arc-regular or s-transitive on Γ, or that Γ is (G,s)*-arc-transitive*, (G,s)*-arc-regular* or (G,s)*-transitive*, respectively, if the group G acts in the corresponding fashion.

Next, a cubic graph is one in which every vertex has degree/valency 3. We now summarise some well known theory of symmetric cubic graphs.

By the seminal work of Tutte [23, 24], it is known that every finite symmetric cubic graph is s-arc-regular for some $s \le 5$. Considerable attention has been paid to analysing, constructing and classifying such graphs; see [5, 7, 9, 19, 26, 27] for example. Many but not all of these graphs are Cayley graphs.

If G is an s-arc-regular group of automorphisms of a symmetric cubic graph, then the vertex-stabiliser G_v is isomorphic to $\mathbb{Z}_3$, S_3, D_6, S_4 or the direct product $S_4 \times \mathbb{Z}_2$, when $s = 1, 2, 3, 4$ or 5, respectively. Also the edge-stabiliser G_e is unique up to isomorphism in the cases $s = 1, 3$ and 5, but there are two different possibilities for G_e when $s = 2$ or 4, depending on whether G_e contains an edge-reversing automorphism of order 2.

Taking into account the isomorphism type of the pair (G_v, G_e) for a given incident vertex-edge pair (v, e) in Γ, this gives seven different classes of possible actions of an arc-transitive group on a finite cubic graph Γ. These classes correspond to seven classes of 'universal' groups acting arc-transitively on the infinite cubic tree with finite vertex-stabiliser (see [9, 11]).

It follows that the automorphism group of any finite symmetric cubic graph Γ is a homomorphic image of one of these seven groups, called G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5 by Conder and Lorimer in [6]. Based on the analysis undertaken in [9, 11], we will use the following presentations for these seven groups, as given in [6]:

G_1 is generated by two elements h and a, subject to the relations $h^3 = a^2 = 1$;

G_2^1 is generated by h, a and p, subject to $h^3 = a^2 = p^2 = 1$, $apa = p$, $php = h^{-1}$;

G_2^2 is generated by h, a and p, subject to $h^3 = p^2 = 1$, $a^2 = p$, $php = h^{-1}$;

G_3 is generated by h, a, p and q, subject to $h^3 = a^2 = p^2 = q^2 = 1$, $apa = q$, $qp = pq$, $ph = hp$, $qhq = h^{-1}$;

G_4^1 is generated by h, a, p, q and r, subject to $h^3 = a^2 = p^2 = q^2 = r^2 = 1$, $apa = p$, $aqa = r$, $h^{-1}ph = q$, $h^{-1}qh = pq$, $rhr = h^{-1}$, $pq = qp$, $pr = rp$, $rq = pqr$;

G_4^2 is generated by h, a, p, q and r, subject to $h^3 = p^2 = q^2 = r^2 = 1$, $a^2 = p$, $a^{-1}qa = r$, $h^{-1}ph = q$, $h^{-1}qh = pq$, $rhr = h^{-1}$, $pq = qp$, $pr = rp$, $rq = pqr$;

G_5 is generated by h, a, p, q, r and s, subject to $h^3 = a^2 = p^2 = q^2 = r^2 = s^2 = 1$, $apa = q$, $ara = s$, $h^{-1}ph = p$, $h^{-1}qh = r$, $h^{-1}rh = pqr$, $shs = h^{-1}$, $pq = qp$, $pr = rp$, $ps = sp$, $qr = rq$, $qs = sq$, $sr = pqrs$.

Also any arc-transitive subgroup G of $\mathrm{Aut}(\Gamma)$ is a homomorphic image of one of these seven groups, and G is said to be of type 1, 2^1, 2^2, 3, 4^1, 4^2 or 5, according to which of the seven groups it comes from. For example, the complete graph K_4 is 2-arc-regular, and its automorphism group S_4 is of type 2^1 (as it contains edge-reversing automorphisms of order 2), while its subgroup A_4 is an arc-transitive subgroup of type 1. In particular, a finite symmetric cubic graph may admit more than one type of arc-transitive group action. The combinations of types that are possible were determined by Conder and Nedela, as follows, with details summarised in Table 1 below.

Proposition 2.1 [7, Theorem 5.1] *Finite cubic symmetric graphs can be classified into* 17 *different families, according to the combinations of arc-transitive actions they admit.*

s	Types	Bipartite?	s	Types	Bipartite?	s	Types	Bipartite?
1	1	Sometimes	3	$2^1, 3$	Never	5	$1, 4^1, 4^2, 5$	Always
2	$1, 2^1$	Sometimes	3	$2^2, 3$	Never	5	$4^1, 4^2, 5$	Always
2	2^1	Sometimes	3	3	Sometimes	5	$4^1, 5$	Never
2	2^2	Sometimes	4	$1, 4^1$	Always	5	$4^2, 5$	Never
3	$1, 2^1, 2^2, 3$	Always	4	4^1	Sometimes	5	5	Sometimes
3	$2^1, 2^2, 3$	Always	4	4^2	Sometimes			

Table 1: The 17 families of arc-transitive group actions on finite cubic symmetric graphs

Next, we turn our attention to quotients and covers of symmetric graphs.

Let Γ be a symmetric graph, and suppose N is a normal subgroup of $\mathrm{Aut}(\Gamma)$. Then the *quotient graph* of Γ relative to N is defined as the graph Γ_N with vertices the orbits of N on $V(\Gamma)$, and with two orbits adjacent in Γ_N whenever there is an edge in Γ between vertices of these two orbits. In particular, if Γ and Γ_N have the same valency, then Γ is called a *normal cover* of Γ_N.

Proposition 2.2 [20, Theorem 9] *Let Γ be a connected (G, s)-transitive graph of prime valency, and suppose N is a normal subgroup of G with at least three orbits on $V(\Gamma)$. Then N is the kernel of the action of G on $V(\Gamma_N)$, and acts semi-regularly on $V(\Gamma)$. Furthermore, Γ_N is $(G/N, s)$-transitive, with $G/N \leq \mathrm{Aut}(\Gamma_N)$.*

Let Υ be the (infinite) cubic tree, denoted in some other places by Γ_3. The normalisers of s-arc-regular subgroups of $\mathrm{Aut}(\Upsilon)$ were given by Djoković and Miller in [9].

Proposition 2.3 [9, Theorem 6] *Let G be an s-arc-regular subgroup of* $\mathrm{Aut}(\Upsilon)$. *Then*

(a) *if $s = 1$, 2 or 4, then the normaliser of G in* $\mathrm{Aut}(\Upsilon)$ *is the unique $(s+1)$-arc-regular subgroup of* $\mathrm{Aut}(\Upsilon)$ *containing G, while*

(b) *if $s = 3$ or 5, then G is self-normalising in* $\mathrm{Aut}(\Upsilon)$.

Now we describe a widely known construction for vertex-transitive and symmetric graphs, part of which is attributed to Sabidussi [22].

Let G be a group, H a subgroup of G, and D a union of some double cosets of H in G such that $D^{-1} = D$. Then the *double-coset graph* $\Gamma = \mathrm{Cos}(G, H, D)$ is defined as the graph with vertex-set $(G : H)$, the set of all right cosets of H in G, and edge-set $E(\Gamma) = \{\{Hg, Hxg\} : g \in G, x \in D\}$. This graph is regular with valency $|D|/|H|$, and is connected if and only if $G = \langle D, H\rangle$, that is, if and only if G is generated by D and H. Moreover, G acts vertex-transitively on Γ (as a group of automorphisms) by right multiplication, and this action of G is arc-transitive if and only if D consists of just one double coset HaH. Also the kernel of this action of G is the core H_G of H in G, and so the action of G on Γ is faithful if and only if H is core-free in G.

Conversely, suppose Γ is any graph on which the group G acts faithfully and vertex-transitively. Then it is easy to show that Γ is isomorphic to the double-coset graph $\mathrm{Cos}(G, H, D)$, where $H = G_v$ is the stabiliser in G of the vertex $v \in V(\Gamma)$, and D is a union of double cosets of H, consisting of all elements of G taking v to one of its neighbours, with $D^{-1} = D$. Moreover, if G is arc-transitive on Γ, and a is an element of G that swaps v with one of its neighbours, then $a^2 \in H$ and $D = HaH$, and the valency of Γ is $|D|/|H| = |H : H \cap H^a|$. Also a can be chosen as a 2-element in G.

Another way to describe this situation is as follows:

Proposition 2.4 *Let Γ be a connected G-arc-transitive graph of valency k, and let $\{v, w\}$ be any edge of Γ. Then Γ is isomorphic to the double-coset graph* $\mathrm{Cos}(G, G_v, G_vaG_v)$, *where a is a 2-element in G such that $v^a = w$, $a^2 \in G_v$, $a^{-1}G_{vw}a = G_{vw}$, $\langle G_v, a\rangle = G$ and $|G_v : G_v \cap G_v^a| = k$.*

For further details about double-coset graphs, see [10, 20, 21, 22]. Also note that $\mathrm{Cos}(G, \{1\}, D)$ is the same as the Cayley graph $\mathrm{Cay}(G, D)$.

Finally, the following is a simple observation about the dual of a permutation group.

Proposition 2.5 *Let G be a transitive permutation group on a set Ω, and let M be a subgroup of G, and α a point of Ω. Then M acts semi-regularly on Ω if and only if G_α acts semi-regularly by right multiplication on the right coset space $(G{:}M)$.*

Proof: Because G is transitive on Ω, we know that M acts semi-regularly on Ω if and only if $M \cap G_{\alpha^g} = \{1\}$ for every $g \in G$. On the other hand, G acts transitively on $(G{:}M)$ by right multiplication, and the stabiliser in G of M under this action is the subgroup M itself, which implies that G_α acts semi-regularly on $(G{:}M)$ if and only if $G_\alpha \cap M^h = \{1\}$ for all $h \in G$. But clearly $G_{\alpha^g} \cap M = \{1\}$ for all $g \in G$ if and only if $G_\alpha \cap M^h = \{1\}$ for all $h \in G$, and so the conclusion holds. □

3 Symmetric cubic core-free m-Cayley graphs

By a theorem of Goldschmidt [11], every arc-transitive subgroup of the automorphism group of the (infinite) cubic tree Υ with finite vertex-stabiliser is isomorphic to one of the seven universal groups for an arc-transitive action on a finite cubic graph, namely $U = G_1$, G_2^1, G_2^2, G_3, G_4^1, G_4^2 or G_5 (as defined in Section 2). In particular, the action is s-arc-regular if $U = G_s$, G_s^1 or G_s^2 for some s, and the stabiliser of any vertex of Υ is isomorphic to the subgroup H of U generated by all the generators of U given in Section 2 except the edge-reversing element a, and is isomorphic to $\mathbb{Z}_3$, S_3, $S_3 \times \mathbb{Z}_2$, S_4 or $S_4 \times \mathbb{Z}_2$ when $s = 1, 2, 3, 4$ or 5, respectively.

In this section, we will explain how to classify connected cubic core-free symmetric m-Cayley graphs based on the group U and its subgroups of finite index, and how to implement such a classification using the computer software Magma [2]. Some of the initial ideas behind this were presented by the second author in [3].

Let Γ be a finite connected cubic core-free s-arc-regular m-Cayley graph over a finite group G, with $s \leq 5$ and $m \geq 1$, and set $A = \mathrm{Aut}(\Gamma)$.

By the work of Goldschmidt [11] and Djoković and Miller [9], we know there exists an s-arc-regular subgroup U of $\mathrm{Aut}(\Upsilon)$ and an epimorphism $\theta : U \to A$. Let K be the kernel of θ, and let M be the pre-image of G in U, so that $U/K \cong A$ and $M/K \cong G$, and also $K = M_U$, since G is core-free in A. Then Γ is isomorphic to the quotient graph Υ_K, and M acts semi-regularly on $V(\Upsilon)$ since G acts semi-regularly on $V(\Gamma)$. Moreover, because G is semi-regular on $V(\Gamma)$ with m orbits, we have

$$|A| = |V(\Gamma)||A_v| = m|G||A_v| = 2^{s-1} \cdot 3 \cdot m|G|$$

and therefore $|U : M| = |A : G| = 2^{s-1} \cdot 3 \cdot m$.

Next consider the natural action of U on the coset space $(U : M)$ by right multiplication. Since $K = M_U$, we see that the induced permutation group of U on $(U : M)$ is $\bar{U} = U/M_U$, which is also the image of U under θ. Moreover, Γ is isomorphic to $\mathrm{Cos}(\bar{U}, \bar{H}, \bar{H}\bar{a}\bar{H})$, where $\bar{H}$ and $\bar{a}$ are the group induced by H and the permutation induced by a on $(U : M)$.

On the other hand, suppose U is any s-arc-regular subgroup of $\mathrm{Aut}(\Upsilon)$, with $1 \leq s \leq 5$, and let M be a subgroup of index $2^{s-1} \cdot 3 \cdot m$ in U that acts semi-regularly on $V(\Upsilon)$. If we let $\bar{U} = U/M_U$, $\bar{M} = M/M_U$, $\bar{H} = HM_U/M_U$ and $\bar{a} = aM_U$, then $\bar{M}$ is core-free in $\bar{U}$ (since M_U is the core of M in U), and $\Upsilon_{M_U} \cong \mathrm{Cos}(\bar{U}, \bar{H}, \bar{H}\bar{a}\bar{H})$ is a connected cubic core-free symmetric m-Cayley graph over $\bar{M}$.

The above observations suggest that one way to classify all connected cubic core-free symmetric m-Cayley graphs is to find all semi-regular subgroups M of each of the s-arc-regular subgroups U of $\mathrm{Aut}(\Upsilon)$ with $|U : M| = 2^{s-1} \cdot 3 \cdot m$. By Proposition 2.5, we know that M acts semi-regularly on $V(\Upsilon)$ if and only if H acts semi-regularly on $(U : M)$, by right multiplication, and because of this, we can look for all such subgroups with the help of the 'low index subgroups' capabilities of Magma. Then for each such M, we need to determine whether or not the quotient graph Υ_{M_U} is s-arc-regular, and whether two such quotient graphs are isomorphic. This can be done with the help of the following theorems.

Theorem 3.1

(a) *If U and V are any arc-transitive subgroups of* $\mathrm{Aut}(\Upsilon)$ *with the same Djoković-Miller type, then U and V are conjugate in* $\mathrm{Aut}(\Upsilon)$.

(b) *If K and L are subgroups of* $\mathrm{Aut}(\Upsilon)$ *with at least three orbits on $V(\Upsilon)$, and are normal in arc-transitive subgroups U and V of* $\mathrm{Aut}(\Upsilon)$ *that have the same Djoković-Miller type, then $\Upsilon_K \cong \Upsilon_L$ if and only if K and L are conjugate in* $\mathrm{Aut}(\Upsilon)$.

Proof: First, part (a) was proved by Djoković and Miller in [9, Theorem 2].

Next, for part (b), suppose that U and V act regularly on the s-arcs of Υ.

Now if K and L are conjugate by an automorphism θ of Υ, then θ induces an isomorphism from U/K to $U^\theta/K^\theta = U^\theta/L$, and this implies that U/K and U^θ/L have a common group presentation in terms of the canonical generating sets for symmetric cubic graphs of the type associated with U, and it then follows that the graphs Υ_K and Υ_L (which are constructible from those presentations) are isomorphic.

Conversely, suppose the graphs Υ_K and Υ_L are isomorphic. By part (a), we know that U and V are conjugate in $\mathrm{Aut}(\Upsilon)$, say $V = U^\theta$. It follows that K^θ and L are subgroups of the same subgroup V of $\mathrm{Aut}(\Upsilon)$, with $\Upsilon_{K^\theta} \cong \Upsilon_K \cong \Upsilon_L$. We can now use material from [9] to deduce that K^θ and L are conjugate in $\mathrm{Aut}(\Upsilon)$. If $s = 3$ or 5, then Theorem 7 of [9] gives $K^\theta = L$, while if $s = 2$ or 4, then by Theorem 8 of [9] we find that either $K^\theta = L$, or K^θ and L are conjugate within the $(s+1)$-arc-regular subgroup $N_{\mathrm{Aut}(\Upsilon)}(V)$ of $\mathrm{Aut}(\Upsilon)$. Finally, if $s = 1$, then we consider the graph Υ_L. If Υ_L is t-arc-regular for some $t > 1$, then V can be replaced by some t-arc-regular subgroup of $\mathrm{Aut}(\Upsilon)$, and so K^θ and L are conjugate in $\mathrm{Aut}(\Upsilon)$ by the same arguments as above. Otherwise Υ_L is 1-arc-regular, and then by Theorem 8 and Proposition 21 of [9], either $K^\theta = L$, or K^θ and L are two normal subgroups of V that are interchanged under conjugation by any element of the 2-arc-regular subgroup $N_{\mathrm{Aut}(\Upsilon)}(V)$ not contained in V. Hence in all cases, K^θ and L are conjugate in $\mathrm{Aut}(\Upsilon)$, and therefore so are K and L, as required. □

Theorem 3.2 *Let U be an s-arc-regular subgroup of the cubic tree Υ, and let M and L be semi-regular subgroups of index $2^{s-1} \cdot 3 \cdot m$ in U such that M_U and L_U have at least three orbits on $V(\Upsilon)$. Then the following hold:*

- (a) *$M_U = L_U$ if and only if the images of the generators of U in U/M_U satisfy the same relations as their images in U/L_U.*
- (b) *The quotient graph Υ_{M_U} is t-arc-regular (for some t with $s \le t \le 5$) if and only if $N_{\mathrm{Aut}(\Upsilon)}(M_U)$ is t-arc-regular on Υ (for the same value of t).*
- (c) *If Υ_{M_U} and Υ_{L_U} are s-arc-regular, then $\Upsilon_{M_U} \cong \Upsilon_{L_U}$ if and only if M_U and L_U are conjugate in $N_{\mathrm{Aut}(\Upsilon)}(U)$.*
- (d) *If Υ_{M_U} and Υ_{L_U} are s-arc-regular, then $\Upsilon_{M_U} \cong \Upsilon_{L_U}$ if and only if M is conjugate to a subgroup S of U in $N_{\mathrm{Aut}(\Upsilon)}(U)$ such that $S_U = L_U$; and moreover, in that case, $|U\!:\!S| = 2^{s-1} \cdot 3 \cdot m$, and S is semi-regular on $V(\Upsilon)$.*

Proof: For part (a), the 'only if' part is obvious, and conversely, if the images of the generators of U in U/M_U and U/L_U satisfy the same relations, then the epimorphisms from U to U/M_U and U/L_U have the same kernel, and therefore $L_U = M_U$.

For part (b), if Υ_{M_U} is t-arc-regular then $\mathrm{Aut}(\Upsilon_{M_U})$ is a quotient of some t-arc-regular subgroup V of $\mathrm{Aut}(\Upsilon)$, say V/K, with $K \le U \le V$, and then by Theorems 7 and 8 of [9] it follows that $K^\theta = M_U$ for some $\theta \in \mathrm{Aut}(\Upsilon)$, and hence $N_{\mathrm{Aut}(\Upsilon)}(M_U) = N_{\mathrm{Aut}(\Upsilon)}(K)^\theta = V^\theta$ is t-arc-regular as well. Conversely, if $V = N_{\mathrm{Aut}(\Upsilon)}(M_U)$ is t-arc-regular on Υ then so is $V/M_U \cong \mathrm{Aut}(\Upsilon_{M_U})$ on Υ_{M_U}, and hence the graph Υ_{M_U} is t-arc-regular.

Next, for part (c), if M_U and L_U are conjugate in $N_{\mathrm{Aut}(\Upsilon)}(U)$, then we find $\Upsilon_{M_U} \cong \Upsilon_{L_U}$ by Theorem 3.1. Conversely, suppose $\Upsilon_{M_U} \cong \Upsilon_{L_U}$. Then by Theorem 3.1 there exists $\theta \in \mathrm{Aut}(\Upsilon)$ such that ${M_U}^\theta = L_U$, and so $N_{\mathrm{Aut}(\Upsilon)}(M_U)^\theta = N_{\mathrm{Aut}(\Upsilon)}(L_U)$. Also since Υ_{M_U} and Υ_{L_U} are s-arc-regular, part (b) shows that both $N_{\mathrm{Aut}(\Upsilon)}(M_U)$ and $N_{\mathrm{Aut}(\Upsilon)}(L_U)$ are s-arc-regular, and hence both coincide with U. Thus, $U^\theta = N_{\mathrm{Aut}(\Upsilon)}(M_U)^\theta = N_{\mathrm{Aut}(\Upsilon)}(L_U) = U$, and so $\theta \in N_{\mathrm{Aut}(\Upsilon)}(U)$, which makes M_U conjugate to L_U in $N_{\mathrm{Aut}(\Upsilon)}(U)$.

Finally, for part (d), if M is conjugate to S in $N_{\mathrm{Aut}(\Upsilon)}(U)$ with $S_U = L_U$, then also M_U is conjugate to $S_U = L_U$, so $\Upsilon_{M_U} \cong \Upsilon_{L_U}$ by part (c). Conversely, if $\Upsilon_{M_U} \cong \Upsilon_{L_U}$ then $(M_U)^{\theta} = L_U$ for some $\theta \in N_{\mathrm{Aut}(\Upsilon)}(U)$, and then $(M^\theta)_U = (M_U)^\theta = L_U$, and we can take $S = M^\theta$, which is a subgroup of $U^\theta = U$. Moreover, when that happens, $S = M^\theta$ is semi-regular (since M is semi-regular), and $|U\!:\!S| = |U\!:\!M^\theta| = |U\!:\!M| = 2^{s-1}\cdot 3\cdot m$. □

Corollary 3.3 *Let U be an s-arc-regular subgroup of the infinite cubic tree Υ, and let M and L be semi-regular subgroups of index $2^{s-1}\cdot 3\cdot m$ in U such that M_U and L_U have at least three orbits on $V(\Upsilon)$. Let $N = N_{\mathrm{Aut}(\Upsilon)}(U)$, and let $\bar{N}$ and $\bar{U}$ be the permutation groups induced by N and U on the right coset spaces $(N:M)$ and $(U:M)$, respectively. Then the following hold:*

(a) *If $s = 3$ or 5, then Υ_{M_U} is s-arc-regular, and $\Upsilon_{M_U} \cong \Upsilon_{L_U}$ if and only if $M_U = L_U$;*

(b) *If $s = 2$ or 4, then N is $(s+1)$-arc-regular, and Υ_{M_U} is s-arc-regular or $(s+1)$-arc-regular, and furthermore, Υ_{M_U} is $(s+1)$-arc-regular if and only if $|\bar{N}| = 2|\bar{U}|$;*

(c) *Let $s = 1$. If $|\bar{N}| = 2|\bar{U}|$ then Υ_{M_U} is 2- or 3-arc-regular, while if $|\bar{N}| \neq 2|\bar{U}|$ then Υ_{M_U} is 1-, 4- or 5-arc-regular, and also if Υ_{M_U} is 4-arc-regular or 5-arc-regular then $\bar{U}$ has a semi-regular normal subgroup of index 42.*

Proof: First, because M_U and L_U have at least three orbits on $V(\Upsilon)$, the graphs Υ_{M_U} and Υ_{L_U} are finite and simple, and by Proposition 2.2, the quotient groups U/M_U and U/L_U are s-arc-regular subgroups of $\mathrm{Aut}(\Upsilon_{M_U})$ and $\mathrm{Aut}(\Upsilon_{L_U})$, respectively.

For part (a), suppose $s = 3$ or 5. Then we know from Proposition 2.1 that Υ_{M_U} is s-arc-regular, and by Proposition 2.3 that $N_{\mathrm{Aut}(\Upsilon)}(U) = U$. It now follows from part (c) of Theorem 3.2 that $\Upsilon_{M_U} \cong \Upsilon_{L_U}$ if and only if $M_U = L_U$ (since conjugation in U preserves its normal subgroups).

Next, for part (b), where $s = 2$ or 4, Proposition 2.1 tells us that Υ_{M_U} is s-arc-regular or $(s+1)$-arc-regular, and by Proposition 2.3, N is $(s+1)$-arc-regular, so that $|N\!:\!U| = 2$. To prove the last part of (b) and some of (c), consider the core M_N of M in N. Note that $M_N \subseteq M_U \subseteq U$ and $M_N \lhd U$, since $M_U \lhd U \subseteq N$.

If Υ_{M_U} is $(s+1)$-arc-regular, then $N_{\mathrm{Aut}(\Upsilon)}(M_U)$ is $(s+1)$-arc-regular by part (b) of Theorem 3.2. It follows that U has index 2 in $N_{\mathrm{Aut}(\Upsilon)}(M_U)$ and hence is normal in $N_{\mathrm{Aut}(\Upsilon)}(M_U)$, and so $N_{\mathrm{Aut}(\Upsilon)}(M_U) = N$ by Proposition 2.2. Thus M_U is normal in N, and hence is the core of M in N, and so $|\bar{N}| = |N/M_N| = |N/M_U| = 2|U/M_U| = 2|\bar{U}|$.

Conversely, if $|\bar{N}| = 2|\bar{U}|$, then $2|U/M_N| = |N/M_N| = |\bar{N}| = 2|\bar{U}| = 2|U/M_U|$, and it follows that $|U/M_N| = |U/M_U|$, and therefore $M_N = M_U$. In particular, $M_U \lhd N$ and then $N/M_U = N/M_N$, which is transitive on the $(s+1)$-arcs of $\Upsilon_{M_N} = \Upsilon_{M_U}$, and so Υ_{M_U} is $(s+1)$-arc-regular, as required.

This leaves us with part (c), where $s = 1$, and N is 2-arc-regular (by Proposition 2.3).

Suppose $|\bar{N}| = 2|\bar{U}|$. Then just as above, $M_U = M_N$ and so $M_U \lhd N$ and Υ_{M_U} is $(N/M_U, 2)$-arc-regular, and by Proposition 2.1 we find that Υ_{M_U} is 2- or 3-arc-regular.

Conversely, suppose that Υ_{M_U} is 2- or 3-arc-regular. Then by part (b) of Theorem 3.2, we know that $N_{\mathrm{Aut}(\Upsilon)}(M_U)$ is 2-arc-regular or 3-arc-regular. If it is 2-arc-regular, then it coincides with N by Proposition 2.3, and in that case $M_U \lhd N$. On the other hand, if it is 3-arc-regular, then U/M_U and $N_{\mathrm{Aut}(\Upsilon)}(M_U)/M_U$ are 1-arc-regular and 3-arc-regular on Υ_{M_U} respectively, and then by Proposition 2.1, there is a 2-arc-regular subgroup V/M_U of index 2 in $N_{\mathrm{Aut}(\Upsilon)}(M_U)/M_U$. Accordingly, V is 2-arc-regular and $|V : U| = 2$, so $U \lhd V$, and therefore $V = N$, by Proposition 2.3. But then also $N = V \subseteq N_{\mathrm{Aut}(\Upsilon)}(M_U)$, and again $M_U \lhd N$. Hence in both cases $M_U \lhd N$, which implies that $M_N = M_U$, and $|\bar{N}| = 2|\bar{U}|$ as in part (b).

In particular, if $|\bar{N}| \neq 2|\bar{U}|$, then Υ_{M_U} must be 1-, 4- or 5-arc-regular.

Finally, suppose Υ_{M_U} is 4- or 5-arc-regular. Then by [7, Propositions 3.2 and 3.4] we know that Υ_{M_U} is a cover of the Heawood graph of order 14, or the Biggs-Conway graph of order 2352, respectively. Also by Proposition 3.4 and the comments in §4.13 of [7], a 1-arc-regular subgroup of automorphisms of one of those two graphs lifts to the 1-arc-regular subgroup $\bar{U}$ of $\mathrm{Aut}(\Upsilon_{M_U})$, and a 4-arc-regular subgroup of type 4^1 lifts to a 4-arc-regular subgroup J of $\mathrm{Aut}(\Upsilon_{M_U})$. The index of $\bar{U}$ in J is 8, and the permutation group induced by J on the coset space $(J : \bar{U})$ is the same as the group induced by G_4^1 on the coset space $(G_4^1 : G_1)$, namely the group $\mathrm{PGL}(2, 7)$, of order 336. The kernel of the former action is therefore a normal subgroup of the form $\bar{K} = K/M_U$ where K is a semi-regular normal subgroup of index 336 in J, and index $336/8 = 42$ in $\bar{U}$. □

In order to apply part (c) of Corollary 3.3, it is helpful to understand the embedding of the group G_1 into the group G_4^1 in a little more detail. The following will be used in the next section.

Proposition 3.4 *The group G_1 contains two normal subgroups of index 42, each of which is free of rank 8. One of them is generated by the elements x_1 to x_8 and the other by the elements y_1 to y_8 defined below:*

$$
\begin{aligned}
x_1 &= (ha)^6, & y_1 &= (h^{-1}a)^6,\\
x_2 &= hah^{-1}ah^{-1}ahahah^{-1}a, & y_2 &= h^{-1}ahahah^{-1}ah^{-1}aha,\\
x_3 &= (h^{-1}a)^6, & y_3 &= (ha)^6,\\
x_4 &= h^{-1}ahah^{-1}ah^{-1}ahaha, & y_4 &= hah^{-1}ahahah^{-1}ah^{-1}a,\\
x_5 &= hah^{-1}ahah^{-1}ah^{-1}aha, & y_5 &= h^{-1}ahah^{-1}ahahah^{-1}a,\\
x_6 &= hahah^{-1}ahah^{-1}ah^{-1}a, & y_6 &= h^{-1}ah^{-1}ahah^{-1}ahaha,\\
x_7 &= h^{-1}ahahah^{-1}ahah^{-1}a, & y_7 &= hah^{-1}ah^{-1}ahah^{-1}aha,\\
x_8 &= h^{-1}ah^{-1}ahahah^{-1}aha, & y_8 &= hahah^{-1}ah^{-1}ahah^{-1}a.
\end{aligned}
$$

These two normal subgroups are interchanged by the automorphism of G_1 that swaps the generating pair (h, a) with the generating pair (h^{-1}, a), or more generally by conjugation by any element of $G_2^1 \backslash G_1$. Moreover, the first one is normal in the group G_4^1, with conjugation *by the additional generator p of G_4^1 inducing the permutation $(x_1, x_5)(x_2, x_6)(x_3, x_7)(x_4, x_8)$ on its generators, while the second one is not normal in G_4^1.*

Proof: Much of this can be found and/or verified using Magma. In particular, the `LowIndexNormalSubgroups` command gives the two normal subgroups of index 42, and then the `Rewrite` command shows that each of them is free of rank 8, and gives generating set for each one. But actually the first normal subgroup is the kernel of the action of G_4^1 on the coset space $(G_4^1 \!:\! G_1)$, with image $\mathrm{PGL}(2,7)$ of order 336 obtainable by adding the relation $(ha)^6 = 1$ to the standard presentation for G_4^1. Using the relations $h^3 = a^2 = 1$ it is easy to see that conjugation of its generators by each of h and a has the following effects:

$$\begin{array}{llll}
{x_1}^h = x_3^{-1}, & {x_2}^h = x_4^{-1}, & {x_3}^h = x_2x_7^{-1}, & {x_4}^h = x_6x_3^{-1}, \\
{x_5}^h = x_8^{-1}, & {x_6}^h = x_7^{-1}, & {x_7}^h = x_1x_4^{-1}, & {x_8}^h = x_5x_8^{-1}, \\
{x_1}^a = x_3^{-1}, & {x_2}^a = x_2^{-1}, & {x_3}^a = x_1^{-1}, & {x_4}^a = x_8^{-1}, \\
{x_5}^a = x_7^{-1}, & {x_6}^a = x_6^{-1}, & {x_7}^a = x_5^{-1}, & {x_8}^a = x_4^{-1}.
\end{array}$$

Similarly, using the relations from G_4^1 it is easy to show that conjugation by p induces the permutation $(x_1, x_5)(x_2, x_6)(x_3, x_7)(x_4, x_8)$. For example,

$$\begin{aligned}
{x_1}^p &= phahahahahahap = hqahahahahahpa = harhahahahapqha \\
&= hah^{-1}rahahahpraha = hah^{-1}aqhahapqrh^{-1}aha = hah^{-1}ahpqahqrah^{-1}aha \\
&= hah^{-1}ahaprhqrah^{-1}aha = aharhrah^{-1}aha = hah^{-1}ahah^{-1}ah^{-1}aha = x_5.
\end{aligned}$$

Next, each y_i is the same as the corresponding x_i but with every occurrence of h replaced by h^{-1} and vice versa, and so the second subgroup is the image of the first under the involutory automorphism of G_1 that takes (h, a) to (h^{-1}, a). This is the same as conjugation within the group G_2^1 by the additional generator of G_2^1 (also called p, but different from the p in G_4^1), and so the second subgroup is also the conjugate of the first by any element of $G_2^1 \setminus G_1$. Finally, the second subgroup is not normal in G_4^1, for otherwise it would contain all the conjugates of $x_1 = y_3$ and hence be the same as the first one. □

4 Computational strategy

In this section, we outline a strategy for determining all connected cubic core-free symmetric m-Cayley graphs. This can be done type-by-type, or in other words, by finding all such graphs with a particular Djoković-Miller type. All the steps involved can be implemented with the help of Magma [2].

First let $U = G_1$, G_2^1, G_2^2, G_3, G_4^1, G_4^2 or G_5, and take the presentation of U as given in Section 2, with H being the subgroup generated by all generators of U other than a. Also for $U = G_1$ let $N = G_2^1$, and for G_2^1 and G_2^2 let $N = G_3$, and for G_4^1 and G_4^2 let $N = G_5$. In each of these cases, we may view U as a subgroup of index 2 in N, and use the command `Rewrite(N,U)` in Magma to simplify the presentation of U.

For a subgroup M of finite index in U, let $\bar{U}$ and $\bar{N}$ be the permutation groups induced by U and N on $(U\!:\!M)$ and $(N\!:\!M)$ respectively, by right multiplication, so that $\bar{U} \cong U/M_U$ and $\bar{N} \cong N/M_N$. Also recall that a finite connected cubic core-free s-arc-regular m-Cayley graph Γ is isomorphic to Υ_{M_U} for some U with the same Djoković-Miller type as Aut(Γ), and some subgroup M of U of index $2^{s-1} \cdot 3 \cdot m$ acting semi-regularly on $V(\Upsilon)$, with M_U having at least three orbits on $V(\Upsilon)$.

To classify finite connected cubic core-free s-arc-regular m-Cayley graphs, we need to find a set T of subgroups of U such that every connected cubic core-free s-arc-regular m-Cayley graph with the same type as U is isomorphic to Υ_{M_U} for some $M \in T$, and $\Upsilon_{L_U} \not\cong \Upsilon_{M_U}$ for any two distinct members L and M of T.

Step 1: *Construct a set T_1 of subgroups M of the s-arc-regular subgroup U such that*
(a) $|U\!:\!M| = 2^{s-1} \cdot 3 \cdot m$, *and no other member of T_1 is conjugate to M in U;*
(b) *M acts semi-regularly on $V(\Upsilon)$; and*
(c) *M_U has at least three orbits on $V(\Upsilon)$.*

These subgroups can be found using the `LowIndexSubgroups` command in Magma, to find all subgroups of index $2^{s-1} \cdot 3 \cdot m$ in U.

Condition (a) requires no checking because this command returns a single representative from each conjugacy class of subgroups (with index up to a given number). There is no direct method to check condition (b) on the semi-regularity of M using Magma, because M is infinite, but instead this can done by using the `IsSemiregular` command to check the semi-regularity of the vertex-stabiliser H (in U) on the right coset space $(U\!:\!M)$; see Proposition 2.5. Finally, for condition (c), note that if M_U has only one or two orbits then $m \leq 2$, and if $m = 2$ then M_U has two orbits and hence M is normal in U, while if $m = 1$ then M_U has one or two orbits and hence $|M\!:\!M_U| = 1$ or 2; and these possibilities can be checked easily using Magma.

Step 2: *Construct the subset T_2 of T_1 consisting of all M for which Υ_{M_U} is s-arc-regular.*

This is equivalent to removing all $M \in T_1$ for which Υ_{M_U} is not s-arc-regular, and the way to do it depends on the value of $s \in \{1, 2, 3, 4, 5\}$, using Corollary 3.3.

If $s = 3$ or 5, then Υ_{M_U} is s-arc-regular and so we may take $T_2 = T_1$.

If $s = 2$ or 4, then Υ_{M_U} is either s-arc-regular or $(s+1)$-arc-regular, and the latter occurs if and only if $|\bar{N}| = 2|\bar{U}|$, where $\bar{U}$ and $\bar{N}$ are the permutation groups induced by U and N on $(U : M)$ and $(N : M)$ respectively. Hence we can consider the $(s+1)$-arc-regular subgroup $N = G_{s+1}$ of Aut(Υ) and take U as a subgroup of index 2 in N, and then delete all those M from T_1 for which $|\bar{N}| = 2|\bar{U}|$. The latter can be done easily by using Magma's `CosetImage` command to get the induced permutation groups, and then comparing their orders.

If $s = 1$, then we may take U as a subgroup of index 2 in $N = G_2^1$, and then delete all those M from T_1 for which Υ_{M_U} is 2-arc-regular or 3-arc-regular, again by checking the permutation groups $\bar{U}$ and $\bar{N}$ induced by U and N on $(U\!:\!M)$ and $(N\!:\!M)$, respectively.

In the case $s = 1$, we also need to check whether or not the graph Υ_{M_U} is 4- or 5-arc-regular. In principle this can be done in a similar way to the check for 2- or 3-

arc-regularity, but the computation is messy (since it involves taking U separately as a non-normal subgroup of index 8 in G_4^1, rather than as a normal subgroup of index 2 in $N = G_2^1$ as earlier), and this takes a lot of additional time and memory. The process becomes much easier by using Proposition 3.4.

If $s = 1$ and Υ_{M_U} is 4- or 5-arc-regular, then we know from Corollary 3.3 that $\bar{U}$ has a semi-regular normal subgroup of index 42, being the image of one of the two normal subgroups of index 42 in $U = G_1$. Now if the image of neither one of those two subgroups is normal of index 42 in $\bar{U}$, then no further consideration is needed. On the other hand, if the image $\bar{K}$ of one of them is, then K could be either one of the two normal subgroups of index 42 in G_1 given in Proposition 3.4. If K is the first one, then $\bar{K}$ is generated by the images $\bar{x}_i$ of the elements x_i given in Proposition 3.4, and all we need to do is check whether there exists an automorphism of $\bar{K}$ that induces the permutation $(\bar{x}_1, \bar{x}_5)(\bar{x}_2, \bar{x}_6)(\bar{x}_3, \bar{x}_7)(\bar{x}_4, \bar{x}_8)$ on these — for example by using the `IsHomomorphism` command in Magma. On the other hand, if $\bar{K}$ is the second one, then we can replace $\bar{h}$ by $\bar{h}^{-1}$ when defining the $\bar{x}_i$ and then repeat the same process.

Step 3: *Construct a subset T_3 of T_2 with the property that for every $M \in T_2$ there is exactly one $L \in T_3$ for which $L_U = M_U$.*

Another way of expressing this is to say that T_3 is a transversal for the equivalence classes of members of T_2 under the relation $\sim$ given by $L \sim M$ if and only if $L_U = M_U$. We can construct T_3 by initialising it as the empty set, then running through the members of T_2 and adding one to T_3 whenever its core in U is 'new'. Note that M_U is the kernel of the natural action of U on the coset space $(U\!:\!M)$, and so two members of T_2 are equivalent if and only if the corresponding kernels are the same. This can be determined with the help of Magma in a number of ways, such as by using the `IsHomomorphism` command to check for an isomorphism from U/L_U to U/M_U that takes the image of the canonical generating set for U in U/L_U to the corresponding image in U/M_U (as required by Theorem 3.2(a)).

Step 4: *Construct a subset T of T_3 with the property that for every $M \in T_3$ there is exactly one $L \in T$ for which $\Upsilon_{L_U} \cong \Upsilon_{M_U}$.*

This can be done in a similar way to Step 3, using Corollary 3.3 as necessary. Note that by part (d) of Theorem 3.2, we know that $\Upsilon_{M_U} \cong \Upsilon_{L_U}$ if and only if M is conjugate to a subgroup S of $N = N_{\mathrm{Aut}(\Upsilon)}(U)$ with $S_U = L_U$. This can be checked using the `IsConjugate` command in Magma, for the subgroup N given by Corollary 3.3.

In particular, if $s = 3$ or 5 then $N = U$ and by part (a) of Corollary 3.3 the check is not needed, and we may take $T = T_3$. On the other hand, if $s = 1$, we may take N as a copy of G_2^1 or G_3, and if $s = 2$ or 4, we may take N as a copy of G_3 or G_5 respectively, and the check is necessary in all three of these cases. □

Once Step 4 is completed, we have the set T, and from this we may construct all connected cubic core-free s-arc-regular m-Cayley graphs associated with the group U as the set of graphs $\mathrm{Cos}(\bar{U}, \bar{H}, \bar{H}\bar{a}\bar{H})$, one for each $M \in T$. These graphs will have the same type as U, and the number of them is $|T|$. (Also if $U = G_3$ or G_5, then $T_2 = T_1$ and

$T = T_3$, and so we need only perform Steps 1 and 3 in those two cases.)

5 Degenerate quotients

Let U be any s-arc-regular subgroup of the infinite cubic tree Υ (with $1 \le s \le 5$), and let M be a semi-regular subgroup of index $2^{s-1} \cdot 3 \cdot m$ in U (for some positive integer m). If the core M_U of M in U has at least three orbits on $V(\Upsilon)$, then by Proposition 2.2 we know that the quotient graph Υ_{M_U} is a finite symmetric cubic graph on which U/M_U acts s-arc-transitively.

In this section, we consider the two special cases when M_U has just one or two orbits on $V(\Upsilon)$. In such cases we will call the corresponding quotient graphs *degenerate quotients*.

We may view U as a permutation group on the arc set $A(\Upsilon)$ of Υ, because the induced action of U on $A(\Upsilon)$ is faithful, and then in the cases where M_U has just one or two orbits on $A(\Upsilon)$, the quotient graph Υ_{M_U} is respectively one of the following graphs: either F_3, a graph with one vertex and three semi-edges, or D_3, a graph with two vertices and three edges joining them. These two graphs are illustrated in Figure 1.

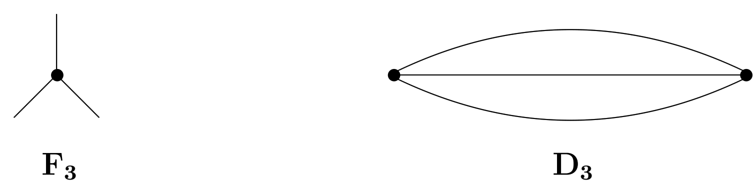

Figure 1: The two degenerate possibilities for the quotient graph Υ_{M_U}

Note that F_3 and D_3 are not simple graphs, but they can still be viewed as more general graphs, which may contain loops, multiple edges, or semi-edges.

In such a general graph X, we can still define arcs, with an arc associated with a semi-edge having the same start- and end-vertex, and then define an *s-arc* in X as a sequence $(a_1, ..., a_s)$ of s arcs of X such that the arcs a_{i-1} and a_i cannot be associated with a single edge, and the end-vertex of the arc a_{i-1} is the start-vertex of the arc a_i, for $1 < i \le s$. (This is consistent with the definition of 's-arc' given for a simple graph in Section 1.) An automorphism of X may then be viewed as a permutation on the arc-set of X that preserves incidence of arcs in X and induces an action on the vertex-set $V(X)$.

Next, for any positive integer k we may define F_k as the general graph consisting of a single vertex and k semi-edges, and D_k as the general graph with two vertices and k edges joining them. Then the graph F_k has k arcs and D_k has $2k$ arcs, and $\mathrm{Aut}(\mathrm{F}_k) \cong S_k$ while $\mathrm{Aut}(\mathrm{D}_k) \cong S_k \times \mathbb{Z}_2$, and both of these graphs are 1- and 2-arc-transitive, but neither of them is 3-arc-transitive when $k \ge 3$, because 3-arcs of the form (a_1, a_2, a_1) lie in a different orbit from any 3-arc of the form (a_1, a_2, a_3) with $a_1 \ne a_3$.

Theorem 5.1 *Let Γ be a connected (G, s)-arc-transitive graph of valency at least 3, for some $s \geq 1$. If N is a normal subgroup of G that acts semi-regularly on $V(\Gamma)$ with one or two orbits, then $s = 1$ or 2.*

Proof: First note that Γ must be a simple graph, with no loops, multiple edges or semi-edges. Let $k \geq 3$ be the valency of Γ, and suppose that the given semi-regular normal subgroup of G has exactly n orbits on the arc-set $A(\Gamma)$, say $O_1, O_2, \cdots, O_n$.

No two arcs with the same start-vertex v can lie in the same orbit O_i, because otherwise some non-trivial element of N moves the first to the second but then fixes v, contradicting semi-regularity of N. Hence $n = k$ when N has a single orbit on $V(\Gamma)$, or $n = 2k$ when N has two orbits on $V(\Gamma)$, and it follows that the quotient graph Γ_N is isomorphic to F_k or D_k, respectively.

Next, G permutes the orbits O_i among themselves, since N is normal in G (with $O_i^g = v^{Ng} = v^{gN} = O_j$ whenever $v \in O_i$ and $v^g \in O_j$). Let K be the kernel of this action of G, so that $N \subseteq K$, and consider the action of K on $V(\Gamma)$.

If v is any vertex in Γ, then since each of the k neighbours of v lies in a different N-orbit, and K preserves all those orbits, the stabiliser K_v fixes every neighbour of v. An easy induction then proves that K_v fixes every vertex at distance t from v, for all $t > 1$, and hence K_v fixes every vertex of Γ. Thus K acts semi-regularly on $V(\Gamma)$.

Also because $N \subseteq K$, we know that K is transitive on each O_i, and then since N is semi-regular on $V(\Gamma)$, it follows that $K = N$ (for otherwise a vertex-stabiliser in K would be non-trivial). In turn, this implies that G/N $(= G/K)$ is isomorphic to a subgroup of $\mathrm{Aut}(\Gamma_N)$, and therefore Γ_N is $(G/N, s)$-arc-transitive. On the other hand, we know that Γ_N is isomorphic to F_k or D_k, and hence is not 3-arc-transitive, and so we conclude that $s = 1$ or 2, as required. □

We can now apply this to the question of Li from [18].

Let Γ be a symmetric Cayley graph of valency $k \geq 3$ over some finite group G. Then Γ is said to be *bi-normal* if the largest normal subgroup of $\mathrm{Aut}(\Gamma)$ contained in G has two orbits on $V(\Gamma)$. In [17, Proposition 2.3] it was shown that there is no normal symmetric Cayley graph that is 3-arc-transitive, and this now follows also from Theorem 5.1 above. But in [18, Question 1.2 (a)], also Li posed the following question: Do there exist any 3-transitive bi-normal Cayley graphs?

Corollary 5.2 *There is no 3-transitive bi-normal finite Cayley graph.*

Proof: This was proved in [8, Theorem 1.1], but it follows easily from Theorem 5.1, since if G is an s-arc-transitive group of automorphisms of the bi-normal finite Cayley graph Γ, then the largest normal subgroup N of $\mathrm{Aut}(\Gamma)$ contained in G that acts semi-regularly on $V(\Gamma)$ has two orbits on $V(\Gamma)$, and so $s = 1$ or 2 by Theorem 5.1. □

Corollary 5.2 can be also be verified directly in the case of valency 3, because it can be shown with the help of Magma that the groups G_3, G_4^1, G_4^2 and G_5 have no semi-regular normal subgroup of index 12, 24, 48 or 96, respectively.

Similarly, G_2^2 has no semi-regular normal subgroup of index 6 or 12. On the other hand, there are many known 1- or 2-arc-regular normal or bi-normal Cayley graphs; see [1, 15, 25] for example. These come from semi-regular normal subgroups in the groups G_1 and G_2^1. Indeed computations with MAGMA show that G_1 has one semi-regular normal subgroup of index 3 (namely $\langle a, h^{-1}ah, hah^{-1}\rangle$) and two semi-regular normal subgroups of index 6 (namely $\langle (ha)^2, (h^{-1}a)^2\rangle$ and $\langle [h, a], [h^{-1}, a]\rangle$), and G_2^1 has two semi-regular normal subgroups of index 6 (namely $\langle a, h^{-1}ah, hah^{-1}\rangle$ and $\langle ap, h^{-1}aph, haph^{-1}\rangle$) and two semi-regular normal subgroups of index 12 (namely the two of index 6 in G_1).

For the reminder of this paper, recall some of the beginning of this section, where U was an s-arc-regular subgroup of the cubic tree Υ, and M was a semi-regular subgroup of index $2^{s-1} \cdot 3 \cdot m$ in U (for some positive integer m). By Theorem 5.1 we may further assume that if M_U has just one or two orbits on $V(\Upsilon)$, then $s = 1$ or 2. Equivalently, if $3 \le s \le 5$, then M_U has at least three orbits on $V(\Upsilon)$.

6 Cubic core-free symmetric Cayley graphs

The main aim of this section is to give a new proof of the following theorem, which originated from the combined work of Xu, Fang, Wang & Xu [26, 27], Conder [3] and Li & Lu [19]. Our proof uses the computational approach described in Section 3.

Theorem 6.1 *Let N_s be the number of connected cubic core-free s-arc-regular Cayley graphs over a finite group, and let N_s^* be the number of such graphs where that group G is a non-abelian finite simple group. Then N_s and N_s^* are given in Table 2, along with some information about the Cayley graph(s) and/or group(s), for $1 \le s \le 5$. The Cayley graphs in the rows for $s = 2$ and $s = 4$ have types 2^1 and 4^1 respectively.*

s	N_s	Graph orders	N_s^*
1	0	None	0
2	2	4 or 8	0
3	3	6, 96 or 110	0
4	4	14, 506, 30618 or 23!	0
5	6	2352, 3072432, $2^4 \cdot 3^{15} \cdot 7^2$, (23)! $\cdot$ (24!)/2 or 47!/2	2 ($G = A_{47}$)

Table 2: Connected cubic core-free s-arc-regular Cayley graphs

Proof: We consider the possibilities for the Djoković-Miller type, in descending order of the value of s.

(a) Let $s = 5$, and $U = G_5$. In Step 1 of our computational approach, we find that G_5 has 6 conjugacy classes of semi-regular subgroups M of index 48 (such that M_U has at least three orbits on $V(\Upsilon)$), with representatives $M_1 = \langle a, h^{-1}ah, hah^{-1}\rangle$, $M_2 = \langle a, pqhah\rangle$, $M_3 = \langle a, hah\rangle$, $M_4 = \langle a, hah^{-1}, prh^{-1}ah\rangle$, $M_5 = \langle a, phah\rangle$ and $M_6 = \langle a, rhah\rangle$. The permutation representations on cosets give rise to six connected cubic core-free 5-arc-regular Cayley graphs, of orders 2352, 3072432, $2^4 \cdot 3^{15} \cdot 7^2$, $(23!) \cdot (24!)/2$, $47!/2$ and $47!/2$, respectively, at Step 3. Moreover, the only cases where M/M_U is a non-abelian simple group are the last two, for which $M/M_U \cong A_{47}$.

(b) Let $s = 4$, and $U = G_4^1$. Here we may view U as a subgroup of $N = G_5$. In Step 1 we have $|T_1| = 8$, meaning that G_4^1 has eight conjugacy classes of semi-regular subgroups M of index 24 (such that M_U has at least three orbits on $V(\Upsilon)$). These occur in pairs (that are conjugate within G_5). At Steps 2 and 3, we have $T_1 = T_2 = T_3$, but in Step 4, the number of possibilities is reduced to $|T| = 4$, with representatives $M_1 = \langle a, h^{-1}ah, hah^{-1}\rangle$, $M_2 = \langle a, phah\rangle$, $M_3 = \langle a, hah\rangle$ and $M_4 = \langle a, hah^{-1}, qh^{-1}ah\rangle$. These give rise to four connected cubic core-free 4-arc-regular Cayley graphs, of orders 14, 506, 30618 and 23!, and the quotient M/M_U is not simple in any of these four cases.

(c) Let $s = 4$, and $U = G_4^2$. Again we may view U as a subgroup of $N = G_5$, but here at Step 1, there are no semi-regular subgroups M of index 24 at all, so $T_1 = \emptyset$, and there is no connected cubic core-free 4-arc-regular Cayley graph admitting an action of type 4^2.

(d) Let $s = 3$, and $U = G_3$. At Step 1, we find that U has five conjugacy classes of semi-regular subgroups M of index 12 (such that M_U has at least three orbits on $V(\Upsilon)$), and at Step 3, the number of possibilities is reduced to $|T_3| = 3$, with representatives $M_1 = \langle a, h^{-1}ah, hah^{-1}\rangle$, $M_2 = \langle a, pqh^{-1}ah\rangle$ and $M_3 = \langle a, phah\rangle$. These give rise to three connected cubic core-free 3-arc-regular Cayley graphs, of orders 6, 96 and 110, and the quotient M/M_U is not simple in any of these three cases.

(e) Let $s = 2$, and $U = G_2^1$. Here we may view U as a subgroup of $N = G_3$. At Step 1, we find that U has four conjugacy classes of semi-regular subgroups M of index 6 such that M_U has at least three orbits on $V(\Upsilon)$, and another four where M_U has only one or two orbits on $V(\Upsilon)$. Thus $|T_1| = 4$. Then at Steps 2 and 3, we find that $T_1 = T_2 = T_3$, but at Step 4, the number of possibilities is halved (thanks to conjugacy within $N = G_3$), and we have $|T|$ =2, with representatives $M_1 = \langle a, phah^{-1}\rangle$ and $M_2 = \langle a, hah^{-1}, phah\rangle$. These give rise to two connected cubic core-free 2-arc-regular Cayley graphs, namely the complete graph K_4 and the 3-cube Q_3, of orders 4 and 8, and the quotient M/M_U is not simple in either of these two cases.

(f) Let $s = 2$, and $U = G_2^2$. Again we may view U as a subgroup of $N = G_3$, but here at Step 1, there are no semi-regular subgroups M of index 6 at all, so $T_1 = \emptyset$, and there is no connected cubic core-free 2-arc-regular Cayley graph admitting an action of type 2^2.

(g) Let $s = 1$, and $U = G_1$. Here we may view U as a subgroup of $N = G_2^1$, but at Step 1, we find that U has no semi-regular subgroups M of index 3, so $T_1 = \emptyset$, and there are no connected cubic core-free 1-arc-regular Cayley graphs at all. □

Remark: In the cases where we do find a cubic core-free symmetric Cayley graph, the graph is isomorphic to $\mathrm{Cos}(\bar{U}, \bar{H}, \bar{H}\bar{a}\bar{H})$, where $\bar{U}$, $\bar{H}$ and $\bar{a}$ denote the images of U, H and a in the natural action of U on the right coset space $(U\!:\!M)$, and is also isomorphic to the Cayley graph $\mathrm{Cay}(\bar{M}, \bar{M} \cap \bar{H}\bar{a}\bar{H})$.

For example, the connected cubic core-free 5-arc-regular Cayley graph of order 2352 is the Biggs-Conway graph, and is the Cayley graph $\mathrm{Cay}(G, S)$ where $G = M_1/(M_1)_{G_5}$ and S is the set of three conjugate involutions $\bar{a}$, $\bar{h}^{-1}\bar{a}\bar{h}$ and $\bar{h}\bar{a}\bar{h}^{-1}$ in G, while the two such graphs of order $47!/2$ are Cayley graphs over A_{47} with generating set S made up of the involution $\bar{a}$ and a non-involution $\bar{b}$ and its inverse, where $b = phah$ or $rhah$ respectively.

7 Cubic core-free symmetric bi-Cayley graphs

In this section, we classify finite connected cubic core-free symmetric bi-Cayley graphs.

Theorem 7.1 *Let N_s be the number of connected cubic core-free s-arc-regular bi-Cayley graphs Γ over a finite group, and let N_s^* be the number of such graphs where that group G is a non-abelian finite simple group. Then N_s and N_s^* are given in Table 3, along with some information about the bi-Cayley graph(s) and/or group(s), for $1 \le s \le 5$. The bi-Cayley graphs in the row for $s = 2$ all have type 2^1, while all of those in the row for $s = 4$ have type 4^1, except for one, which has type 4^2 and automorphism group A_{48}.*

s	N_s	Graph orders	N_s^*
1	0	None	0
2	5	4, 8, 20, 32 or 220	0
3	15	6, 10, 20, 40, 96, 110, 220, 506, 1152, 1536, 145200, 19906560 or 23! (3 graphs)	3 ($G = A_{23}$)
4	30	14, 506, 2162, 3584, 30618, $2^{17} \cdot 3^7 \cdot 7$, $2^{24} \cdot 11 \cdot 23$, $2^{17} \cdot 3^8 \cdot 5^8 \cdot 7$, 23!, $2^{23} \cdot 23!$ (3 graphs) or 47! (18 graphs)	19 ($G = A_{23}$ or A_{47})
5	59	2352, $2^4 \cdot 3 \cdot 11^2 \cdot 23^2$, $2^5 \cdot 3 \cdot 23^2 \cdot 47^2$, $2^{20} \cdot 3 \cdot 7^2$, $2^4 \cdot 3^{15} \cdot 7^2$, $2^{36} \cdot 3^{15} \cdot 7^2$, $2^{50} \cdot 3 \cdot 11^2 \cdot 23^2$, $2^{36} \cdot 3^{17} \cdot 5^{16} \cdot 7^2$, $12 \cdot (23!)^2$, 47! (6 graphs), $2^{45} \cdot 23! \cdot 24!$ (3 graphs), $24 \cdot (47!)^2$ (21 graphs) or 95! (20 graphs)	26 ($G = A_{47}$ or A_{95})

Table 3: Connected cubic core-free s-arc-regular bi-Cayley graphs

Proof: Again we consider the possibilities for the Djoković-Miller type, in descending order of the value of s.

(a) Let $s = 5$, and $U = G_5$. At Step 1, we find that U has 65 conjugacy classes of semi-regular subgroups M of index 96 (such that M_U has at least three orbits on $V(\Upsilon)$), and then Step 3 reduces this number to 59. These give rise to 59 connected cubic core-free 5-arc-regular bi-Cayley graphs, of orders 2352, $2^4 \cdot 3 \cdot 11^2 \cdot 23^2$, $2^5 \cdot 3 \cdot 23^2 \cdot 47^2$, $2^{20} \cdot 3 \cdot 7^2$, $2^4 \cdot 3^{15} \cdot 7^2$, $2^{36} \cdot 3^{15} \cdot 7^2$, $2^{50} \cdot 3 \cdot 11^2 \cdot 23^2$, $2^{36} \cdot 3^{17} \cdot 5^{16} \cdot 7^2$, $12 \cdot (23!)^2$, 47! (6 graphs), $2^{45} \cdot 23! \cdot 24!$ (3 graphs), $24 \cdot (47!)^2$ (21 graphs) and 95! (20 graphs). Also the computation shows that the bi-Cayley group $G \cong M/M_U$ is a non-abelian simple group only in 26 of these 59 cases: 6 with $G \cong A_{47}$, and 20 with $G \cong A_{95}$.

(b) Let $s = 4$, and $U = G_4^1$. Here we view U as a subgroup of $N = G_5$, when considering subgroups of index 48 in U. At Step 1 we have $|T_1| = 72$, and then Steps 2 and 3 give $|T_2| = 70$ and $|T_3| = 58$, which reduces to $|T| = 29$ at Step 4. From this we find there are

29 connected cubic core-free 4-arc-regular bi-Cayley graphs, of orders 14, 506, 2162, 3584, 30618, $2^{17} \cdot 3^7 \cdot 7$, $2^{17} \cdot 3^8 \cdot 5^8 \cdot 7$, $2^{24} \cdot 11 \cdot 23$, 23!, $2^{23} \cdot (23!)$ (3 graphs) or 47! (17 graphs) admitting action of type 4^1. Also M/M_U is a non-abelian simple group in only the 9th and last cases, where $M/M_U \cong A_{23}$ for one graph, and $M/M_U \cong A_{47}$ for another 17.

(c) Let $s = 4$, and $U = G_4^2$. Again here we may view U as a subgroup of $N = G_5$. At Step 1 we have $|T_1| = 8$, and at Steps 2 and 3 we find $|T_2| = |T_3| = 2$. This halves to $|T| = 1$ at Step 4, and we obtain just one connected cubic core-free 4-arc-regular bi-Cayley graph admitting action of type 4^2, with order 47!, and $M/M_U \cong A_{47}$. This adds just 1 to the contribution to each of the numbers N_4 and N_4^* from case (b).

(d) Let $s = 3$, and $U = G_3$. Here we consider subgroups of index 24 in U. At Step 1 we have $|T_1| = 24$, and then Step 3 reduces this number to 15. The members of the final set $T = T_3$ give rise to 15 connected cubic core-free 3-arc-regular bi-Cayley graphs, consisting of one graph of each of the orders 6, 10, 20, 40, 96, 110, 220, 506, 1152, 1536, 145200 and 19906560, plus three graphs of order 23!. Moreover, M/M_U is a non-abelian simple group only in this last case, where $M/M_U \cong A_{23}$ (for three non-isomorphic graphs).

(e) Let $s = 2$, and $U = G_2^1$. Here we may view U as a subgroup of $N = G_3$, when considering subgroups of index 12 in U. At Step 1 we have $|T_1| = 27$, and by Steps 2 and 3 we find $|T_2| = 22$ and $|T_3| = 10$. By Step 4 this reduces to $|T| = 5$, and the members of T give rise to five connected cubic core-free 2-arc-regular bi-Cayley graphs of orders 4, 8, 20, 32 and 220, all admitting an action of type 2^1. In all of these five cases, M/M_U is not a non-abelian simple group.

(f) Let $s = 2$, and $U = G_2^2$. Again here we may view U as a subgroup of $N = G_3$. At Step 1 we have $|T_1| = 3$, but at Step 2 we find $T_2 = \emptyset$, and hence there is no connected cubic core-free 2-arc-regular bi-Cayley graph admitting action of type 2^2.

(g) Let $s = 1$, and $U = G_1$. Again here we may view U as a subgroup of $N = G_2^1$, but consider subgroups of index 6 in U. At Step 1 we have $|T_1| = 5$ (with two possibilities discarded because they are normal in U), but at Step 2 we find $T_2 = \emptyset$, without even needing to check if the Heawood graph or the Biggs-Conway graph is involved. In particular, this shows there are no connected cubic core-free 1-arc-regular bi-Cayley graphs at all. □

8 Cubic core-free 1-arc-regular m-Cayley graphs

In this final section, we describe the classification of the connected cubic core-free 1-arc-regular m-Cayley graphs for some small values of m, namely $m = 3$ to 7.

For $m = 3$ and 4, we give the following two theorems:

Theorem 8.1 *There is a unique connected cubic core-free* 1*-arc-regular* 3*-Cayley graph over a finite group, and the graph is* F144A *of order 144 from* [5, Table 1]*, with automorphism group of order* 432 *and* 3*-Cayley group* $G = Q_8 \rtimes D_3$ *of order* 48.

Proof: We have $s = 1$ and we take $U = G_1$, but we view this as a subgroup of $N = G_2^1$. At Step 1 we have 12 conjugacy classes of semi-regular subgroups M of index 9 in U, and for all of these, M_U has at least three orbits on $V(\Upsilon)$. Then at Steps 2 and 3 we find $|T_2| = |T_3| = 2$ (without removing any member of T_1 that would give rise to a cover of the Heawood graph or the Biggs-Conway graph), at Step 4 we find $|T| = 1$. The single remaining member M of T gives rise to a connected cubic core-free 1-arc-regular 3-Cayley graph of order 144, and by [5] we know this is the graph F144A. Further computation shows that the group $G = M/M_U$ is isomorphic to a semi-direct product $Q_8 \rtimes D_3$. □

Theorem 8.2 *A connected cubic core-free* 1*-arc-regular* 4*-Cayley graph is isomorphic to one of the following*:

(a) *the graph* F144A *in Theorem* 8.1;

(b) *the graph* C6912.4 *from* [4], *with automorphism group* $(\mathbb{Z}_2^6 \rtimes \mathbb{Z}_3^3) \rtimes A_4$ *of order* 20736 *and* 4*-Cayley group* $G = \mathbb{Z}_2^6 \rtimes \mathbb{Z}_3^3$ *of order* 1728;

(c) *a cubic* 4*-Cayley graph over* $G = A_{11}$ *of order* 2.11!, *with automorphism group* A_{12};

(d) *three non-isomorphic cubic* 4*-Cayley graphs over* $G = S_{11}$ *of order* 4.11!, *with automorphism group* S_{12}.

Proof: Again we have $s = 1$ and take $U = G_1$, and view this as a subgroup of $N = G_2^1$. At Step 1, we have 64 conjugacy classes of semi-regular subgroups M of index 12 in U such that M_U has at least three orbits on $V(\Upsilon)$. Then at Steps 2 and 3 we find $|T_2| = |T_3| = 12$ (without removing any member of T_1 that would give rise to a cover of the Heawood graph or the Biggs-Conway graph), and at Step 4 we have $|T| = 6$. The six resulting connected cubic core-free 1-arc-regular 4-Cayley graphs have orders 144, 6912, $2 \cdot 11!$ and $4 \cdot 11!$ (three graphs), and have the other properties stated. □

Finally we state the following additional theorem without giving similar details. It can be obtained by the analogous approach, via conjugacy classes of subgroups of index 15, 18 and 21 in $U = G_1$.

Theorem 8.3 *There are* 81, 462 *and* 3267 *connected cubic core-free* 1*-arc-regular* m*-Cayley graphs for* $m = 5$, 6 *and* 7, *respectively.*

In the first two cases (where $m = 5$ or 6), at Step 2 there is no $M \in T_1$ for which the image $\bar{K}$ of one of the two normal subgroups of index 42 in $U = G_1$ has index 42 in $\bar{U} = U/M_U$. On the other hand, in the third case (where $m = 7$) there are 20 such M, and for exactly six of those, the image $\bar{K}$ admits an automorphism that makes the graph Γ_{M_U} a cover of the Heawood graph.

Also this theorem can be extended to larger values of m, but of course the numbers and the computation times involved increase quite rapidly.

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