Accepted for publication in Computers and Chemical Engineering, 6 October 2018

Logarithmic mean: Chen's approximation or explicit solution?

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Keywords: Logarithmic mean, approximation, HEN, Lambert W-function

Abstract

An explicit solution has been obtained for the logarithmic mean temperature difference method of heat exchanger calculation by making use of the Lambert W-function. The results might be of use where an explicit solution involving the logarithmic mean is required.

Chen's Approximation

An approximation to the logarithmic mean (LMTD) between two numbers θ_1 and θ_2 attributed to Chen (1987) is given in Eq. (1).

$$LMTD = \frac{\theta_1 - \theta_2}{\ln\left(\frac{\theta_1}{\theta_2}\right)} \approx \left\{\theta_1 . \theta_2 . \left(\frac{\theta_1 + \theta_2}{2}\right)\right\}^{\frac{1}{3}} \qquad \text{Eq. 1}$$

Eq. (1) has been widely used in optimization models for heat exchanger network and other engineering equipment (see, for example, Yee et al. 1990, Yee & Grossman, 1990, Amarger et al, 1992, Lewin, 1998, Adjiman et al, 2000, Jackson & Grossmann, 2001, Davis & Sandall, 2003, Ponce-Ortega et al, 2008, Gabriel, et al., 2016, Bongartz & Mitsos, 2017, Pavao et al., 2017). The Chen approximation slightly overestimates the area requirement, but it avoids numerical problems associated with the logarithmic term, and it also has the important advantage that when either θ_1 or θ_2 equals zero the driving force will be approximated to be zero (Yee et al. 1990, Yee & Grossmann, 1990, Lewin, 1998, Floudas et al., 1999, Adjiman et al., 2000, Davis & Sandall, 2003).

Lambert W-function

The earliest mention of the problem stated in Eq. 2 is attributed to Euler (1779), but Euler himself credited Lambert (1758) with it (see also Corless et al. 1996, Hayes 2005). Corless et al. (1996) gave a detailed analysis of the Lambert W-function while Barry et al. (2000) gave analytical approximations for it. Hayes (2005) and Stewart (2005) gave simplified accounts of the properties of the Lambert W-function. Others, including Keady (1998) who applied the Lambert W-function to the Colebrook-White equation, Valluri et al. (2000) who discussed its possible applications in physics, and Golicnik (2012) applied the function to enzyme-catalysed biochemical reactions, while Disney & Warburton (2012) applied it to certain economic problems. Hayes (2005) suggests that scientific calculators should have a built in Lambert key, and on the *American Scientist* Website, Hayes (2005) asked: "Should Lambert W be added to the canon of standard textbook functions?" Stewart (2005) also suggested that the Lambert W function should be included in the mathematical curricula.

In Eq. 2, it is easy to calculate y when the value of x is given.

$$xe^{x} = y \qquad \text{Eq. 2}$$

However, in the inverse case, i.e. given y, the value of x is not readily obtained. This is where the Lambert W-function comes in because it is the inverse of Eq. 2, i.e. if y is the input, the Lambert W-function gives x = W(y) such that Eq. 2 is satisfied.

For example, with reference to Fig. 1, for y=e, W(e) gives x=W(y)=1 because e = $1*e^1$, and this is plotted as (e, 1); and for y=$\left(-\frac{1}{e}\right)$, W$\left(-\frac{1}{e}\right)$ gives x=W(y)=-1 because $-1*e^{-1}=-e^{-1}$, and this is plotted as ($-\frac{1}{e}$, -1).

The definition of W(y) may be written as

$$x = W(y) = W(xe^x) \qquad \text{Eq. 3}$$

The inverse W function given by Eq. 3 has two values of W when y is between $-\frac{1}{e}$ and 0. When y is greater than 0, W is single valued (Hayes, 2005).

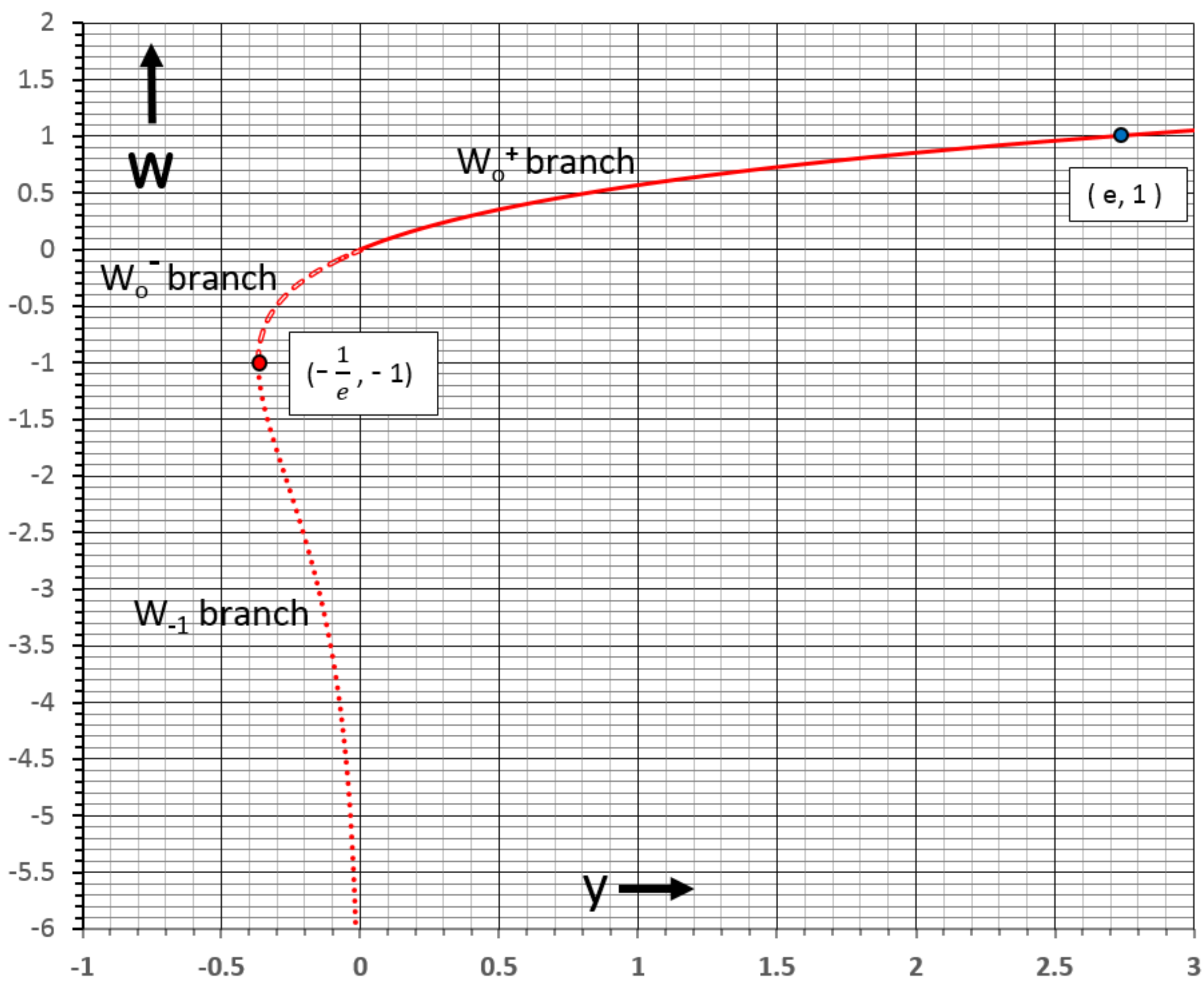

Fig. 1. The Lambert W-function.

The curve may be divided into three regions (Golicnik, 2012, Barry et al., 2000) and the branches are:

1. Region 1: y>0, the W_0^+ branch;

2. Region 2: $-e^{-1} < y < 0$, and $0 > W > -1$, the W_0^- branch;

3. Region 3: $-e^{-1} < y < 0$, $W < -1$, the W_{-1} branch.

Application of the Lambert W-function to the Logarithmic Mean

Writing the heat exchanger equation as

$$Q = UA\frac{\theta_1 - \theta_2}{\ln\left(\frac{\theta_1}{\theta_2}\right)} \qquad \text{Eq. 4}$$

Dividing by θ_2 and re-arranging

$$\frac{Q}{UA\theta_2} = \frac{\frac{\theta_1}{\theta_2} - 1}{ln\frac{\theta_1}{\theta_2}} \qquad \text{Eq. 5}$$

Consider θ_2 as known and θ_1 the unknown to be solved,

Let $$\frac{\theta_1}{\theta_2} = \beta \qquad \text{Eq. 6}$$

Let $$\frac{Q}{UA} = K, \quad such\ that\ \frac{K}{\theta_2} = \frac{\beta - 1}{\ln\beta} \qquad \text{Eq. 7}$$

Thus, $$\ln\beta = \frac{\theta_2}{K}(\beta - 1) = \frac{\theta_2}{K}\beta - \frac{\theta_2}{K} \qquad \text{Eq. 8}$$

Giving $$\beta = e^{\left(\frac{\theta_2}{K}\beta - \frac{\theta_2}{K}\right)} = \frac{e^{\frac{\theta_2}{K}\beta}}{e^{\frac{\theta_2}{K}}} \qquad \text{Eq. 9}$$

Eq. 9 may be re-cast as

$$\beta e^{-\frac{\theta_2\beta}{K}} = e^{-\frac{\theta_2}{K}} \qquad \text{Eq. 10}$$

Multiplying both sides by $-\frac{\theta_2}{K}$

$$-\frac{\theta_2}{K}\beta e^{-\frac{\theta_2}{K}\beta} = -\frac{\theta_2}{K}e^{-\frac{\theta_2}{K}} \qquad \text{Eq. 11}$$

In Eq. 11, θ_2 is known, while β the unknown to be solved appears on the left-hand-side of the equation only.

This is where the Lambert W function comes in. Taking the W of both sides of Eq. 11, the LHS is of the form xe^x, where $x = -\frac{\theta_2}{K}\beta$ according to Eq. 3.

Thus, Eq. 11 becomes

$$-\frac{\theta_2}{K}\beta = W\left(-\frac{\theta_2}{K}\, e^{-\frac{\theta_2}{K}}\right) \qquad \text{Eq. 12}$$

All terms on the RHS of Eq. 12 that are within the W function are known and hence $W\left(-\frac{\theta_2}{K}\, e^{-\frac{\theta_2}{K}}\right)$ can be evaluated. In fact, by inspection with reference to Eq. 3, it should give a numerical value of $-\frac{\theta_2}{K}$. However, from Fig. 1, W of a negative number for $-e^{-1} < y < 0$ will yield two roots, and the value of $-\frac{\theta_2}{K}$ is expected to be one of them. The two roots are obtained as one each from the W_o^- branch and the W_{-1} branch. Thus β can be evaluated.

Example 1 (for the general case when $\theta_1 \neq \theta_2$)

To illustrate the application of Eq. 12, using the same problem considered in Patterson (1984):

θ_2 = (50-30)=20°C; θ_1 = (125-t_1)°C; $\beta = \theta_1 / \theta_2$

Q = 7500kW; UA = 175 kW/°C

K = Q/UA = 7500/175 = 42.857°C

θ_2 /K = 20/42.857 = 0.4667

Thus, if β is solved, θ_1 and t_1 can be readily evaluated.

Applying Eq. 12, the LHS gives -0.4667β, and the RHS gives W(-0.4667e$^{-0.4667}$), i.e. W(-0.2927). From Figure 1, W(-0.2927) has two roots, one from the W_o^- branch and one from the W_{-1} branch.

The exact values of the two roots may be obtained using a standard mathematical software package such as Wolfram MathWorld, Matlab, Macsyma, Maple, Mathematica etc. (Hayes 2005). Note that the Lambert W-function is also referred to as the Product Log function in some software packages. Barry et al. (2000) and Golicnik (2012), among others, have provided equations for the evaluation of W, and tabulated values are also given in Disney & Warburton (2012).

W(-0.2927) gives -0.46 from the W_o^- branch (which is $-\frac{\theta_2}{K}$= -0.4667 as discussed earlier) and -1.84 from the W_{-1} branch. Thus

$$-0.4667\beta = -0.46; \quad and \quad -0.4667\beta = -1.84$$

The first solution is 'trivial' as this occurs when β=1, but $\theta_1 \neq \theta_2$, and hence this root can be discarded. The second solution gives β=3.94, resulting in θ_1 = 78.9 and t_1 = 46.1°C. As expected, this solution is exactly the same as that obtained by iteration using the actual LMTD.

Example 2 (for the case when $\theta_1 = \theta_2$)

Basically the Paterson (1984) case but setting θ_2 to be equal to K.

θ_2 = (72.86-30)= 42.86°C; θ_1 = (125-t_1)°C; $\beta = \theta_1/\theta_2$

Q = 7500kW; UA = 175 kW/°C

K = Q/UA = 7500/175 = 42.857°C

θ_2 /K = 42.86/42.857 ≈ 1.

Applying Eq. 12, the LHS gives – β, and the RHS gives W(-e^{-1}). From Figure 1, this occurs at the junction of W_o^- and W_{-1}, i.e. there is only one root, and W(-e^{-1}) = -1, i.e. β=1 giving $\theta_1 = \theta_2$.

Discussion

The two examples shown above are based on the case considered by Paterson(1984) and Chen(1987) using the transformed equation given in Eq. 12. Eq. 12 is equivalent to Eq. 4, but it allows for the exact evaluation θ_2 without a trial-and-error process which is necessary when using Eq. 4 because of the appearance of θ_2 in two places on the right-hand-side of the equation, inside and outside of the logarithmic term. In the case where Q (or A) is the only unknown, Q may be evaluated from an energy balance, and Eq. 4 will provide an explicit solution.

The above examples considered θ_2 as the unknown. If θ_1 is the unknown instead and is required to be solved, it is simply a matter of interchanging θ_1 and θ_2 in Eq. 4, 5, and 6, and the solution will follow immediately.

It is also noted that in the case where $\theta_1 = \theta_2$ as illustrated in Example 2, Eq. 4 will become $Q = UA\frac{\theta_1 - \theta_2}{\ln\left(\frac{\theta_1}{\theta_2}\right)} = UA\theta_1 = UA\theta_2$.

Conclusion

An explicit solution has been shown for heat exchanger calculations using the logarithmic mean temperature method by employing the Lambert W-function due originally to Lambert (1758) and Euler (1779). It remains to be seen if it will replace the use of approximations for the logarithmic term in the research on optimization such as those referenced in this note.

Declaration of Interest

None. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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