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Coherence in Atom and Quantum Optics

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University of Auckland 1998
Acknowledgments

I would very much like to thank my supervisors Professor Dan Walls and Dr. Sze Tan for their support and encouragement. Dan has superb leadership ability and has been especially helpful in providing me with achievable goals. Sze is an excellent source of advice with his broad field of expertise including physics, mathematics and computing problems that he endows with his seemingly unlimited supply of enthusiasm. Dr. Matthew Collett possesses a clarity in thinking, which in his advisory role has been extremely valuable in leading the way in complex and confusing problems that I have encountered.

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As a final acknowledgement, the credit for any mistakes in this work, belongs entirely to the author.

Tony Wong
March 1998
Abstract

This thesis is the result of three years research in the Physics department at the University of Auckland. This research can be roughly divided into three projects. In the first year, Professor Dan Walls brought to my attention an experiment performed by Eichmann and co-workers. They observed interference of resonance fluorescence from two trapped ions using a polarization-sensitive detection scheme. We applied traditional quantum optics methods to analyze this experiment. A numerical technique, the Monte Carlo wave function method was used to simulate the classical motion of the ions and various orientations of the incident light. The results of Eichmann’s experiment has been claimed by Scully and co-workers to support their view that only complementarity is required to explain the loss of interference in which-way schemes. However, our analysis shows to the contrary that Eichmann’s experiment cannot be regarded as a which-way experiment and its results can be understood without resorting to these which-way arguments.

The experimental Bose–Einstein condensation of dilute gases and the avalanche of theoretical work that followed it provoked our second project. Following a simplistic model of interference between two bose–condensates suggested by Javanainen we explore the establishment of the relative phase. We extend his work by including the effect of collisions (atom–atom interactions). Although we treat the condensate with a single mode model so that the formulation looks similar to quantum optics, these condensates consist of atoms and the resultant atom–atom interactions introduces an additional complexity; that of non-linear effects. Viewpoints must also be altered, absolute phase in a matter wave such as a bose–condensate is unphysical thus motivating an analysis of relative phases between two condensates. This project lasted for about one and a half years with injections of fresh input from several collaborators. Firstly, Professor Ewan Wright became involved in the collapse and revivals of the phase. His work on the collapse using a multi-mode single condensate was connected with our single-mode two condensate material. Secondly, during Professor Robert Graham’s short stay at
the University of Auckland, he performed analytic calculations of the establishment of the relative phase for the cases of thermal or Poissonian initial states. This work fitted nicely with our numerical efforts in this regime where our knowledge of the initial atom number is uncertain.

Finally, Dr Karl Vogel and Bernhard Kneer, a Ph.D. student studying under Professor Wolfgang Schleich, started the third project with their atom laser proposal. The dynamics of a condensate confined in a trap was modelled by using a modified Gross–Pitaevski equation that included both a pump and a loss term. This represents an atom laser since the confined condensate can be regarded as a "laser mode". My input into this model has been concentrated upon a spatially dependent loss. This has allowed us to study the influence of the pump and the spatially dependent loss on the shape and phase of the confined condensate in the steady–state regime.

The scientific programming language Matlab© was used to do the numerical component of my research. Mathematica© has been useful in the analytical work, while the typesetting was done using Latex©. The computers that were used ranged in power from a Macintosh IIfx to a Pentium 166, and the Sun Sparc 10 and 20 workstations.
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Part I

Interference of Resonance Fluorescence from two four–level atoms
Chapter 1

Introduction

Young's two slit experiment is the canonical experiment demonstrating the wave nature of light. With the addition of a measuring device to monitor which slit the photon has passed through, the phenomenon of wave–particle duality may be investigated. The position measurement used to find which path the photon has taken destroys the interference fringes. Such path detection schemes are known as welcher Weg (which–way) measurements. This is the archetypal example of Bohr's complementarity principle [1] which states that the wave and particle nature of matter are complementary aspects, both cannot be observed under the same conditions. For example, Young's two slit experiment demonstrates the wave nature of matter but when path detectors are introduced, its particle nature is then observed. Thus, both descriptions are essential to fully predict the experiment. Heisenberg's uncertainty principle has also been used to explain the destruction of the interference pattern in the well known which–way schemes proposed in the past, such as Einstein's recoiling slit [2] and Feynman's light microscope [3]. In this more fundamental view, the uncertainty principle enforces the complementarity principle via random momentum kicks to the particle.

These which–way measurement schemes have in recent years undergone a revival in interest. This renewal of attention was sparked by the proposal of a which–way measurement scheme by Scully and co–workers [4, 5, 6]. They
claim that no transverse momentum is imparted to the particle in the destruction of the interference fringes in their scheme. Storey and co-workers [5, 6, 7] have challenged this and have argued that transverse momentum is transferred whenever interference is destroyed, according to the uncertainty principle. More recently, Wiseman and co-workers [8, 9] have attempted to resolve the issue by differentiating between “local” and “non-local” momentum transfer in the proposed which-way measurement schemes.

In the first part of this thesis we will concentrate on both a theoretical and numerical study of the experiment performed by Eichmann and co-workers [10] at NIST\(^1\). We are motivated to investigate this experiment by Scully’s claim that it supports his view that no momentum is transferred. After performing an analysis we conclude that Eichmann’s experiment is not a which-way measurement scheme, and that the experimental results can be understood without resorting to which-way interpretations. In the remaining sections of this introductory chapter we will give a very brief introduction to Einstein’s recoiling slit and Feynman’s light microscope which-way measurement schemes, followed by a description of Eichmann’s experiment.

1.1 Einstein’s recoiling slit and Feynman’s light microscope

Einstein’s recoiling slit configuration is shown in fig. 1.1. A single slit is placed before the double slits and is mounted on rollers such that it is free to move along the \(x\)-axis. This screen plays the role of a path detector and is kicked up or down depending on the transfer of momentum it receives from the passage of a photon through its single slit. Note that the configuration we present here was the one that was treated by Wooters and Zurek [11]; the original configuration proposed by Einstein consists of the slit attached to a spring instead of rollers (see Jammer [2]). Tan and Walls [12] have also analyzed this configuration but from a different viewpoint. They explain

\(^1\)National Institute of Standards and Technology, Boulder, Colorado, USA.
1.2. EICHMANN'S EXPERIMENT

Figure 1.1: Einstein recoiling slit configuration where the first slit is used as a path detector.

the loss of coherence by using a random-averaged model. This model shows that loss of coherence may always be described in terms of a "stochastic disturbance to the system due to coupling with an environment".

Tan and Walls have also considered Feynman's light microscope scheme, which is shown in fig. 1.2. A electron travels along the $z$-axis, through the double slits and onto a screen placed at a distance $L$ away. The path detector consists of a travelling light field along the $x$-axis which would be scattered off the electron as it leaves the double slit. Photon detectors are used to measure the scattering angles to determine which slit the electron has passed through. Cohen-Tannoudji, Bardou and Aspect [13] have studied Feynman's light microscope by considering the elastic scattering by using the $s$-matrix method.

1.2 Eichmann's experiment

Advances in trapped ion technology have given experimenters the opportunity to perform Young's double slit experiment with a new twist. Two trapped atomic ions play the role of the double slits. An incoming laser light illuminates them. A classical picture of this interference setup would be to imagine the incoming light scattering off these ions to form an interference
CHAPTER 1. INTRODUCTION

Figure 1.2: Feynman’s light microscope configuration where light is scattered of an electron as it exits the double slit.

pattern on a screen far away. The separation of the fringes will depend on the distance between the ions in the same manner as the conventional Young’s double slit experiment. Thus the separation between the ions needs to be well-defined and each ion is required to be well localized within the trap in order that the interference pattern is not washed out. This situation is not easily achievable in atomic ion traps but a degree of localization is obtainable by using laser-cooling methods\(^2\). Another achievement is the ability to fix the number of atomic ions in these traps, that is, one is able to state that a trap contains only two ions. The advantage of using trapped atomic ions instead of slits is that they have an internal electronic structure which is responsive to laser light.

A quantum description of the interference setup would be as follows; the laser light is tuned close to the resonance frequency of some chosen transitions in the atomic ion’s electronic structure. The ion is excited to a higher energy level (pumped by the laser light) and eventually decays back to its ground states, emitting a photon in the process. The emission of photons, i.e. the resonance fluorescence, is collected at a detector far away. Because of the internal structure of the atomic ions, the incident laser light is not only used

to pump them but also to cool them. Their kinetic energies are reduced by the laser light field since the field preferentially transfers photon momentum to the ions in a direction that opposes their motion. The technique used to achieve this is called Doppler laser-cooling. This laser cooling of the ions is important as the lowering of their temperature strongly localizes them in the trap.

Another possibility due to the internal structure of the ions is its utilization as a path detector. The emission of a photon in resonance fluorescence may leave behind traces of its passage in the electronic configuration of the ion that has undergone the emission. Thus, in certain situations it is possible in principle to tell which ion has emitted a photon from the pair of ions by looking at their electronic configurations for tell-tale signs. In this new class of which-way measurement schemes, the ions act as both slits and path detectors, distinguishing them from that earlier schemes such as Einstein’s recoiling slit and Feynman’s light microscope schemes where the two are separate.

The observation of interference between fluorescent light from two atomic $^{198}\text{Hg}^+$ ions has been achieved by Eichmann and co-workers [10]. The setup of their experiment is shown in fig. 1.3. The two $^{198}\text{Hg}^+$ ions are confined in a linear Paul trap [19] and are well-localized due to Doppler cooling [15] from the incident coherent light field which is also used to pump them. In this configuration, the incident light field is a linearly-polarized travelling wave and the resonance fluorescence from the ions was collected with a polarization sensitive detection scheme. The polarization of an emitted photon is intimately related to the electronic configuration of the ion that it originated from. An interference pattern is observed in one choice of polarization but not in the other. These results were explained in two ways, firstly in terms of a which-way argument and secondly in terms of a theoretical analysis by Polder and Schuurmans [20] who considered coherent driving of a single four-level atom. Further theoretical analysis on Eichmann experiment have since been performed by Itano [22].
Figure 1.3: Schematic of Eichmann's experiment. Incident light from a laser is scattered of the two ions in the trap. Two lenses are used to image the scattered light to two detectors.

The which-way argument is presented diagrammatically in fig. 1.4 for one possible initial configuration of the ions. Each ellipse denotes the combined two ion system with a vertical dashed line to partition the left ion from the right ion. The ions themselves are represented as four-level atoms (see chapter 2); the left and the right ground levels corresponds to the $|1\rangle$ and $|2\rangle$ ground states respectively, whereas the left and the right excited levels are related to the $|3\rangle$ and $|4\rangle$ states respectively. The top ellipse of fig. 1.4 represents the initial configuration where each ion has its $|1\rangle$ ground state occupied only. The middle ellipse describes the excitation of the left ion from the $|1\rangle$ to the $|3\rangle$ state via the linearly-polarized incident laser light. The ion then has two decay options, it can either emit a $\pi$ or $\sigma_-$ polarized photon to the $|1\rangle$ or $|2\rangle$ ground states respectively. Option $\pi$ (to the $|1\rangle$ state) results in the initial and final states being the same whereas option $\sigma_-$ (to the $|2\rangle$ state) they differ. Thus, in principle one can determine which ion the photon originated from if the emitted light is $\sigma_-$ polarized. This is not possible in principle with $\pi$-polarized light. Other initial configurations follow a similar argument so that we can only tell which ion the photon came from if it is...
circularly ($\sigma_{\pm}$) polarized. Since the experimental results show interference in the $\pi$-polarized light, but not in the circularly polarized light, it seems to support this argument. However, since the incident laser light is continuous and the experiment is not re-prepared to an initial configuration *whenever* a photon is detected, it is operating in the steady-state regime where all memory of the initial configuration is quickly lost. Thus an argument based on comparing initial and final states of the system should not be used to explain this experiment.
Figure 1.4: Diagrammatic representation of the which-way argument. The four states are labelled in the top "initial state" ellipse but are neglected in the other ellipses in the interest of clarity. The dashed line in each ellipse is used to partition the left and right ions of the system.
Chapter 2

Theoretical analysis of two trapped atoms

Various situations have been theoretically studied, using both two-level and four-level atoms. The possibility of placing the two atoms in an optical cavity has also been considered [16, 17, 18]. We will however concentrate on a model of Eichmann’s experiment. In this chapter, we carry out a theoretical analysis which considers the presence of both atoms and stress that for the coherent steady-state excitation as used in the experiment (continuous laser light illuminating the two ions), the which-path argument is not applicable. We also extend the analysis of Polder and Schuurmans [20] to the experimental situation. In brief, the atomic dipoles are only coherently driven in a specific direction given by the polarization of the incident light field, and so interference is only expected for scattered light of that polarization. Scattered light of the orthogonal polarization is also present, since the atoms can decay spontaneously, but this is incoherent with the driving field and does not lead to interference. In chapter 3, we will introduce and discuss results of a numerical simulation of the experiment. This simulation will model the classical motion of the atoms in a trap. It will also free us from some of the other assumptions that we will present in this current chapter.
CHAPTER 2. INTERFERENCE OF TWO TRAPPED ATOMS

2.1 The Model

The experiment consists of two $^{198}\text{Hg}^+$ ions trapped along the axis of a linear trap illuminated by a linearly polarized, travelling wave laser beam tuned below the resonance frequency of the $^{198}\text{Hg}^+$ $6s^2S_{1/2} - 6p^2P_{1/2}$ transition at 194 nm. We model the ions as four-level atoms (see fig. 2.1) interacting with a linearly polarized light field polarized along the $z$ axis and travelling along the $y$ direction. All other energy levels of $^{198}\text{Hg}^+$ are ignored as they are far from resonance of the 198 nm $j = 1/2$ to $j = 1/2$ transition. We treat the incident laser light as a classical field and the external motion of the atoms will be neglected, motion will be re-instated in a numerical model introduced in chapter 3. The separation between the atoms is considered to be large (many wavelengths of the laser light) so that dipole–dipole interactions between the atoms can be ignored. The experiment used various separations ranging from 3 to 5 $\mu$m. This corresponds to about 20 wavelengths for the 194 nm incident laser light. In this approximation, the atoms can be considered as independent fixed sources except that their radiation is synchronized by the definite phase of the incident laser light. The atomic dipole operator $\vec{\mu}$ is the
2.1. THE MODEL

sum of atomic raising $\mu^+_{\text{k}}$ and lowering $\mu^-_{\text{k}}$ operators whose components are given by [20]

$$
\mu^+_{\text{x}} = \mu(|1\rangle\langle 4| + |2\rangle\langle 3|)\hat{x} \quad (2.1)
$$

$$
\mu^+_{\text{y}} = -i\mu(|1\rangle\langle 4| - |2\rangle\langle 3|)\hat{y} \quad (2.2)
$$

$$
\mu^+_{\text{z}} = \mu(|2\rangle\langle 4| - |1\rangle\langle 3|)\hat{z} \quad (2.3)
$$

where $\mu^+_{\text{k}}$ is the $\text{k}$ component of the atomic dipole, $\mu$ is the dipole matrix element, $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the usual Cartesian unit vectors. The atomic operator $|i\rangle\langle j|$ couples the $i$'th and $j$'th levels of the atom. Because of our initial assumption of the atoms as independent fixed sources, our theoretical model is immensely simplified to that of solving a single atom system. We now proceed with the system Hamiltonian for one of the atoms in the interaction picture,

$$
H_s = H_{\text{atom}} + \vec{\mu} \cdot \vec{E}(\vec{r})
$$

$$
= \frac{1}{2} \hbar \Delta \left( \sigma_{44} + \sigma_{33} - \sigma_{22} - \sigma_{11} \right) + \hbar \Omega \left( \sigma_{24} + \sigma_{42} - \sigma_{13} - \sigma_{31} \right), \quad (2.4)
$$

where $\Delta$ is the detuning between the atomic transitions and the incident laser light. For simplicity, the ground states (and the excited states) are considered to be degenerate in energy (see fig. 2.1). We define the Rabi frequency $\Omega = \mu E_o / 2\hbar$ where $E_o$ is the electric field amplitude of the incident laser and we use the notation $\sigma_{ij} \equiv |i\rangle\langle j|$. The equation of motion for the reduced density operator $\rho_s$ for a single atom is governed by the master equation

$$
\frac{d\rho_s}{dt} = \frac{i}{\hbar} [\rho_s, H_s] + \mathcal{L}_{\text{relax}}(\rho_s), \quad (2.5)
$$

where $\mathcal{L}_{\text{relax}}$ is the relaxation superoperator

$$
\mathcal{L}_{\text{relax}}(\rho_s) = -\gamma \left( \sigma_{44}\rho_s + \rho_s\sigma_{44} + \sigma_{33}\rho_s + \rho_s\sigma_{33} \right) + \gamma \left( \sigma_{14}\rho_s\sigma_{41} + (\sigma_{24} - \sigma_{13}) \rho_s (\sigma_{42} - \sigma_{31}) + \sigma_{23}\rho_s\sigma_{32} \right).
$$
We have assumed that both excited states have the same decay rate $\gamma$. Note that the decay channel $\sigma_{24} - \sigma_{13}$ in the above superoperator corresponds to the emission of a photon linearly polarized along the $z$-axis, see Eq. 2.3, i.e. the $\pi$-polarization. The other two decay channels $\sigma_{14}$ and $\sigma_{23}$ are related to the circular polarizations via the relations $\mu_{\sigma_{14}} = (\mu_{z}^{+} + i\mu_{y}^{+})/2$ and $\mu_{\sigma_{23}} = (\mu_{z}^{+} - i\mu_{y}^{+})/2$, hence they correspond to the $\sigma_{+}$ and $\sigma_{-}$ polarizations respectively. Figure 2.2 graphically displays the relationships between the decay channels and the polarizations of the emitted photon.

2.2 The Optical Bloch Equations

We can derive sixteen Optical Bloch equations from the master equation corresponding to the sixteen atomic operators. However, due to the geometry chosen for the incident light, it is linearly polarized along the $z$-axis, the sixteen equations de-couple into two sets of eight. The first set of coupled equations involve the populations $\langle \sigma_{4i} \rangle$ and the "linear coherences" $\langle \sigma_{13} \rangle$ and $\langle \sigma_{24} \rangle$ terms which relate to the $z$ component of the atomic dipole (see Eq. 2.3).

Using

$$\frac{d}{dt}\langle \sigma_{ij} \rangle = \text{Tr} \{\sigma_{ij}\hat{\rho}_{s}\}$$

(2.6)
2.2. THE OPTICAL BLOCH EQUATIONS

and the master equation (Eq. 2.5) we obtain the following

\[
\frac{d}{dt} \langle \sigma_{13} \rangle = (-\gamma - i\Delta) \langle \sigma_{13} \rangle - i\Omega (\langle \sigma_{33} \rangle - \langle \sigma_{11} \rangle) \tag{2.7}
\]

\[
\frac{d}{dt} \langle \sigma_{31} \rangle = (-\gamma + i\Delta) \langle \sigma_{31} \rangle + i\Omega (\langle \sigma_{33} \rangle - \langle \sigma_{11} \rangle) \tag{2.8}
\]

\[
\frac{d}{dt} \langle \sigma_{24} \rangle = (-\gamma - i\Delta) \langle \sigma_{24} \rangle - i\Omega (\langle \sigma_{22} \rangle - \langle \sigma_{44} \rangle) \tag{2.9}
\]

\[
\frac{d}{dt} \langle \sigma_{42} \rangle = (-\gamma + i\Delta) \langle \sigma_{42} \rangle + i\Omega (\langle \sigma_{22} \rangle - \langle \sigma_{44} \rangle) \tag{2.10}
\]

\[
\frac{d}{dt} \langle \sigma_{11} \rangle = -i\Omega (\langle \sigma_{31} \rangle - \langle \sigma_{13} \rangle) + \gamma (\langle \sigma_{33} \rangle + \langle \sigma_{44} \rangle) \tag{2.11}
\]

\[
\frac{d}{dt} \langle \sigma_{22} \rangle = -i\Omega (\langle \sigma_{24} \rangle - \langle \sigma_{42} \rangle) + \gamma (\langle \sigma_{33} \rangle + \langle \sigma_{44} \rangle) \tag{2.12}
\]

\[
\frac{d}{dt} \langle \sigma_{33} \rangle = -i\Omega (\langle \sigma_{13} \rangle - \langle \sigma_{31} \rangle) - 2\gamma \langle \sigma_{33} \rangle \tag{2.13}
\]

\[
\frac{d}{dt} \langle \sigma_{44} \rangle = -i\Omega (\langle \sigma_{42} \rangle - \langle \sigma_{24} \rangle) - 2\gamma \langle \sigma_{44} \rangle. \tag{2.14}
\]

Since normalisation requires that

\[
\sum_{i=1}^{4} \langle \sigma_{ii} \rangle = 1, \tag{2.15}
\]

the number of independent equations may be reduced to seven. The problem now becomes a linear system of equations and it is useful to express it in a matrix form

\[
\frac{d}{dt} X_{\text{lin}} = A_{\text{lin}} X_{\text{lin}} + C, \tag{2.16}
\]

where the matrix connecting the linear coherences is defined as

\[
A_{\text{lin}} = \begin{bmatrix}
-i\Delta - \gamma & 0 & 0 & 0 & i\Omega & -i\Omega & 0 \\
0 & i\Delta - \gamma & 0 & 0 & -i\Omega & i\Omega & 0 \\
0 & 0 & -i\Delta - \gamma & 0 & i\Omega & i\Omega & 2i\Omega \\
0 & 0 & 0 & i\Delta - \gamma & -i\Omega & -i\Omega & -2i\Omega \\
i\Omega & -i\Omega & 0 & 0 & 0 & \gamma & \gamma \\
-i\Omega & i\Omega & 0 & 0 & 0 & -2\gamma & 0 \\
0 & 0 & i\Omega & -i\Omega & 0 & 0 & -2\gamma
\end{bmatrix}, \tag{2.17}
\]
where the vector \( X_{\text{lin}} = (\langle \sigma_{13} \rangle, \langle \sigma_{31} \rangle, \langle \sigma_{24} \rangle, \langle \sigma_{42} \rangle, \langle \sigma_{11} \rangle, \langle \sigma_{33} \rangle, \langle \sigma_{44} \rangle)^T \) and the inhomogenous term \( C = (0, 0, -i\Omega, i\Omega, 0, 0, 0)^T \). The steady–state is obtained by setting the time derivatives to zero so that the time independent solution is

\[
X_{\text{lin}}^{ss} = -A_{\text{lin}}^{-1}C. \tag{2.18}
\]

The steady–state situation is the experimental regime of interest, where the transient dynamics have damped away to reach the long time behaviour. Here we solve for this steady–state behaviour of the linear coherences in order to calculate the degree of coherence of the \( \pi \)–polarized fluorescent light. We re–write Eq. 2.18 as

\[
X_{\text{lin}}^{ss} = \frac{-1}{\text{det} [A_{\text{lin}}]} B \cdot C
\]

where \( B = \text{det} [A_{\text{lin}}] \cdot A_{\text{lin}}^{-1} \) to obtain the optimal expression to input into the symbolic software package of Mathematica\(^\text{\textregistered}\) to yield the following solutions\(^1\)

\[
\begin{align*}
\langle \sigma_{13} \rangle_{ss} &= (\bar{\Delta} + i\bar{\gamma}) \langle \sigma_{33} \rangle_{ss} \tag{2.19} \\
\langle \sigma_{31} \rangle_{ss} &= \langle \sigma_{13} \rangle_{ss}^* \tag{2.20} \\
\langle \sigma_{24} \rangle_{ss} &= (-\bar{\Delta} - i\bar{\gamma}) \langle \sigma_{44} \rangle_{ss} \tag{2.21} \\
\langle \sigma_{42} \rangle_{ss} &= \langle \sigma_{24} \rangle_{ss}^* \tag{2.22} \\
\langle \sigma_{11} \rangle_{ss} &= \frac{1}{2} - \langle \sigma_{33} \rangle_{ss} \tag{2.23} \\
\langle \sigma_{22} \rangle_{ss} &= \langle \sigma_{11} \rangle_{ss} \tag{2.24} \\
\langle \sigma_{33} \rangle_{ss} &= \frac{1}{2}(\bar{\Delta}^2 + \bar{\gamma}^2 + 2)^{-1} \tag{2.25} \\
\langle \sigma_{44} \rangle_{ss} &= \langle \sigma_{33} \rangle_{ss} \tag{2.26}
\end{align*}
\]

where we have scaled the detuning and decay rates by the Rabi frequency, i.e. \( \bar{\Delta} = \Delta/\Omega \) and \( \bar{\gamma} = \gamma/\Omega \). As expected from the symmetry of our model of the atomic ions as four–level atoms where the ground and the excited levels are degenerate, the populations of the two ground states are equal as are the populations of the two excited states. In the limit of infinite

\(^1\)These solutions are compatible to those of ref [20] since only one atom has been included at this time.
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driving strength, \( \Omega \to \infty \), we find that the coherences vanish, \( \langle \sigma_{13} \rangle_{ss} = \langle \sigma_{31} \rangle_{ss} = \langle \sigma_{24} \rangle_{ss} = \langle \sigma_{42} \rangle_{ss} = 0 \), and that the populations are all equal, \( \langle \sigma_{11} \rangle_{ss} = \langle \sigma_{22} \rangle_{ss} = \langle \sigma_{33} \rangle_{ss} = \langle \sigma_{44} \rangle_{ss} = 1/4 \). This is expected since the atoms have become saturated by the infinite driving.

A graph of the populations versus laser light strength is shown in fig. 2.3(a). For strong fields the atom is saturated with all the populations equal to one quarter. For weak fields, the excited states are close to zero with the ground states close to one half. The linear coherences are shown in fig. 2.3 (b). Since the coherences are purely imaginary we plot the imaginary part of \( \langle \sigma_{13} \rangle \) and \( \langle \sigma_{24} \rangle \). We see that as the field strength increases the coherences grow and reach a peak for \( \Omega/\gamma \approx 2/3 \). There is a gradual loss of coherence as the field is further increased due to the saturation of the atom. At low field strengths the atom spends most of its time in the ground state and is very rarely pumped to the excited states. It behaves like a coherently driven linear oscillator and scatters coherently. In the strong field case, the atom undergoes Rabi oscillations and is frequently pumped to the excited states from which it spontaneously decays. These spontaneous emission events destroy the coherences.

The second set of equations consist of the "cross coherences" and the coherences between the ground \( \langle \sigma_{12} \rangle \) and between the excited states \( \langle \sigma_{34} \rangle \). These cross coherences, \( \langle \sigma_{14} \rangle \) and \( \langle \sigma_{23} \rangle \), as mentioned previously corresponds to the \( \sigma_+ \) and \( \sigma_- \) polarizations respectively. Thus they will determine the degree of coherence that the circularly (\( \sigma_\pm \)) polarized photons possess. The set of equations are

\[
\frac{d}{dt} \langle \sigma_{14} \rangle = (-\gamma - i\Delta)\langle \sigma_{14} \rangle - i\Omega(\langle \sigma_{12} \rangle + \langle \sigma_{34} \rangle) \tag{2.27}
\]

\[
\frac{d}{dt} \langle \sigma_{23} \rangle = (-\gamma - i\Delta)\langle \sigma_{23} \rangle + i\Omega(\langle \sigma_{21} \rangle + \langle \sigma_{43} \rangle) \tag{2.28}
\]

\[
\frac{d}{dt} \langle \sigma_{12} \rangle = -i\Omega(\langle \sigma_{14} \rangle + \langle \sigma_{32} \rangle) - \gamma\langle \sigma_{34} \rangle \tag{2.29}
\]

\[
\frac{d}{dt} \langle \sigma_{34} \rangle = -i\Omega(\langle \sigma_{32} \rangle + \langle \sigma_{14} \rangle) - 2\gamma\langle \sigma_{34} \rangle. \tag{2.30}
\]

The other four equations for \( \langle \sigma_{41} \rangle \), \( \langle \sigma_{32} \rangle \), \( \langle \sigma_{21} \rangle \) and \( \langle \sigma_{43} \rangle \) are just the Hermi-
Figure 2.3: (a) Atomic populations versus field strength. The $\langle \sigma_{11} \rangle$ and $\langle \sigma_{22} \rangle$ curves graphed as the solid line (they coincide) with the $\langle \sigma_{33} \rangle$ and $\langle \sigma_{44} \rangle$ curves graphed as the dashed line (again they coincide). (b) The atomic linear coherences versus field strength. The $\langle \sigma_{13} \rangle$ and $\langle \sigma_{42} \rangle$ curves are graphed as the solid line (they coincide) with the $\langle \sigma_{31} \rangle$ and $\langle \sigma_{24} \rangle$ curves graphed as the dashed line (again they coincide). Detuning is set at zero for both graphs.
2.2. THE OPTICAL BLOCH EQUATIONS

tian conjugates of these equations. The corresponding matrix equation for this system of linear equations is then

$$\frac{d}{dt}X_{\text{cross}} = A_{\text{cross}}X_{\text{cross}},$$  \hspace{1cm} (2.31)

where the matrix connecting the cross-coherences is defined by

$$A_{\text{cross}} = \begin{bmatrix}
-i\Delta - \gamma & 0 & 0 & 0 & -i\Omega & 0 & -i\Omega \\
0 & i\Delta - \gamma & 0 & 0 & i\Omega & 0 & i\Omega \\
0 & 0 & -i\Delta - \gamma & 0 & 0 & i\Omega & 0 \\
0 & 0 & 0 & i\Delta - \gamma & -i\Omega & 0 & -i\Omega \\
-i\Omega & 0 & 0 & -i\Omega & 0 & 0 & -\gamma \\
0 & i\Omega & i\Omega & 0 & 0 & 0 & -\gamma \\
-i\Omega & 0 & 0 & -i\Omega & 0 & 0 & -2\gamma \\
0 & i\Omega & i\Omega & 0 & 0 & 0 & -2\gamma 
\end{bmatrix},$$  \hspace{1cm} (2.32)

and the vector $X_{\text{cross}} = (\langle \sigma_{14} \rangle, \langle \sigma_{41} \rangle, \langle \sigma_{23} \rangle, \langle \sigma_{32} \rangle, \langle \sigma_{12} \rangle, \langle \sigma_{21} \rangle, \langle \sigma_{34} \rangle, \langle \sigma_{43} \rangle)^T$. Since the system is homogenous, the trivial solution $X_{\text{cross}} = 0$ is valid if $A_{\text{cross}}$ is a non-singular matrix. We need to calculate its determinant in order to discover its nature. By using Mathematica©, we find

$$\text{det}[A_{\text{cross}}] = 16\gamma^4\Omega^4$$  \hspace{1cm} (2.33)

which is non-zero whenever the pumping and decay rate is non-zero. Obviously we need a finite pumping strength in order to pump the atoms as well as a finite decay rate to observe the fluorescence, therefore $X_{\text{cross}} = 0$ is a valid solution for the parameters that are used in an experiment.

Thus in the steady-state we find that only the trivial solution remains so that all these coherences ultimately vanish. Any initial coherences involving these terms are transient and are damped away over the time scale of this transient regime. Thus the cross coherences, $\langle \sigma_{14} \rangle$ and $\langle \sigma_{23} \rangle$, which determines the $x$ and $y$ components of the atomic dipole (see Eq. 2.1–2.3), show no interference from light polarized in the $xy$ plane. That is, the circularly polarized light is not coherent.
2.3 Transient coherences

It is possible to calculate the time dependent coherences for special situations. Although this is not experimentally observable, nevertheless it is interesting to consider dynamics which are conditioned on the initial state of the two atom system. We would like to know whether there is any interesting short-time behaviour of the cross coherences; they vanish in the steady-state but there may be transient effects. In order to make the calculations tractable, certain simplifying conditions are assumed. In addition to the fixed independent radiators assumption we have already utilized we also require that the laser light is tuned to resonance ($\Delta = 0$) and the atoms are restricted to specific initial conditions.

The linear system of differential equations which include the $\sigma_{14}$ and $\sigma_{23}$ coherences was introduced previously in Eq. 2.31, which we write down again as

$$\frac{d}{dt}X_{\text{cross}}(t) = A_{\text{cross}}X_{\text{cross}}(t).$$

A well-known method of solving such a system of equations is to use Laplace transforms. The Laplace transform of the above equations is

$$sX_{\text{cross}}(s) - X_{\text{cross}}(0) = A_{\text{cross}}X_{\text{cross}}(s),$$

rearranging we obtain

$$X_{\text{cross}}(s) = (sI - A_{\text{cross}})^{-1}X_{\text{cross}}(0)$$

where we define $I$ as the identity matrix of the same size as $A_{\text{cross}}$. The formal solution is now obtained by applying the inverse Laplace transform

$$X_{\text{cross}}(t) = L^{-1}\left\{B^{-1}\right\}X_{\text{cross}}(0)$$

where we have defined the matrix $B = sI - A_{\text{cross}}$ for notational convenience. We now need to choose the initial conditions to plug into $X_{\text{cross}}(0)$. A first choice would be the initial wave function

$$|\psi(t = 0)\rangle = |k\rangle$$
with \( k \in \{1, 2, 3, 4\} \) corresponding to the single atom occupying one of the states with no superpositions between the states. Remember that we have assumed that the two atoms are independent so there are no superpositions between the atoms. However since \( X_{\text{cross}}(0) \) consist only of the cross coherence operators, \( \sigma_{ij} \) where \( i \neq j \) the expectation values are all zero so that \( X_{\text{cross}}(0) = 0 \). The trivial solution is valid if the inverse Laplace transform of \( B^{-1} \) is non-singular. We choose to consider a more interesting initial state, one that allows superpositions between the ground state

\[
|\psi(t = 0)\rangle = \alpha_0|1\rangle + \beta_0|2\rangle.
\]

(2.38)

The only non-zero initial cross coherences are \( \langle \sigma_{12} \rangle = \alpha_0 \beta_0^* = \langle \sigma_{12} \rangle^* \). It is convenient to write

\[
\begin{align*}
\alpha_0 &= r_1e^{-i\phi_1} \\
\beta_0 &= r_2e^{-i\phi_2},
\end{align*}
\]

and define the product of the amplitude \( r = r_1r_2 \) and the relative phase \( \phi = \phi_2 - \phi_1 \). The initial condition is thus \( X_{\text{cross}}(0) = \{0, 0, 0, 0, re^{-i\phi}, re^{i\phi}, 0, 0\}^T \).

Note that since \( |\alpha_0|^2 + |\beta_0|^2 = 1 \Rightarrow r_1^2 + r_2^2 = 1 \Rightarrow 0 \leq r \leq 1/2 \). The calculation of the inverse Laplace transform of \( B^{-1} \) is long and un-instructive and has been placed in appendix A.1. The time dependent solution is

\[
\langle \sigma_{14} \rangle (t) = -i\Omega re^{-i\phi}e^{-\gamma t} \left[ -\frac{\gamma}{\alpha} + \left( \frac{1}{2\gamma} + \frac{2\Omega^2}{\gamma\alpha} \right) (e^{\sqrt{\alpha}t} + e^{-\sqrt{\alpha}t}) + \frac{1}{2\sqrt{\alpha}} (e^{\sqrt{\alpha}t} - e^{-\sqrt{\alpha}t}) \right],
\]

(2.39)

where

\[
\alpha = \gamma^2 - 4\Omega^2
\]

(2.40)

whenever \( \alpha \neq 0 \), otherwise

\[
\langle \sigma_{14} \rangle (t) = -i\Omega re^{-i\phi}te^{-\gamma t} \left( 1 + \frac{\gamma t}{2} \right).
\]

(2.41)

The other solutions of interest are derived from the relations

\[
\langle \sigma_{32} \rangle (t) = \langle \sigma_{14} \rangle (t)
\]

(2.42)
and

\[ \langle \sigma_{ij} \rangle (t) = \langle \sigma_{ji} \rangle^* (t). \]  

(2.43)

We see that the initial conditions only influence the coherences by scaling them with the amplitude \( r \) and a rotation in the Argand plane. We will rewrite the solution as a product of the initial conditions and the function \( F \),

\[ \langle \sigma_{14} \rangle (t) = -ire^{-i\phi} F (\Omega, \gamma, t) \]  

(2.44)

with \( F \) defined as

\[
F (\Omega, \gamma, t) = \begin{cases} 
\Omega t e^{-\gamma t} \left( 1 + \frac{n t}{2} \right), & \text{if } \alpha = 0 \\
\Omega e^{-\gamma t} \left[ -\frac{1}{\alpha} + \left( \frac{1}{2\gamma} + \frac{2n^2}{\gamma^2} \right) \left( e^{\sqrt{\alpha} t} + e^{-\sqrt{\alpha} t} \right) + \frac{1}{2\sqrt{\alpha}} \left( e^{\sqrt{\alpha} t} - e^{-\sqrt{\alpha} t} \right) \right], & \text{otherwise.}
\end{cases}
\]

Equation 2.44 tells us that the product of the two amplitudes of the initial ground states plays a dominant role in the size of the cross coherences. When the only one of the ground states is initially occupied we have \( r = 0 \) so that there is no cross coherences at any time, Rabi frequency and decay rate, i.e. independent of \( F \). However we obtain the maximal \( r \) value of 1/2 for the symmetric initial condition of equal amplitudes for each ground state. This can be readily understood when it is pointed out that the pumping mechanism, the laser light is linearly polarized along the \( z \)-axis, so that only the \( |1\rangle \rightarrow |3\rangle \) and \( |2\rangle \rightarrow |4\rangle \) transitions are driven coherently. In order to build up for instance the cross coherence \( \langle \sigma_{14} \rangle \), the \( |4\rangle \) excited state needs to be populated from the \( |2\rangle \rightarrow |4\rangle \) transitions, with the level of occupation depending on the relative occupation of the \( |2\rangle \) ground state with the \( |1\rangle \) ground state. When the ground state amplitudes are initially equal we obtain the largest build up of cross coherence. Conversely when the ground state \( |1\rangle \ (|2\rangle) \) is initially un-occupied then there is no coherent pumping of its corresponding excited state \( |3\rangle \ (|4\rangle) \) so that no coherence is built up.

Figure 2.4 shows a mesh plot of \( F \) as a function of time and pump strength, cross sections are also displayed in fig. 2.5 at fixed pump strengths of \( \Omega/\gamma = 1/8, 1/4, 3/8 \) and 1/2 for clarity. We see that the general behaviour is a
2.3. TRANSIENT COHERENCES

Figure 2.4: Mesh plot of $F(\Omega, \gamma, t)$ for weak pump strengths. The time scale are in units of $\gamma^{-1}$. Note the general damping behaviour with a hint of an oscillations around $\Omega/\gamma = 0.9.$

rapid rise followed by a gentle decay, with the peak value higher as the pump strength increases. The peak is reached earlier and the subsequent decay is quicker at higher pump strengths. Note that at $\Omega/\gamma = 0$, $F$ is zero for all time.

Looking at the mesh plot we see that there is a hint of oscillations appearing from about $\Omega/\gamma = 0.8$ to 1.0. The boundary between the critical damping and oscillatory behaviours occurs at $\alpha = 0$, that is, when $\Omega/\gamma = 1/2$. The oscillatory nature can be shown in the case of $\alpha < 0$ by writing $\sqrt\alpha = i\sqrt{|\alpha|}$ so that

$$F(\Omega, \gamma, t) = \Omega e^{-\gamma t} \left[ \frac{\gamma}{|\alpha|} + \frac{1}{\gamma} \left( 1 - \frac{4\Omega^2}{|\alpha|} \right) \cos \left( \sqrt{|\alpha|} t \right) + \frac{1}{\sqrt{|\alpha|}} \sin \left( \sqrt{|\alpha|} t \right) \right],$$

for higher pump strengths (larger $|\alpha|$), the oscillations become more rapid, this can be seen in fig. 2.6.

It is also of interest to calculate the populations of the excited states $|3\rangle$ and $|4\rangle$. Note that we will now fix $r$ to its maximal value of 1/2 to consider the case of symmetric initial amplitudes of the ground state, the relative phase is however still unrestricted. Performing a similar calculation as above
Figure 2.5: Plot of $F$ versus time (in units of $\gamma^{-1}$) for various pumping strengths. The solid, dashed, dash–dotted and dotted curves correspond to $\Omega/\gamma$ equal to $1/2$, $3/8$, $1/4$ and $1/8$ respectively. These pump rates of less than $\gamma/2$, exhibit critical damping behaviour.

starting with the inhomogeneous equations (Eq. 2.16) instead, and referring to appendix A.2 for the details we arrive at the following solutions

\[
\langle \sigma_{33} \rangle (t) = \frac{\Omega^2}{4\Omega^2 + 2\gamma^2} \left\{ 1 - \frac{e^{-3\gamma t/2}}{2} \left[ e^{\sqrt{\beta}t} + e^{-\sqrt{\beta}t} + \frac{3\gamma}{2\sqrt{\beta}} \left( e^{\sqrt{\beta}t} - e^{-\sqrt{\beta}t} \right) \right] \right\}
= \langle \sigma_{44} \rangle (t)
\]

(2.45)

with

\[
\beta = \frac{\gamma^2}{4} - 4\Omega^2
\]

(2.46)

when $\beta$ is non–zero and

\[
\langle \sigma_{33} \rangle (t) = \langle \sigma_{44} \rangle (t)
= \frac{4\Omega^2}{9\gamma^2} \left[ 1 - e^{-3\gamma t/2} \left( 1 + \frac{3\gamma t}{2\Omega^2} \right) \right].
\]

(2.47)
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Figure 2.6: Mesh plot of $F(\Omega, \gamma, t)$ for a higher range of pump strengths. Beyond pump values of $1/2$ oscillations can be seen, with the frequency increasing as the pump strength is increased. Note that the time range (in units of $\gamma^{-1}$) is shorter than that of fig. 2.4 in order to resolve the more rapid oscillations at $\Omega/\gamma = 5$. 
Figure 2.7: Plot of $F$ versus time (in units of $\gamma^{-1}$) for various pumping strengths. The thick solid, dashed and thin solid lines correspond to values of $\Omega/\gamma$ equal to $1/2$, $2$ and $10$ respectively. Pump rates greater than $\gamma/2$ exhibit oscillations.

for $\beta = 0$. A mesh plot of $\langle \sigma_{33} \rangle (t)$ is shown in fig. 2.8. It has the same range of pump strengths and time scale as fig. 2.6, but we have rotated the plot so that we can see the case of zero $\Omega/\gamma$ closest to us, the population quickly increases as we increase the pump strength to a plateau (at long times) where the value is close to $1/4$ consistent with the previous steady-state solutions. At short times, however, oscillations appear when $\Omega/\gamma > 1/4$ ($\beta = 0$) with values above $1/4$ and below zero are possible. This behaviour is transient and damps away. We will return to transient behaviour in chapter 3 with a numerical treatment of the problem where we are able to study the transient interference pattern for various initial states.
2.3. TRANSIENT COHERENCES

Figure 2.8: Mesh plot of $\langle \sigma_{33} \rangle$ as a function of the pump strength and time (in units of $\gamma^{-1}$). Note that this plot is rotated by about 180 degrees from fig. 2.4 in order to show clearly the shape at low pump strengths (small $\Omega/\gamma$). We see similar oscillatory behaviour as the function $F$ but the oscillations are centred on a plateau of about $1/4$. 
CHAPTER 2. INTERFERENCE OF TWO TRAPPED ATOMS

2.4 The Interference Pattern

We have calculated in section 2.2 the steady-state solutions for the atomic coherences and populations for the case of a single four-level atom interacting with a classical laser light field linearly polarized along the \( z \) axis. In this section we wish to use these results to calculate the far-field interference pattern from two such atoms in the case when their separation is large enough that they may be treated independently.

Let us consider a screen placed in the far-field (large \( y \)) and oriented in the \( xz \) plane. In the far-field, and in the paraxial approximation, the electric field at a point on the screen at time \( t \) due to an oscillating dipole is proportional to the projection onto the plane of the screen of the acceleration of the dipole moment at the retarded time \( t - \tau \), where \( \tau \) is the light travel time from the dipole to the observation point. We shall label a point on the screen by \( (\tau_1, \tau_2) \) where these are the light travel times from each of the atoms to the point on the screen. The intensity of the light at this point is

\[
I(\tau_1, \tau_2) \propto \left( E_x^\dagger E_x + E_z^\dagger E_z \right), \tag{2.48}
\]

where

\[
E_k^\dagger(t; \tau_1, \tau_2) \propto e^{-i\omega(t-\tau_1)}\mu_k^\dagger + e^{-i\omega(t-\tau_2)}U_k^\dagger, \tag{2.49}
\]

for \( k \in \{x, z\} \), \( \mu \) and \( U \) are the atomic dipoles of the first and second atoms respectively and \( \omega \) is the angular frequency of the laser light. If we use the notation \( \sigma_{ij} \) for the atomic operators of the first atom and \( \Sigma_{ij} \) for those the second, we note that the cross terms like \( \langle \mu_2^\dagger U_2^\dagger \rangle \) are zero in the steady-state since

\[
\langle \mu_2^\dagger U_2^\dagger \rangle_{ss} \propto \langle (\sigma_{41} + \sigma_{32})(\Sigma_{14} + \Sigma_{23}) \rangle_{ss}
= \langle (\sigma_{41})_{ss} + (\sigma_{32})_{ss}((\Sigma_{14})_{ss} + (\Sigma_{23})_{ss}) \rangle
= 0. \tag{2.50}
\]

We have factorized the products above since we have assumed that the atoms are independent. The intensity of the interference pattern when all the light
2.4. THE INTERFERENCE PATTERN

is detected is given by

\[ I_{\text{unpol}}(\tau_1, \tau_2) \propto \left( \mu_2^+ \mu_2^- + \mathcal{U}_2^+ \mathcal{U}_2^- + \mu_2^+ \mu_2^- + \mathcal{U}_2^+ \mathcal{U}_2^- \right) \]
\[ + \left( \mu_2^+ \mathcal{U}_2^+ \right) \exp(i[\omega(\tau_1 - \tau_2) - \phi_0]) \]
\[ + \left( \mathcal{U}_2^+ \mu_2^- \right) \exp(-i[\omega(\tau_1 - \tau_2) - \phi_0]), \] (2.51)

where the additional phase term \( \phi_0 \) has been included to allow for the phase of the incident light to differ at the positions of the two ions. If this is normalised by twice the intensity due to single atom fluorescence, we obtain

\[ I_{N,\text{unpol}}(\tau_1, \tau_2) = 1 + V_{\text{unpol}} \cos[\omega(\tau_1 - \tau_2) - \phi_0], \] (2.52)

where the visibility is

\[ V_{\text{unpol}} = \frac{1}{2} \left( \frac{\Delta^2 + \gamma^2}{\Delta^2 + \gamma^2 + 2} \right) \] (2.53)

We note that this interference pattern is the sum of an incoherent term and a coherent term and so the visibility is always less than one half. If we introduce a polarizer at angle \( \eta \) to the z axis in front of the screen, it is possible to separate out the two components. The expression for the intensity is then

\[ I_{\text{pol}}(\tau_1, \tau_2; \eta) \propto \left( \mu_2^+ \mu_2^- + \mathcal{U}_2^+ \mathcal{U}_2^- \right) \sin^2 \eta + \left( \mu_2^+ \mu_2^- + \mathcal{U}_2^+ \mathcal{U}_2^- \right) \cos^2 \eta \]
\[ + \left( \mu_2^+ \mathcal{U}_2^+ \right) \cos^2(\omega(\tau_1 - \tau_2) - \phi_0) \]
\[ + \left( \mathcal{U}_2^+ \mu_2^- \right) \sin^2(\omega(\tau_1 - \tau_2) - \phi_0) \] (2.54)

normalising this as before yields

\[ I_{N,\text{pol}}(\tau_1, \tau_2; \eta) = 1 + V_{\text{pol}}(\eta) \cos[\omega(\tau_1 - \tau_2) - \phi_0] \] (2.55)

where the visibility is now given by

\[ V_{\text{pol}}(\eta) = \left( \frac{\Delta^2 + \gamma^2}{\Delta^2 + \gamma^2 + 2} \right) \cos^2 \eta. \] (2.56)

Thus if we detect the z polarized light by setting \( \eta = 90^\circ \), we see that the visibility is zero for any field strength and detuning. On the other hand if
Figure 2.9: Visibility plotted against field strength for three different detunings. We have the axis of the polarizer aligned with the $z$ axis so that the angle $\eta$ is equal to zero. The solid line (a) is the case where the detuning is zero, the dashed line (b) is when $\Delta = 5\gamma$ and the dashed-dotted line (c) is for $\Delta = 10\gamma$.

If the polarizer is set along the $z$ axis ($\eta = 0$) then the visibility is non-zero. This visibility is plotted against field strength in fig. 2.9 for various detunings with the polarizer aligned with the $z$ axis ($\eta = 0$). The maximum theoretical visibility of one occurs for all detunings in the limit of zero field strength. As the field strength is increased the visibility falls towards zero, with the detuning determining the rate of this drop.
2.5 Comparison with experimental parameters

In the experiment of Eichmann et al. [10] the incident laser is linearly polarized at an angle of 62° degrees to the z axis in the yz plane. This gives a y component to the atomic dipole which is not considered in the above calculation, but this does not affect the field on a screen in the xz plane. The inclination also introduces a non-zero phase difference $\phi_o$ between the two atoms which has be included in our calculations. For more general orientations of the incident field, we resort to numerical simulations as described later. Another difference between our analysis and the experiment is that instead of rotating a polarizer in the output field, a fixed Brewster plate is used and the polarization of the incident light is changed. Our analysis is not valid for incident light polarized along the x direction, however our use of a rotating polarizer gives some insight into the polarization-sensitive measurement.

Figure 2.10 is the generic Young's double slit configuration, using elementary trigonometry one finds the following expression for the path difference

$$|r_1 - r_2| = \sqrt{2L} \sqrt{1 + \frac{d^2}{4L^2} - \sqrt{\left(1 + \frac{d^2}{4L^2}\right)^2 - \frac{d^2}{L^2} \sin^2 \theta}}$$

which in the limit of $d^2/L^2 \ll 1$, i.e. in the Fraunhofer approximation we obtain the usual expression

$$|r_1 - r_2| \approx d \sin \theta.$$
Figure 2.10: Generic configuration of Young’s double slit; vectors $\vec{r}_1$ and $\vec{r}_2$ are defined from their respective slits to the point $P$ on the screen.

$\phi$ is simply

$$\theta = \phi - 28^\circ. \quad (2.58)$$

The path difference is then

$$|\vec{r}_1 - \vec{r}_2| \approx d \sin (\phi - 28^\circ). \quad (2.59)$$

We use this expression to now write

$$\omega(\tau_1 - \tau_2) = \frac{\omega}{c} |\vec{r}_1 - \vec{r}_2|$$

$$\approx \frac{2\pi d}{\lambda} \sin (\phi - 28^\circ) \quad (2.60)$$

where $c$ is the speed of light and $\lambda$ is the wavelength of the fluorescence. Thus by taking into account the geometry of the experiment and using Eq. 2.55, we expect that the measured interference pattern should be

$$I_{N,\text{unpol}} = 1 + \frac{1}{2} \left( \frac{\Delta^2 + \gamma^2}{\Delta^2 + \gamma^2 + 2} \right) \cos \left[ \left( \frac{2\pi d}{\lambda} \sin (\phi - 28^\circ) - \phi_0 \right) \right] \quad (2.61)$$

when all the light is collected and

$$I_{N,\text{pol}} = 1 + \cos^2 \eta \left( \frac{\Delta^2 + \gamma^2}{\Delta^2 + \gamma^2 + 2} \right) \cos \left[ \left( \frac{2\pi d}{\lambda} \sin (\phi - 28^\circ) - \phi_0 \right) \right] \quad (2.62)$$
2.6. SUMMARY

Figure 2.11: Relationship between angles $\theta$ and $\phi$ with the incident laser light.

when a polarization-sensitive measurement is made. $d$ is the distance between the atoms, $\phi$ is angle relative to the incident light and $\lambda$ is the wavelength of this light.

The visibility measured in the experiment when all the light was collected had a value of 0.2. The maximum visibility from Eq. 2.61 is one half. This less than "ideal" measured visibility is not surprising since many detrimental factors have not been included in the analytical treatment. Factors such as stray light entering the detector and quantum jumps to other levels in the mercury ions so that we may observe only the fluorescence from one ion [21]. The motion of the ions in the trap also smear out the interference pattern, this will be discussed in detail in chapter 3.

2.6 Summary

We have analyzed the experiment by Eichmann et al. and presented theoretical models for understanding the degree of interference observed for different incident and detected polarizations. For a simple geometry of the incident field, it is possible to solve the optical Bloch equations analytically for a single four-level atom in the steady-state, and to extend this result to the case
of two atoms, assuming that they radiate independently. From this analysis, the fluorescent light is seen to consist of a coherent component which exhibits interference and an incoherent component which does not. These components may be separated out by polarizers before the light is detected.
Chapter 3

Numerical simulation of the system

3.1 Introduction

In this chapter, we describe the use of a Monte Carlo wave function method for numerical simulation of the system. This approach allows greater flexibility in specifying the geometry of the incident light and also allows us to include the photodetection process as part of the simulation. By doing this, we no longer simply relate the far field intensity of the fluorescent light to the moments of the atomic variables, but explicitly consider how atomic quantum jumps introduce photons into electromagnetic field modes propagating in various directions. The advantage of this approach is that it allows the atoms to be separated by arbitrary distances and takes into account the dipole–dipole correlations between the atoms. This simulation approach is very closely related to the actual experimental situation as it gives a time-resolved classical record of photocounts at different angles which gradually build up into an interference pattern.
Figure 3.1: The spherical coordinate system used in the Monte Carlo simulations with the $\varepsilon_\theta$ and $\varepsilon_\phi$ unit vectors defined. We also have atoms one and two at positions $\vec{r}_1$ and $\vec{r}_2$ respectively.

### 3.2 The numerical model

For simplicity, we assume idealized unit efficiency photodetectors in the far field covering the entire $4\pi$ steradians surrounding the two atoms. Each detector is assumed to be polarization-sensitive and resolves detected photons into two orthogonal linear polarizations. Working in spherical polar coordinates with polar angle $\theta$ and azimuthal angle $\phi$ (see fig. 3.1), we use the unit vectors $\varepsilon_\theta$ and $\varepsilon_\phi$ as a basis for the polarization of the emitted photon. These unit vectors are related to Cartesian coordinates by

$$
\varepsilon_\theta = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z} \\
\varepsilon_\phi = -\sin \phi \hat{x} + \cos \phi \hat{y}
$$

where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the Cartesian unit vectors. The system Hamiltonian for the two atoms is just the sum of the individual Hamiltonians for each atom

$$
H_{\text{sys}} = \frac{1}{2} \hbar \Delta (\sigma_{44} + \sigma_{33} - \sigma_{22} - \sigma_{11} + \Sigma_{44} + \Sigma_{33} - \Sigma_{22} - \Sigma_{11})
$$
3.2. THE NUMERICAL MODEL

\[ +\hbar\Omega_2 (\sigma_{24} + \sigma_{42} - \sigma_{13} - \sigma_{31} + \Sigma_{24} + \Sigma_{42} - \Sigma_{13} - \Sigma_{31}) \]
\[ +\hbar\Omega_3 (\sigma_{41} + \sigma_{23} + \Sigma_{41} + \Sigma_{23}) + \hbar\Omega_4 (\sigma_{14} + \sigma_{32} + \Sigma_{14} + \Sigma_{32}), \]

(3.3)

where we have set \( \Omega_{\pm} = \Omega_x \pm i\Omega_y \), and where \( \Omega_x, \Omega_y \) and \( \Omega_z \) are the Rabi frequencies corresponding to the \( x, y, \) and \( z \) components of the electric field

\[ \Omega_k = \frac{\mu E_k}{2\hbar}, \quad k = x, y, z, \]

(3.4)

where \( \mu \) is the dipole matrix element and each atom is considered to feel the same electric field \( \vec{E} \). We are assuming here that the incident light is propagating at right angles to the line joining the two atoms. The relaxation superoperator is in Lindblad form

\[ \mathcal{L}_{\text{relax}}(\rho_s) = -\frac{1}{2} \sum_m \left( C_m^\dagger \rho_m C_m + \rho_s C_m^\dagger C_m \right) + \sum_m C_m \rho_s C_m^\dagger, \]

(3.5)

where the collapse operators \( C_m \) correspond to the couplings between system and the baths. When the atoms are far apart, they may be regarded as coupling to independent baths and so collapses associated with one atom may be distinguished from those associated with the other. If we explicitly include the photodetection scheme, the appropriate collapse operators are parameterized by the direction and polarization of the outgoing photon. For spontaneous emission from a single two-level atom, the appropriate collapse operators (including recoil) are [23]

\[ C_{\Omega,\epsilon} = \left( \frac{3\gamma}{8\pi} \right)^{1/2} \exp \left( -i\vec{k} \cdot \vec{R} \right) \left( \epsilon^* \cdot \vec{S}^\dagger \right), \]

(3.6)

where \( \vec{R} \) is the atomic position operator, \( \vec{k} \) is the wave number of the outgoing photon travelling in direction \( \Omega \), \( \vec{S}^\dagger \) is an operator proportional to the atomic lowering operator \( (\vec{S}^\dagger \propto \sigma^-) \) and \( \epsilon \) labels the outgoing polarization. For our situation with two fixed atoms, \( \vec{R} \) may be replaced by the locations of the two atoms \( \vec{r}_1 \) and \( \vec{r}_2 \) (see fig. 3.1) and both atoms couple to each mode so that

\[ C_{\Omega,\epsilon} = \mathcal{N} \left\{ \exp \left( -i\vec{k} \cdot \vec{r}_1 \right) \left( \epsilon^* \cdot \vec{S}_1^\dagger \right) + \exp \left( -i\vec{k} \cdot \vec{r}_2 \right) \left( \epsilon^* \cdot \vec{S}_2^\dagger \right) \right\} \]

(3.7)
where $N$ is a normalisation factor. $\vec{S}_1^\dagger$ and $\vec{S}_2^\dagger$ are proportional to the atomic dipole operators for the first and second atoms respectively. The Cartesian components of these operators are

$$\vec{S}_1 \propto \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z} \quad (3.8)$$

$$\vec{S}_2 \propto \Sigma_x \hat{x} + \Sigma_y \hat{y} + \Sigma_z \hat{z} \quad (3.9)$$

where the $\sigma_i$'s are given by

$$\sigma_x = \sigma_{14} + \sigma_{23} \quad (3.10)$$

$$\sigma_y = -i (\sigma_{14} - \sigma_{23}) \quad (3.11)$$

$$\sigma_z = \sigma_{24} - \sigma_{13}, \quad (3.12)$$

and the $\Sigma_i$'s related to the $\Sigma_{ij}$'s in a similar way. We see explicitly from these expressions that when a quantum jump (i.e., a photodetection) occurs, this may come from either atom, the coefficient being dependent on the polarization and the appropriate phase factor.

Between the times of the quantum jumps, the atomic wave function is evolved using the non-Hermitian effective Hamiltonian given by

$$H_{eff} = H_{sys} - \frac{i\hbar}{2} \sum_m C_m^\dagger C_m. \quad (3.13)$$

On the right-hand-side, the summation is responsible for the decrease in the norm of the Monte Carlo wave function. Its expectation value is proportional to the intensity of the fluorescent light and the probability that a quantum jump occurs. For our photodetection model, this eventually evaluates to

$$\int \sum_{\vec{k},\ell} C_{\ell,\ell}^\dagger C_{\ell,\ell} d\Omega = \frac{8\pi}{3} N^2 \{ \left( \sigma_x^1 \sigma_x + \sigma_y^1 \sigma_y + \sigma_z^1 \sigma_z + \Sigma_x^1 \Sigma_x + \Sigma_y^1 \Sigma_y + \Sigma_z^1 \Sigma_z \right)$$

$$+ A(\alpha) \left( \sigma_x^1 \Sigma_x + \sigma_y^1 \Sigma_y + \Sigma_x^1 \sigma_x + \Sigma_y^1 \sigma_y \right)$$

$$B(\alpha) \left( \sigma_z^1 \Sigma_z + \Sigma_z^1 \sigma_z \right) \} \quad (3.14)$$

\(^1\)see Mølmer et al. [23]

\(^2\)See appendix B for evaluation of this integral.
3.2. THE NUMERICAL MODEL

where

\[ A(\alpha) = \frac{3}{8} \left( \frac{4}{\alpha} \sin \alpha - T(\alpha) \right) \]  
(3.15)

\[ B(\alpha) = \frac{3}{4} T(\alpha) \]  
(3.16)

and \( \alpha = |\vec{k}| \cdot |\vec{r}_2 - \vec{r}_1| \) is \( 2\pi \) times the separation between the two atoms in optical wavelengths. The function \( T \) is given by

\[ T(\alpha) = \frac{4}{\alpha^2} \left( \frac{1}{\alpha} \sin \frac{\alpha}{\alpha} - \cos \alpha \right). \]  
(3.17)

The graph of the functions \( A \) and \( B \) are shown in fig. 3.2. At \( \alpha \) equal to zero where the two atoms coincide, both functions are equal to one. As the separation between the atoms is increased, both display damped oscillatory behaviour decaying to zero in the limit of large separations. \( B \) is damped more rapidly and is almost zero for separations of more than three wavelengths, whereas \( A \) has larger amplitude oscillations which can still be clearly seen even after six wavelengths. Notice that \( B \) corresponds to the size of the correlations between the \( z \) components of the atomic dipoles whereas \( A \) determines the size of the correlations between the \( x \) and \( y \) components.

In the limit of large separation between the two atoms, only the first term of Eq. 3.14 remains and we have

\[
\left[ \int \sum_{\ell \perp k} C^\dagger_{\Omega,\ell} C_{\Omega,k} d\Omega \right]_{\alpha \to \infty} = N^2 \frac{8\pi}{3} \left( \sigma_x^1 \sigma_x + \sigma_y^1 \sigma_y + \sigma_z^1 \sigma_z + \Sigma_x^1 \Sigma_x + \Sigma_y^1 \Sigma_y + \Sigma_z^1 \Sigma_z \right) \\
\equiv N^2 \frac{8\pi}{3} \left( \sigma_{44} + \sigma_{33} + \Sigma_{44} + \Sigma_{33} \right). \quad (3.18)
\]

This may be compared with the form expected for two atoms coupling to independent baths for which

\[ \sum_m C^\dagger_m C_m = 2\gamma (\sigma_{44} + \sigma_{33} + \Sigma_{44} + \Sigma_{33}) \]

We see that in the limit of large separation, the atoms behave independently, and that the normalisation condition is \( N = (\gamma/4\pi)^{1/2} \). In the opposite limit

\(^3\text{The function } T(\alpha) \text{ has similar structural form to the dipole-dipole term used by T.G. Rudolph et al. [24] and J. Guo et al. [25] to modify the decay rate of the atoms.}\)
Figure 3.2: The function $A$ is shown by the solid line. The dashed line is the plot of the function $B$ and the scaled parameter $\bar{\alpha} = \alpha/(2\pi)$, is the number of wavelengths (of the incident light) separating the two atoms.

of the two atoms coinciding

\[
\left[ \int \sum_{\ell, k} C_{\ell, k}^\dagger C_{\ell, k} d\Omega \right]_{\alpha \to 0} = N^2 \frac{8\pi}{3} \left( \sigma_x^1 \sigma_x + \sigma_y^1 \sigma_y + \sigma_z^1 \sigma_z + \Sigma_x^1 \Sigma_x + \Sigma_y^1 \Sigma_y + \Sigma_z^1 \Sigma_z \right.
\]
\[
+ \sigma_x^1 \Sigma_x + \Sigma_x^1 \sigma_x + \sigma_y^1 \Sigma_y + \Sigma_y^1 \sigma_y + \sigma_z^1 \Sigma_z + \Sigma_z^1 \sigma_z \bigg)
\]
\[
= N^2 \frac{8\pi}{3} \left( |\sigma_x + \Sigma_x|^2 + |\sigma_y + \Sigma_y|^2 + |\sigma_z + \Sigma_z|^2 \right).
\]

This gives four times the decay rate of a single atom instead of two times for the case of large separation since the interactions between the atoms grow as the separation decreases until at zero separation the size of these dipole–dipole terms become of the same order as the population terms. This gives rise to superfluorescence where the two atoms behave as one. This is clearly illustrated by the factorization of the above expression into the coherent sum.
3.3. RESULTS

of the two atoms.

In order to carry out the Monte Carlo simulation, a wave function is evolved using the Schrödinger equation with the non-Hermitian effective Hamiltonian until the square of the norm of the wave function reaches a threshold drawn from a uniform distribution lying between zero and one. At this time, it is necessary to select one of the possible quantum jumps. The probability density of detecting a fluorescence photon travelling in direction $(\theta, \phi)$ with polarization $\epsilon_i$ is given by

$$P_{\epsilon_i}(\theta, \phi) = \frac{\langle \hat{C}_{\Omega,\epsilon_i} \hat{C}_{\Omega,\epsilon_i} \rangle}{\int \sum_{\epsilon_\text{\theta}} \hat{C}_{\Omega,\epsilon_\text{\theta}} \hat{C}_{\Omega,\epsilon_\text{\theta}} d\Omega} \quad i = \theta \text{ or } \phi.$$  \hspace{1cm} (3.20)

Conventionally, a Monte Carlo simulation requires many runs to be carried out to obtain the desired statistics of the system from an ensemble of quantum trajectories. For the problem being considered, the result of one quantum trajectory gives the photo-emission record for the two atom system, and we can let this single trajectory run as long as it is necessary to obtain the required number of photons.

3.3 Results

The raw output of these simulations is a list of the emission times, directions and polarizations of each emitted photon. The direction is represented by the two angles $\theta$ and $\phi$, with the polarization aligned either with the unit vector $\epsilon_\theta$ or $\epsilon_\phi$.

The wavelength of the incident laser light used in all of the following numerical simulations was 194 nm. In the first simulation we have chosen parameters which gave maximum visibility of the interference pattern. The laser field was weak, the ratio of Rabi frequency to decay rate being 0.2. The atomic separation was set at 0.5 $\mu$m so that five intensity peaks would be observed as $\theta$ is varied through 180°. This gives a sufficient number of points in order to clearly distinguish each peak with the number of photodetections.
Figure 3.3: Plots of intensity as a function of both angles $\theta$ and $\phi$. The incident light is linearly polarized along the $z$ axis. These plots represent the light pattern distributed over the entire sphere around the two atoms.

The incident light is propagating along the $y$ axis.

The intensity pattern as seen on a sphere around the ions is shown in figures 3.3 and 3.4 for incident light polarized in the $z$ direction. Since the system is symmetric about the $z$ axis, fig. 3.3 shows an interference pattern as $\theta$ (but not $\phi$) is varied. Fig. 3.4 is another simulation in which the incident light is linearly polarized along the $x$ axis. Even though the system is no longer symmetric about the $z$ axis, there is little dependence of the interference on the angle $\phi$. We shall henceforth only plot the interference patterns as functions of $\theta$, and integrate over the angle $\phi$. This integration over $\phi$ modulates the intensity distribution with a sinusoidal envelope since the solid angle integrated over at the equator ($\theta = 90^\circ$) is greater than at the poles ($\theta = 0^\circ$ or $180^\circ$).
3.3. RESULTS

Figure 3.4: Similar plot as fig. 3.3 but incident light is linearly polarized along the $x$ axis.

For the case of incident light polarized in the $z$ direction, fig. 3.5(a) shows the intensity distribution for $\epsilon_\theta$ polarized light as a function of angle $\theta$ from a simulation consisting of one hundred thousand quantum jumps. Out of these, about half were detected with $\epsilon_\theta$ polarization. The histogram\(^4\) is computed using a bin width of $1^\circ$. An interference pattern is clearly visible. From Eq. 2.56, we expect a visibility of $25/27$ or about 0.9 but the visibility measured from the simulation is approximately 0.7. However Eq. 2.56 is only valid in the paraxial approximation and only considers photons polarized in the $z$ direction. When this approximation breaks down (i.e., when $\theta$ is no longer close to $90^\circ$), the $\epsilon_\theta$ polarization will include contributions from the components of the atomic dipole other than $\sigma_z$. These will reduce the total

\(^4\)We have plotted the intensity distributions as histograms without the internal lines for clarity.
visibility as we do not expect interference from the $x$ and $y$ components of the atomic dipole since they are not coherently driven. If we now look at the orthogonal polarization (the $\epsilon_\phi$ polarization) as shown in fig. 3.5(b), there is no interference pattern. The $\epsilon_\phi$ polarization is always perpendicular to the $z$ axis for any angle $\theta$ so only the $x$ and $y$ components of the atomic dipoles contribute.

We now consider the case where the incident laser light is polarized along the $x$ axis instead of the $z$ axis. The histogram for the intensity of $\epsilon_\theta$ polarized light versus $\theta$ is shown in fig. 3.6(a). There is no obvious interference pattern now since the $\theta$ polarization consists mainly of the $z$ component of the atomic dipole for angles around $\theta = 90^\circ$. Small peaks can be seen at about 40 and 140 degrees, which are due to the $x$ component of the atomic dipole which contribute in larger amounts as the $\theta$ deviates significantly from 90 degrees.

Fig. 3.6(b) displays interference as expected since the $\phi$ polarization only consists of the $x$ and $y$ components of the atomic dipole. Since the incident light is linearly polarized along the $x$ axis, the $y$ component of the scattered light is not coherent and the visibility should be less than seen in 3.5 (a). From the central peaks the visibility is about one third, approximately half that for the case of light polarized in the $z$ direction.

### 3.4 Effects of motion

We have so far ignored the effects of the motion of the ions as this has been secondary to our goal of demonstrating how the visibility of the interference pattern varies with polarization. However it is a significant effect in the experiment performed by Eichmann et al. We shall model the motion of the ions as harmonic oscillators for each of the three main modes, one involving stretching and two involving tilting [10, 19]. The stretching mode involves motion of the atoms along the $z$ axis whereas the two tilting modes involve motion in the $xy$ plane. Thus the stretching mode alters the distance between the ions and the period of the interference pattern while the tilting modes
3.4. EFFECTS OF MOTION

Figure 3.5: Histogram of the number of detected photons with (a) $\theta$ and (b) $\phi$ polarizations as a function of the angle $\theta$. The incident laser light was linearly polarized along the $z$ axis. The detuning was set to zero, the atomic separation was 0.5 $\mu$m and the decay rate $\gamma = 5\Omega$. The number of photons detected was 50,086 and 49,914 for the $\theta$ and $\phi$ polarizations respectively.
Figure 3.6: Histogram of the number of detected photons with (a) $\theta$ and (b) $\phi$ polarizations as a function of the angle $\theta$. The incident laser light was linearly polarized along the $x$ axis. The detuning was set to zero, the atomic separation was 0.5 $\mu m$ and the decay rate $\gamma = 5\Omega$. The number of photons detected was 49,770 and 50,230 for the $\theta$ and $\phi$ polarizations respectively.
3.4. EFFECTS OF MOTION

change the angle $\theta$ of the photon emission displacing the interference pattern. Both of these effects smear the interference pattern, reducing the visibility.

The temperature of the ions in the trap were around the milli–Kelvin region and the frequency of oscillation of the modes were of order of a MHz. By comparing the thermal kinetic energy of an ion $k_B T/2$ with the separation of the trap energy levels, $\hbar \omega$, we find that $k_B T/(2\hbar \omega) \approx 10^2$ and so the motion may be treated classically. The ions experience micromotion within the trap and the recoil momentum of the photon emissions may be neglected.

In the actual experiment, the values used for the separation between the ions was around $5 \mu m$. These values are about ten times larger than the values used in the simulations. Thus we expect to see over fifty peaks over a range of one hundred and eighty degrees. We carried out a simulation with an ion separation of $3.35 \mu m$, where one hundred thousand jumps was performed. In fig. 3.7(a), we show the results for incident light which is linearly polarized along the $z$ axis, with no motion of the ions. The peaks are poorly resolved and an excessive length of time would be required to obtain sharper resolution for such finely separated peaks. Note that in this graph, the detector is taken to be polarization insensitive and we count all the photons. Fig. 3.7(a) shows the correct period (nine peaks from 75 to 105 degrees) with a visibility of about 0.35 which is lower than the predicted value of 0.46 using Eq. 2.53.

The geometry of the incident light in the simulation differs from that of the experiment. The light is propagating along the $g$ axis in the simulation whereas the experiment has the incident light at an angle of $62^o$ relative to the $z$ axis. However, the experiment does not detect light over the entire $4\pi$ steradians, only a small portion is actually observed. This observation region is a small disk perpendicular to the $y$ axis. By summing over the $\phi$ angle in the simulation and considering only $\theta$ angles close to the $y$ axis we have an approximation of the experimentally observed region of interest.

The micromotion is simulated in two ways. We keep track of the oscillatory motion of the stretch mode starting with some arbitrary initial value.
Since we are interested in the long time (steady-state) regime the actual initial value we choose is unimportant. The previous numerical procedure is re-used but the atomic separation is altered for every photon emission according to the displacement of the stretch mode. This is valid because the characteristic time scale of the stretch mode $\tau_{\text{stretch}} \sim 10^{-6}s$ is far longer than the atomic half-life $\tau_{1/2} \sim 10^{-8}s$. The steady-state regime is satisfied since the duration of a simulation, $\tau_{\text{sim}}$, producing one hundred thousand photodetections is approximately $10^{-3}s$ which is much longer than the time scale of the stretch mode. The effects of the tilting motion is to shift the interference pattern. Thus during the simulation whenever we calculate a photodetection, the direction of the photon needs to be shifted according to the current displacement of the tilting modes. The $x$ and $y$ displacements are randomly generated from a harmonic oscillator probability distribution. In hindsight we could have generated $z$ displacements from a harmonic oscillator distribution as well, however in the simulations we have chosen to keep track of the stretch mode starting from some arbitrary value. Both methods are equivalent for the steady-state regime. The requirements on the time scales involved is

$$\tau_{\text{sim}} \gg \tau_{\text{stretch}} \gg \tau_{1/2}. \quad (3.21)$$

As expected, when classical motion is included in the simulation, the visibility of the interference is reduced, see fig. 3.7(b). We see an interference pattern with the peaks placed in the same positions as fig. 3.7(a) but with a reduced visibility of about 0.28. The visibility for individual peaks degrades as one deviates from the central peak at ninety degrees. This maybe explained by the fact that we have modelled the motion along the $z$ axis as a stretching where the centre-of-mass remains stationary, thus the central peak is in the same position whatever the period. As one deviates to the sides, a small change in period has an increasingly larger effect on the position of the peak. The parameters used in this simulation for the motion were: stretch frequency $\omega_{\text{stretch}}/2\pi = \sqrt{3}$ MHz, tilt frequencies $\omega_{\text{tilt}}/2\pi = \sqrt{5}/2$ MHz (for
3.4. EFFECTS OF MOTION

Figure 3.7: Histogram of the number of detected photons irrespective of their polarizations as a function of $\theta$ (a). Classical motion is included for (b). Both simulations used atomic separations of 3.35 \( \mu m \) and decay rates of $\gamma = 5\Omega$. 
both modes), radial confinement was 30 nm and the amplitude of the stretch mode was 150 nm. Experimental results were shown in the paper by Eichmann et al. for this set of parameters. The visibility was about 0.2 which is less than the value obtained in the simulation. The greater visibility in the simulation is not unexpected since the effects of stray light and the fact that $^{198}\text{Hg}^+$ ions are not ideal four-level atoms would further reduce the visibility.

### 3.5 The Transient Regime

So far we have investigated the steady-state regime which corresponds to the experimental situation. However, in the transient regime of a modified version of the experiment it is possible to consider a valid which-path argument which explains the disappearance of the interference pattern. This loss of the fringe visibility is connected to the information of which atom has undergone an emission being somehow recorded in the system. This knowledge can be obtained for some initial atomic states but not others.

In this section, we assume that the two atoms are initialized to some known state and then are exposed to the incident light. When the first fluorescence photon is detected, this "thought experiment" is stopped and the atoms are re-initialized to the original state. Over many trials, an interference pattern may develop in the far field, and it is this pattern which is of interest. Note this is not possible in the current experiment. The procedure we have outlined above represents a high degree of control on the preparation of the initial electronic configuration of the atoms and its repeatability which is not easily obtainable in an experimental situation.

Let us first consider the case where the initial joint atomic state is $|\psi\rangle = |1, 1\rangle$ and the incident light is polarized in the $z$ direction so that the $1 \leftrightarrow 3$ transition is driven. When one of the atoms has been excited to the state 3, it may decay either to the state 1 or to the state 2. If the decay is to state 1, the final state of the atoms is again $|1, 1\rangle$ and it is not possible to tell in principle from which atom the light came from. We thus expect that the
3.5. THE TRANSIENT REGIME

detection of light polarized in the $\epsilon_\theta$ direction which is sensitive to radiation from the $3 \rightarrow 1$ transition will exhibit interference. On the other hand, if the excited atom decays to state 2, the final state of the atoms is either $|2,1\rangle$ or $|1,2\rangle$, leaving a record of the atom responsible for the emission. We thus expect light detected with polarization in the $\epsilon_\phi$ direction to not exhibit any interference at all.

The results of a simulation with this initial state are shown in fig. 3.8(a) for $\epsilon_\theta$ polarization and in fig. 3.8(b) for $\epsilon_\phi$ polarization. The atomic separation was 500 nm and the ratio of Rabi frequency to decay rate is 0.2 as before. As expected, a very distinct interference pattern can be seen in (a) consisting of five peaks. The visibility is about 0.88, this is quite close to the expected theoretical value of about 0.9. For the same set of parameters this transient simulation gives higher visibility than the corresponding steady-state simulation. Also as expected, no interference is seen in (b).

Let us now consider the initial joint atomic state $|\psi\rangle = (|1,1\rangle + |1,2\rangle + |2,1\rangle + |2,2\rangle)/2$. In this case, it is not possible to distinguish the atom responsible for emitting the photon whether it is detected in the $\epsilon_\theta$ or the $\epsilon_\phi$ polarizations. We thus expect to see interference for both polarizations. The result of the simulation for the $\epsilon_\phi$ polarization is shown in fig. 3.9, confirming that this interference does occur in the transient regime. As discussed previously however, since the driving field is polarized in the $z$ direction, this coherence must approach zero in the steady-state and there can be no steady-state interference pattern, although interference does occur in the transient regime.

We have used extreme parameters so far, for the case of the $|1,1\rangle$ state we know for certain that each atom is initially in a particular ground state and for the other initial state $(|\psi\rangle = (|1,1\rangle + |1,2\rangle + |2,1\rangle + |2,2\rangle)/2$) we have no initial preference for which ground state each of the atoms are in. In fig. 3.10 results for the $\epsilon_\phi$ polarization are shown for an initial state $|\psi\rangle = (2|1,1\rangle^2 + |1,2\rangle + |2,1\rangle + |2,2\rangle)/\sqrt{7}$. In this intermediate case, there is some visibility in the transient interference pattern, but this is only about a third,
Figure 3.8: (a) The $\theta$ and (b) $\phi$ polarized light distribution versus the angle $\theta$.

We have calculated 100,000 quantum trajectories in this transient simulation using the initial state $|\psi\rangle = (|1, 1\rangle + |1, 2\rangle + |2, 1\rangle + |2, 2\rangle)/2$. There were 49,988 $\theta$ and 50,012 $\phi$ polarized photons detected respectively.
Figure 3.9: The $\phi$ polarized light distribution versus the angle $\theta$ from the transient simulation using the initial state $|\psi\rangle = |1, 1\rangle$. There were 50,335 $\phi$ polarized photons from the 100,000 quantum trajectories.

lower than that seen in the previous case.

In the transient regime we expect interference in the undriven polarization which does not appear in the steady-state regime for the majority of the possible initial states. The interference we see here is due to the presence of initial coherences. It is similar to the case of two two-level atoms prepared in the excited states, one would eventually decay emitting a photon in the process. The atoms are then re-prepared in the excited states and the process is repeated to build up an intensity pattern.

3.6 Summary

In order to model the actual geometry of the incident laser light's polarization used in the experiment, a numerical technique based on the Monte Carlo
Figure 3.10: The $\phi$ polarized light distribution versus the angle $\theta$ from the transient simulation using the initial state $\ket{\psi} = (2|1,1\rangle + |1,2\rangle + |2,1\rangle + |2,2\rangle)/\sqrt{7}$. There were 50,175 $\phi$ polarized photons from the 100,000 quantum trajectories.
3.6. SUMMARY

wave function method was used. In this method, the photodetection process is simulated so that the interference pattern builds up gradually from a series of detections at various angles. This qualitatively reproduces the experimental results and also allowed us to consider dipole–dipole interactions which become important when the atomic separation is reduced.

To more closely simulate the experimental configuration, classical motion of the ions was included in the simulation. This reduces the visibility of the interference to a value closer to that of the experiment. The residual discrepancy is probably due to the approximation of modelling the ions as four–level atoms and experimental factors such as the presence of stray light on the observation screen.

Finally, we demonstrated the formation of a transient interference pattern in situations where there is no steady–state pattern by resetting the atomic state to a specified initial condition after each photodetection. In this case, a which–path argument can be used to understand the visibility of the transient patterns.
Part II

Trapped dilute-gas Bose-Einstein condensates
Chapter 4

Introduction

Bose–Einstein condensation (BEC) has been the source of various spectacular phenomena, such as superconductivity, superfluidity and laser action. It is one of the most dramatic consequences of quantum statistical mechanics where bosons, particles of integral spin, undergo a phase transition. This occurs when the thermal de Broglie wavelength of the bosons become larger than the spacing between them. Bosons are then stimulated by the presence of other bosons to macroscopically occupy the lowest energy level.

A new class of BEC was discovered by the atom optics community in 1995, with the first experimental condensation of a dilute atomic gas demonstrated by Anderson and co-workers [26] at JILA. Being dilute gases, these condensates are weakly–interacting systems and have generated great interest in both experimenters and theorists as the earlier traditional condensates were interaction–dominated systems. Condensation in dilute atomic gases is a purer system where the quantum statistical effects would be easier to separate from the interactions.

Anderson and co–workers produced a condensate of approximately 2000 spin–polarized $^{87}$Rb atoms in a cylindrically symmetric magnetic trap. A finite condensate fraction first appeared at a temperature of 170 nK with a corresponding density of $2.6 \times 10^{12}\text{cm}^{-3}$. This achievement was quickly

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followed by condensations of $^{23}$Na and $^7$Li at MIT\textsuperscript{2} by Davis and co-workers [28]. Bradley, Sackett and Hulet [27] also observed Bose condensation of $^7$Li at Rice University.

### 4.1 Theory

We will give a brief outline of the commonly used theory in treating dilute atomic BEC. A binary collisional model of the condensation will be introduced and applied to a second-quantized Hamiltonian. A mean-field theory is then utilized to obtain a non-linear Schröedinger equation. A strongly repulsive limit is then introduced. The interested reader is referred to the review article by Parkins and Walls [29] which covers in more detail and scope the body of theoretical work done so far.

Interactions between particles are in general very complicated in BEC, in the case of dilute atomic gases they can be modelled in a more simpler manner than traditional BEC. At very low temperatures, the de Broglie wavelengths of the atoms are very large compared with the interaction potential range. Since the gas is dilute, the atoms are well-separated and rarely meet so that the interactions are weak and a s-wave scattering treatment can be used. That is, only the two-body collisions are important. The interaction potential can then be well-approximated by a hard sphere potential

$$U (\mathbf{r} - \mathbf{r}') = U_0 \delta (\mathbf{r} - \mathbf{r}') ,$$

where $U_0$ is the "interaction strength" defined by

$$U_0 = \frac{4 \pi \hbar^2 a}{m} ,$$

with $m$ being the mass of an atom. The s-wave scattering length is denoted by $a$. We start with the second-quantized Hamiltonian of a system of bosons confined in a trap potential $V_{\text{trap}} (\mathbf{r})$

$$\hat{H} = \int d^3 r \hat{\psi}^\dagger (\mathbf{r}) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} (\mathbf{r}) \right] \hat{\psi} (\mathbf{r})$$

\textsuperscript{2}Massachusetts Institute of Technology.
where $\hat{\psi}(r)$ and $\hat{\psi}^\dagger(r)$ are the boson field operators that annihilate and create a particle at the position $r$ respectively. In order for this $s$-wave scattering two-body collisional model to be valid, one requires that $a^3 \rho \ll 1$. The dilute gas BEC experiments by Anderson and Davis has densities of order $10^{12} - 10^{13}$ cm$^{-3}$, since $^{87}$Rb has a scattering length of about 6 nm we find that $a^3 \rho \simeq 10^{-7} - 10^{-6}$ which satisfies this requirement. The Heisenberg equation of motion for the field operator $\hat{\psi}(r)$ may be derived from Eq. 4.1 as

$$im\hbar \frac{\partial \hat{\psi}(r,t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(r)\right] \hat{\psi}(r,t) + U_0 \hat{\psi}^\dagger(r,t) \hat{\psi}(r,t) \hat{\psi}(r,t),$$

which cannot in general be solved, but in the mean-field treatment, we can write the operator as a mean-field component plus a small correction term

$$\hat{\psi}(r,t) = \langle \hat{\psi}(r,t) \rangle + \tilde{\psi}(r,t)$$

where $\langle \hat{\psi}(r,t) \rangle$ is the condensate wave function corresponding to the mean (classical) field. This field possesses a well-defined phase which can be interpreted as being established from the condensation process via bose-broken symmetry. Corrections to this are left in the operator $\tilde{\psi}(r,t)$ which has an expectation value of zero. This operator describes the quantum and thermal fluctuations about the classical (mean) field. At zero temperature we can assume it effects are negligible and by approximating $\psi(r,t) = \langle \hat{\psi}(r,t) \rangle$ we neglect $\tilde{\psi}(r,t)$, and obtain the non-linear Schröedinger equation or the so-called "Gross–Pitaevski" (GP) equation for the mean field

$$i\hbar \frac{\partial \psi_N(r,t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(r) + NU_0 |\psi_N(r,t)|^2\right] \psi_N(r,t),$$

where $N$ represents the number of atoms in the condensate. Note that there are two conventions for the normalisation of $\psi$, either to $N$ or to unity.

\[3\] For a comprehensive discussion of bose-broken symmetry see, for example, Griffen
Equation 4.2 is normalised such that

$$\int d^3 r |\psi_N (r, t)|^2 = 1.$$  

A time independent form of the GP equation is obtained by substituting $\psi (r, t) = \exp (-i \mu t / \hbar)$ into Eq. 4.2 where $\mu$ is the chemical potential of the condensate to obtain

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} (r) + NU_0 |\psi_N (r)|^2 \right] \psi_N (r) = \mu \psi_N (r).$$  \hspace{1cm} (4.3)

In the non-interacting limit where $U_0$ is set to zero, we see that Eq. 4.3 is just the time independent Schröedinger equation. The chemical potential $\mu$ can be interpreted as the energy required to add a further atom into the condensate. In the strongly–interacting limit, simplifying assumptions can be made to obtain analytical solutions under the so–called Thomas–Fermi (TF) approximation. Under these assumptions, the non–linear term is very much larger than the kinetic energy term such that the latter can be dropped from Eq. 4.3. Thus we are considering an interaction–dominated regime which is a much more accurate description of current condensates than the non–interacting case as these condensates possess millions of atoms. We find the density in the Thomas–Fermi approximation to be

$$|\psi_N (r)|^2 = \frac{1}{NU_0} (\mu - V_{\text{trap}})$$  \hspace{1cm} (4.4)

where $\mu$ is determined from the normalisation condition of Eq. 4.1. The TF solution is therefore

$$\psi_N (r, t) = \sqrt\frac{\mu - V_{\text{trap}}}{NU_0} \exp (-i \mu t / \hbar).$$  \hspace{1cm} (4.5)

The Gross–Pitaevski equation has been very successful in predicting the behaviour of dilute atomic gas BECs. It has been used to model the shape and size of a condensate [34]–[40], the expansion of a condensate [39], and interference of a condensate released from a trap [41]. This model has also produced accurate predictions for the excitations of a condensate [42]–[47]. The transition temperature, condensate fraction and heat capacity has also been calculated using the GP equation [48, 49].
4.1. THEORY

4.1.1 Single-mode approximation

The Gross–Pitaevski equation is a very good description of the mean-field behaviour of a condensate. By using the GP equation, we assume that the condensate has characteristics of a classical field such as a well-defined phase. All the quantum and thermal fluctuations have been neglected in the zero temperature approximation, thus the GP equation (Eq. 4.2) cannot describe the quantum state dynamics of the condensate; the state always remains as a coherence state. We will be interested in the establishment of the quantum phase via a measurement scheme in chapter 5 where the quantum system changes from a state of undefined phase to one where the phase is well defined. This quantum system consists of two BECs. The choice of two condensates instead of one allows us to consider the relative phase between the two. This avoids the unphysical problem of dealing with absolute phase of a single condensate. We will now introduce a single-mode approximation that can incorporate such state dynamics.

The full description is reduced by using a mode expansion and truncating. In particular, we write the Heisenberg field annihilation operator as a mode expansion over single-particle states

\[ \psi(r, t) = \sum_\alpha \hat{a}_\alpha(t) \varphi_\alpha(r) \]

\[ = \hat{a}(t) \psi_R(r) + \tilde{\psi}(r, t), \quad (4.6) \]

where \( \{\varphi_\alpha(r)\} \) are a complete orthonormal basis set. Here in the last line we have chosen \( \varphi_0(r) = \psi_R(r) \) as one member of the complete orthonormal set, and identified \( \hat{a} = \hat{a}_0 \). Then by construction the first term in the mode expansion acts only on the condensate state vector, whereas the second term \( \tilde{\psi}(r, t) \) accounts for the non-condensate atoms. Substituting the mode expansion in the second-quantized Hamiltonian (Eq. 4.1), retaining only the first term representing the condensate, and using the Gross–Pitaevski equation (Eq. 4.3), we obtain the following single-mode quantum Hamiltonian for the condensate in the Schröedinger picture

\[ \hat{H}_S = \hat{H}(0) = \hat{a}^\dagger \hat{a} \varphi_R + \frac{\hbar \kappa}{2} \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a}, \quad (4.7) \]
where

\[
\hat{H}_I(t) = \hat{U}^\dagger(t) \left[ \hat{H}_S - i\hbar \frac{\partial \hat{U}(t)}{\partial t} \right] \hat{U}(t)
\]

(4.10)

and κ is the collision rate between condensate atoms. The first term on the right-hand-side, which gives rise to an energy shift, is the number operator \( \hat{a}^\dagger \hat{a} \) times the single-particle contribution to the energy per particle,

\[
e_N = \int dr \psi_N^*(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(r) \right] \psi_N(r),
\]

(4.9)

whereas the second term accounts for many-body interactions. To proceed we introduce an interaction picture defined by the transformation

\[
\hat{H}(t) = \frac{\hbar \kappa}{2} (\hat{a}^\dagger \hat{a})^2,
\]

(4.11)

which yields the single-mode Hamiltonian

A basic assumption underlying this model is that the quantum state describing the condensate involves only particle numbers \( n \approx \bar{N} \), that is the particle number remains close to the mean number.
Chapter 5
Quantum phase between two Fock states

5.1 Introduction

The experimental realization of weakly-interacting Bose-Einstein condensates has stimulated a large amount of theoretical work on their properties. In particular, there has been a great interest in the establishment of the relative phase [50, 51, 52, 53, 54, 55, 58, 59, 60] between two Bose-Einstein condensates. It is important to consider the relative phase between two condensates instead of the phase of one. A description of the phase of a single condensate would imply that one is dealing with superpositions of atom number which is unphysical. On the other hand, describing the relative phase between two condensates is possible by using superpositions of atom number–difference eigenstates between the two condensates, which is physically reasonable. Thus in this chapter we study, by numerical and analytical methods, how the relative phase is established between the two independent condensates via measurement. Investigations of the effect of collisions (atom–atom interactions) has led to further work on the collapses and revivals of the phase.

Let us start by introducing the system studied by Javanainen and Yoo [50,
Consider the interference between two condensates, which are dropped on top of one another. Each of the condensates consist of $N/2$ atoms with momentum $k_1$ and $k_2$ respectively. An array of atom detectors are placed below the two falling condensates and the field operator corresponding to an atom detected at position $x$ is

$$\hat{\psi}(x) = \hat{a}_1 + \hat{a}_2 e^{i\phi(x)},$$

where $\phi(x) = (k_2 - k_1)x$, and $\hat{a}_1$ and $\hat{a}_2$ are the atom annihilation operators for the first and second condensate respectively. That is, we are using a single-mode model of the condensates since we need to consider the state dynamics of the condensates. It is not unreasonable to consider that the condensates are in Fock states after condensation but if we are unaware of how many atoms the condensates are composed of then we need to describe them as some mixture of Fock states. We will do this in chapter 6. To illustrate the effect of atom detections on the condensates let us consider the simplest case where, the initial state is a Fock state

$$|\varphi_0\rangle = |N/2, N/2\rangle.$$  

After one atom detection at $x$, the un-normalised wave function of the condensates becomes

$$|\varphi_1\rangle = \hat{\psi}(x)|\varphi_0\rangle = \sqrt{N/2} \left(|N/2 - 1, N/2\rangle + e^{i\phi(x)}|N/2, N/2 - 1\rangle\right),$$

which now possesses some phase information due to the detection at $x$ and is an entangled state of the number difference between the two condensates. Note that the position of this first atom detection is random since the initial state is independent of $x$ (has no phase information) whereas when we consider the conditional probability of detecting an atom at $x$ given we have already detected one at $x_1$, we find that

$$p(x|x_1) = 1 + \frac{N}{2(N-1)} \cos [\phi(x_1) - \phi(x)].$$
which has the form of an interference pattern. If we decide to generate a number of \( x \) positions all with \( x_1 \) as their first measurements then we do indeed obtain an interference pattern with visibility \( N/(2(N-1)) \) and phase \( \phi(x_1) \). We shall refer to these parameters as the *conditional visibility* and *conditional phase* because they are dependent on the previous detection or detections. Additional measurements also preserve the form of Eq. 5.4, that is, the conditional probability \( p(x|x_1,\ldots,x_m) \) of detecting an atom at \( x \) given we have already detected \( m \) atoms at \( x_1,\ldots,x_m \) continues to have the form of an interference pattern. It is useful to talk of these conditional visibilities and phases after a varying number of measurements as they serve to describe the establishment of the relative phase between the condensates. We will show in chapter 6 that in the absence of interactions between the atoms, the conditional visibility increases as we perform more and more measurements and approaches a value of one corresponding to perfect visibility. At the same time, the conditional phase stabilizes to a particular phase with fluctuations inversely proportional to the number of measurements. In an experimental situation, one would simply look at the spatial interference pattern generated from detecting \( m \) atoms. However when \( m \gg 1 \) the conditional visibility and conditional phase become good estimates of the actual visibility and phase of the spatial interference pattern as seen after \( m \) detections because both the conditional visibility and conditional phase have become well-defined with little difference between their values at \( m \) and \( m+1 \) detections. Therefore graphs of the conditional visibility and conditional phase as functions of the number of detections are useful tools to display the establishment of the relative phase between the condensates.

So far we have only considered the effects of atom detections on the condensates and have not included the evolution of the wave function during the time intervals between these detections. The next step is thus to include the effects of atom–atom interactions (collisions) by treating the time evolution. To do this, we numerically simulate the time evolution between the atom detections via a Monte Carlo wave function method. The competing
effects of the atom detections that establish phase coherence and the decohering effects of collisions can be understood by the following picture. The atom detections generate an entangled state vector that is a superposition of eigenstates of number difference between the condensates. The neighbouring eigenstates have certain phase relations which determine the degree of coherence that the state vector possesses, in comparison, a coherent state has a fixed phase relation between its number states corresponding to perfect coherence. The effect of collisions is to impart a phase rotation to each eigenstate. The size of this rotation is dependent on the number difference, so that each number-difference eigenstate has its phase shifted differently. There is therefore a dephasing of the coherence which is dependent on the duration of this rotation, i.e. the time evolution of the state vector between atom detections.

5.2 The role of collisions

The simplest system to begin with is the situation where the condensates are initially in Fock states and where the momentum is arbitrarily set to $\pm \pi$ so that $\phi(x) = 2\pi x$. Borrowing from the theory of photon detection the joint counting rate for an $m$ atom detector is

$$p^m(x_1, \ldots, x_m) = \frac{(N - m)!}{N!} \langle \hat{\psi}^\dagger(x_1) \ldots \hat{\psi}^\dagger(x_m) \hat{\psi}(x_m) \ldots \hat{\psi}(x_1) \rangle.$$  (5.5)

Note that the joint probability expressed above in Eq. 5.5 does not include any time-evolution of the condensates between different counting events. The inclusion of collisions by simulating the time evolution of the wave function is not too difficult in the case of initial Fock states so we begin with this simple system but we also consider the time evolution. A Monte Carlo wave function (MCWF) simulation is used to model both the effects of this evolution and the detection process. The effective Hamiltonian used in the MCWF simulation [61] is

$$H_{\text{eff}} = \frac{\hbar K}{2} \left( (a_1^\dagger a_1)^2 + (a_2^\dagger a_2)^2 \right) - \frac{i\hbar \gamma}{2} \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right).$$  (5.6)
5.2. THE ROLE OF COLLISIONS

The detection and collision rates are denoted by $\gamma$ and $\kappa$ respectively. We only consider collisions occurring between atoms within the same condensate. The collisions between the two condensates (a $a_1^*a_1a_2^*a_2$ term) are not included since they are dependent on the actual geometry of the physical situation. The coefficient of this term could be anywhere between zero and $\hbar \kappa$ depending on the overlap between the two condensates. By setting this coefficient to zero we are taking the worse case scenario where the decoherence due to collisions is the largest. Alternatively setting this to $\hbar \kappa$ completes the square so that the term only depends on the total atom number. The effect of the collisions would then be to rotate the phase of the entire state vector and thus the coherence between the individual entangled number states is preserved. After $m$ detections, the state vector of the condensates will be

$$|\varphi_m\rangle = \hat{\psi}(x_m)e^{-iH_\sigma t_m/\hbar} \ldots \hat{\psi}(x_1)e^{-iH_\sigma t_1/\hbar}|\varphi_0\rangle,$$  \hspace{1cm} (5.7)

where $|\varphi_0\rangle$ is the initial state vector and $\{t_1, \ldots, t_m\}$ is the sequence of time intervals between the detections.

The calculation of the joint probability is made possible by the fact that the conditional probability densities all have the form

$$p(x) = 1 + \beta \cos(2\pi x + \varphi).$$  \hspace{1cm} (5.8)

This can be seen from the fact that each successive measurement acts on the previous result as

$$\langle \varphi_m | \hat{\psi}^*(x)\hat{\psi}(x)|\varphi_m\rangle = \frac{N - m}{2} + A \cos[\theta - \phi(x)],$$  \hspace{1cm} (5.9)

giving the required form. We have rewritten $\langle \varphi_m | a_1^*a_2|\varphi_m\rangle$ as the complex number $A \exp(-i\theta)$.

Let us consider a realistic experimental situation and see what sort of constraints are imposed by the collisions. In the case of a condensate composed of rubidium-87 atoms at a temperature of 180nK with a density $\rho = 10^{12} \text{cm}^{-3}$ [26, 28], we can estimate the collision rate by

$$\kappa \sim \rho \pi a_v^2 v_{RMS}$$  \hspace{1cm} (5.10)
where $a$ is the scattering length and $v_{\text{RMS}}$ is the root mean square speed of the rubidium atoms. We obtain a collision rate of about one collision per second for these parameters. Thus each atom on average will experience one collision during a time interval of one second. An interference experiment with these parameters would require the detection process to be completed on a time scale of about one second in order to avoid any significant degradation of visibility due to collisions.

Our simulation uses the Monte Carlo method to determine stochastically the time intervals between atomic detections and calculate the corresponding evolution of the system over the given time; Javanainen and Yoo's method is incorporated in the calculations of the joint probabilities that determine where the atoms are detected. The $m$ atomic detections, \{\(x_1, \ldots, x_m\)\}, are sorted into 50 bins and graphed for one run in fig 5.1. This has no collisions giving perfect visibility with a good fit to the cosine function. Fig 5.2 shows the corresponding graph where we have set the collision rate equal to twice the detection rate. In this case the collisions reduce the visibility to 0.50. The phase of the two interference patterns are not expected to be the same, since the relative phase between the condensates varies randomly from simulation to simulation.

The numerical simulations used to generate the graphs in fig 5.1 and fig 5.2 also calculate the conditional probability distributions before each detection (they are necessary to calculate the location of the detected atom). The evolution of these conditional distributions give an insight into the build up of the interference pattern. By graphing these conditional visibilities and phases, one obtains a history of the build up of interference fringes for that particular sequence of \(\{x_1, \ldots, x_m\}\) detections. To look at the properties of all such sequences for a given collision rate we need to run many such simulations and average over the conditional visibilities. This is shown in fig. 5.3. In the case of no collisions ($\kappa = 0$), we see that the curve peaks at the limit of large $N$ with the visibility always increasing, each additional measurement giving more information about the phase. However for non-
5.2. THE ROLE OF COLLISIONS

Figure 5.1: Numerically simulated histogram of 5000 atomic detections plotted as circles. The total atom number $N$ was 10,000. The solid curve is a least-squares fit using the function $1 + \beta \cos(2\pi x + \varphi)$. The free parameters are the visibility $\beta$ and the phase $\varphi$. The atomic detections are sorted into 50 equally spaced bins.

Zero collision rates, the curves peak at a small fraction of the total number of detections with a declining visibility thereafter since coherence is gradually lost due to the cumulative effects of the collisions. Note that the graphs are averages over a thousand runs, so that the steady decline in visibility reflects the fact that for any particular run, the visibility is fluctuating wildly. This occurs when the present measurement does not support the past evidence. Note that this does not happen for the case of no collisions since the coherence is never reduced so that all measurements support the previous ones.

To look at the properties of the phase is a little bit more tricky, since each run produces a fixed phase for the interference pattern that is random. The conditional phases of two simulations are shown in fig. 5.4(a) and (b); both have their collision rate set to zero with initial atom number $N = 200$. After
Figure 5.2: A histogram of 5000 position detections as in fig. 5.1. We include the effects of collisions (setting $\kappa = 2\gamma$), giving a visibility of about one-half compared with a value close to 1 for the case of no collisions.

Half the atoms are detected, the conditional phase has stabilized to a steady value. This value would be close to the actual phase measured (phase of the interference as displayed by the histograms) in their respective interference patterns. When the collision rate is non-zero this conditional phase may never settle down, but continues to wander due to collisions. However for low collision rates there is a recognizable steady conditional phase. We can then average over many simulations by shifting each curve of fig. 5.4 so that this steady value is zero. By considering the conditional phases from each simulation for a fixed number of measurements we can calculate the variance for each $m$. The variances for a few collisions rates are shown in fig. 5.5. The larger values of the variances for the higher collision rate curves reflect the larger phase fluctuations in their individual simulations. The highest collision rate used here is 0.2 times the detection rate, so the conditional phases are much more sensitive to the collisions than are the conditional visibilities.
5.2. THE ROLE OF COLLISIONS

Figure 5.3: In order to produce high visibility in the spatial interference, the visibilities of the conditional probability distributions need to be high ($\beta$ close to 1). The averaged conditional visibilities are shown with various collision rates $\kappa$ in units of the detection rate $\gamma$. A single run will generate a conditional visibility curve dependent on the $m$ positions measured; however, the average gives a curve that is dependent on the properties of the joint probability distribution $p^m(x_1, \ldots, x_m)$. The visibility curves shown here are averages over 1000 simulations, each starting with 200 atoms.
Figure 5.4: Conditional phases for two simulations without the effect of collisions. The number of initial atoms $N$ was 200.

The dashed curve for which the collision rate is $0.1\gamma$ has larger variance over most of the detections. Note, however that for the initial atom detections the conditional visibility is not greatly modified since the collisions have a small effect for states of the system, which contains only a few number states. The same dependence can be seen for the dash-dotted curve (collision rate $\kappa = 0.2\gamma$), with the collisions increasing greatly the variance over a large number of detections.

We conclude that by the introduction of collisions in the condensates, we see the coherence (built up by the detections) is increasingly degraded as we detect more and more atoms. Thus there is a competition between the growth of coherence between the condensates and the degradation of coherence as we detect a sequence of atoms. The accuracy of a phase standard for the relative phase between Bose-condensates will be limited by collisions. The
loss of coherence due to collisions is not too severe since for parameters where the collision rate is twice the detection rate, the simulation generated an interference pattern with a visibility of approximately one half.

5.2.1 A closer look at the effect of collisions

The effect of collisions can be studied in greater detail if we consider the state of the system after $m$ detections. The initial Fock state, say $|n, n\rangle$, becomes an expansion of $m + 1$ entangled Fock states after the $m$ detections. Let us write the state vector after $m$ detections as

$$|\varphi_m\rangle = \sum_{k=0}^{m} c_k |n - m + k, n - k\rangle,$$

(5.11)

which is normalised so that $\sum |c_k|^2 = 1$. The effects of the collisions are included in the time evolution operator for the two condensates

$$\hat{U}(t) = e^{-i(\hat{H}_1 + \hat{H}_2)t/\hbar}$$

$$= \exp \left( -\frac{ik}{2} \left[ (\hat{a}_1^\dagger \hat{a}_1)^2 + (\hat{a}_2^\dagger \hat{a}_2)^2 \right] t \right),$$

(5.12)

with $\hat{H}_i$ the Hamiltonian for the $i^{th}$ condensate. Thus after $m$ detections we have prepared a state $|\varphi_m\rangle$ which then experiences the above evolution operator for a time $t$ followed by a further detection. The conditional visibility of this detection is

$$p(x|x_1, \ldots, x_m, t) \propto \langle \varphi_m|\hat{U}^\dagger(t)\hat{\Psi}^\dagger(x)\hat{\Psi}(x)\hat{U}(t)|\varphi_m\rangle.$$

(5.13)

The expectation value for the total number operator is

$$\langle \varphi_m|\hat{U}^\dagger(t)(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2)\hat{U}(t)|\varphi_m\rangle = 2n - m,$$

(5.14)

and, as expected, it is just the total initial number of atoms minus the number of detections. The expectation value of the cross term which gives the size of the conditional visibility is

$$\langle \varphi_m|\hat{U}^\dagger(t)\hat{a}_1^\dagger \hat{a}_2 \hat{U}(t)|\varphi_m\rangle = e^{-i\phi(x)} \sum_{k=1}^{m} c_k^* c_{k-1} \sqrt{(n - k + 1)(n - m + k)} \times \exp \{ikt [2k - (m + 1) + i\phi(x)] \}.$$

(5.15)
Figure 5.5: Plot of the variance of the conditional phases of 1000 simulations ($N = 200$) as a function of the number of detections $m$. The solid curve has no collisions included, whereas the dashed and dash-dotted curves have collision rates $\kappa = 0.1\gamma$ and $0.2\gamma$, respectively, with $\gamma$ being the atom detection rate.
5.3. COLLAPSES AND REVIVALS

Putting Eqs. 5.14 and 5.15 into Eq. 5.13 we obtain

\[ p(x|x_1, \ldots, x_m, t) \propto n - m/2 + \sum_{k=1}^{m} A(k) \cos[\phi(x) + \kappa t(2k - m - 1) - \Theta_k], \]

(5.16)

where we have defined the phase \( \Theta_k \) from the coefficients of the state vector as \( c_k c_{k-1} = A_k e^{-i\Theta_k} \) and the weighting function \( A \) is

\[ A(k) = A_k \sqrt{(n - k + 1)(n - m + k)}. \]

(5.17)

The effects of collisions can be seen from eqn. 5.16. The conditional probabilities are proportional to the expectation value described above. For non-zero collision rates (\( \kappa \neq 0 \)) they are a weighted sum over cosines of differing phase shifts \( 2\kappa t k \). Since the state vector for the condensates after \( m \) detections is represented by an expansion number states, evolution by the collision term differs between these number states. The phases of the number state coefficients rotate at different frequencies, leading to a loss of coherence. The state vector is initially in one number state so the collision term has no effect and has little influence for the first couple of detections', however for later detections (\( m \) large) the coherence is significantly reduced.

Looking at fig. 5.3 again, we see that initially the conditional visibility curves for the various collision rates are the same until each curve reaches its peak where the collisions become significant. For large collision rates the collisions kick in early giving a lower peak, while at smaller rates the collisions kick in much later giving a higher peak. In the case of no collisions the conditional visibility effectively peaks at infinite detections with a visibility of one.

5.3 Collapses and Revivals

The de-phasing effect of the collisions are not only responsible for the loss of coherence or “collapse” of the phase but also its “revival”. This can be clearly seen if we rewrite Eq. 5.16 by making the following assumption. We may take \( \Theta_k \) outside the summation in Eq. 5.16 since it is the relative phase
between neighbouring number states. This is assumed to be fairly constant for large \( m \) because the resulting entangled state approaches something that resembles a coherent state as we detect more and more atoms. In a coherent state, the relative phase between neighbouring number states is identical to the phase of that state. Let us define this fairly constant relative phase to be \( \Theta \) which is a good estimate of the relative phase between the two condensates. By expanding the cosine terms in Eq. 5.16 to separate the time dependence from the phase terms, and noting that the resulting summation over sine functions vanishes due to the cancellation of positive and negative frequency components, we obtain

\[
p(x|x_1, \ldots, x_m, t) \propto n - m/2 + \sum_{k=1}^{m} A(k) \cos \left[ (2k - m - 1) \kappa t \right] \cos \left[ \phi(x) - \Theta \right].
\]

(5.18)

The visibility of the interference pattern is therefore determined from a weighted sum over cosines of differing frequencies. These frequencies depend on the parity of the expression enclosed by the round brackets, \( 2k - m - 1 \). When this is odd (\( m \) is even) the frequencies are \( \{1 - m, \ldots, -3, -1, 1, 3, \ldots, m - 1\} \) whereas when it is even (\( m \) is odd) they are \( \{1 - m, \ldots, -2, 0, 2, \ldots, m - 1\} \). In both cases the revival period of the visibility is

\[
T = \frac{\pi}{\kappa}.
\]

(5.19)

For the case where \( m \) is even so that we have a sum over odd frequencies, the cosine term in Eq. 5.18 alternates between plus or minus 1 at subsequent revival times. Since the visibility is by definition a positive quantity, this alternate sign change in the cosine term represents a \( \pi \) phase shift of the interference pattern at alternate revivals.

### 5.3.1 Numerical simulation

We now observe collapses and revivals by applying the same numerical methods that we have used previously to investigate the effect of collisions. In each run of the numerical simulation the state vector experiences three different regimes. Initially a sequence of detections are accumulated to prepare
the entangled state, after which the detections are turned off. During the co-
herent evolution stage, free of detections, the conditional visibility undergoes
collapses and revivals due to collisions. Finally, the detections are turned on
again. If the detections are turned on when the visibility is zero, during a
collapse, it is quickly re-established by the second sequence of detections. If,
however, the detections are turn on when the visibility is in a revival phase
the visibility starts from this non-zero value and then quickly increases to
one. This behaviour is seen in Fig. 5.6 where the detection process occurs at
a much faster rate than the collapse and revivals. We plot a single run on
three separate set of axes: The top graph shows the initial growth of the con-
ditional visibility due to the detections of atoms from the condensates. The
visibility quickly increases to a value close to unity after 100 atoms are de-
tected. (In these simulations the total number of atoms is $10^5$ and 200 atomic
detections are made, so the assumption underlying the quantum anharmonic
oscillator model that the mean number of atoms varies little is well obeyed.)
The middle graph shows what happens to the visibility once the detection
process is turned off. The state vector undergoes coherent evolution due to
the unitary operator Eq. 5.13, and collapses and revivals of the visibility are
clearly evident. This behaviour is reminiscent of the collapses and revivals
in the Jaynes–Cummings model of quantum optics for a two-photon process
which also displays the periodic revivals of Fig. 5.6. The bottom graph dis-
plays the subsequent evolution of the visibility when the detection process is
turned on again. Due to the collapses and revivals during the coherent evo-
lution, the initial visibility for the final stage depends on the time at which
the detections are re-initiated. Whatever the initial visibility, this second
detection proceeds very rapidly to increase the visibility to a value closes to
unity.

The cleanliness and exhibition of full revivals in the middle graph of
Fig. 5.6 can be understood by noticing that the two condensate system is a
closed system (no loss mechanism) undergoing coherent evolution. The state
vector after the initial sequence of detections is an expansion of entangled
Figure 5.6: The top graph shows the growth in the conditional visibility as a function of time corresponding to 100 atomic detections. The middle displays the collapse and revivals of this conditional visibility when the detection process is turned off and the bottom graph shows the growth of this visibility when the detection process is turned on again. The total number of atoms in the two condensates was 100,000 and we have used a collision-to-detection rate ratio of one ($\kappa = \gamma$) for the initial and final detection periods.
number states. The effect of collisions during the coherent evolution is to rotate the phase of the coefficients of each entangled state by an amount proportional to the sum of the squares of the number of atoms in each condensate. Thus the phase of the coefficients of this state vector rotate at differing frequencies.

So far we have shown collapses and revivals in the conditional visibility of the interference pattern. For something more relevant to an experimental situation we would like to look at variables associated with the actual observed interference patterns. The phase shift of an interference pattern is a direct measure of the relative phase established between the two condensates. Thus, let us consider the following procedure: Firstly, we prepare a state vector of the two condensates with an established relative phase via measurements, and consider this entangled state between the two condensates after the detections to be our initial state which possesses some degree of coherence. This state is then allowed to undergo coherent evolution with no detections, and finally we turn on the measurement process after an elapsed time and collect our second sequence of measurements. The phase of the resulting interference pattern is calculated with respect to the phase of the interference pattern we observed previously from the first sequence of measurements. Now we re-prepare the initial state and repeat the second sequence of measurements and subsequent calculation of phase. Repeating this many times we obtain a set of relative phases between the first and second set of measurements, the idea being that if the time elapsed between the two detection regimes corresponds to some multiple of the full revival time then this set of relative phases should be sharply peaked at zero. For other elapsed times we may expect to see partial revivals. We do not need to numerically calculate the coherent evolution but what we need from the numerical simulation is the coefficients of the prepared state $|\varphi_m\rangle$. Its coherent evolution due to the Hamiltonian $\hat{H}$ describing the collisions previously is

$$ |\varphi_m(t)\rangle = \exp\left[-i\left(\hat{H}_1 + \hat{H}_2\right)t/\hbar\right]|\varphi_m\rangle $$
\[
\exp \left( -\frac{i}{2} \kappa \left[ n^2 + (n - m)^2 \right] t \right)
\times \sum_{k=0}^{m} c_k \exp \left( -i \kappa \left[ -mk + k^2 \right] t \right) |n - m + k, n - k \rangle . \tag{5.20}
\]

We use the phase eigenstates for the atom number difference between the two condensates
\[
|\phi \rangle = \sum_{n_1, n_2} \exp \left[ -\frac{i}{2} (n_1 - n_2) \phi \right] |n_1, n_2 \rangle , \tag{5.21}
\]
which has a factor of one-half in the exponential since the state \( |\varphi_m \rangle \) has a fixed total atom number of \( 2n - m \) so that the atom number difference \( (n_1 - n_2) \) is quantized in units of \( 2 \). Thus this factor is required so that \( \phi \) is the relative phase between the condensates. The probability distribution of the phase \( \phi \) after elapsed time \( t \) is
\[
|\langle \phi | \varphi_m (t) \rangle |^2 = \left| \sum_{k=0}^{m} c_k \exp \left( -i [k (k - m) \kappa t + 2k\phi] \right) \right|^2 . \tag{5.22}
\]

This probability is a function of two variables, phase and elapsed time, and is evaluated numerically. We display the probability distributions in "birds-eye" plots via the "image" command using Matlab\textsuperscript{©}. Figure 5.7(a) displays the probability distribution when we have made an even number of detections \( (m \text{ even}) \). The white regions denotes the peaks with the black ones corresponding to the valleys. We see a sharp peak at zero time about zero phase difference between the two interference patterns, this is not surprising since no coherent evolution has occurred. By the time we have evolved for one revival period we obtain a sharp peak at \( \pi \) radians corresponding to the first revival time with \( \pi \) phase shift due to detecting an even number of atoms described in the previous section. The second revival occurs at a phase difference of zero radians as predicted. Away from these revival times, the phase distribution is not flat but displays many partial revivals. Figure 5.7(b) is a zoomed in view of Fig. 5.7(a) between 0.2 and 0.5 revival periods, we can clearly see the partial revival at 0.2 which consists of five peaks. In fact we see partial revivals at every integer fraction of a revival period provided the
Figure 5.7: Plot of the phase distribution when the number of detections \((m)\) is even as a function of the turning on time. The phase is the relative phase between the first and second sequence of measurements in units of radians while the turning on time is in units of the revival period. The brightness of a region corresponds to the relative probability of obtaining a particular phase for a particular turning on time. The bright regions denote peaks while the darker ones correspond to valleys. The entire plot is shown in (a) with a zoomed in plot between 0.2 – 0.5 times displayed in (b).
resolution is good enough. We illustrate this by labelling to the left of the graph with the appropriate integer fractions corresponding to the particular partial revival. Figure 5.8 shows the distribution for the other case where the number of detections is odd. As predicted there is no \( \pi \) phase shifts. Again, we see partial revivals when we zoom in between 0.2 and 0.5 revival periods as shown in Fig. 5.8(b).

The beautifully ordered features of figures 5.7 and 5.8 has been observed in many areas of wave physics. Schleich and co–workers [62]–[66] have investigated this fundamental interference phenomenon in a variety of systems: space–time structures (canals and ridges) observed in the probability density of a particle in a box [62, 63], and the fractional and full revivals in multilevel quantum systems [64, 65]. They explain such general phenomena succinctly by stating: “Highly regular spatio–temporal or multi–dimensional patterns in the quantum mechanical probability or classical field intensity distributions can appear due to pair interference between individual eigen–modes of the system forming the so called intermode traces.” [66]

5.3.2 Quantum oscillator model with bose–broken symmetry

We will show in this brief section that collapses and revivals also arises in a quantum anharmonic oscillator model. We choose initial coherent states instead of preparing an entangled state from an initial detection process. In the language of the previous sections we are considering the interference between two coherent states including the effects of collisions. Thus we treat the Bose–Einstein condensates as coherent states, imposing on them a relative phase whereas previously we establish this phase via measurements. By treating the condensates as coherent states we are invoking bose–broken symmetry; an arbitrary phase is selected during condensation. A number state may be considered as a continuous superposition of coherent states all with different phases; spontaneous symmetry breaking then selects just one (arbitrary) phase. Thus in this section we investigate the consequences of
Figure 5.8: Plot of the phase distribution when the number of detections ($m$) is odd as a function of the turning on time. The entire plot is shown in (a) with a zoomed in plot between 0.2 – 0.5 times displayed in (b).
invoking bose–broken symmetry as opposed to establishing the phase via measurements.

The Hamiltonian for the two condensates is again

\[ H = \frac{1}{2} \sum_{i=1}^{2} \hbar \kappa \left( \hat{a}_i^\dagger \hat{a}_i \right)^2. \]

The Heisenberg equation of motion for each field annihilation operator is then

\[ \frac{d\hat{a}_i}{dt} = \frac{1}{\hbar} \left[ \hat{a}_i, H \right] = \frac{\kappa}{2i} \left( 1 + 2\hat{a}_i^\dagger \hat{a}_i \right) \hat{a}_i. \]

By inspection the time dependence of the \( \hat{a}_i \) operator is

\[ \hat{a}_i (t) = \exp \left[ -\frac{i}{2} \left( 1 + 2\hat{a}_i^\dagger \hat{a}_i \right) \kappa t \right] \hat{a}_i, \]

which yields the Heisenberg picture field operator for the sum of the two modes

\[ \hat{\Psi} (t) = \frac{1}{\sqrt{2}} \left\{ \exp \left[ -\frac{i}{2} \left( 1 + 2\hat{a}_1^\dagger \hat{a}_1 \right) \kappa t \right] \hat{a}_1 + \exp \left[ -\frac{i}{2} \left( 1 + 2\hat{a}_2^\dagger \hat{a}_2 \right) \kappa t \right] \hat{a}_2 \right\}, \]

where we have suppressed the spatial dependence. This yields the operator for the intensity of the atomic pattern as

\[ \hat{\Psi}^\dagger (t) \hat{\Psi} (t) = \frac{1}{2} \left\{ \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_1^\dagger \exp \left[ i \left( \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2 \right) \kappa t \right] \hat{a}_2 + \text{h.c.} \right\}. \]

Thus, if we define the initial coherent states as \( |\alpha\rangle \) and \( |\beta\rangle \) for the modes \( \hat{a}_1 \) and \( \hat{a}_2 \) respectively, the intensity is evaluated to be

\[ I \propto \langle \alpha, \beta | \hat{\Psi}^\dagger (t) \hat{\Psi} (t) |\alpha, \beta\rangle \]

\[ = \frac{1}{2} \left\{ |\alpha|^2 + |\beta|^2 + \alpha^* \beta \exp \left[ \left( e^{i\kappa t} - 1 \right) |\alpha|^2 + \left( e^{-i\kappa t} - 1 \right) |\beta|^2 \right] \right. \]

\[ + \beta^* \alpha \exp \left[ \left( e^{-i\kappa t} - 1 \right) |\alpha|^2 + \left( e^{i\kappa t} - 1 \right) |\beta|^2 \right] \right\}, \]

with the aid of the following identity [67]

\[ \langle \alpha | e^{-\alpha^\dagger a} | \alpha \rangle = \exp \left[ \left( e^{-x} - 1 \right) |\alpha|^2 \right]. \]
5.4. SUMMARY

In the case of maximum visibility for the interference pattern we need the modes to be equal in amplitude, thus we set $\beta = \alpha e^{-i\phi}$ where $\phi$ is the phase difference between the modes. Substituting this into Eq. 5.28 we obtain

$$I \propto |\alpha|^2 \left\{ 1 + \exp \left[ 2 |\alpha|^2 (\cos \kappa t - 1) \cos \phi \right] \right\}.$$  \hspace{1cm} (5.29)

The characteristics of collapses and revivals can be clearly seen in Eq. 5.29. Revivals occur at times $t = 2\pi n/\kappa$ (where $n$ is an integer) with the shape of the collapses described by the exponential term whenever $t$ is no longer a multiple of $2\pi/\kappa$. The period of these revivals is then

$$T = \frac{2\pi}{\kappa}. \hspace{1cm} (5.30)$$

This gives twice the period of the previous case of revivals from a detection process (see Eq. 5.19), even though we have used identical Hamiltonians. This difference arises because we have two independent coherent states whereas in the case of detecting atoms from two initial Fock states the total atom number is always fixed at the initial total number minus the number of atomic detections. This additional constraint on the total number means that the difference atom number operator inside the exponential in Eq. 5.27 is quantized in units of 2. The above expression for the period is only applicable when we are free to superpose total atom numbers, as in the case of two coherent states for which the difference atom number operator is quantized in units of 1. This gives a factor of 2 difference in the revival times between assuming an initial relative phase and establishing this phase in the dynamics of the revivals. The visibility, as described by Eq. 5.29, smoothly drops to its minimal value halfway between subsequent revivals, exactly where the state induced by detection would have an additional revival.

5.4 Summary

In summary we have seen that each additional detection of an atom from the condensates alter our knowledge of their relative phase. If the detection
is where we have expected it to be then our estimate of the phase is consolidated. However if the detection is where we do not expect it to be then our estimate is degraded. After many detections the relative phase is very precisely defined.

With the introduction of collisions in the condensates we see that the coherence (built up by the detections) is increasingly degraded as we detect more and more atoms. Thus there is a competition between the growth of coherence between the condensates and the degradation of coherence as we detect a sequence of atoms. The accuracy of a phase standard for the relative phase between bose-condensates will be limited by collisions. The lost of coherence due to collisions is not too severe since for parameters where the collision rate is twice the detection rate, we obtain a visibility of approximately one half from the numerical simulation.

We also consider collapse and revivals in the case of two interfering condensates that are initially in number states; an extreme example for which there is no relative phase before atomic detections. These condensates are then prepared into an entangled state vector composed of entangled number-difference eigenstates via the measurement process. The effects of collisions in the time evolution is to rotate the phases of the individual entangled eigenstates of the state vector. We observe the collapse and revivals of this state under coherent evolution when the measurement process is turned off. Predictions of the period of these collapses and revivals were obtained. The simple anharmonic model of interference between two condensates also displays collapses and revivals. The period of these revivals is twice the time required for the previous case since we have neglected the constraint of the total atom number being fixed. Thus the consequence of invoking bose-broken symmetry is to change the revival time by a factor of two.
Chapter 6

Quantum phase between mixed states

6.1 Introduction

We now return to the possibility mentioned earlier of the situation where we do not have perfect knowledge of the number of atoms in the condensates. In this situation we consider a mixed initial state of the condensates and in particular we look at an initial Poisson or thermal state. The time evolution in between the atom counts is neglected for simplicity and a numerical algorithm is devised which separates the stochastic sequence of atom detections \( x_1, \ldots, x_m \) from the necessary averaging over the Poisson and thermal distributions. This allows us to calculate all the moments of the distributions analytically and perform numerics only on the detection process. We find that the visibility of the interference pattern is close to one for the initial Poisson state but it becomes a stochastic variable for the thermal state, for which the visibility will vary from run to run around an average visibility of \( \pi/4 \). We have defined a single "run" as a sequence of spatial atom detections corresponding to an experimental situation of observing an interference pattern after a certain number of atoms has been detected. To perform another run one would reset the initials states of the condensates and perform the
same number of detections. The usefulness of averaging over many runs is that it enables ensemble properties of the system to be calculated. Note that the relative phase we are studying is itself a stochastic variable from run to run. It is not hard to see why if one considers the fact that we are starting from Fock states (or mixtures of them) which do not possess any coherence, so the phase is random from run to run and the average over the interference patterns have zero visibility.

Analytical results are obtained in the limit where the number of detections is large. We write the density operator for the system in a P-representation, thereby describing the pure Fock state as a distribution over coherent states. The initial distribution of phase is uniform since the initial state has no coherence but this phase distribution quickly narrows as we make atom detections. This use of the P-distribution has been found by Cirac et. al. \[53\]. From numerical results we know that this distribution of phase is narrow for large number of atom detections and we can approximate it by a Gaussian. The fluctuations of the phase for initial Fock, Poisson and thermal states have been calculated and are in good agreement with the numerical results; the variance of the phase is inversely proportional to the number of measurements. The average conditional visibility is one for the Fock and Poisson states, and is $\pi/4$ for the thermal state, agreeing well with the numerical results.

### 6.2 Mixture of number states

Let us start our treatment by considering a mixture of number states. We shall take this mixture to have a number distribution $P_n$ giving an initial density operator of the form

$$\rho = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P_{n_1} |n_1\rangle \langle n_1| \otimes P_{n_2} |n_2\rangle \langle n_2|, \quad (6.1)$$

where $n_1$ and $n_2$ refer to the first and second condensate respectively. If we denote the field operator for the detection of an atom at $x$ as $\hat{\psi}(x)$ then the
6.2. MIXTURE OF NUMBER STATES

The joint probability of \( m \) detections at the \( \{x_1, \ldots, x_m\} \) positions is

\[
p^{(m)}(x_1, \ldots, x_m) = \mathcal{N}^{(m)} \text{Tr} \left\{ \rho \hat{\psi}^\dagger(x_1) \ldots \hat{\psi}^\dagger(x_m) \hat{\psi}(x_m) \ldots \hat{\psi}(x_1) \right\},
\]

where the symbol \( \text{Tr} \) denotes the trace over the \( n_1 \) and \( n_2 \) number states. The normalisation \( \mathcal{N}^{(m)} \) is defined by

\[
\mathcal{N}^{(m)} = \left[ \int dx_1 \ldots dx_m \text{Tr} \left\{ \rho \hat{\psi}^\dagger(x_1) \ldots \hat{\psi}^\dagger(x_m) \hat{\psi}(x_m) \ldots \hat{\psi}(x_1) \right\} \right]^{-1}
\]

which is independent of \( x_1, \ldots, x_m \) but will in general depend on \( \rho \). In order to allow for the possibility of different detection rates of atoms from the two condensates (e.g. by letting them leak out of their traps at different rates) we define a more general field operator from what we have used previously

\[
\hat{\psi}(x) = \hat{a}_1 + \sqrt{\Gamma} \hat{a}_2 e^{-i\varphi(x)},
\]

with the ratio \( \Gamma = \gamma_2/\gamma_1 \) where \( \gamma_1 \) and \( \gamma_2 \) are the detection rates for each of the condensates. We shall assume \( \gamma_1 \geq \gamma_2 \) without restriction of generality.

Substituting equation (6.1) into (6.2), we obtain the following expression for the joint probability.

\[
p^{(m)}(x_1, \ldots, x_m) = \mathcal{N}^{(m)} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P_{n_1, n_2}(n_1, n_2) \hat{\psi}^\dagger(x_1) \ldots \hat{\psi}^\dagger(x_m) \hat{\psi}(x_m) \ldots \hat{\psi}(x_1)|n_1, n_2\rangle
\]

where we have written \( P_{n_1, n_2} = P_{n_1} P_{n_2} \) for convenience. The expression for the joint probability above shows that it is a weighted sum of probabilities over fixed numbers \( n_1 \) and \( n_2 \) of initial numbers of atoms in each condensate. The weighting is determined by the probability distribution of the number of atoms. For very narrow distributions, corresponding to an initial fixed number of atoms in each condensate, we see that this will simplify down to the expression used previously by Javanainen and Yoo [50, 60]. However for broader distributions, for example a thermal distribution, this sum will affect the joint probability and hence the spatial interference.
6.3 Visibility conditioned on 1 detection

Before we show how to numerically generate this interference pattern, we look at the build up of interference for the first two detections. Let us start by considering a joint Fock state. After one detection in which an atom is observed at $x_1$, the un-normalised state vector for the system is

$$\hat{\psi}(x_1) |n_1, n_2\rangle = \frac{\sqrt{n_1} |n_1 - 1, n_2\rangle + \sqrt{\Gamma n_2 e^{-i\phi(x_1)}} |n_1, n_2 - 1\rangle}{\sqrt{n_1} + \sqrt{\Gamma n_2}}.$$ (6.6)

The joint probability density for detecting atoms at $x$ and $x_1$ starting from $|n_1, n_2\rangle$ is then proportional to

$$p(x_1) = \langle n_1, n_2 | \hat{\psi}^\dagger(x_1) \hat{\psi}(x) \hat{\psi}(x) \hat{\psi}(x_1) | n_1, n_2\rangle = n_1(n_1 - 1) + \Gamma^2 n_2(n_2 - 1) + 2\Gamma n_1 n_2 \{1 + \cos [\phi(x) - \phi(x_1)]\}. \quad (6.7)$$

Since the initial state of interest is a mixture of Fock states with weights $P_{n_1, n_2}$, the joint probability for this initial state by Eq. 6.5 is

$$p^{(2)}(x, x_1) = \mathcal{N}^{(2)} \sum_{n_1, n_2} P_{n_1, n_2} \langle n_1, n_2 | \hat{\psi}^\dagger(x_1) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\psi}(x_1) | n_1, n_2\rangle$$

$$= \mathcal{N}^{(2)} \sum_{n_1, n_2} P_{n_1, n_2} \left(n_1(n_1 - 1) + \Gamma^2 n_2(n_2 - 1) + 2\Gamma n_1 n_2 \{1 + \cos [\phi(x) - \phi(x_1)]\}\right)$$

$$= \mathcal{N}^{(2)} \left[\langle n_1^2 \rangle - \langle n_1 \rangle \right] + \Gamma^2 \left[\langle n_2^2 \rangle - \langle n_2 \rangle\right] + 2\Gamma \langle n_1 \rangle \langle n_2 \rangle \{1 + \cos [\phi(x) - \phi(x_1)]\}\right)$$ (6.8)

where we have used angle brackets to denote averages taken over $P_{n_1, n_2}$. These factorize as we assume that $P_{n_1, n_2} = P_{n_1} P_{n_2}$. The conditional probability $p(x|x_1) = p^{(2)}(x, x_1)/p^{(1)}(x_1)$ differs from the above only by a $x$-independent factor and we may write

$$p(x|x_1) = \frac{\mathcal{N}^{(2)}}{\mathcal{N}^{(1)}} \{1 + \nu \cos [\phi(x) - \phi(x_1)]\} \quad (6.9)$$
6.3. VISIBILITY CONDITIONED ON 1 DETECTION

where

\[ V = \frac{2\Gamma \langle n_1 \rangle \langle n_2 \rangle}{\left[ \langle n_1^2 \rangle - \langle n_1 \rangle \right] + \Gamma^2 \left[ \langle n_2^2 \rangle - \langle n_2 \rangle \right] + 2\Gamma \langle n_1 \rangle \langle n_2 \rangle} \]  

(6.10)

may be interpreted as the conditional visibility of the interference pattern.

For a thermal distribution we use the following relationship between the second moment and the mean \( \overline{m} \equiv \langle m \rangle \),

\[ \langle m^2 \rangle = 2\overline{m}^2 + \overline{m}. \]  

(6.11)

Substituting the above equations into Eq. 6.10 we obtain an expression for the conditional visibility in terms of the means

\[ V_{\text{thermal}}(\overline{n}_1, \overline{n}_2) = \frac{\Gamma \overline{n}_1 \overline{n}_2}{\overline{n}_1^2 + (\Gamma \overline{n}_2)^2 + \Gamma \overline{n}_1 \overline{n}_2} \]  

(6.12)

which gives a maximum visibility for equal net detection rate in each condensate, i.e. giving a maximum value of one third for \( \overline{n}_1 = \Gamma \overline{n}_2 \).

Alternatively, for the case of a Poissonian number distribution we use the relationship

\[ \langle m^2 \rangle = \overline{m}^2 + \overline{m}. \]  

(6.13)

Proceeding in the same manner as in the thermal case we obtain

\[ V_{\text{Poisson}}(\overline{n}_1, \overline{n}_2) = \frac{2\Gamma \overline{n}_1 \overline{n}_2}{(\overline{n}_1 + \Gamma \overline{n}_2)^2}, \]  

(6.14)

which has a maximum value of one half for \( \overline{n}_1 = \Gamma \overline{n}_2 \). The values of one third and one half that we have obtained have also been seen in optical experiments where intensity correlations were measured for Poissonian and thermal light sources by Rarity et. al. [56]

Let us also consider the limiting case of fixed initial number, where we know that there are exactly \( n_1 \) atoms in one condensate and \( n_2 \) in the other. Then using Eq. 6.10 the conditional visibility is

\[ V_{\text{Fock}}(n_1, n_2) = \frac{2\Gamma n_1 n_2}{(n_1 + \Gamma n_2)^2 - (n_1 + \Gamma^2 n_2)}. \]  

(6.15)

The maximum occurs again when \( n_1 = \Gamma n_2 \) where the net detection rates for both condensates are equal. This is not surprising since the size of the
interference term depends on our lack of knowledge of which condensate a
given detected atom comes from. If the net detection rate of one condensate
is larger than the other, then we know that the detected atom is more likely
to have originated from this particular condensate. Unlike the previous two
cases, the maximum visibility depends on \( N \) the initial total number of atoms
in the condensates. The maximum visibility for the special case of equal
number and detection rate for each condensate we have

\[
[V_{\text{Fock}}]_{\gamma_1=\gamma_2,n_1=n_2} = \frac{1}{2(1-1/N)}.
\]  

(6.16)

In the limit of large \( N \) the conditional visibility approaches that of the Pois-
sonian mixture which has the value of one half. For practical purposes where
the number of atoms in each condensate is well over one thousand, the cases of
a Poissonian mixture and of initial Fock states are indistinguishable. We plot
these conditional visibilities for the different initial conditions as a function of
the ratio of initial net detection rates, \( \Gamma \bar{n}_1/\bar{n}_2 \) between the two condensates
in Fig. (6.1).

To see how the maximum visibility changes as we alter the width of these
initial distributions, let us also consider some arbitrary Gaussian distribu-
tion with a variance \( \sigma^2 \) and mean \( \bar{n} \). The conditional visibility when both
condensates start with this Gaussian distribution and equal detection rates
is

\[
V_{\text{Gaussian}}(\bar{n}, \sigma^2) = \frac{\bar{n}^2}{\sigma^2 + 2\bar{n}^2 - \bar{n}}.
\]  

(6.17)

This is approximately equal to \((2 + \sigma^2/\bar{n}^2)^{-1}\) for large \( \bar{n} \). For wide distribu-
tions \( \sigma^2 \gg \bar{n}^2 \), we see that the conditional visibility tends to zero. Conversely
for narrow distributions where \( \sigma^2 \ll \bar{n}^2 \), the visibility becomes approximately
one half for large \( \bar{n} \). In the special case of \( \sigma^2 = \bar{n} \) where we approximate
the Poisson distribution by a Gaussian, Eq. 6.17 yields a visibility of one
half which is consistent with the value obtained from the expression for the
Poissonian visibility given by Eq. 6.14.
Figure 6.1: The conditional visibility curves for an initial number state \( n = 20 \), Poisson and thermal are states plotted as dashed, solid and dash–dotted curves respectively. The visibility is dependent upon the ratio of the mean counting rates between the condensates \( \Gamma n_2/n_1 \).

### 6.4 Conditional probability density after \( m \) detections

The conditional probability density \( p(x|\tau) \) displayed by Eq. 6.9 can be generalized to an expression governing the probability density of \( x \) given the previous \( m \) measurements \( \{x_1 \ldots x_m\} \). We can write the operator representing the cumulative effect of \( m \) detections as

\[
\hat{\Psi}(x_m) \hat{\Psi}(x_{m-1}) \ldots \hat{\Psi}(x_1) = \prod_{j=1}^{m} \left( \hat{a}_1 + \sqrt{\Gamma} \hat{a}_2 e^{-i\phi_j} \right) \\
= \sum_{k=0}^{m} n_k^{(m)}(\phi_1, \ldots, \phi_m) \hat{a}_1^{m-k} \hat{a}_2^{k} \Gamma^{k/2} \quad (6.19)
\]
where we define $\phi_k \equiv \phi(x_k)$ for notational convenience and the coefficients $\pi_k^{(m)}(\phi_1, \ldots, \phi_m)$ can be found by computing the power series expansion

$$\prod_{j=1}^{m} \left(1 + z e^{-i\phi_j}\right) = \sum_{k=0}^{m} \pi_k^{(m)}(\phi_1, \ldots, \phi_m) z^k. \quad (6.20)$$

They satisfy the recursion relation

$$\pi_k^{(m+1)} = \pi_k^{(m)}(1 - \delta_{k,m+1}) + \pi_{k-1}^{(m)}(1 - \delta_{k,0}) e^{-i\phi_{m+1}} \quad (6.21)$$

where we have used the notation $\pi_k^{(m)} \equiv \pi_k^{(m)}(\phi_1, \ldots, \phi_m)$ for brevity. In a numerical simulation, the product (6.20) can be updated after every atomic detection by carrying out polynomial multiplication. The un-normalised state vector after applying the above operator to the initial state $|n_1, n_2\rangle$ is

$$|\varphi_m\rangle = \sum_{k=0}^{m} \sqrt{\frac{n_1! n_2!}{(n_1 - m + k)! (n_2 - k)!}} \pi_k^{(m)} \Gamma^{k/2} |n_1 - m + k, n_2 - k\rangle. \quad (6.22)$$

Let us now consider the un-normalised wave function after the $(m+1)'th$ detection

$$|\varphi_{m+1}(\phi)\rangle = (\hat{a}_1 + \sqrt{\Gamma} \hat{a}_2 e^{-i\phi}) |\varphi_m\rangle \quad (6.23)$$

where we have explicitly shown the $\phi$ dependence. The joint probability of $m + 1$ detections at the $\{x_1, \ldots, x_m, x\}$ positions is

$$p^{(m+1)}(x_1, \ldots, x_m, x) = \mathcal{N}^{(m+1)} \text{Tr} \left\{ \rho \hat{\psi}^\dagger(x_1) \cdots \hat{\psi}^\dagger(x_m) \hat{\psi}^\dagger(x) \hat{\psi}(x_m) \cdots \hat{\psi}(x_1) \right\} \quad (6.24)$$

$$= \mathcal{N}^{(m+1)} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P_{n_1, n_2} \langle \varphi_{m+1}(\phi) | \varphi_{m+1}(\phi) \rangle, \quad (6.25)$$

where

$$\langle \varphi_{m+1}(\phi) | \varphi_{m+1}(\phi) \rangle = \frac{n_1!}{(n_1 - m - 1)!} \left[ (\frac{\Gamma^{k+1} n_1! n_2!}{(n_1 - m + k)! (n_2 - k - 1)!} |\pi_k^{(m)} e^{-i\phi} + \pi_{k+1}^{(m)}|^2 \right] \quad (6.26)$$
The conditional probability density is thus

\[ p(x|x_1, \ldots, x_m) = N^{(m)} \left[ \langle n_1 (n_1 - 1) \ldots (n_1 - m) \rangle \right. \]
\[ + \sum_{k=0}^{m-1} \langle n_1 (n_1 - 1) \ldots (n_1 - m + k + 1) \rangle \]
\[ \times \langle n_2 (n_2 - 1) \ldots (n_2 - k) \rangle \Gamma^{k+1} \left| \eta_k^{(m)} e^{-i\phi} + \pi_k^{(m)} \right|^2 \]
\[ + \Gamma^{m+1} \langle n_2 (n_2 - 1) \ldots (n_2 - m) \rangle \]  
\[ (6.27) \]

where

\[ N^{(m)} = \frac{N^{(m+1)}}{p^{(m)}(x_1, \ldots, x_m)} \]  
\[ (6.28) \]

is a \( x \) independent normalisation factor. The angle brackets denotes the sum over the probability distribution \( P_{n_1, n_2} \).

For different initial mixtures, either a thermal or a Poissonian distribution, we obtain different relationships between the higher order moments and the first (i.e. the mean). The relationship for the thermal case is

\[ \langle n (n - 1) \ldots (n - k) \rangle_{\text{thermal}} = (k + 1)! \pi^{k+1} \]  
\[ (6.29) \]

and

\[ \langle n (n - 1) \ldots (n - k) \rangle_{\text{Poisson}} = \pi^{k+1} \]  
\[ (6.30) \]

for the Poissonian distribution. Using these properties we obtain the conditional probability density for the Poisson distribution

\[ p_{\text{Poisson}}(x|x_1, \ldots, x_m) = N_{\text{Poisson}}^{(m)} \left[ \eta_1^{m+1} + (\Gamma \eta_2)^{m+1} \right. \]
\[ + \sum_{k=0}^{m-1} A_k^{(\text{Poisson})} \left| \eta_k^{(m)} e^{-i\phi} + \pi_k^{(m)} \right|^2 \]  
\[ (6.31) \]

and

\[ p_{\text{thermal}}(x|x_1, \ldots, x_m) = N_{\text{thermal}}^{(m)} \left\{ (m + 1)! \left[ \eta_1^{m+1} + (\Gamma \eta_2)^{m+1} \right] \right. \]
\[ + \sum_{k=0}^{m-1} A_k^{(\text{thermal})} \left| \eta_k^{(m)} e^{-i\phi} + \pi_k^{(m)} \right|^2 \} \]  
\[ (6.32) \]
for the thermal distribution. Note that the $x$ independent normalisation factors are now calculated with respect to the relevant density operator, i.e. Poisson for $N^{(m)}_{\text{Poisson}}$ and thermal for $N^{(m)}_{\text{thermal}}$. Both the thermal and Poisson distribution's conditional probability densities have a similar form with different weighting factors $A_k$ given by

$$A_k^{(\text{Poisson})} = \frac{n_1^{m-k} n_2^{k+1} \Gamma^{k+1}}{(m-k)! (k+1)!}$$

(6.33)

and

$$A_k^{(\text{thermal})} = (m-k)! (k+1)! A_k^{(\text{Poisson})}.$$  

(6.34)

The generalized conditional probability distributions Eq. 6.31 and Eq. 6.32 groups all the measurement dependent terms, the $\{\phi_1, \ldots, \phi_m\}$ detections, within the $| \ldots \rangle$ brackets. Numerical calculations are readily obtainable since we need only to simulate the expression within the $| \ldots \rangle$ brackets. We shall come back to this point in the numerical simulation section.

### 6.5 Analytical results in the limit of large number of detections

R. Graham\(^1\) has calculated analytical expressions of the spread in phase and visibility during his short stay here at the University of Auckland. These results are part of a collaboration, which produced a PRA paper [57] and are reproduced in the appendix C.

R. Graham uses the P-distribution to write the density operator for the system as a P-representation, thereby describing the pure Fock state as a distribution over coherent states. The initial distribution of phase is uniform since the initial state have no coherence but this phase distribution quickly narrows as we make atom detections. From numerical results we know that this distribution of phase is indeed narrow for large number of atom detec-

\(^1\)Prof. Robert Graham of Universität GH Essen, Fachbereich Physik, D45117 Essen, Germany.
6.6. **NUMERICAL RESULTS**

He calculates analytical expressions by approximating the spread in phase and visibility by a Gaussian.

For a Poissonian mixture, the average inverse variance has the following form as a function of the number of measurements $m$

$$\left\langle \frac{1}{\sigma_m^2} \right\rangle = \left[ (1 - \sqrt{1 - \lambda^2}) + O \left( \sigma_m^2 \right) \right] m + \text{const.} \quad (6.35)$$

where $\sigma_m^2$ is the variance of the phase distribution and $\lambda$ is defined by

$$\lambda = \frac{2\sqrt{n_1 n_2}}{n_1 + n_2}.$$  

Equation 6.35 shows explicitly how the inverse variance grows on average and becomes large when $m$ is large. The average visibility as a function of $m$ decays via

$$\lambda' = \lambda e^{-\frac{1}{2}\sigma_m^2}. \quad (6.36)$$

In the case of thermal mixtures, the average visibility is predicted to have the value

$$\bar{\lambda} = \frac{\pi}{4} \quad (6.37)$$

in the limit of large $m$.

**6.6 Numerical results**

The form of the expressions for the conditional probabilities, Eq. 6.31 and Eq. 6.32 can be readily applied to stochastic simulations of atom detection which generates interference patterns. The procedure follows the spirit of Javanainen and Yoo’s work; a random number $\phi_1$ is generated for the initial atom detection (since we know the initial conditional probability distribution is uniform), then it is used to calculate the conditional probability density $p(\phi|\phi_1)$ so that the second detection $\phi_2$ can be generated by randomly selecting a value according to $p(\phi|\phi_1)$. This process is repeated to generate $m$ atom detections which are binned and displayed as an interference pattern.

Examples of these stochastically generated interference patterns are displayed in fig. 6.2 as histograms of the raw output from the simulations, the
Figure 6.2: Histogram of 500 numerically generated atomic detections plotted as circles. The solid curve is a least-square fit of the form \( 1 + \beta \cos (2\pi x + \phi) \). A thermal initial state is shown in (a) and an initial Poisson state is shown in (b). The mean counting rates are equal \((\bar{n}_1 = \Gamma \bar{n}_2)\) in these simulations.

A sequence of detected positions \(\{\phi_1, \ldots, \phi_m\}\) are sorted into 25 bins plotted as circles. Figure 6.2(a) shows the interference pattern for an initial thermal mixture which has a visibility of 0.79 whereas in fig. 6.2(b) we have a visibility of 0.97 for the Poissonian mixture. These visibilities are calculated via a least-squares fit of the form \( 1 + \beta \cos (2\pi + \phi) \) shown as the solid curves in fig. 6.2(a) and (b). In both cases we have simulated 500 detections.

The numerical simulations used to generate the histograms in fig.6.2(a) and (b) also calculate the conditional probability distributions before each detection as these are necessary to calculate the location of the detected atom. The evolution of these conditional distributions give an insight into the build up of the interference pattern [58]. The visibility of the complete interference pattern (as calculated from the least-square fit) can be considered...
as an average over these conditional visibilities. Graphs of the conditional
visibility as a function of atom detections typically approach a value close to 1
within 100 detections and stay at that value thereafter. Thus the conditional
visibility after \( m \) detections can be thought of as a good approximation of
the visibility of the complete interference pattern for values of \( m > 100 \). We
have not displayed these graphs of stochastically generated sequences since
they possess fluctuations about a generic shape which is a property of all
such sequences. We plot the average conditional visibility over many such
sequences in fig. 6.3 and fig. 6.4 for Poisson and thermal mixtures respec-
tively. The ratio \( \Gamma \bar{n}/\bar{n} \) was set at 1 with the average performed over 1000
runs (we shall hereby refer to an individual sequence of detections as a single
“run” for convenience). The shaded regions around the average conditional
visibility displayed as a solid line depict the extent of the fluctuations for
an individual run. The boundaries of this shaded region corresponds to the
upper and lower quartiles respectively, 25% of the data lies below the lower
quartile whereas 25% lies above the upper quartile, thus the probability that
the fluctuations lie within this region is 50%. These fluctuations are much
larger for the thermal mixtures in comparison with those of the Poissonian
mixtures.

The maximum value of the visibility in fig. 6.3 occurs at 500 atoms de-
tected with a value of 0.999 (3 sig. fig.). The value of the visibility obtained
from Eq. 6.36 using \( \sigma_m^2 \sim 1/m \) is 0.999 (3 sig. fig.) which agrees very well
with the simulation. In the case of the thermal mixture, fig. 6.4, the maxi-
mum value of the visibility is 0.777 again at 500 detections. This is close to
the analytical value of \( \pi/4 \approx 0.785 \) predicted by Eq. 6.37.

The large difference in the fluctuations between the two cases is due to
their differing degree of sensitivity to particular runs. In the Poisson case, it
is relatively insensitive since the majority of the terms in the \( |...| \) brackets
of Eq. 6.31 depends on the \( \{\phi_1, \ldots, \phi_m\} \) detections fairly equally whereas in
the thermal case we see there is an additional factorial factor in front of the
\( |...| \) brackets which favours \( k \) values at the ends (close to zero and \( m - 1 \)).
Chapter 6. Quantum Phase Between Mixed States

Figure 6.3: Plot of the conditional visibility averaged over one thousand runs versus the number of atoms detected for an initial Poisson state where $\bar{n}_1 = \Gamma \bar{n}_2$. The shaded region corresponds to the interquartile range of the individual runs with the mean over all runs depicted by the solid line. Therefore 50% of the runs lies within this shaded region.

This enhances the sensitivity of the visibility upon particular combinations of $\phi_k$ thus the sensitivity is high. Alternatively, the large fluctuations of the visibility in the thermal case is not surprising when we consider the analytic treatment of section 6.5. The thermal state was represented as a mixture of Poissonian states so that the averaged conditional visibility has an additional average over the mixture of Poissonian states.

The variance for the Poisson case can be estimated from fig. 6.3 by using Eq. 6.36 since the fluctuations are small. When the variance is small compared to 1, the variance can be approximated by

$$\sigma_m^2 \approx 2(\lambda - \lambda')$$  \hspace{1cm} (6.38)
6.6. NUMERICAL RESULTS

Figure 6.4: Plot of the conditional visibility averaged over one thousand runs versus the number of atoms detected for an initial thermal state where \( \bar{n}_1 = \Gamma \bar{n}_2 \). The shaded region corresponds to the interquartile range of the individual runs with the mean over all runs depicted by the solid line. Therefore 50% of the runs lies within the shaded region. We also plot a dashed line at \( \pi/4 \) corresponding to the value of the average visibility predicted from the analytical work.

where \( \lambda' \) is the numerical averaged visibility and in the case of equal counting rates \( \lambda = 1 \). The variance for the Poisson case is graphed as the solid curve in fig. 6.5 with the dashed curve displaying the \( \sigma_m^2 \sim 1/m \) relationship predicted by Eq. 6.35. Note that we have not plotted variances below 50 detections since Eq. 6.36 is only valid for number of detections \( m \gg 1 \). As expected, the agreement between the two curves becomes better as more atoms are detected. We cannot obtain a good estimate of the variance in the same manner for the thermal case because the fluctuations in the visibility are large.
6.6.1 Quantum Phase between initial Fock States

In this final short section we will re-visit the simpler case of initial Fock states. We are motivated to present these results because numerical simulations of the wave function of the condensates have been performed [58] where the phase variance can be calculated directly from this wave function. This gives us an alternative method of calculating the phase variance independent of Eq. 6.38 for the specific case of Fock states. The phase dynamics for this specific case can be used as a check on the dynamics already observed in the more general situation of initial mixture states since we know that the analytical predictions for the Poissonian ensemble also holds true for the simpler case of Fock states. The details of the proof of this relationship are left in appendix C.3.

The calculation of the phase variance is complicated by the difficulties of
defining a Hermitian phase operator, many of these difficulties are discussed in the work of Susskind and Glogower [68]. We choose to use the Susskind-Glogower (SG) operator. Although it is not Hermitian it is nonetheless useful, and is defined for a single mode as

\[ \hat{e}^{i\phi} = (aa\dagger)^{-1/2} a. \] (6.39)

Generalizing to two modes, the relative phase can be written in the number state basis as

\[ \hat{e}^{i\Delta\phi} = \sum_{n,m=0}^{\infty} |n, m\rangle \langle n - 1, m + 1| + |n + 1, m - 1\rangle \langle n, m|. \] (6.40)

It is convenient to use the following measure of the spread of the phase distribution:

\[ \delta\phi = 1 - \langle \cos\Delta\phi \rangle^2 - \langle \sin\Delta\phi \rangle^2 \] (6.41)

where the trigonometric operators are defined in terms of the SG phase operators as

\[ \cos\Delta\phi = \frac{1}{2} \left( \hat{e}^{i\Delta\phi} + \hat{e}^{-i\Delta\phi} \right) \] (6.42)

and

\[ \sin\Delta\phi = \frac{1}{2i} \left( \hat{e}^{i\Delta\phi} - \hat{e}^{-i\Delta\phi} \right). \] (6.43)

Note that, we denote the Hermitian conjugate of \( \hat{e}^{i\Delta\phi} \) as \( \hat{e}^{-i\Delta\phi} \), i.e. \( \hat{e}^{-i\Delta\phi} = (\hat{e}^{i\Delta\phi})^\dagger \). This measure ranges from zero to one with values close to zero agreeing well with the actual variance; a value of zero corresponds to perfectly defined phase. Since our expressions are only valid for small variances, this measure is very useful as an estimate of the variance.

The numerical results are displayed in fig. 6.6 as three curves of \( \delta\phi \), each one with differing relative count rates between the condensates \( (\Gamma N_2/N_1) \) where \( N_1 \) and \( N_2 \) are the initial atom number in the first and second condensate. Predictions of the gradient for each relative count rate are obtained from Eq. 6.35. The solid, dashed and dash–dotted curves of fig. 6.6 corresponds to \( \Gamma N_2/N_1 \) ratios of 1, 1/2, and 1/4 which are predicted to have
Figure 6.6: Plot of inverse $\delta\phi$ versus number of atoms detected with $\Gamma N_2/N_1 = 1, 1/2$ and $1/4$ for the solid, dashed and dash-dotted curves respectively. These results are from simulations of initial Fock states where $\delta\phi$ is a measure of the spread of the phase.

gradients 1, 2/3, and 2/5 respectively. We obtained numerical gradients of 0.97, 0.66, and 0.41 for the three curves. Note that the numerical gradient was calculated for points from 20 detections onwards, points below 20 were ignored since the predictions are only valid for $m \gg 1$. This is clearly seen in fig. 6.6 by the larger curvature of the lines in the region below 20 detections in comparison to points after 20. This good agreement between the analytic and numerical gradients verifies the relationship of the visibility $\lambda$ with the relative counting rates, Eq. C.5, C.19, and C.24. Note that if we had graphed $\delta\phi$ itself instead of its inverse, we would had seen a curve shaped like that of fig. 6.5. Hence the accuracy of the $1/m$ prediction of the previous section is supported by the straightness of the curves of fig. 6.6.
6.7 Summary

We have analyzed in detail the build up in quantum coherence between two Bose–Einstein condensates which are initially in a thermal, Poisson or Fock state. Interference patterns are produced via spatial atom detections which establishes an arbitrary but fixed relative phase between the condensates. In the regime where the total number of atoms detected is only a negligible fraction of the atoms in the condensates although the actual number of detections is much greater than 1, we find that the visibility of the interference pattern for the Poisson distribution depends on the relative counting rates of each condensate with a maximum of one for equal rates. In the thermal case, the visibility becomes a stochastic variable, which varies from run to run around an averaged value determined again by the relative counting rates but the average has a maximum of \( \pi/4 \) for equal rates in good agreement with our numerical simulations. The inverse variance of the phase distribution grows linearly with the number of detected atoms \( m \) in the regime where \( 1 \ll m \ll N_1, N_2 \). This has been shown analytically for the Poisson state and this relationship also holds true for the thermal state since we may write the thermal state as a mixture of Poisson states. We have shown in the special case of initial Fock states that the results follow those derived for the Poisson state. In particular, the inverse variance is proportional to \( m \), which has been numerically verified.
Part III

Generic model of an atom laser
Chapter 7

Introduction

The explosion in interest concerning Bose–Einstein condensation in dilute atomic gases has stimulated a similar level of interest in the invention of a generator of coherent matter waves. Such a device or source is analogous to an optical laser and has been called an “atom laser”. An alternative, more accurate name “bosser” has also been coined\(^1\) but “atom laser” has gained widespread usage. Perhaps the acronym “laser” (light amplification by stimulated emission of radiation) with its huge impact on physics since its invention has acquired too many positive connotations to past up in the naming of a device of possibly similar potential. A measure of the laser’s influence in physics is its broad range of applications outside the traditional physics fields. This is possibly best illustrated by considering the various areas where the laser is taken for granted in everyday life: listening to a favourite piece music recorded on a compact disc, enjoying a movie played from a laser disc, or watching a laser light show. Of course its applications are not limited to entertainment, for example in medicine, it is used in corrective eye surgery, and we should not forget to mention its humble role in the checkout scanner in the local supermarket. However, the observation of a coherent matter wave generator could possibly have the greatest influence in the fields of atom optics, atom lithography and precision measurements. We

\(^1\)Other names such as “atomaser” and “ataser” has also been coined!
shall refer to such a device as an atom laser.

The field of quantum optics can trace its origins from the invention of the laser. It is natural for workers in this field with their close links to the new fields of atom optics and atomic traps to propose schemes for an atom laser. Atom optics is the study of the wave nature of atoms; the regime where their de Broglie wavelength becomes significant compared to the relevant length scale. Atomic traps with their laser cooling schemes and evaporative cooling methods has led to the experimental observation of BEC in dilute gases. This achievement has dramatically increased the feasibility of atom lasers since BEC of a dilute atomic gas possesses many properties of a laser. These similarities between the condensate mode and the electromagnetic field of an optical laser has led to atom laser proposals which consist of modifications of present BEC experiments where the condensate mode is used as a source of coherent matter waves. A basic pulsed “atom laser” has already been demonstrated by Ketterle and co-workers at MIT [69, 70]. They have successfully condensed atomic sodium and have used this as a starting point by exploiting the coherence of the condensate. The major modification to their BEC experiment was to incorporate an output coupler; this enable them to release atoms of the trapped coherent field in a controlled manner. Since a magnetic trap confines atoms who’s magnetic moments are aligned anti-parallel to the magnetic field, they used a short radio-frequency pulse to tilt the magnetic moments of the atoms. By changing the amplitude of this radio-frequency field they can obtained remarkable control over the fraction of atoms that is released from the trap (0 to 100% transfer of atoms are possible). This is however only a “crude atom laser” in the sense that there is no pumping so that the condensate will eventually be depleted. Note that this device emits pulses of coherent matter waves; atom laser proposals usually refer to a continuous coherent matter wave output.

Before we discuss these schemes, we need to define first what an atom laser is. Both Holland et al. [71] and Wiseman [72] have provided definitions of an atom laser. We choose to follow the latter definition since we prefer one
7.1. \textit{WHAT IS AN ATOM LASER?}

where only the properties of the output field of the atom laser are referred to. Holland's definition includes the internal state of the atom laser since one of his statements requires that the linewidth should be inversely proportional to the intracavity boson number.

7.1 What is an atom laser?

We have described an atom laser as a device which emits coherent matter waves, but what do we mean by coherent? Let us consider what we mean by coherence in case of the output field of an optical laser. Interference experiments clearly illustrate some degree of coherence, but is this all that is required? Young's double slit experiments demonstrates \textit{first order} coherence (as defined by Glauber \[73\]). However, in order to observe interference in these experiments the coherence time of the output field needs to be longer than the time associated with the path difference. The longer the coherence time, the closer the output field is to monochromaticity. One does not need laser light to perform Young's double slit experiment because other sources of light can be monochromatic. For example, one can filter thermal radiation to obtain a field as close to being monochromatic as required; this beam will possess first order coherence. Thus, we need higher than first order coherence to define the properties of laser light. Hanbury Brown–Twiss \[74\] experiments are capable of testing \textit{second order} coherence as they can observe second order correlations (also called intensity correlations). Thus, these experiments can be used to distinguish between laser light and filtered thermal radiation since they have differing values of the second order correlation function; $g^{(2)}(0) = 2$ for thermal radiation but $g^{(2)}(0) \approx 1$ for laser light.

In his definition, Wiseman states "The definition I propose uses the fundamental principle that the output of the laser is well approximated by a classical wave of fixed intensity and phase. This principle yields four quantitative conditions that the output of a device must satisfy in order for the device to be considered a laser." \[72\] His criteria can be summarized as:
1. Directionality: The output is highly directional.

2. Monochromaticity: The longitudinal spatial frequency of the output beam has a small spread.

3. Well-defined intensity: The output intensity fluctuations are small.

4. Well-defined phase: The output phase fluctuations are small.

The first condition is just to define the direction of propagation and directions of diffraction. Note that this does not rule out the possibility of the output propagating in a waveguide, such as an optical laser feeding into an optical fibre. Waveguides for atom lasers can compensate for gravity and dispersion problems in matter fields.

The second condition is really a mono-energeticity condition on the matter wave and can be described in terms of the power spectrum

\[ P(\omega) = \frac{1}{2\pi I} \int d\tau e^{-i\omega \tau} \langle \dot{\psi}^* (t + \tau) \dot{\psi}^* (t) \rangle \]

\[ \sim (2\pi)^{-1} \frac{\gamma}{(\omega - \bar{\omega})^2 + (\gamma/2)^2}, \]

where \( I = \langle \dot{\psi}^* \dot{\psi} \rangle \) is the intensity. The second line above suggests that this power spectrum may typically resemble a Lorentzian shape of spectral width \( \gamma \). The output can be regarded as monochromatic if the spectral width is sufficiently small, \( \gamma \ll \bar{\omega} \), corresponding to a long coherence time \( \tau_{coh} = \gamma^{-1} \). This brings us to the fourth condition of well-defined phase; the coherence time gives us the time scale over which the phase of the field is approximately constant, i.e. well-defined. One must perform phase measurements within this time scale in order for the concept of phase to make any sense. Let us return to the MIT realization of a "crude atom laser" and note that this criterion on monochromaticity is not directly applicable since the output was pulsed. We can try to apply it to each individual pulse where the phase is constant, but the intensity of each pulse changes as it expands away from the trap. Also each subsequent pulse is reduced in size due to the depletion of the condensate. Thus we cannot call the MIT experiment an atom laser according
to Wiseman's definition. However, we should not be too harsh, as his criteria is really meant to be used for continuous lasers. The third condition, which we have skipped over, requires that the intensity correlations (or the second order correlation function) has the same form as optical lasers (\(g^{(2)}(0) \approx 1\)) corresponding to nearly Poissonian statistics. This is an important condition as it distinguishes an atom laser from a filtered thermal source of atoms as mentioned previously.

### 7.2 Atom laser models

It is useful in comparing models to highlight the important features of a typical laser. Wiseman [72] has suggested the following list:

1. A *cavity* supporting the *laser* mode of the boson field and allowing an *output beam* to form.

2. A *source* of bosons that is *coupled* to the laser mode.

3. An *irreversibility* that favours the transfer of bosons from source to laser mode.

4. A *sink* that also takes bosons from the source.

5. A *pump* to replenish the source.

A schematic of a typical atom laser model is depicted in fig. 7.1. A cold thermal source of atoms is required unlike the optical laser which typically has energy pumped into the system; photons being created from the energy supplied. For atoms which possess mass, this is not possible so that a source of atoms must be supplied from another mode or modes of an atomic trap. In typical cooling models, the source of atoms are a higher mode or modes that are coupled to the ground state of the trap which corresponds to the laser mode. This laser mode is assumed to become macroscopically occupied and is coupled to the outside world to allow for an output beam to form.
Atom laser models can be divided into several categories. We choose to distinguish them via their mathematical approaches: whether a rate or master equation is used. Master equation models [71]-[79] represents a full quantum mechanical treatment but suffers from grossly simplified models in order to make the analysis tractable. Rate equation models [80]-[83] are more realistic but cannot describe coherence properties. Note that Bordé [82] considers an atom laser model in the context of molecular dissociation instead of BEC. Moy, Hope and Savage had used a model based on rate equations with emphasis on the output coupling in the past [83] but have now extended their model to include the external modes of the matter field [84]. Thus, they can treat the coherence properties of the output without resorting to a master equation approach. This method has the advantage over the master equation treatment as it avoids the Markov approximation which is invalid in most physically relevant parameter regimes; strong output coupling. However, they have yet to include atom-atom interactions in their model. There have been models of output couplers [85, 86] where the BEC is described by using the Gross–Pitaevski (GP) equation. They are not atom
laser models according to Wiseman’s list of important features since there is no pumping. Atom–atom interactions are however implicitly included in their output coupler treatment since the GP equation contains the effect of s-wave scattering. Note that the coherence properties of the output beam cannot be calculated as the GP equation assumes that the state of the condensate is coherent (and will always remain so) so that no loss of coherence is possible in this formulation.

The early atom laser models [71, 75, 79, 80, 81] were analogous to optical laser models. The concept of an optical laser was extended to consider a coherent source of atomic bosons. Optical cavities were replaced with atomic traps; pumping from the source to the laser mode and the process of stimulated emission was replaced with an atom cooling mechanism and Bose enhancement. These early models can be classified according to the type of mechanism that is used to transfer atoms from the source to the laser mode: either evaporative cooling [71, 79] or optical cooling [75, 80, 81].

Optical cooling reduces the kinetic energy of atoms by transferring their energies and momenta to optical laser beams, spontaneously emitted photons and other atoms. The net effect is the transference of atoms from higher energy modes to the laser mode. Irreversibility arises from the spontaneous emission. One needs the rate of spontaneous emission from the laser mode to be much lower than all the other modes to achieve macroscopic occupation of the laser mode. The chief “bad” mechanism is photon re-absorption. In this undesirable process, atoms are removed from the laser mode via the absorption of the emitted photons.

Evaporative cooling works by selectively removing atoms with higher than average energy; when the remaining atoms re-thermalize by the process of elastic collisions, they are reduced in temperature. Thus, one can imagine the high-energy tail of the initial thermal distribution of atoms being sliced away, re-thermalization reforms the shape of the distribution but with a lower average energy which in turn has its high-energy tail sliced away. The process continues as long as the re-thermalization can keep up with the removal
and ultimately the lifetime of the atoms in the confining trap. Evaporative cooling has already been successfully used to produce BEC whereas optical cooling has yet to do this. However the vital collisions which re-thermalizes the atoms also results in the mechanism’s greatest problem. Collisions between atoms in the laser mode lead to phase fluctuations in the output beam (third condition of our atom laser definition). That is, the linewidth of the output may be broaden to such an extent that the device can barely meet the coherence requirements of an atom laser.

It is now instructive to briefly outline two proposals$^2$: an evaporative cooling model which uses a master equation approach, and an optical cooling scheme which uses a rate equation approach. Only the form of the model will be given for brevity.

### 7.2.1 Atom laser based on evaporative cooling

The evaporative cooling model used by Wiseman, Martins and Walls [79] is depicted in fig. 7.2. This scheme is based on a very rudimentary three-level model of evaporative cooling. The three levels have energies $E_0$, $E_1$, and $E_2$ corresponding to the ground, source and excited modes of the trap where $E_2 > E_1 > E_0$. The source mode is coupled to a broadband thermal reservoir of atoms with the flux per bandwidth being $N$. A binary collision between atoms in this source mode may result in the cooling of one atom into the ground mode while the other is heated into the excited mode. The ground and the excited modes are both coupled to vacuum reservoirs so that atoms can escape to the outside world. The excited mode coupling is assumed to be very high in order to model the rapid evaporation of the high-energy atoms. So that the net effect of binary collisions in the source is to populate the ground mode since heated atoms are quickly loss from the system. A master equation approach is now applied to the system. By making a rotating wave approximation, they were able to simplify their collisional Hamiltonian.

$^2$These subsections closely follows the concise introductory outlines from the review article by Parkins and Walls [29].
7.2. ATOM LASER MODELS

Figure 7.2: Schematic of the atom laser model of Wiseman, Martins and Walls [79] which uses an evaporative cooling mechanism. The 0 and 2 modes are coupled to vacuum reservoirs with coupling rates $\kappa_0$ and $\kappa_2$ respectively. The 1 mode is coupled to a thermal reservoir $T$. Atoms are redistributed within the trap via binary collisions.

Because the excited mode coupling was assumed to be larger than the other rates (rapid evaporation regime) it was possible to adiabatically eliminate the excited mode from the dynamics yielding the master equation of the reduced density operator $\rho$

$$
\dot{\rho} = i [\hat{V}, \rho] + \sum_{i=0}^{1} \kappa_i \mathcal{D} [\hat{a}_i] \rho + \kappa_1 N \left( \mathcal{D} [\hat{a}_1] + \mathcal{D} [\hat{a}_1^\dagger] \right) \rho + \Gamma \mathcal{D} [\hat{a}_1^\dagger \hat{a}_1^2] \rho
$$

where

$$
\mathcal{D} [\hat{c}] \rho = \dot{\rho} \hat{c}^\dagger - \frac{1}{2} \left( \hat{c}^\dagger \hat{c} \rho + \rho \hat{c}^\dagger \hat{c} \right)
$$

$$
\Gamma = \frac{\kappa_2 |V_{0211}|^2}{(\kappa_2/2)^2 + \Delta^2}
$$

$$
\hat{V} = V_{0000} \hat{a}_0^\dagger \hat{a}_0 + V_{0101} \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 + V_{1211} \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 + v \hat{a}_0 \hat{a}_1^\dagger^2,
$$

with $\Delta = E_2 + E_0 - 2E_1$ and $v = (\Delta/\kappa_2) \Gamma$. Each mode has an associated annihilation operator $\hat{a}_i$ and $V_{ijkl}$ is the collision amplitude for $\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l$. The
first term is just the coherent evolution with effective Hamiltonian $\hat{V}$, the second and third terms models both the coupling to the thermal reservoir for the source ($a_1$) mode and the coupling of the ground ($a_0$) mode to a vacuum reservoir. The last term models the evaporative cooling; two source atoms are lost to create one in the ground with the other lost to the system (since the excited mode has been adiabatically eliminated).

7.2.2 Atom laser based on optical cooling

Olshanii, Castin and Dalibard [81] consider a simple yet general model based on optical cooling. Atoms in an internal state $a$ are injected into the atom laser system at a rate $R_a$ with initial momenta of the order or smaller than $\hbar k$. Undergoing spontaneous emission, they decay down to another state $b$ where they are trapped in a "box" of volume $V$. The density of states in momentum space in this "box" is then $V / (2\pi \hbar)^3$. The magnitude of momenta for atoms in state $b$ must be less than or equal to $p_0 = 2\hbar k$ so that in momentum space they must lie within a sphere of radius $p_0$. We now can calculate the number of levels (momentum states) $N_{\text{lev}}$ in the $b$ mode that are obtainable from the incoming atoms. Thus we have $N_{\text{lev}}$ number of levels $|b, p\rangle$, \footnote{We consider the regime where $V \gg \lambda^3$, i.e. when the number of levels is very large. However Spreeuw et al. [80] operate in the opposite limit in a similar atom laser scheme.}

$$N_{\text{lev}} = \frac{V}{(2\pi \hbar)^3} \frac{4\pi p_0^3}{3} = \frac{32\pi V}{3} \lambda^3,$$

where $\lambda = 2\pi/k$. The rate equation for the mean occupation number $n_b (p)$ of the state $|b, p\rangle$ is

$$\dot{n}_b (p) = -\gamma_b (p) n_b (p) + \frac{\gamma_a}{N_{\text{lev}}} N_a [1 + n_b (p)] - \frac{\sigma c}{V} N_v n_b (p),$$

where $\gamma_b (p)$ is the lost rate of atoms in the $b$ mode from the cavity for a particle momenta $p$. The first term describes this loss from mode $b$.

The second term describes the pumping into the $b$ mode due to the decay from the $a$ mode

$$a \rightarrow b + \text{photon},$$
where \( N_a \) is the population in the \( a \) mode; note that they have assumed that each decay rate out of \( |a, p\rangle \) is equal to \( \gamma_a/N_{\text{lev}} \). The 1 and the \( n_b(p) \) in the square brackets represents the spontaneous and stimulated emission respectively.

The third term describes the undesirable process of re-absorption of a photon by an atom in mode \( b \) back to mode \( a \) i.e.

\[
b + \text{photon} \rightarrow a.
\]

This term is proportional to the absorption cross-section \( \sigma \) of a photon multiplied by the speed of light \( c \) over the volume \( V \), giving a probability of re-absorption scaled by the product of the number of photons in \( V \) and the number of atoms in mode \( b \).

### 7.3 Our generic model of an atom laser

We present a generic model of an atom laser by including a pump and loss term in the Gross-Pitaevski equation. This approach shares some common features with the output coupler models mentioned previously [85, 86] since a condensate in a trap is regarded as a laser mode and is modelled by the GP equation. However, we concentrate on the effect of introducing pump and losses to this condensate mode rather than the output coupler.

Our model can be regarded as the analog of a semi-classical description of the conventional laser where the electromagnetic field is treated classically. We describe our model as generic because we do not attempt to model the pump or loss mechanisms in detail, unlike the previous evaporative and laser cooling models.

In chapter 8 we will study this model in detail. The introductory section, stability analysis and the numerical simulation of the time dependence is

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\( ^{4} \)In general \( \gamma_{a}(p) \) depends on the Franck-Condon factors which describes the overlap between the atomic wave functions of state \( a \) and states \( b, p \). A more precise treatment is given by Spreeuw [80] and, Moy and Savage [87].
the work of Bernhard Kneer under the supervision of Dr. Karl Vogel and Professor Wolfgang Schleich. This work is included in here for the benefit of completeness. My input into this model is the development of the modified Thomas–Fermi solution which, includes the effect of the pump and (position dependent) loss on the spatial phase and density of the condensate. This solution is used to predict the stationary–state condensate population for a given pump strength.
Chapter 8

Generic model of an atom laser

8.1 Introduction

There is a close analogy between the atom laser and the conventional laser, therefore there should also be a similar analogy concerning the theory of an atom laser. We have introduced atom laser proposals in the previous chapter and have classified the majority of them basically into two different approaches, based on either rate equations or master equations. With rate equations one cannot describe any coherence properties, whereas master equations represent a full quantum mechanical treatment. However, in conventional laser theory a semi-classical treatment [88, 89] was very successful. Therefore, one may ask if there is a similar semi-classical theory for the atom laser?

In semi-classical laser theory the electromagnetic field in the cavity is treated classically. Therefore in a semi-classical theory of an atom laser the matter-wave field which can be represented by the field operator $\hat{\Psi}(r, t)$ should be dealt with in a similar way. In our theory we take the mean-field $\psi(r, t) = \langle \hat{\Psi}(r, t) \rangle$ as such a "classical" matter-wave field. The corresponding wave equation for the matter field is the Gross-Pitaevski equation [31, 32, 33] that was successfully used to investigate the properties of a Bose-Einstein condensate. This equation also includes the collisions between the atoms, a
A crucial part of an atom laser as for any laser is a stimulated amplification process. So far two different mechanisms for this matter-wave amplification have been discussed: elastic collisions by evaporative cooling and spontaneous emission of photons by optical cooling. Further, different schemes of pumping a condensate have been discussed [90, 91, 92]. In this chapter we will refer to the second mechanism, but in general our model can be adapted to any mechanism where Bose enhancement is present. Furthermore, different loss mechanisms can influence the behaviour of the atom laser. Therefore, our idea is to generalize the Gross–Pitaevski equation as the equation for the matter-wave field by including gain and loss terms.

We do not consider any output coupler [85, 86, 87, 93, 94, 95] as we are only interested in the field inside the "cavity".

### 8.2 Model

The Gross–Pitaevski equation (GPE) for condensed atoms in a trap is [96]

\[
\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(r)\psi + U_0 |\psi|^2 \psi,
\]

(8.1)

where \(V(r)\) is the trap potential and the non-linear term \(U_0 |\psi|^2 \psi\) takes into account two-particle interactions. The parameter \(U_0\) can be expressed in terms of the \(s\)-wave scattering length \(a\) via the relation \(U_0 = 4\pi \hbar^2 a / m\). The GPE has been used successfully to describe the properties of Bose–Einstein condensates. However, with the interpretation of \(\int \psi^\ast \psi d^3r\) as the number of atoms in the condensate, it does not allow one to describe the growth of a condensate or the loss of atoms in a condensate as \(\int \psi^\ast \psi d^3r\) remains constant where \(\psi\) satisfies Eq. 8.1.

In our model, we introduce two additional terms to the GPE: a loss term and a gain term. We use the loss term

\[
H_{\text{loss}} \psi = -\frac{i\hbar}{2} \gamma \psi
\]

(8.2)
which leads to an exponential decay of the number of atoms in the condensate. As a gain term we introduce

\[ H_{\text{gain}} \psi = \frac{i\hbar}{2} \Gamma N_e \psi, \]  

(8.3)

where \( N_e \) is the number of atoms outside the condensate and \( \Gamma \) is the transition rate for these atoms into the condensate. This might be regarded as a generic pump mechanism of an atom laser as it already contains the Bose enhancement factor as we will see later. A possible mechanism is the transition of untrapped, excited two-level atoms which spontaneously emit photons and make a transition to their ground state that is trapped, see for example the discussion of Ref. [80].

If we generalize the GPE (8.1) by adding the loss term (8.2) and the gain term (8.3), we arrive at the modified GPE

\[
\begin{align*}
\frac{i\hbar}{m} \frac{\partial \psi}{\partial t} & = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(r) \psi + U_0 |\psi|^2 \psi \\
& - \frac{i}{2} \hbar \gamma_g \psi + \frac{i}{2} \hbar \Gamma N_e \psi.
\end{align*}
\]

(8.4)

For the number of uncondensed atoms \( N_e \) we assume the rate equation

\[ \dot{N}_e = R_e - \gamma_e N_e - \Gamma N_e N_g, \]  

(8.5)

where \( R_e \) is a pump rate that models an infinite reservoir of atoms and the term \(-\gamma_e N_e\) takes into account the fact that atoms can escape from our system without being trapped in the ground state. For \( N_g = \int \psi^* \psi d^3r \), the number of atoms in the condensate, we obtain from Eq. 8.4

\[ \dot{N}_g = -\gamma_g N_g + \Gamma N_e N_g. \]  

(8.6)

In Eqs. (8.5) and (8.6) we can clearly identify the bose-enhanced transition of atoms into the condensate. Note, that rate equations similar to Eqs. (8.5) and (8.6) have already been discussed in the literature [80]–[83]. A schematic of our generic model is shown in fig. 8.1.
Figure 8.1: Schematic of our generic atom laser model. Atoms are pumped at a rate $R_e$ into the source modes $|e\rangle$ which are then (optically) cooled into the laser mode $|g\rangle$. Output beam is generated from atoms leaving $|g\rangle$ at rate $\gamma_g$. Atoms can also leave directly from the source modes via $\gamma_e$. 
8.3 Stationary solution and stability analysis

Before we discuss the stationary solution of Eq. 8.4 we will first discuss the stationary solutions of the rate equations Eq. 8.5 and Eq. 8.6: (i) One stationary solution is

\[ N_g^s = 0, \]
\[ N_e^s = \frac{R_e}{\gamma_e}. \]  

(8.7)

By introducing small deviations

\[ n_e(t) = N_e(t) - N_e^s, \]
\[ n_g(t) = N_g(t) - N_g^s, \]  

(8.8)

we arrive at the linearized equations

\[ \dot{n}_e = -\gamma_e n_e - \Gamma N_e^s n_g, \]
\[ \dot{n}_g = -\gamma_g n_g + \Gamma N_e^s n_g \]
\[ = \frac{\Gamma}{\gamma_e} \left( R_e - \frac{\gamma_g \gamma_e}{\Gamma} \right) n_g. \]  

(8.9)

A stability analysis of these equations shows that the pair of stationary solutions in Eq. 8.7 is stable when

\[ R_e < \frac{\gamma_g \gamma_e}{\Gamma}. \]  

(8.10)

The corresponding stable stationary solution of Eq. 8.4 is

\[ \psi_s \equiv 0. \]  

(8.11)

We therefore do not have any laser activity; that is, we are below threshold.

(ii) The other stationary solution is

\[ N_g^s = \frac{R_e}{\gamma_g} - \frac{\gamma_e}{\Gamma}, \]
\[ N_e^s = \frac{\gamma_g}{\Gamma}. \]  

(8.12)
Again we consider small deviations from this solution and obtain the linearized equations

\[ \dot{n}_e = -\left(\gamma_e + \Gamma N_g^2\right)n_e - \gamma_g n_g, \]
\[ \dot{n}_g = \Gamma N_g^2 n_e. \]  

(8.13)

Here a stability analysis shows that Eq. 8.12 are a stable stationary solution for

\[ R_e > \frac{\gamma_g \gamma_e}{\Gamma}. \]  

(8.14)

Therefore we will have laser activity if the pump rate \( R_e \) is above the threshold value given in Eq. 8.14. In that case the steady-state number of atoms in the condensate grows linearly with the pump rate \( R_e \).

The stationary solution \( \psi_s \) of Eq. 8.4 is the stationary solution of the conventional GPE for \( N_g^s \) atoms in a trap \([34, 40]\) since the gain and loss term compensate each other

\[ \mu \psi_s = -\frac{\hbar^2}{2m} \nabla^2 \psi_s + V(\mathbf{r}) \psi_s + U_0|\psi_s|^2 \psi_s, \]  

(8.15)

where \( \mu \) is the chemical potential. A stability analysis of this equation is in general not possible analytically. However, within the framework of the Thomas–Fermi approximation, Stringari \([98]\) has already calculated the excitation spectrum of a condensate in a three-dimensional isotropic harmonic trap. Since our numerical solution of Eq. 8.4 is done for a one-dimensional harmonic trap, we apply in Appendix A Stringari’s method to a one-dimensional harmonic trap and find

\[ \delta \rho(x, t) = \sum_{n=1}^{\infty} A_n \sin(\Omega_n t + \varphi_n) P_n(x/R). \]  

(8.16)

Here we have introduced the small deviation

\[ \delta \rho(x, t) = \psi(x, t)^* \psi(x, t) - \rho_{0}^{\text{TF}}(x) \]  

(8.17)

of \( \psi^*(x, t)\psi(x, t) \) from the Thomas–Fermi solution \([99]\)

\[ \rho_{0}^{\text{TF}}(x) = \frac{\mu - \frac{m \omega^2}{2} x^2}{U_0} \]  

(8.18)
and define the Thomas–Fermi radius $R = \sqrt{2\mu/m\omega_z^2}$. Furthermore, $P_n(\xi)$ are the well-known Legendre polynomials defined by

$$P_n(\xi) = \frac{1}{2^n n!} \frac{d^n}{d\xi^n} (\xi^2 - 1)^n.$$  \hspace{1cm} (8.19)

$A_n$ and $\varphi_n$ follow from the initial deviation of $\psi^* \psi$ from the stationary solution, and the frequencies $\Omega_n$ are given by

$$\Omega_n^2 = \frac{\omega_z^2}{2} n(n + 1),$$  \hspace{1cm} (8.20)

where $n$ is an integer. Our discussion shows that there are excitations which do not decay, even in the presence of a loss term in our modified Gross–Pitaevski equation, Eq. 8.4. However, they do not grow but rather oscillate. Our numerical solution of Eq. 8.4 in the next section will confirm this result.

## 8.4 Numerical results

The important question is, if and how the condensate evolves above threshold into a stationary state from an arbitrary initial condition. To answer this question we have to examine the time dependent behaviour of our atom laser equations (8.4) and (8.5). A “natural” initial condition would be $N_g = 0$ and $N_e = 0$, that is to start without any atoms. However, this would lead to the unstable solution $N_g = 0$ and $N_e = R_e/\gamma_e$, but any small perturbation in $N_g$ leads to a completely different behaviour and approaches the stable solution $N_g^* = R_e/\gamma_g - \gamma_e/\Gamma$ and $N_e^* = \gamma_g/\Gamma$, as discussed in Sec. 8.3. Therefore, we use $N_e(0) = 0$ and $N_g(0) \ll 1$ for our numerical simulations.

Furthermore, we restrict ourselves to the one-dimensional case and a harmonic potential. Hence, Eq. 8.4 changes to

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega_z^2 x^2 + U_x|\psi(x, t)|^2 - \frac{i}{2} \hbar \gamma_g + \frac{i}{2} \hbar \Gamma N_e \right] \psi(x, t).$$  \hspace{1cm} (8.21)
This is already sufficient to get insight into the fundamental aspects of our model. To calculate the time dependent solution of Eq. 8.21 we use the split-operator-method [100] that has been used successfully for the ordinary GPE [51].

For our numerical calculations we took the parameter \( U_x/(\hbar \omega_x a_x) = 1/125 \), where \( a_x \) is the harmonic oscillator length. For \( R_e, \Gamma, \gamma_e, \text{ and } \gamma_g \) there are no experimental values available. As in conventional laser theory [89], where one usually has a high-Q cavity, we require \( \gamma_g \ll \omega_x \). Further, we set \( \Gamma = \gamma_e = \gamma_g \) to have comparable losses. Furthermore, we choose \( R_e \) such that we have a sufficiently large number of atoms in the condensate, that is, \( N_g^s = R_e/\gamma_g \gg 1 \).

Figure 8.2 shows the time dependent solution of the rate equations (8.5) and (8.6) for \( R_e/\omega_x = 10^2 \), \( \Gamma/\omega_x = \gamma_e/\omega_x = \gamma_g/\omega_x = 10^{-2} \). The initial condition was \( N_e(0) = 0 \) and \( N_g(0) = 10^{-3} \). For very small times the atoms accumulate in the excited state since they cannot make a transition into the ground state. During these small times, the occupation number of the ground state is low so that the bose-enhancement factor does not contribute to the transition rate. As soon as we have a noticeable number of atoms in the ground state, \( N_e \) rapidly approaches its stationary value \( N_e^s = \gamma_g/\Gamma \). Additional atoms then essentially end up in the condensate where the number of atoms slowly approaches its stationary value \( N_g^s = R_e/\gamma_g - \gamma_e/\Gamma \).

For the numerical analysis of Eq. 8.5 and Eq. 8.21 we use the same parameters, and for the initial wave function \( \psi(x, t = 0) \) we have used the quantum mechanical ground state of the trap, normalised in such a way that
\[
N_g(0) = \int |\psi(x, t = 0)|^2 dx = 10^{-3}.
\]
We find that the number of atoms, given by \( N_g(t) = \int |\psi(x, t)|^2 dx \), increases as a function of time in the correct way as given by the integration of the rate equations in Fig. 8.2. \( |\psi(x, t)|^2 \) approaches a stationary value. However, some oscillations are left. In order to see these oscillations clearly, we show in Fig. 8.3 the width of \( |\psi(x, t)|^2 \) as a function of time. As a measure of the width we use
\[
\Delta x = \sqrt{I_2 - I_1^2}, \tag{8.22}
\]
Figure 8.2: Number of atoms $N_g$ in the ground state (a) and number of atoms $N_e$ in the excited state (b) as a function of the scaled time $\omega_x t$. The parameters are $R_e/\omega_x = 10^2$, $\Gamma/\omega_x = 10^{-2}$, $\gamma_e/\omega_x = 10^{-2}$, and $\gamma_g/\omega_x = 10^{-2}$. As initial condition, we have use $N_e(0) = 0$ and $N_g(0) = 10^{-3}$. 
Figure 8.3: Width $\Delta x/a_x$ as a function of the scaled time $\omega_x t$. Here we have used $U_x/(\hbar \omega_x a_x) = 1/125$. The initial number atoms in the excited state was zero, i.e. $N_e(0) = 0$. For our initial wave function, $\psi(x,t=0)$ we used the quantum mechanical ground state of the trap, normalised in such a way that $N_g(0) = \int |\psi(x,t)|^2 dx = 10^{-3}$.

where $I_1$ and $I_2$ are given by

$$I_1 = \int \psi^*(x,t)x\psi(x,t) \, dx \tag{8.23}$$

and

$$I_2 = \int \psi^*(x,t)x^2\psi(x,t) \, dx, \tag{8.24}$$

respectively. Note, that $\Delta x$ would be the variance of the position operator, if $\psi(x,t)$ was a true single–particle wave function. The average value of the width $\Delta x^2$ approaches a stationary value as time becomes larger. However, we can see oscillations around this average value which do not decay. The
frequency of these oscillations is $\Omega/\omega_x \approx 1.76$ which is in good agreement of our prediction $\Omega_2/\omega_x = \sqrt{3} \approx 1.732$ of Sec. 8.3. It has been shown that in three dimensions the oscillations can collapse and revive again [101, 102, 103] but only for an anisotropic trap due to mode beating. In the next section, we will introduce a space dependent loss rate $\gamma_g(x)$ and show that these collective excitations will be damped.

Finally, we compare in Fig. 8.4 the stationary solution of the Gross–Pitaevski equation (Eq. 8.15) in one dimension to the solution of Eq. 8.5 and Eq. 8.21. We have plotted $|\psi(x, t)|^2$ for a time where the width is maximal and where it is minimal and find that the solution of Eq. 8.5 and Eq. 8.21 indeed oscillates around the stationary solution of the Gross–Pitaevski equation. In addition to the collective excitations that occur, our simple theory has another disadvantage which is a sensitivity of the solution to the initial condition $\psi(x, t = 0)$. Both problems can be circumvented by the introduction of a space dependent loss, which makes the theory only slightly more complicated. Another way would be to model a trap that has only one bound state, not infinitely many as the harmonic one we have used.

### 8.5 Space dependent loss

Real losses in a trapped condensate are most likely to be spatially dependent, that is, collisions with uncondensed atoms occur rather at the edges of the condensate than in its centre or atoms are “lost” due to output coupling as an emission of a matter wave localized in some region. In our one-dimensional model, this region consists of the two ends of the harmonic trap where the density of the condensate tails off. The pump rate remains spatially independent as the condensate is assumed to be pumped from a cloud of cold thermal atoms which is larger in size and may enclose the condensate.

We introduce a spatial dependent loss $\gamma_g(r)$ to our modified GPE

$$ih \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(r)\psi + U_0|\psi|^2\psi$$
Figure 8.4: The stationary solution of the Gross–Pitaevski equation (diamonds) as a function of the scaled position $x/a_x$ in comparison with the "stationary" solution (solid line) of Eq. 8.21. We have also plotted $|\psi(x,t)|^2$ for times where the width is maximal (dashed) and where it is minimal (dotted).

$$- \frac{i}{2} \hbar \gamma_g(r) \psi + \frac{i}{2} \hbar \Gamma N_c \psi. \quad (8.25)$$

Therefore the rate equation (8.6) for the number $N_g$ of condensed atoms is changed to

$$\dot{N}_g = \Gamma N_c N_g - \int_{-\infty}^{\infty} \gamma_g(r)|\psi(r,t)|^2 \, d^3r. \quad (8.26)$$

The rate equation (8.5) for the number $N_e$ of uncondensed atoms remains the same. Note that the rate equations (Eq. 8.26) and (Eq. 8.5) cannot be solved on their own, because Eq. 8.26 is coupled to our modified GPE (Eq. 8.25).
8.5. SPACE DEPENDENT LOSS

8.5.1 Stability analysis

We find two sets of stationary solutions of the Eqs. 8.5, 8.25, and 8.26, similar to Sec. 8.3: (i) The first set reads

\[ N_e^s = \frac{\int \gamma_g(r)|\psi_1^s(r)|^2 \, d^3r}{\Gamma} \]

\[ N_g^s = \frac{\int \gamma_g(r)|\psi_1^s(r)|^2 \, d^3r}{\Gamma} - \gamma_e \]

where \( \psi_1^s(r) \) denotes the stationary solution of Eq. 8.25 normalised to 1. We show how to find the solution \( \psi_s(r) \) in subsection 8.5.3.

(ii) The second set of solutions is provided by

\[ N_e^s = \frac{R_e}{\gamma_e} \]

\[ \psi_s = 0 \]

\[ N_g^s = 0 \]  \hspace{1cm} (8.28)

In order to find the parameter regimes where either one of the solutions is stable, we need to perform a stability analysis similar to that of Sec. 8.3, but because of the spatial dependent loss rate \( \gamma_g(r) \) we have to introduce the small deviation \( \phi \) from the stationary mean-field \( \psi_s \)

\[ \phi(r,t) = \phi(r,t) - \psi_s(r). \hspace{1cm} (8.29) \]

We arrive at the linearized equations

\[ \dot{n}_e = -(\gamma_e + \Gamma N^s_g)n_e - \Gamma N^s_g n_g, \hspace{1cm} (8.30) \]

\[ \dot{n}_g = \Gamma N^s_g n_e + \Gamma N^s_e n_g - \int \gamma_g(r)(\phi\psi_s^* + \phi^*\psi_s) \, d^3r, \hspace{1cm} (8.31) \]

\[ i\hbar \dot{\phi} = -\frac{\hbar^2}{2m} \nabla^2 \phi + V(r)\phi + U_0\psi_s^2 \phi^* + U_0|\psi_s|^2 \phi \]

\[ - \frac{i}{2}\hbar\gamma_g(r)\phi + \frac{i}{2}\hbar \Gamma(N^s_e \phi + n_e\psi_s), \hspace{1cm} (8.32) \]

and the conjugate equation for \( \phi^* \). We note that \( n_g \) is provided by

\[ n_g = \int (\phi\psi_s^* + \phi^*\psi_s) \, d^3r. \hspace{1cm} (8.33) \]
To find the laser threshold, we note that close to threshold \( N_g^0 \) is small and therefore \( \psi_s \) is also small. Thus we can assume that the deviation \( \phi \) is localized in a small region around zero at the centre of the trap. Therefore we can pull out \( \gamma_g(0) \) from the integral in Eq. 8.30 and obtain the linearized equations

\[
\begin{align*}
\dot{n}_e &= -(\gamma_e + \Gamma N_g^0) n_e - \Gamma N_e^s n_e, \\
\dot{n}_g &= \Gamma N_g^s n_e + (\Gamma N_e^s - \gamma_g(0)) n_g,
\end{align*}
\]

which are decoupled from Eq. 8.32. A stability analysis of these two equations provides the lasing threshold

\[
R_e = \frac{\gamma_e \gamma_g(0)}{\Gamma}
\]

which is equal to the one in Sec. 8.3, if one replaces \( \gamma_g \) by \( \gamma_g(0) \). Therefore, Eqs. 8.27 are the stable solutions for \( R_e > \gamma_e \gamma_g(0)/\Gamma \), and Eqs. 8.28 are the stable solutions for \( R_e < \gamma_e \gamma_g(0)/\Gamma \). Note that we obtain the solutions for the laser threshold and the particle numbers of the spatially independent case when we set \( \gamma_g(r) = \gamma_g \).

### 8.5.2 Time dependent solution

In our numerical simulation we restrict ourselves to one dimension and a harmonic trap again. The loss shall be localized at the edges of the condensate denoted by \( x_0 \) and have a Gaussian shape of width \( \sigma \)

\[
\gamma_g(x) = \frac{\gamma_g'}{2 \sqrt{\pi} \sigma} \left( e^{-(x-x_0)^2/\sigma^2} + e^{-(x+x_0)^2/\sigma^2} \right).
\]

The loss is normalised such that

\[
\int_{-\infty}^{\infty} \gamma_g(x) \, dx = \gamma_g'.
\]

We have performed a numerical simulation for the parameters \( \gamma_e/\omega_x = \Gamma/\omega_x = 10^{-2}, \gamma'_g/\omega_x = 6.8 \times 10^{-1}, R_e/\omega_x = 10^2, x_0/a_x = 5, \) and \( \sigma/a_x = 1 \). In Fig. 8.5 we compare the time evolution for the number of uncondensed...
8.5. SPACE DEPENDENT LOSS

Figure 8.5: (a) Comparison of time behaviour for the number of condensed atoms with spatial dependent (solid line) and independent (dashed line) loss. (b) The same for uncondensed atoms.
Figure 8.6: Scaled width $\Delta x/a_x$ of the mean-field wave function versus the scaled time $\omega_x t$ for space dependent loss. The collective excitations are damped away.

atoms for the position dependent and independent loss. In the case of the spatial independent loss we have taken the parameters of Fig. 8.2. In order to obtain the same final number of atoms in the condensate for comparison purposes, we adjusted $\gamma'$. We see that the overall behaviour is not that different from the spatial independent case and therefore our original rate equations are still well approximated by the modified ones. We have plotted the scaled width in Fig. 8.6 where we see that the collective excitations are damped away, and a steady state is reached. This is true in general, that is, for any initial mean-field. In Fig. 8.7 we have plotted the final mean-field in comparison to the loss-function, without scaling. The overlap between the two is essential in the formula for the number of atoms in the condensate, see Eq. 8.27. The mean-field obtained by evolving the modified GPE over a
sufficiently long enough time such that the transient oscillations are damped out is equivalent to the stationary ground state energy solution of both the modified and conventional GPE where their steady-state atom numbers are equal and the loss is localized at the edges. If the loss is localized closer to the centre, we still find that the mean-field converges to the stationary ground state of the modified GPE but not of the conventional GPE. The way to find this stationary solution is shown in the next subsection. This is an essential result that the collective excitations can be damped by introduction of a spatially dependent loss term which is located in two peaks at the side of the condensate. In our theory of an atom laser, this plays the role of mode selection, i.e., the ground state energy solution survives whereas the excited states, i.e., the collective excitations, are damped away.

8.5.3 Stationary solution

The stationary regime of an atom laser where the transients have damped away is of interest because this would be readily accessible in an operational atom laser. We have shown in the previous section the numerical evolution of the wave function through these transients and confirmed that a stationary state is reached. The goal of this section is to calculate the stationary state directly. Let us start by writing down the following rate equations, Eq. 8.5 and Eq. 8.26 again for clarity

\[ \dot{N}_e = R_e - \gamma_e N_e - \Gamma N_e N_g, \]

\[ \dot{N}_g = \Gamma N_e N_g - \int \gamma_g (\mathbf{r}) |\psi(\mathbf{r}, t)|^2 d^3r, \]

and find a expression for the condensate population in the stationary state regime. From the first equation we have \( R_e = \gamma_e N_e^s + \Gamma N_e N_g^s \). This is a statement on number conservation of the uncondensed atoms, the total loss rate of atoms going into the condensate mode and atoms leaving the system is equal to the inward pump rate \( R_e \). Using this expression and the second equation we find that the stationary state condensate population satisfies the
Figure 8.7: Mean-field in the steady-state regime (solid line), as a function of scaled position, compared to the space dependent loss-function $\gamma_g(x)$ (dashed line) with arbitrary units on $y$-axis.

The equation

$$0 = \frac{R_e N^s}{N^g + \gamma_e / \Gamma} - \int \gamma_g(x) |\psi_s(x)|^2 dx$$

(8.41)

where $\psi_s$ is the solution of the time independent GPE with pumping and losses. We have not solved this equation for $\psi_s$ but may use the Thomas–Fermi (TF) solution $\psi_{TF}$ to approximate $\psi_s$ [99]. The TF solution should be good for large atom numbers, however it does not take into account the pump and losses. Thus, it is inconsistent to include the effects of the pump and losses in the rate equations but leave them out in using the TF solution. We shall now introduce a modified TF solution which does take into account the pump and losses. Our goal is to obtain a consistent approximation of $\psi_s$. 
8.5. SPACE DEPENDENT LOSS

Modified Thomas–Fermi solution

We start by writing down the time independent form of the generalized GPE

\[ \mu \psi_s = -\frac{i \hbar^2}{2m} \nabla^2 \psi_s + V(\mathbf{r}) \psi_s + U_0 |\psi_s|^2 \psi_s \]

\[ - \frac{i}{2} \hbar \gamma_g (\mathbf{r}) \psi_s + \frac{i}{2} \hbar \Gamma N_e \psi_s, \]  

which can be obtained from Eq. 8.4 by looking for stationary solutions of the form \( \psi(\mathbf{r}, t) = \psi_s(\mathbf{r}) \exp(-i\mu t/\hbar) \) where \( \mu \) denotes the chemical potential. Because we have included pumping and losses, the stationary solution corresponds to an equilibrium condition where the rate of atoms into and out of the condensate are equal. At this equilibrium, the number of atoms in the condensate does not change so that the chemical potential is real. Thus to consider both the density and velocity of the condensate we substitute

\[ \psi_s(\mathbf{r}) = \sqrt{\rho(\mathbf{r})} e^{i\phi(\mathbf{r})} \]  

into Eq. 8.42 and separating into its real and imaginary parts to obtain

\[ \frac{\hbar^2}{2m} \nabla^2 \sqrt{\rho} = \left[ -\mu + \frac{1}{2} mv^2 + V(\mathbf{r}) + U_0 \rho \right] \sqrt{\rho}, \]  

and

\[ \frac{v}{2} \nabla \sqrt{\rho} = [ -\nabla v - \gamma_g (\mathbf{r}) + \Gamma N_e ] \sqrt{\rho} \]  

where we have defined the velocity \( v \equiv (m/\hbar) \nabla \phi \). We make the assumption that the density profile is slowly varying compared to the flow rate of atoms into and out of the condensate. The \( \nabla^2 \sqrt{\rho} \) and \( \nabla \sqrt{\rho} \) terms may then be neglected but the velocity term is retained. These assumptions are not unreasonable since we are only interested in the stationary state regime. This is where we make the Thomas–Fermi approximation. Setting these terms to zero we obtain

\[ \rho_{MTF}(\mathbf{r}) = \frac{\mu - \frac{1}{2} m [v_{MTF}(\mathbf{r})]^2 - V(\mathbf{r})}{U_0} \]  

with

\[ \nabla v_{MTF}(\mathbf{r}) = -\gamma_g (\mathbf{r}) + \Gamma N_e, \]
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giving the modified Thomas–Fermi solution

\[ \psi_{MTF}(r) = \sqrt{\rho_{MTF}(r)} e^{i\phi(r)}. \] (8.48)

Note that the phase \( \phi \) depends on the pump, uncondensed atom number, and shape of the loss rate. The chemical potential \( \mu \) is determined from the normalisation condition \( \int \rho_{MTF} d^3r = N_g \). Our modified solution consists of a spatial alteration of the density and a spatial phase shift. The square of the velocity plays a similar role as the trap potential in shaping the density. It is altered according to the steepness of the loss shape. The spatial phase illustrates the equilibrium nature of the stationary state. The atoms are flowing in and out at the same rate (integrated over all \( r \)) so that the ground state population is stationary. Because of the spatial loss, the outward flow is heavily concentrated in regions where the loss rate is high, and thus the velocity (the gradient of the phase) is also high in these regions. The spatial phase reflects the space dependence of the outward flow.

To proceed further we now need to specify the trap and loss shapes. We only consider a one-dimensional model and choose a harmonic trap potential with the shape of the loss rate consisting of two Lorentzians centred at each tail of the condensate. The potential and the loss–function are defined as

\[ V(x) = \frac{1}{2} m \omega_x^2 x^2 \] (8.49)

and

\[ \gamma_s(x) = \gamma_s' \left[ \frac{\alpha^2}{(x-z_0)^2 + \alpha^2} + \frac{\alpha^2}{(x+z_0)^2 + \alpha^2} \right]. \] (8.50)

The Lorentzians are centred at \( \pm z_0 \) with their FWHM equal to \( 2\alpha \), and peak heights of \( \gamma_s' \). The choice of a Lorentzian shape rather than a Gaussian was motivated by the calculative convenience of definite integrals involving the Lorentzian over the Gaussian function. Using these specific shapes, we find that the velocity is

\[ u_{MTF}(x) = \Gamma N_e x - \gamma_s' \alpha \left[ \arctan \left( \frac{x+z_0}{\alpha} \right) + \arctan \left( \frac{x-z_0}{\alpha} \right) \right]. \] (8.51)
where we have set \( v(0) = 0 \) to remove the constant of integration. The value of the velocity at the centre is arbitrary because only the relative velocity profile is physically describing the flow of atoms from one region of the condensate to another region. The expression for \( \phi \) is then

\[
\phi_{MTF}(x) = \frac{\gamma_e N_e}{2} x^2 - \gamma_g' \alpha^2 [\mathcal{F}_+(x) + \mathcal{F}_-(x)] + C
\]

where

\[
\mathcal{F}_\pm(x) = \left( \frac{x \pm z_0}{\alpha} \right) \arctan \left( \frac{x \pm z_0}{\alpha} \right) - \log \sqrt{1 + \left( \frac{x \pm z_0}{\alpha} \right)^2}
\]

and \( C \) is determined by setting \( \phi(0) = 0 \). The phase at the centre is also arbitrary since only the gradient of the phase has any physical meaning; it describes the velocity profile.

Figure 8.8 shows both the TF and modified TF solution for the following set of parameters: \( R_x/\omega_x = 5 \times 10^4 \), \( \Gamma/\omega_x = 0.5 \), \( \gamma_e/\omega_x = 0 \), \( \gamma_g'/\omega_x = 5 \), \( \alpha/a_x = 1 \), \( z_0/a_x = 10 \), \( N^{e*}_e = 2/3 \), and \( N^{s*}_g = 1.5 \times 10^5 \). We have set the parameter \( U_x/(\hbar \omega_x a_x) = 0.0078 \). The modified TF solution is denoted by the solid curve, the TF solution by the dashed one, and the loss-function \( \gamma_g(x) \) by the dash-dotted curve. Note that the loss curve is not to scale but is displayed to show where the peaks lie.

The density is shown in Fig. 8.8(a), we see that the modified TF solution is more squeezed and higher at the centre with a steeper decline at the tails (where the loss is at its greatest) than the TF solution. These slight alterations in the density are not unexpected, but note that these effects are small since the pump rate of \( R_x/\omega_x = 0.5 \times 10^5 \) is a significant fraction of the atom number \( N^{s*}_g = 1.5 \times 10^5 \). Surprisingly, at regions close to the peaks of the loss where the loss rate is at its highest, the density for the modified TF can be higher than the TF solution. This is understandable when we note that at these regions of high loss, the velocity of the atoms are correspondingly high. The high relative kinetic energy of the atoms at these points permits greater concentrations in the same way as angular momentum in vortex states allow greater densities away from the centre of a trap potential. Pump rates of a
Figure 8.8: (a) Density of the modified (solid line) and usual (dashed line) TF solution. The spatial loss is shown by the dashed–dotted line for comparison, note the position of the peaks, the height is not to scale. (b) Velocity of the TF and modified TF solutions.
few percent of the atom number would be more sensible as it is expected that only a small portion of the condensate atoms are lost and gained during a trap period. For smaller pump rates the differences between the two solutions would be very small, however these differences can become important when the peaks of the spatial loss are very sharp.

The velocity is shown in Fig. 8.8(b), displaying a linear curve around the central region of the condensate, illustrating the flow of atoms to the ends where they are predominately lost from the peaks. Since the usual TF solution does not take into account the pump and losses, the velocity is zero as shown by the dashed curve.

We can proceed with a consistent approximation of $\psi$, by using the modified TF solution, substituting $\psi_{MTF}$ into Eq. 8.41 we obtain

$$\frac{R_e N_g^s}{N_g^s + \gamma_e/\Gamma} = \int \frac{\gamma_g(x)}{U_x} \left\{ \mu - \frac{1}{2} m [v_{MTF}(x)]^2 - \frac{1}{2} m \omega_x^2 x^2 \right\} dx. \quad (8.54)$$

Note that the chemical potential $\mu$ is a function of the stationary state population $N_g^s$ via the normalisation condition

$$\int \left\{ \mu - \frac{1}{2} m [v_{MTF}(x)]^2 - \frac{1}{2} m \omega_x^2 x^2 \right\} dx = U_x N_g^s. \quad (8.55)$$

The velocity $v_{MTF}$ depends on $N_g^s$ which is equal to $R_e/ (\gamma_e + \Gamma N_g^s)$ in the stationary state, therefore it is also a function of $N_g^s$. Thus Eq. 8.54 is a transcendental equation which we need to solve to find the stationary state population $N_g^s$. We solve for $N_g^s$ numerically for various pump strengths and display this as the solid curve in Fig. 8.9(a), the dashed curve corresponds to performing the same calculation using the usual TF solution instead of the modified one. Both curves possess a threshold pump rate at around 200 and deviate from each other as the pump strength increases. The parameters used were: $\Gamma/\omega_x = 10^{-3}$, $\gamma_e/\omega_x = 10$, $\gamma_g/\omega_x = 1$, $\alpha/a_x = 1$ and $z_0/a_x = 10$.

The dash–dotted curve denotes the solution of the spatial independent case where we know $N_g^s = R_e/ \gamma_g - \gamma_e/\Gamma$ above threshold and zero below. For $\gamma_g$
we choose the value at the centre, i.e. $\gamma_0 (x = 0)$ for this comparison since for weak pump strengths, the wave function is far away from the peaks $z_0$ of the loss and only experience the relatively flat part of the loss–function about the central region.

Fig. 8.9(b) zooms in on the region around the threshold where all three curves cross at $R e / \omega_x = 198$. Above and close to threshold, the atom number is low, so that the spatial extent of the wave function is predominately around the relatively flat region between the two peaks of the spatial loss–function. Hence for these pump strengths corresponding to low atom number, the losses appear to be position independent. Thus the threshold is at $\gamma_0 (0) \gamma_e / \Gamma = (0.0198 \times 10) / 10^{-3} = 198$ for all three curves.

8.5.4 Numerical solution

The significance of the kinetic term which has so far been neglected in the analytical treatment via the Thomas–Fermi approximation can be determined by putting it back into our model and using a numerical integration method. We use the split–operator technique [51, 100] that evolves the wave function with the kinetic term separately from the rest of the Hamiltonian.

In our semi–classical treatment we are also interested in the stationary state behaviour. An imaginary time method is employed to find the ground state wave function of the trap including the effects of the pump and spatial losses for a fixed atom number. This evolution in imaginary time [41] attenuates the differences between the arbitrary initial wave function and the eigenfunction of the trap, pump and spatial loss system. The inclusion of the pump and spatial loss gives a space dependent phase to the ground state wave function representing the velocity of the atoms in the stationary state. Thus we obtain a solution which includes the effects of the pump and losses without resorting to a Thomas–Fermi approximation. This solves half of the problem, the stationary state of the modified GPE, without regard to the

\footnote{The interested reader is referred to Olsen [104] who has applied various numerical methods to the GPE, including the split–operator and imaginary time methods}
Figure 8.9: (a) Stationary state condensate population calculated from using the TF (dashed line) and modified TF solution (solid line). The dashed-dotted line is the solution for the spatial independent loss case. (b) Same as (a) but closer to threshold.
dynamics of the uncondensed population. Thus it is inconsistent with the rate equation governing the uncondensed atoms. The full problem, however requires that the atom number is itself a variable in order for the solution to be consistent with the uncondensed part of the system.

To allow for this additional degree of freedom we follow the method that was used previously where we inserted either the TF or modified TF solution into Eq. 8.41 as an approximation of \( \psi_s \). We now use the imaginary time procedure to approximate \( \psi_s \) for a fixed atom number up to some predetermined accuracy. A modified false position method was then used to find the zero of the R.H.S. of Eq. 8.41 as a function of the atom number \( N_e \). Note that other numerical methods for finding zeros of functions may be used. In each step of this algorithm, \( \psi_s \) must be re-calculated for a new atom number. This procedure finds the particular atom number that is consistent with the uncondensed part of the system, i.e. the rate equation governing \( N_e \). Since we are effectively finding the zero of a function that we know only numerically, we need the accuracy of \( \psi_s \) to be higher than the precision required from the modified false position algorithm.

The resulting atom numbers for given pump strengths are plotted as circles in Fig. 8.10(a) and (b). The system parameters are the same as those used in the analytical situation. The analytical curves from Fig. 8.9, the modified TF and TF solution corresponding to the solid and dashed curves respectively have been re-drawn on Fig. 8.10. The numerical results agree well with the modified TF curve for pump rates away from threshold. At pump rates just above threshold as shown in Fig. 8.10(b) we see that the agreement is not as good. The Thomas–Fermi approximation is not expected to be accurate at low atom numbers of less than 500. Both the TF and modified TF curves over-estimate the numerical result since the approximation truncates the "true" wave function past the Thomas–Fermi radius.
Figure 8.10: (a) Numerical calculation of the stationary state condensate population (circles) versus pump strength. The analytical curves for the TF (dashed line) and modified TF (solid line) are shown for comparison. (b) Same as (a) but closer to threshold.
8.6 Summary

We have presented a theory of an atom laser that is very close to semi-classical laser theory where the matter-wave equation is a modified Gross-Pitaevski equation with additional loss and gain terms. We can describe the build-up of the mean-field of a condensate where its coherence properties are maintained. However, in the case of a spatial independent loss, we find collective excitations and a dependence on the initial mean-field, i.e. the known stationary ground state of the GPE which is the desired lasing mode cannot be reached in general. A natural mode selection can be introduced by using a spatial dependent loss. This spatial loss is localized at the edges of the condensate.

A modified Thomas-Fermi solution was developed to consider the effects of introducing a pump and a spatial dependent loss to a condensate in a trap. We find that the density is slightly altered from the usual TF solution with the possibility that higher densities can occur where the gradient of the loss-function is high. This counter-intuitive behaviour is clearly understood when the velocity (the gradient of the position dependent phase) is observed. This velocity field depicts the flow of atoms out of the condensate, so it is not surprising that in these regions the atom density is higher than in the usual TF situation. An equilibrium state is established where atoms are pumped in equally at every location, but the loss is concentrated heavily at certain local regions. Whereas in the usual TF solution there is a zero velocity field corresponding to a spatially independent phase. That is, there is no flow pattern established since pumping and losses are not considered. We compare the usual TF solution to the modified TF solutions and find that the latter has better agreement to the numerical results.
Part IV
Appendices
Appendix A

Time dependent coherences

A.1 Time dependent solutions for the cross coherences

The formal solution for the homogeneous system of differential equations

\[ \frac{d}{dt} X_{\text{cross}}(t) = A_{\text{cross}} X_{\text{cross}}(t) \]  \hspace{1cm} (A.1)

is

\[ X_{\text{cross}}(t) = \mathcal{L}^{-1} \left\{ B_{\text{cross}}^{-1} X_{\text{cross}}(0) \right\} \]

where \( B_{\text{cross}} = sI - A_{\text{cross}} \), \( I \) is the identity matrix, and \( \mathcal{L}^{-1} \) denotes the inverse Laplace transform. We choose the initial condition

\[ X_{\text{cross}}(t = 0) = \{0, 0, 0, 0, re^{-i\phi}, re^{i\phi}, 0, 0\}^T. \]

Writing out the matrix \( B_{\text{cross}} \) out in full

\[
B_{\text{cross}} = 
\begin{pmatrix}
    s + i\Delta + \gamma & 0 & 0 & i\Omega & 0 & i\Omega & 0 \\
    0 & s - i\Delta + \gamma & 0 & 0 & -i\Omega & 0 & -i\Omega \\
    0 & 0 & s + i\Delta + \gamma & 0 & 0 & -i\Omega & 0 & -i\Omega \\
    0 & 0 & 0 & s - i\Delta + \gamma & i\Omega & 0 & i\Omega & 0 \\
    i\Omega & 0 & 0 & i\Omega & s & 0 & 0 & 0 \\
    0 & -i\Omega & -i\Omega & 0 & 0 & s & 0 & 0 \\
    i\Omega & 0 & 0 & i\Omega & 0 & 0 & s + 2\gamma & 0 \\
    0 & -i\Omega & -i\Omega & 0 & 0 & 0 & 0 & s + 2\gamma \\
\end{pmatrix},
\]

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and calculating its determinant by using Mathematica© and setting $\Delta = 0$
for the special case of zero detuning, we obtain

$$D_{\text{cross}} = \det (B_{\text{cross}})$$

$$= s^4 + 4\gamma s^3 + \left(5\gamma^2 + 4\Omega^2\right)s^2 + 2\gamma \left(\gamma^2 + 4\Omega^2\right)s + 4\gamma^2\Omega^2.$$

We also calculate its inverse using Mathematica©. The solution in the
Laplace transform basis is

$$\left(\begin{array}{c}
\langle \widetilde{\sigma}_{14} \rangle (s) \\
\langle \widetilde{\sigma}_{41} \rangle (s) \\
\langle \widetilde{\sigma}_{23} \rangle (s) \\
\langle \widetilde{\sigma}_{32} \rangle (s) \\
\langle \widetilde{\sigma}_{12} \rangle (s) \\
\langle \widetilde{\sigma}_{21} \rangle (s) \\
\langle \widetilde{\sigma}_{34} \rangle (s) \\
\langle \widetilde{\sigma}_{43} \rangle (s)
\end{array}\right) = \frac{1}{D_{\text{cross}}} \left(\begin{array}{c}
-i\Omega (s + 2\gamma) (s + \gamma) e^{-i\phi} \\
i\Omega (s + 2\gamma) (s + \gamma) e^{i\phi} \\
i\Omega (s + 2\gamma) (s + \gamma) e^{i\phi} \\
i\Omega (s + 2\gamma) (s + \gamma) e^{-i\phi} \\
\mathcal{N}re^{-i\phi} \\
\mathcal{N}re^{i\phi} \\
-2\Omega^2 (s + \gamma) e^{-i\phi} \\
-2\Omega^2 (s + \gamma) e^{i\phi}
\end{array}\right), \quad (A.2)$$

where we define

$$\mathcal{N} = s^3 + 4\gamma s^2 + \left(5\gamma^2 + 2\Omega^2\right)s + 2\gamma\Omega^2 + 2\gamma^3. \quad (A.3)$$

We should note that

$$\langle \widetilde{\sigma}_{ij} \rangle (s) = \langle \widetilde{\sigma}_{ji} \rangle^* (s), \quad (A.4)$$

$$\langle \widetilde{\sigma}_{14} \rangle (s) = \langle \widetilde{\sigma}_{32} \rangle (s), \quad (A.5)$$

and

$$\langle \widetilde{\sigma}_{41} \rangle (s) = \langle \widetilde{\sigma}_{23} \rangle (s)$$

which means that we need only work out the inverse transform of $\langle \widetilde{\sigma}_{14} \rangle (s)$
to obtain $\langle \sigma_{14} \rangle (t)$, with all the other expectation values in the time domain
calculable from these relations.

By factorizing the expression for the determinant, we can write

$$\langle \widetilde{\sigma}_{14} \rangle (s) = \frac{-i\Omega re^{-i\phi} (s + 2\gamma)}{(s + \gamma) (s - z_+) (s - z_-)}, \quad (A.6)$$
with the roots $z_\pm$ defined by

$$z_\pm = -\gamma \pm \sqrt{\alpha} \quad (A.7)$$

and

$$\alpha = \gamma^2 - 4\Omega^2. \quad (A.8)$$

Equation A.6 can be expressed in the partial fraction form

$$\langle \sigma_{14} \rangle (s) = -i\Omega re^{-i\phi} \left( \frac{A}{s + \gamma} + \frac{B}{s - z_+} + \frac{C}{s - z_-} \right) \quad (A.9)$$

where $A$, $B$ and $C$ are functions of $\Omega$ and $\gamma$ defined by

$$A = \frac{\gamma}{\alpha}, \quad (A.10)$$
$$B = \frac{1}{2\sqrt{\alpha}} + \frac{1}{2\gamma} + \frac{2\Omega^2}{\gamma\alpha}, \quad (A.11)$$
$$C = -\frac{1}{2\sqrt{\alpha}} + \frac{1}{2\gamma} + \frac{2\Omega^2}{\gamma\alpha}. \quad (A.12)$$

Note that this is valid only if $\alpha$ is non-zero. Transforming back to the time domain we obtain

$$\langle \sigma_{14} \rangle (t) = -i\Omega re^{-i\phi} [A \exp(-\gamma t) + B \exp(z_+ t) + C \exp(z_- t)] \quad (A.13)$$

which can be re-written in the form of Eq. 2.39

$$\langle \sigma_{14} \rangle (t) = -i\Omega re^{-i\phi} e^{-\gamma t} \left[ \frac{\gamma}{\alpha} \left( \frac{1}{2\gamma} + \frac{2\Omega^2}{\gamma\alpha} \right) (e^{\sqrt{\alpha}t} + e^{-\sqrt{\alpha}t}) \right.$$\]
$$\left. + \frac{1}{2\sqrt{\alpha}} (e^{\sqrt{\alpha}t} - e^{-\sqrt{\alpha}t}) \right]. \quad (A.14)$$

For the special case where $\alpha = 0$ we have instead

$$\langle \sigma_{14} \rangle (s) = -i\Omega r e^{-i\phi} \left( \frac{s + 2\gamma}{(s + \gamma)^3} \right)$$
$$= -i\Omega r e^{-i\phi} \left( \frac{1!}{(s - (\gamma))^2} \right) \left( \frac{\gamma}{2} \right) \left( \frac{2!}{(s - (\gamma))^3} \right),$$

so that in the time domain we derive Eq. 2.41

$$\langle \sigma_{14} \rangle (t) = -i\Omega r e^{-i\phi} \left\{ t e^{-\gamma t} + \frac{\gamma t^2 e^{-\gamma t}}{2} \right\}$$
$$= -i\Omega r e^{-i\phi} t e^{-\gamma t} \left( 1 + \frac{\gamma t}{2} \right). \quad (A.15)$$
APPENDIX A. TIME DEPENDENT COHERENCES

A.2 Time dependent solutions for the linear coherences

The formal solution for the inhomogeneous system of differential equations

\[
\frac{d}{dt} X_{\text{lin}} (t) = A_{\text{lin}} X_{\text{lin}} (t) + C, \quad (A.16)
\]

is

\[
X_{\text{lin}} (t) = \mathcal{L}^{-1} \left\{ B^{-1} \cdot \left[ X_{\text{lin}} (0) + C s^{-1} \right] \right\} \quad (A.17)
\]

where we define \( B_{\text{lin}} = -s I - A_{\text{lin}} \). We choose the initial condition

\[
X_{\text{cross}} (t = 0) = \{0, 0, 0, 0, r_1^2, 0, 0\}^T
\]

which corresponds to the initial wave function

\[
|\psi (t = 0)\rangle = r_1 e^{-i\phi_1} |1\rangle + r_2 e^{-i\phi_2} |2\rangle.
\]

Writing out the matrix \( B_{\text{lin}} \) out in full

\[
B_{\text{lin}} = \begin{pmatrix}
  s + i\Delta + \gamma & 0 & 0 & 0 & -i\Omega & i\Omega & 0 \\
  0 & s - i\Delta + \gamma & 0 & 0 & i\Omega & -i\Omega & 0 \\
  0 & 0 & s + i\Delta + \gamma & 0 & -i\Omega & -i\Omega & -2i\Omega \\
  0 & 0 & 0 & s - i\Delta + \gamma & i\Omega & i\Omega & 2i\Omega \\
  -i\Omega & i\Omega & 0 & 0 & s & -\gamma & -\gamma \\
  i\Omega & -i\Omega & 0 & 0 & 0 & s + 2\gamma & 0 \\
  0 & 0 & -i\Omega & i\Omega & 0 & 0 & s + 2\gamma 
\end{pmatrix},
\]

and calculating its determinant by using Mathematica© and setting \( \Delta = 0 \) for the special case of zero detuning, we obtain

\[
D_{\text{lin}} = \det (B_{\text{lin}})
\]

\[
= (2\gamma^3 + 4\gamma\Omega^2 + 5\gamma^2 s + 4\Omega^2 s + 4\gamma s^2 + s^3)
\times (4\gamma^2\Omega^2 + 2\gamma^3 s + 8\gamma\Omega^2 s + 5\gamma^2 s^2 + 4\Omega^2 s^2 + 4\gamma s^3 + s^4)
\]

\[
= (s + \gamma)^3 (s - c_-) (s - c_+) (s - z_-) (s - z_+).
\]  \hspace{1cm} (A.18)

where

\[
\gamma = -\gamma \pm \sqrt{\alpha}
\]
and
\[ c_\pm = \frac{-3\gamma}{2} \pm \sqrt{\beta}. \]

We define \( \alpha \) and \( \beta \) by
\[ \alpha = \gamma^2 - 4\Omega^2 \]
and
\[ \beta = \gamma^2/4 - 4\Omega^2. \]

In the special initial condition where \( r_1 = \sqrt{2} \), i.e., equal amplitude in each of the ground states, we are able to obtain the relatively simple expressions
\[ \langle \sigma_{33} \rangle(s) = \langle \sigma_{44} \rangle(s) = \frac{\Omega^2}{s(s - c_-)(s - c_+)} \]
\[ = \Omega^2 \left\{ \frac{A}{s} + \frac{B_+}{s - c_+} + \frac{B_-}{s - c_-} \right\} , \quad (A.19) \]
where the functions \( A \) and \( B_\pm \) are defined as
\[ A = \frac{\Omega^2}{4\Omega^2 + 2\gamma^2} , \quad (A.20) \]
and
\[ B_\pm = \frac{A}{2} \left( 1 \pm \frac{3\gamma}{2\sqrt{\beta}} \right) . \quad (A.21) \]
This is valid for \( \beta \) non-zero, note that the expression is independent of \( \alpha \).

The solution in the time domain is therefore
\[ \langle \sigma_{33} \rangle(t) = \langle \sigma_{44} \rangle(t) \]
\[ = A + B_+ \exp(c_+t) + B_- \exp(c_-t) \]
\[ = \frac{\Omega^2}{4\Omega^2 + 2\gamma^2} \left\{ 1 - \frac{e^{-3\gamma t/2}}{2} \left[ e^{\sqrt{\beta}t} + e^{-\sqrt{\beta}t} + \frac{3\gamma}{2\sqrt{\beta}} \left( e^{\sqrt{\beta}t} - e^{-\sqrt{\beta}t} \right) \right] \right\} \]
corresponding to Eq. 2.45.

For the case where \( \beta = 0 \) we have
\[ \langle \sigma_{33} \rangle(s) = \langle \sigma_{44} \rangle(s) = \frac{\Omega^2}{s(s - \left(-\frac{3\gamma}{2}\right))^2} \]
\[ = \Omega^2 \left\{ \frac{4}{9\gamma^2} \cdot \frac{1}{s} + \frac{4}{9\gamma^2} \cdot \frac{1}{s - \left(-\frac{3\gamma}{2}\right)} + \frac{2}{3\gamma} \cdot \frac{1}{s - \left(-\frac{3\gamma}{2}\right)} \right\} , \]
transforming to the time domain we derive Eq. 2.47

\[
\langle \sigma_{33} \rangle (t) = \langle \sigma_{44} \rangle (t) \\
= \Omega^2 \left( \frac{4}{9 \gamma^2} - \frac{4}{9 \gamma^2} e^{-3\gamma t/2} - \frac{2}{3\gamma} te^{-3\gamma t/2} \right) \\
= \frac{4\Omega^2}{9\gamma^2} \left[ 1 - e^{-3\gamma t/2} \left( 1 + \frac{3\gamma t}{2\Omega^2} \right) \right]. \tag{A.22}
\]
Appendix B

Evaluation of $I(\alpha)$

We need to evaluate the following integral

$$I(\alpha) = \int \sum_{\ell,k} C^\ell_k C_{\ell,k} d\Omega.$$  \hspace{1cm} (B.1)

in order to perform the Monte Carlo simulation in chapter 3.

Let us split up this integral into two parts, one for each polarization. Starting with \(\theta\)-polarization we have

$$\int C^\ell_k C_{\ell,k} d\Omega = N^2 \left\{ \int d\Omega \left( \hat{\epsilon}_\theta \cdot \hat{S}_1 \right) \left( \hat{\epsilon}_\theta \cdot \hat{S}_2 \right) + \left( \hat{\epsilon}_\theta \cdot \hat{S}_1 \right) \left( \hat{\epsilon}_\theta \cdot \hat{S}_2 \right) \exp(-i\alpha \cos \theta) \right\}$$  \hspace{1cm} (B.2)

where \(\alpha = |\mathbf{k}| \cdot |\mathbf{F}_2 - \mathbf{F}_1|\). Using Eq. 3.1, 3.2, and equations 3.8 to 3.12 we obtain for the first integral of Eq. B.2

$$\int d\Omega \left( \hat{\epsilon}_\theta \cdot \hat{S}_1 \right) \left( \hat{\epsilon}_\theta \cdot \hat{S}_2 \right)$$

\[= \int_0^\pi \int_0^{2\pi} \left( \sigma_3 \sigma_3 \cos^2 \theta \cos^2 \phi + \sigma_y \sigma_y \cos^2 \theta \sin^2 \phi + \sigma_z \sigma_z \sin^2 \theta \right) \sin \theta d\theta d\phi \]

\[= \frac{2\pi}{3} \left( \sigma_3 \sigma_3 + \sigma_y \sigma_y \right) + \frac{8\pi}{3} \sigma_z \sigma_z \]

\[= 4\pi (\sigma_{33} + \sigma_{44}). \]  \hspace{1cm} (B.3)
Note that all the cross terms (like \( \sigma^+_z\sigma_y \)) integrate to zero since they are multiplied by sine and cosine functions. Similarly we find

\[
\int d\Omega \left( \hat{\xi}_\theta \cdot \vec{S}_2^* \right) \left( \hat{\xi}_\theta^* \cdot \vec{S}_2 \right) = 4\pi \left( \Sigma_{33} + \Sigma_{44} \right). \tag{B.4}
\]

The “cross–atom” integral

\[
\int d\Omega \left( \hat{\xi}_\theta \cdot \vec{S}_2^* \right) \left( \hat{\xi}_\theta^* \cdot \vec{S}_2 \right) \exp \left( -i\alpha \cos \theta \right)
\]

\[
= \left( \sigma^+_z \cos \theta \cos \phi + \sigma^+_y \cos \theta \sin \phi - \sigma^-_z \sin \theta \right)
\]

\[
\times \left( \Sigma_x \cos \theta \cos \phi + \Sigma_y \cos \theta \sin \phi - \Sigma_z \sin \theta \right) \exp \left( -i\alpha \cos \theta \right)
\]

\[
= \int_0^\pi \int_0^{2\pi} \left( \sigma^+_z \Sigma_x \cos^2 \theta \cos \phi + \sigma^+_y \Sigma_y \cos^2 \theta \sin \phi + \sigma^-_z \Sigma_z \sin^2 \theta \right)
\]

\[
\times \sin \theta \exp \left( -i\alpha \cos \theta \right) d\theta d\phi.
\]

Again, the cross terms integrate to zero, so that

\[
\int d\Omega \left( \hat{\xi}_\theta \cdot \vec{S}_2^* \right) \left( \hat{\xi}_\theta^* \cdot \vec{S}_2 \right) \exp \left( -i\alpha \cos \theta \right)
\]

\[
= \pi \left( \sigma^+_z \Sigma_x I_z + \sigma^+_y \Sigma_y I_x + 2\sigma^-_z \Sigma_z I_z \right) \tag{B.5}
\]

where

\[
I_x = \frac{2}{\alpha} \sin \alpha - I_z (\alpha)
\]

\[
I_y = I_x,
\]

\[
I_z = \frac{4}{\alpha^2} \left( \frac{1}{\alpha} \sin \alpha - \cos \alpha \right).
\]

The integral for \( \left( \hat{\xi}_\theta \cdot \vec{S}_2^* \right) \left( \hat{\xi}_\theta^* \cdot \vec{S}_2 \right) \exp \left( i\alpha \cos \theta \right) \) has the same form as above but with \( \sigma \) and \( \Sigma \) exchanged. Finally we obtain the expression

\[
\int C_{\Omega,\alpha}^1 C_{\Omega,\alpha}^1 d\Omega
\]

\[
= \frac{2\pi}{3} \left( \sigma^+_z \sigma_x + \sigma^+_y \sigma_y + \Sigma^+_z \Sigma_x + \Sigma^+_y \Sigma_y \right) + \frac{8\pi}{3} \left( \sigma^+_z \sigma_y + \Sigma^+_z \Sigma_y \right)
\]

\[
+ \pi \left( \frac{2}{\alpha} \sin \alpha - T (\alpha) \right) \left( \sigma^+_z \Sigma_x + \sigma^+_y \Sigma_y + \Sigma^+_x \sigma_x + \Sigma^+_y \sigma_y \right)
\]

\[
+ 2\pi T (\alpha) \left( \sigma^+_z \Sigma_x + \Sigma^+_z \sigma_z \right) \tag{B.6}
\]

where we have re-labelled the integral \( I_z \) by \( T (\alpha) \) to stress its \( \alpha \) dependence.
We now need to evaluate the second part of the integral, that include \( \phi \)-polarization terms. We find for atom 1,

\[
\int d\Omega \left( \vec{\epsilon}_\phi \cdot \vec{S}_1^\dagger \right) \left( \vec{\epsilon}_\phi^* \cdot \vec{S}_1^\dagger \right) = \left( -\sigma_x^1 \sin \phi + \sigma_y^1 \cos \phi \right) \left( -\sigma_x \sin \phi + \sigma_y \cos \phi \right) \\
= \int_0^{2\pi} \int_0^{2\pi} \left[ \sigma_x^1 \sigma_x \sin^2 \phi + \sigma_y^1 \sigma_y \cos^2 \phi + \left( \sigma_x^1 \sigma_y + \sigma_y^1 \sigma_x \right) \sin \phi \cos \phi \right] \\
\times \sin \theta d\theta d\phi \\
= 2\pi \left( \sigma_x^1 \sigma_x + \sigma_y^1 \sigma_y \right) \\
= 4\pi \left( \sigma_x^1 \sigma_x + \sigma_y^1 \sigma_y \right). \tag{B.7}
\]

Similarly for atom 2,

\[
\int d\Omega \left( \vec{\epsilon}_\phi \cdot \vec{S}_2^\dagger \right) \left( \vec{\epsilon}_\phi^* \cdot \vec{S}_2^\dagger \right) = 4\pi \left( \Sigma_x^1 \Sigma_x + \Sigma_y^1 \Sigma_y \right). \tag{B.8}
\]

The "cross-atom" term

\[
\int d\Omega \left( \vec{\epsilon}_\phi \cdot \vec{S}_1^\dagger \right) \left( \vec{\epsilon}_\phi^* \cdot \vec{S}_2^\dagger \right) \exp(-i\alpha \cos \theta) \\
= \int_0^{2\pi} \int_0^{2\pi} \left[ \sigma_x^1 \Sigma_x \sin^2 \phi + \sigma_y^1 \Sigma_y \cos^2 \phi + \left( \sigma_x^1 \Sigma_y + \sigma_y^1 \Sigma_x \right) \sin \phi \cos \phi \right] \\
\times \sin \theta \exp(-i\alpha \cos \theta) d\theta d\phi \\
= \frac{2\pi}{\alpha} \sin \alpha \left( \sigma_x^1 \Sigma_x + \sigma_y^1 \Sigma_y \right), \tag{B.9}
\]

with its Hermitian conjugate

\[
\int d\Omega \left( \vec{\epsilon}_\phi \cdot \vec{S}_2^\dagger \right) \left( \vec{\epsilon}_\phi^* \cdot \vec{S}_1^\dagger \right) \exp(i\alpha \cos \theta) = \frac{2\pi}{\alpha} \sin \alpha \left( \Sigma_x^1 \sigma_x + \Sigma_y^1 \sigma_y \right). \tag{B.10}
\]

The integration over the \( \phi \)-polarization is then

\[
\int C_{i,\iota}^\dagger C_{\Omega,\iota} d\Omega \\
= 2\pi \left( \sigma_x^1 \sigma_x + \sigma_y^1 \sigma_y + \Sigma_x^1 \Sigma_x + \Sigma_y^1 \Sigma_y \right) \\
+ \frac{2\pi}{\alpha} \sin \alpha \left( \sigma_x^1 \Sigma_x + \sigma_y^1 \Sigma_y + \Sigma_x^1 \sigma_x + \Sigma_y^1 \sigma_y \right). \tag{B.11}
\]
Combining the expressions for the two polarizations we derive Eq. 3.14

\[
\int \sum_{\ell, k} C_{\ell \kappa}^r C_{\kappa \ell}^r d\Omega \\
= \mathcal{N}^2 \left\{ \frac{8\pi}{3} \left( \sigma_z^1 \sigma_z + \sigma_y^1 \sigma_y + \sigma_z^1 \Sigma_z + \Sigma_y^1 \Sigma_y + \Sigma_z^1 \Sigma_z \right) \\
+ \pi \left[ \frac{4}{\alpha} \sin \alpha - \mathcal{T}(\alpha) \right] \left( \sigma_z^1 \Sigma_x + \sigma_y^1 \Sigma_y + \Sigma_z^1 \sigma_x + \Sigma_y^1 \sigma_y \right) \\
+ 2\pi \mathcal{T}(\alpha) \left( \sigma_z^1 \Sigma_z + \Sigma_z^1 \sigma_z \right) \right\}.
\]
Appendix C

Analytical results for a large number of detections

C.1 Poissonian mixtures

In the case of Poissonian mixtures the expression (6.31) can be made more explicit and analytical results can be extracted. As shown by Cirac et. al. [53] it is useful to represent the state after \( m \) detections in the form of a P-representation

\[
\rho_m = \int \frac{d\psi}{2\pi} f_m(\psi) \int \frac{d\varphi_1}{2\pi} \left| \alpha_1 e^{i\varphi_1} \right\rangle \langle \alpha_1 e^{i\varphi_1} \right| \otimes \left| \alpha_2 e^{i(\varphi_1 + \psi)} \right\rangle \langle \alpha_2 e^{i(\varphi_1 + \psi)} \right|
\]

(C.1)

with

\[
\alpha_1 = \sqrt{n_1}, \quad \alpha_2 = \sqrt{n_2}.
\]

(C.2)

In writing Eq. C.1 we are assuming that the observation time \( t_m \) satisfies \( \gamma_1 t_m \ll 1 \), so that only a negligible fraction of the atoms in the condensates are counted. That is, the number of atoms counted is small compared with the mean of the Poisson distribution. The initial Poissonian mixture \( \rho_0 \) is also of the form of Eq. C.1 with

\[
f_0(\psi) = 1.
\]

(C.3)
The effect of an additional counting event on \((C.1)\) is to change

\[
 f_m (\psi) \rightarrow f_{m+1} (\psi)
\]

with

\[
 f_{m+1} (\psi) = \frac{1}{N_{m+1}} \left[ 1 + \lambda \cos (\phi_{m+1} - \psi) \right] f_m (\psi) \tag{C.4}
\]

where

\[
 \lambda = \frac{2\sqrt{\bar{n}_1 \bar{n}_2}}{\bar{n}_1 + \Gamma \bar{n}_2} \tag{C.5}
\]

and \(N_{m+1}\) is determined by normalizing \(\int \frac{d\psi}{2\pi} f_{m+1} (\psi) = 1\) after each counting event. We note that \(0 < \lambda \leq 1\). In the following we shall assume that \(\lambda < 1\), as the case \(\lambda = 1\) requires a separate mathematical treatment. The physical meaning of \(f_m (\psi)\) is clear from \(\text{Eq. C.1}\): it is the probability distribution of the relative phase between the two condensate modes. The explicit form of \(f_m (\psi)\) as a function of \(\psi\) is easily obtained from \(\text{Eq. C.3}\) and \(\text{Eq. C.4}\) as

\[
 f_m (\psi) = \prod_{k=1}^{m} \left[ 1 + \lambda \cos (\phi_k - \psi) \right] \frac{1}{N_k} \tag{C.6}
\]

The normalized joint probability distribution to observe the phases \(\phi_1, \ldots, \phi_{m+1}\) is equal to

\[
 \int \frac{d\psi}{2\pi} \prod_{k=1}^{m+1} \left[ 1 + \lambda \cos (\phi_k - \psi) \right] = \prod_{k=1}^{m+1} N_k \tag{C.7}
\]

Therefore the conditional probability \((6.31)\) determined by the ratio of 2 joint probabilities is equal to \(N_{m+1}\) and given by

\[
 p_{\text{Poisson}} (x | x_1, \ldots, x_m) = \int \frac{d\psi}{2\pi} \left[ 1 + \lambda \cos (\phi - \psi) \right] f_m (\psi) \tag{C.8}
\]

where \(\phi = \phi (x)\) and \(f_m (\psi)\) depends on the previous \(m\) detections. Now we investigate the behaviour of \(f_m (\psi)\) as a function of \(\psi\) for large \(m\). We shall assume that it becomes a narrow distribution centred around some \(m\)-dependent maximum \(\psi_m\) with a variance \(\sigma_m^2\), i.e. we put

\[
 f_m (\psi) = \frac{1}{\sqrt{2\pi} \sigma_m} \exp \left[ -\frac{(\psi - \psi_m)^2}{2\sigma_m^2} \right] \tag{C.9}
\]

\[\text{1This formula permits us to appreciate why the case } \lambda = 1 \text{ is very special: For } \lambda < 1 \text{ the distribution } (C.6) \text{ is a positive function } f_m (\psi) > 0 \text{ everywhere. For } \lambda = 1 \text{ it has } m \text{ 2-fold degenerate zeros, i.e. } f_m (\psi) \text{ cannot approach a smooth function for } m \rightarrow \infty.\]
We assume here that \( \sigma_m^2 \ll 1 \), so that the \( 2\pi \)-periodicity of \( f_m(\psi) \) is not in noticeable conflict with (C.9).

We shall now determine the time evolution of \( \psi_m \) and \( \sigma_m^2 \), using Eq. C.4, and show that our assumptions are self-consistent in that indeed \( \sigma_m^2 \ll 1 \) for \( m \gg 1 \). From Eq. C.6 we obtain by taking the logarithm

\[
\ln f_m(\psi) = \sum_{k=1}^{m} \ln [1 + \lambda \cos (\phi_k - \psi)] + c \tag{C.10}
\]

where the constant \( c \) depends on the \( \phi_m \) but not on \( \psi \). The maximum \( \psi_m \) of \( \ln f_m(\psi) \) must therefore satisfy

\[
\sum_{k=1}^{m} \frac{\lambda \sin (\phi_k - \psi_m)}{1 + \lambda \cos (\phi_k - \psi_m)} = 0. \tag{C.11}
\]

The evolution of the maximum is obtained similarly by taking the logarithm of Eq. C.4 and using Eq. C.9. We find

\[
-\frac{(\psi_{m+1} - \psi_m)}{\sigma_{m+1}^2} + \frac{\lambda \sin (\phi_{m+1} - \psi_m)}{1 + \lambda \cos (\phi_{m+1} - \psi_m)} = 0. \tag{C.12}
\]

It is clear that \( (\psi_{m+1} - \psi_m) \) depends on the outcome of the \( (m + 1) \)th measurement. If \( f_m(\psi) \) is, in fact, a narrow distribution on the scale \( 2\pi \), as we have assumed, then the probability distribution to find \( \phi_{m+1} \) in that measurement can be estimated as

\[
P_{m+1}(\phi_{m+1}) \simeq 1 + \lambda e^{-\frac{1}{2} \sigma_m^2} \cos (\phi_{m+1} - \psi_m) \tag{C.13}
\]

as follows from Eq. C.8 and Eq. C.9. Based on this we find

\[
\left\langle \frac{\psi_{m+1} - \psi_m}{\sigma_{m+1}^2} \right\rangle = 0,
\]

\[
\left\langle \frac{(\psi_{m+1} - \psi_m)^2}{\sigma_{m+1}^4} \right\rangle = \lambda^2 \frac{\lambda^2}{2\pi} \frac{\sin^2 \phi}{1 + \lambda \cos \phi} - \frac{\lambda^3}{2} \frac{\sigma_m^2}{2\pi} \frac{\cos \phi \sin^2 \phi}{(1 + \lambda \cos \phi)^2}
\]

\[
= 1 - \sqrt{1 - \lambda^2} + \frac{1}{2} \sigma_m^2 \frac{(1 - \sqrt{1 - \lambda^2})^2}{\sqrt{1 - \lambda^2}} \tag{C.14}
\]

where we used \( \sigma_m^2 \ll 1 \) to expand to first order \( \exp (-\sigma_m^2/2) = 1 - \sigma_m^2/2 \). We conclude that, if \( \lambda \) is bounded away from 1, for small variance of the phase
distribution the variance in the jitter of the average phase $\psi_m$ in subsequent measurements is of the order of the square of the variance $\sigma_m^2$ in the phase distribution, i.e. the position of the maximum is very stable and may be considered as fixed in the limit we consider.

Turning to the time evolution of the variance we obtain again using Eq. C.9 on the right hand side of Eq. C.4, taking the logarithm and the second derivative with respect to $\psi$

$$\frac{1}{\sigma_{m+1}^2} = \frac{1}{\sigma_m^2} + \frac{\lambda^2 + \lambda \cos (\phi_{m+1} - \psi_m)}{[1 + \lambda \cos (\phi_{m+1} - \psi_m)]^2}. \quad (C.15)$$

It shows that typically the inverse variance grows according to

$$\frac{1}{\sigma_{m+1}^2} - \frac{1}{\sigma_m^2} = O(1) \quad (C.16)$$

which is the reason why the variance itself indeed becomes small. Averaging as before we obtain for $\lambda$ bounded away from 1

$$\left\langle \frac{1}{\sigma_m^2} \right\rangle = \left[ (1 - \sqrt{1 - \lambda^2}) + O(\sigma_m^2) \right] m + \text{const.}$$

which shows more explicitly how the inverse variance grows on the average and becomes large for large $m$.

Finally, we relate the phase distribution $f_m(\psi)$ after $m$ measurements to the observed interference pattern. Experimentally, a phase distribution may be extracted by fitting the observed interference pattern, normalized with respect to its constant part, to the expected density Eq. C.8 after $m$ counts

$$\int \frac{d\psi}{2\pi} [1 + \lambda \cos (\phi - \psi)] f_m(\psi) = 1 + \lambda' \cos (\phi - \psi_m) \quad (C.17)$$

with

$$\lambda' = \lambda e^{-\frac{1}{2} \sigma_m^2}.$$

Here we used the Gaussian form of $f_m(\psi)$ assumed in Eq. C.9. Equations C.17 and 6.36 allow us to extract numbers for $\psi_m$ and $\sigma_m^2$. A weak link in the argument leading to Eq. C.17 might seem to be the fact that the conditional probability (C.8) is proportional to the expectation value of the
density in the condensates, while what we really need is a normalized measure of the density of the counted atoms. However, all that is really required is that these two quantities should be proportional to each other, which is an assumption implicitly already made when assuming that the counting rate is proportional to the number operator in the condensate.

The results we have presented are also valid for case of initial Fock states, this is shown at the end of this appendix.

C.2 Thermal mixtures

If the initial states of the two condensates are thermal mixtures we may adapt the results for the Poissonian mixtures as follows. The thermal initial state can be represented as a mixture of Poissonian states via

$$
\rho_0 = \frac{1}{Z_1 Z_2} \sum_{n_1, n_2} e^{-(\beta - \mu)(n_1 + n_2)} |n_1\rangle \langle n_1| \otimes |n_2\rangle \langle n_2|
$$

$$
= \frac{1}{N} \int d^2 \alpha_1 \int d^2 \alpha_2 \exp \left( -\frac{\left| \alpha_1 \right|^2}{\bar{n}_1} - \frac{\left| \alpha_2 \right|^2}{\bar{n}_2} \right) |\alpha_1\rangle \langle \alpha_1| \otimes |\alpha_2\rangle \langle \alpha_2|
$$

$$
= \frac{1}{N'} \int_0^\infty dx_1 \int_0^\infty dx_2 \exp \left( -\frac{x_1}{\bar{n}_1} - \frac{x_2}{\bar{n}_2} \right) 
\times \int \frac{d\psi}{2\pi} f_0 (\psi) \int \frac{d\varphi_1}{2\pi} |\sqrt{x_1} e^{i\varphi_1}\rangle \langle \sqrt{x_1} e^{i\varphi_1}| \otimes |\sqrt{x_2} e^{i\varphi_2+\psi}\rangle \langle \sqrt{x_2} e^{i\varphi_2+\psi}|, \tag{C.18}
$$

with the inverse temperature proportional to \( \beta, \mu \) the chemical potential, and normalization factors \( N \) and \( N' \). The evolution of this initial state due to the counting of atoms can now be obtained by using the evolution for Poissonian mixtures for fixed Poissonian averages \( x_1, x_2 \) under the integrals over \( x_1, x_2 \). We obtain with \( f_0 (\psi) = 1 \), using Eq. C.6 for

$$
\lambda = \lambda (x_1, x_2) = \frac{2\sqrt{x_1 x_2}}{x_1 + \Gamma x_2} \tag{C.19}
$$

$$
f_m (\psi) \sim \prod_{k=1}^{m} \left[ x_1 + \Gamma x_2 + 2\sqrt{\Gamma x_1 x_2 \cos (\phi_k - \psi)} \right] , \tag{C.20}
$$
The optimal visibility $\lambda$ thus varies according to Eq. C.19 for different members of the ensemble whose $x_1, x_2$ values are exponentially distributed according to Eq. C.18. The average visibility becomes

$$\bar{\lambda} = \frac{2\sqrt{\Gamma} \langle \sqrt{x_1 x_2} \rangle}{\langle x_1 \rangle + \Gamma \langle x_2 \rangle} = \frac{\pi 2\sqrt{\Gamma} \langle x_1 \rangle \langle x_2 \rangle}{4 \langle x_1 \rangle + \Gamma \langle x_2 \rangle}$$

which may be written as\(^2\)

$$\bar{\lambda} = \frac{\pi}{4} \lambda.$$  \hspace{1cm} (C.21)

For equal average counting rates

$$\bar{n}_1 = \Gamma \bar{n}_2$$

the average visibility for initially thermal mixtures is simply

$$\bar{\lambda} = \frac{\pi}{4},$$

which agrees well with the result of the numerical simulations of chapter 5. However, the case of equal average counting rates corresponds to where $\lambda = 1$ where the results are no longer valid, but we can be as close to 1 as we like predicting average visibility bounded by $\pi/4$. This not a physical problem but due to the approximations we have made.

### C.3 Initial number states

Let us consider the case of initial number states and compare the results with those of initial Poissonian mixtures. The initial number state

$$|\psi_0\rangle = |N_1\rangle |N_2\rangle$$  \hspace{1cm} (C.22)

evolves into

$$|\psi_m\rangle = \sum_{n=0}^{m} C_n (m) |N_1 - n\rangle |N_2 - m + n\rangle$$  \hspace{1cm} (C.23)

\(^2\)This is consistent with the visibility defined by Eq. C.5 previously since $\lambda(\langle x_1 \rangle, \langle x_2 \rangle) \equiv \lambda$. 

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C.3. INITIAL NUMBER STATES

where \( C_n(0) = \delta_{n,0} \). We shall assume \( \gamma_1 N_1 > \gamma_2 N_2 \) and that the parameter

\[
\lambda_N = \frac{2 \sqrt{\Gamma N_1 N_2}}{N_1 + \Gamma N_2}
\]  

(C.24)

is bounded away from 1. We shall only consider the case

\[
1 \ll m \ll N_1, N_2.
\]  

(C.25)

Then the evolution of the \( C_n(m) \) under the atom counting process is

\[
C_n(m + 1) \simeq \sqrt{N_1 (1 - \delta_{n,0})} C_{n-1}(m) + \sqrt{\Gamma N_2 (1 - \delta_{n,m+1})} e^{-i\phi_{m+1}} C_n(m).
\]  

(C.26)

We introduce a phase-representation by the Fourier transform

\[
F_m(\varphi) = \sum_{n=0}^{m} C_n(m) e^{-in\varphi},
\]

\[
C_n(m) = \int \frac{d\varphi}{2\pi} e^{in\varphi} F_m(\varphi).
\]  

(C.27)

The number density after \( m \) measurements is then

\[
P_m(x) \simeq \langle \psi_m | \hat{\psi}^\dagger(x) \hat{\psi}(x) | \psi_m \rangle
\]

\[
\simeq 1 + \lambda_N \Re \left\{ \sum_{n=0}^{m} (1 - \delta_{n,0})
\times \int \frac{d\varphi}{2\pi} \int \frac{d\varphi'}{2\pi} F_m^*(\varphi) F_m(\varphi') e^{-in(\varphi - \varphi')} e^{i(\varphi - \varphi(x))} \right\}
\]

which for \( m \gg 1 \) can be approximated by

\[
P_m(x) \simeq 1 + \lambda_N \int \frac{d\varphi}{2\pi} |F_m(\varphi)|^2 \cos[\varphi - \varphi(x)].
\]  

(C.28)

This formula should be compared with Eq. C.8 for Poissonian mixtures. It can then be seen that \( |F_m(\varphi)|^2 \) and \( f_m(\varphi) \) play identical roles. The evolution of \( F_m(\varphi) \) under the atom counting process follows from Eq. C.26 and C.27

\[
F_{m+1}(\varphi) \simeq \left( \sqrt{N_1} e^{-i\varphi} + \sqrt{\Gamma N_2} e^{-i\phi_{m+1}} \right) F_m(\varphi)
\]  

(C.29)
and hence, providing a normalization factor $N_{m+1}$ to keep $|F_{m+1}(\varphi)|^2$ normalized if $|F_{m+1}(\varphi)|$ is

$$
|F_{m+1}(\varphi)|^2 \simeq [1 + \lambda_N \cos(\phi_{m+1} - \varphi)] |F_m(\varphi)|^2 \frac{1}{N_{m+1}} \tag{C.30}
$$

which, to the accuracy we have considered here, is the same as Eq. C.4 for Poissonian mixtures. Hence, the results for the measurement induced phase distribution for number states in the limit $1 \ll m \ll N_1, N_2$ are the same as the results for Poissonian mixtures obtained in section C.
Appendix D

Collective excitations in one dimension

Collective excitations are usually discussed in three dimensions [42, 43, 44, 97, 98]. In order to explain our numerical results, we have to investigate collective excitations in one dimension. We follow the method of Stringari [98] and apply his idea to our one-dimensional model. We substitute

$$\psi(x, t) = \sqrt{\rho(x, t)} e^{i\phi(x, t)}$$

into Eq. 8.4 and obtain after some algebra the hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho \nu) + (\gamma_g - \Gamma N_e) \rho = 0$$

and

$$m \frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x} \left( \frac{m}{2} \nu^2 + V + U_0 \rho - \mu \right) - \frac{\hbar^2}{2m} \frac{\partial}{\partial x} \frac{1}{\sqrt{\rho}} \frac{\partial^2}{\partial x^2} \sqrt{\rho} = 0$$

for the "density"

$$\rho(x, t) = \psi^*(x, t) \psi(x, t)$$

and the "velocity"

$$\nu(x, t) = \frac{\hbar}{m} \frac{\partial \phi(x, t)}{\partial x}.$$
Note, that we have also introduced the chemical potential $\mu$ which is space independent.

We now consider small deviations of $\rho$, $v$, and $N_e$ from their stationary values $\rho_0$, $v \equiv 0$, and $N^* = \gamma_\sigma/\Gamma$, that is,

$$\rho = \rho_0 + \delta \rho$$
$$N_e = \gamma_\sigma/\Gamma + n_e.$$  \hspace{1cm} (D.6)

Furthermore, we make the Thomas–Fermi approximation, that is, we neglect the last term in Eq. D.3 and approximate the stationary solution $\rho_0(x)$ by the Thomas–Fermi solution $[99]$ 

$$\rho_{0,TF}(x) = \frac{\mu - V(x)}{U_0}.$$  \hspace{1cm} (D.7)

We then arrive at the linearized equations 

$$\frac{\partial}{\partial t} \delta \rho + \frac{\partial}{\partial x} \left( \rho_{0,TF} \nu \right) - \Gamma \rho_{0,TF} n_e = 0.$$  \hspace{1cm} (D.8)

and 

$$m \frac{\partial v}{\partial t} + U_0 \frac{\partial}{\partial x} \delta \rho = 0.$$  \hspace{1cm} (D.9)

We combine these two equations to eliminate $v$ and arrive at 

$$\frac{\partial^2}{\partial t^2} \delta \rho - \frac{U_0}{m} \left( \rho_{0,TF} \frac{\partial}{\partial x} \delta \rho \right) = \Gamma \rho_{0,TF} \dot{n}_e.$$  \hspace{1cm} (D.10)

For large times, when $N_e(t)$ and $N_\nu(t)$ have already reached their stationary value, we can neglect the inhomogeneous term $\Gamma \rho_{0,TF} \dot{n}_e$.

Equation (D.10) cannot be solved without knowledge of the potential $V(x)$. We restrict ourselves to the case of a harmonic trap, that is,

$$V(x) = \frac{m \omega^2}{2} x^2.$$  \hspace{1cm} (D.11)

In order to solve Eq. D.10, we introduce the scaled variable $\xi = x/R$, where $R = \sqrt{2\mu/m\omega^2}$ is the Thomas–Fermi radius of the condensate. Using the ansatz 

$$\delta \rho(x, t) = A \sin(\Omega t + \varphi) y(x/R)$$  \hspace{1cm} (D.12)
we obtain an ordinary differential equation for \( y(\xi) \) which reads

\[
\frac{d}{d\xi} \left( (1 - \xi^2) \frac{dy(\xi)}{d\xi} \right) + \frac{2\Omega^2}{\omega_x^2} y(\xi) = 0. \tag{D.13}
\]

In general it only has solutions which are singular at \( \xi = \pm 1 \), that is at \( x = \pm R \). The only exception is

\[
\frac{2\Omega^2}{\omega_x^2} = n(n + 1), \tag{D.14}
\]

where \( n \) is an integer. In this case, it has the well-known Legendre polynomials

\[
P_n(\xi) = \frac{1}{2^n n!} \frac{d^n}{d\xi^n} (\xi^2 - 1)^n \tag{D.15}
\]

as a solution, which fulfills the orthogonality relation

\[
\int_{-1}^{+1} P_n(\xi) P_m(\xi) d\xi = 0 \quad \text{for} \quad n \neq m. \tag{D.16}
\]

The frequencies of the elementary excitations are therefore given by Eq. D.14. The solution of the homogeneous part of Eq. D.10 reads

\[
\delta \rho(x, t) = \sum_{n=1}^{\infty} A_n \sin(\omega_n t + \varphi_n) P_n(x/R), \tag{D.17}
\]

where \( A_n \) and \( \varphi_n \) follow from the initial deviation from the stationary solution.

However, we do not find all these frequencies in our numerical solution of Eqs. (8.4) and (8.5). The reason is that if we start with a symmetric or antisymmetric \( \psi(x, t = 0) \), Eqs. (8.4) and (8.5) do not destroy this symmetry. Therefore \( \rho(x, t) \) as well as \( \delta \rho(x, t) \), the deviation from the (symmetric) stationary solution of the GPE, will always be symmetric. We therefore can only find excitations which correspond to Legendre polynomials of even order if we start with a symmetric or antisymmetric \( \psi(x, 0) \). The corresponding frequencies are

\[
\Omega_{2n} = \omega_x \sqrt{n(2n + 1)}. \tag{D.18}
\]
Two other facts are worth mentioning: (i) the excitations discussed above do not change the number of atoms. Using $P_0(\xi) = 1$ we find with the help of the orthogonality relation Eq. D.16

$$\int \delta \rho(x, t) \, dx = 0. \quad (D.19)$$

(ii) In

$$I_2 = \int \psi^*(x, t)x^2\psi(x, t) \, dx \quad (D.20)$$

we can only find oscillations with frequency $\Omega_2 = \sqrt{3}\omega_x$. This follows from the relation

$$\xi^2 = \frac{2}{3} P_2(\xi) + \frac{1}{3} P_6(\xi) \quad (D.21)$$

together with the orthogonality relation Eq. D.16. A generalization is the statement that

$$\overline{P_k(x/R)} = \int \psi^*(x, t) P_k(x/R)\psi(x, t) \, dx \quad (D.22)$$

shows oscillations with frequency $\Omega_k$. 
Bibliography


[73] see for example, D. F. Walls and G. J. Milburn, Quantum Optics, (Springer–Verlag, Berlin, 1994).

[74] see for example, W. Vogel and D.-G. Welsch, Lectures on Quantum Optics, (Akademie Verlag, Berlin, 1994).


