Stability Analysis and Stabilization of Linear Systems with Distributed Delays

by

Qian Feng

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical and Electronic Engineering, The University of Auckland, July 2019.

Abstract

This thesis is devoted to the methods for the stability (dissipativity) analysis and stabilization of linear systems with non-trivial distributed delays based on the application of the Liapunov-Krasovskii functional (LKF) approach. We first propose methods for designing a dissipative state feedback controller for linear distributed delay systems, where the delay is known and the distributed delay kernels belong to a class of functions. The problem is solved by constructing a functional related to the distributed delay kernels via using a novel integral inequality. We subsequently extend the previous results to handle uncertain linear distributed delay systems, where the presence of linear fractional uncertainties is handled by a novel proposed lemma. Since integral inequalities play a vital role in utilizing the LKF approach, we then propose three general classes of novel integral inequalities, where relations and other properties are established in terms of inequality bound gaps. The proposed inequalities possess very general structures and generalize many existing integral inequalities in the delay-related literature. Next we propose a new method for the dissipativity and stability analysis of linear coupled differential-difference systems (CDDSs) with distributed delays of arbitrary $\mathbb{L}^2$ functions as its kernels. The distributed delay kernels are approximated by a class of functions including the option of Legendre polynomials. In addition, approximate errors are included by the resulting dissipativity (stability) condition via a matrix framework thanks to the application of a novel proposed integral inequality via the construction of an LKF. The previous results are then followed by a study of delay range analysis where the problem of dissipativity and stability analysis of a CDDS with distributed delays is considered with an unknown but bounded delay. By constructing a functional whose matrix parameters are dependent polynomially on the delay value, a dissipativity and stability condition can be derived in terms of sum-of-squares constraints. Finally, we present new methods for the dissipative synthesis of linear systems with non-trivial time-varying distributed delay terms where the value of the time-varying delay is only required to be bounded. The problem is solved by the LKF approach thanks to a novel proposed integral inequality.

Dedication

Δόξα Πατρί και Ὑιῷ και Ἁγίῳ Πνεύματι και νῦν και αεί και εις τούς αιῶνας τῶν αιώνων. Αμήν.

Acknowledgment

First and foremost, all thanksgiving, honor and glory to God the Father and my Lord Jesus Christ and Holy Spirit. You are the only reason that I can go through this research project. I thank you my God for everything you have bestowed on me, your grace, your wisdom, your power, and especially, your LOVE.

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Chapter 2 is an extension of some of the results in ''Dissipative delay range analysis of coupled differential–difference delay systems with distributed delays'' https://www.sciencedirect.com/science/article/pii/S0167691118300732

Nature of contribution by PhD candidate	Research idea, derivation of new methods, literature survey, writing
Extent of contribution by PhD candidate (%)	90

CO-AUTHORS

Name	Nature of Contribution
Sing Kiong Nguang	critical review, proofreading and comments

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate’s contribution to this work, and the nature of the contribution of each of the co-authors; and
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Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work. The contents in Chapter 4 are based on the paper "Two general classes of integral inequalities including weight functions" https://arxiv.org/abs/1806.01514 submitted to IEEE Transaction on Automatic control.	
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Chapter 5 is a refined version of the paper ''Dissipative analysis of linear coupled differential-difference systems with distributed delays'' A short and simplied version of this paper has been submitted to IEEE Transaction on Automatic Control.

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List of Notations

The following list of symbols is applied throughout the entire thesis without overloading of notations. Note that in this thesis overloading of other types of notations, such as the symbols representing general scalar, functions, or matrices, can happen among different chapters or in a single chapter without causing ambiguity.

universal quantifier $\forall$

existential quantifier $\exists$

unique existential quantifier $!\exists$

$\mathbb{N} := \{1, 2, 3 \cdots\}$

$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

$\mathbb{R} := \{\text{All Real Numbers}\}$

$\mathbb{R}_{\geq a} := \{x \in \mathbb{R} : x \geq a\}$

$\mathbb{C} := \{\text{All Complex Numbers}\}$

$\mathbb{R}^{n \times m} := \{\text{All Real } n \times m \text{ Matrices}\}$

$\mathbb{C}^{n \times m} := \{\text{All Complex } n \times m \text{ Matrices}\}$

$\mathbb{S}^n := \{X \in \mathbb{R}^{n \times n} : X = X^\top\}$

$\mathbb{R}^{n \times m}_{[r]} := \{X \in \mathbb{R}^{n \times m} : \operatorname{rank}(X) = r\}$

$\mathbf{Sy}(X) := X + X^\top$ where $X \in \mathbb{R}^{n \times n}$

$\mathcal{Y}^{\mathcal{X}} := \{f(\cdot) : f(\cdot) \text{ is a function from } \mathcal{X} \text{ onto } \mathcal{Y}\}$

$\mathbb{L}_f(\mathcal{X} ; \mathcal{Y}) := \left\{f(\cdot) \in \mathcal{Y}^{\mathcal{X}} : f(\cdot) \text{ is measurable}\right\}$

$\|\mathbf{x}\|_q := (\sum_{i=1}^n |x_i|^q)^{\frac{1}{q}}$ p-norm of $\mathbf{x} \in \mathbb{R}^n$

$\|f(\cdot)\|_p := \left(\int_{\mathcal{X}} |f(x)|^p \mathsf{d}x\right)^{\frac{1}{p}}$ p-seminorm of $f(\cdot)$

$\|\boldsymbol{f}(\cdot)\|_p := \left(\int_{\mathcal{X}} \|\boldsymbol{f}(x)\|_2^p \mathsf{d}x\right)^{\frac{1}{p}}$ p-seminorm of $\boldsymbol{f}(\cdot)$

$\|\boldsymbol{f}(\cdot)\|_\infty := \sup_{x \in \mathcal{X}} \|\boldsymbol{f}(x)\|_2$

$\mathbb{L}^p(\mathcal{X} ; \mathbb{R}) := \{f(\cdot) \in \mathbb{L}_f(\mathcal{X} ; \mathbb{R}) : \|f(\cdot)\|_p < +\infty\}$

$\mathbb{L}^p(\mathcal{X} ; \mathbb{R}^n) := \{\boldsymbol{f}(\cdot) \in \mathbb{L}_f(\mathcal{X} ; \mathbb{R}) : \|\boldsymbol{f}(\cdot)\|_p < +\infty\}$

$\widehat{\mathbb{L}}_p(\mathcal{X} ; \mathbb{R}^n)$ The local integrable version of $\mathbb{L}^p(\mathcal{X} ; \mathbb{R}^n)$

$\mathbb{C}(\mathcal{X} ; \mathbb{R}^n) := \left\{\boldsymbol{f}(\cdot) \in (\mathbb{R}^n)^{\mathcal{X}} : \boldsymbol{f}(\cdot) \text{ is continuous on } \mathcal{X}\right\}$

$\mathbb{C}^k([a,b]\,\mathring{,}\,\mathbb{R}^n) := \left\{ \boldsymbol{f}(\cdot) \in \mathbb{C}([a,b]\,\mathring{,}\,\mathbb{R}^n) : \dfrac{\mathsf{d}^k \boldsymbol{f}(x)}{\mathsf{d}x^k} \in \mathbb{C}([a,b]\,\mathring{,}\,\mathbb{R}^n) \right\}$, the derivatives at a and b are one sided.

$\gamma(\cdot)$ Gamma function

$\textsf{\textbf{Col}}_{i=1}^{n}\, x_i := \left[\textsf{\textbf{Row}}_{i=1}^{n}\, x_i^\top\right]^\top = \left[x_1^\top \cdots x_i^\top \cdots x_n^\top\right]^\top$

$[*]YX = X^\top YX$ or $X^\top Y[*] = X^\top YX$

$\mathsf{O}_{n\times m}$ $n \times m$ zero matrix

$\mathbf{vec}(A) = \mathbf{vec}\left(\textsf{\textbf{Row}}_{i=1}^{m}(\mathbf{a}_i)\right) := \textsf{\textbf{Col}}_{i=1}^{m}\, \mathbf{a}_i,$ vectorization of $A = \textsf{\textbf{Row}}_{i=1}^{m}\, \mathbf{a}_i \in \mathbb{R}^{n\times m}$

O_n $n \times n$ zero matrix

$\mathbf{0}_n$ zeros column vector with dimension n.

$x \vee y =: \max(x, y)$

$x \wedge y =: \min(x, y)$

$$X \oplus Y := \begin{bmatrix} X & \mathsf{O} \\ \mathsf{O} & Y \end{bmatrix}$$

$$\bigoplus_{i=1}^{n} X_i = \begin{bmatrix} X_1 & \mathsf{O} & \cdots & \mathsf{O} \\ \mathsf{O} & X_2 & \cdots & \mathsf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{O} & \mathsf{O} & \mathsf{O} & X_n \end{bmatrix}$$

$[*]YX := X^\top YX$ or $X^\top Y[*] := X^\top YX$

$\otimes$ the Kronecker product

$A \succeq 0 \;:\Longleftrightarrow\; \forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}_n\},\;\; \mathbf{x}^\top A \mathbf{x} \geq 0, \quad A = A^\top$

$A \preceq 0 \;:\Longleftrightarrow\; \forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}_n\},\;\; \mathbf{x}^\top A \mathbf{x} \leq 0, \quad A = A^\top$

$X \succeq Y \;:\Longleftrightarrow\; X - Y \succeq 0$

$X \preceq Y \;:\Longleftrightarrow\; X - Y \preceq 0$

$A \succ 0 \;:\Longleftrightarrow\; \forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}_n\},\;\; \mathbf{x}^\top A \mathbf{x} > 0, \quad A = A^\top$

$A \prec 0 \;:\Longleftrightarrow\; \forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}_n\},\;\; \mathbf{x}^\top A \mathbf{x} < 0, \quad A = A^\top$

$X \succ Y \;:\Longleftrightarrow\; X - Y \succ 0$

$X \prec Y \;:\Longleftrightarrow\; X - Y \prec 0$

$\mathbb{S}^n_{\preceq 0} := \{X \in \mathbb{S}^n : X \preceq 0\}$

$\mathbb{S}^n_{\prec 0} := \{X \in \mathbb{S}^n : X \prec 0\}$

$\mathbb{S}^n_{\succeq 0} := \{X \in \mathbb{S}^n : X \succeq 0\}$

$\mathbb{S}^n_{\succ 0} := \{X \in \mathbb{S}^n : X \succ 0\}$

For block symmetric matrices with many blocks throughout the entire thesis, $*$ stands for the blocks which are symmetric with respect to the main diagonal. Moreover, the order of matrix operations in this thesis is assumed to be *matrix (scalars) multiplications* $> \otimes > \oplus > +$. In addition, the notion of empty matrices, which follows the same definition in Matlab (see `https://au.mathworks.com/help/matlab/ref/zeros.html?requestedDomain=www.mathworks.com`), is applied in some of the chapters in this thesis to render our results more adaptable to the handling of different problems in the context of control and stability analysis. All the matrix operations concerning empty matrices follow the same rules in Matlab. Finally, we define $\mathsf{Col}_{i=1}^{n} = []$ when $n < 1$, where $[]$ is an empty matrix with an appropriate column dimension based on specific contexts.

Chapter 1

Introduction

1.1 Background

Dynamical systems with time delays [1–3] are capable to characterize real-time processes influenced by transport, propagation or aftereffects [3–5]. Mathematically, systems with delays generally can be expressed via functional differential equations [6, 7], coupled differential-difference equations [8–10], or posed as infinite dimensional systems [11–14]. Many examples concerning modeling aftereffects can be found among different research areas such as viscoelasticity [15], biological processes [16–18], SIR epidemic model [19–21], financial market [22, 23] and network control systems concerning internet congestion [24, 25][1]. For systems operating in real-time environment, delay effects can be introduced by engineering devices such as actuator, sensors, wires or communication channel etc. It has been shown that the presence of delay in a system can lead to positive [27, 28] or negative [29] system behaviors. As a result, the analysis and control of dynamical systems with delays play a vital role in the context of system engineering.

There are different types of delays one may consider in a system model such as discrete and distributed delays. A good example to illustrate the nature of a discrete delay model can be found in [3] where it is characterized via a classical transport equation with appropriate boundary conditions. This may intuitively explain the connections between delays and transport phenomena, and why the presence of delays in a system may render the system's dimension to be infinite dimensional. On the other hand, delays can be introduced by media of propagation with more complex structures. A distributed delay is denoted via an integral over a "delay interval" which takes into account a segment of the information of past dynamics. Hence one might argue that the information contained by a distributed delay term is richer than a simple discrete delay. However, this may also make distributed delays more difficult to be analyzed mathematically compared to its discrete counterpart due to the presence of different integral kernels. Systems with distributed delays are encountered among the models of biological processes [18–20, 30–35], population dynamics [36, 37], traffic flows [38–40], neural networks [41–43], matching dynamics [44, 45], shimmy dynamics of wheels [46] and milling processes [47]. Distributed delays can also exist at the input of a system [48, 49], or the feedback loop of a control mechanism [50–54] which may produce positive results [54] to the behavior of closed-loop systems.

The stability of systems is of cardinal importance to qualitatively analyze the behavior of system's dynamics. The presence of delays in a system [55] can have fundamental influences to its stability. Even for the case of linear delay systems [5], stability analysis and synthesis still require sophisticated mathematical instruments. This is quite different from linear time-invariant (LTI) systems with finite dimensions for which stability analysis and state-feedback control can be straightforwardly achieved by constructing a quadratic Liapunov function [56]. On the other hand, the presence of distributed delays in a linear system may further

[1]For further models of real-time applications, see the examples listed in [26] and the chapter 2 of [7]

complicate the problems of stability analysis and stabilization. Thus it is of great interest to propose effective solutions for the problems of stability analysis and synthesis of linear systems with distributed delays.

This thesis is devoted to present new methods for the stability analysis and stabilization of linear systems with distributed delays with finite delay values considering dissipativity constraints. Specifically, the distributed delays considered in this thesis possess non-trivial kernels and the proposed methods are based on the construction of Liapunov-Krasovskii functionals (LKFs). For the rest of this chapter, backgrounds of the mathematics preliminaries concerning systems with delays will be first introduced with the emphasis on the Liapunov-Krasovskii stability criterion. Next, a section of a literature review is presented which includes existing methods on the stability analysis and stabilization of linear systems with delays based on the construction of LKFs. Finally, research motivations and the outline of the rest of the chapters are presented in the last section. Some contents in this thesis are based on or related to the results reported by the author in [57–60].

1.2 A review of the general mathematical representations and stability analysis of systems with delays

In this section, general models for systems with delays are presented based on the framework of coupled differential-functional equations (CDFEs). The expression of coupled differential-functional equations is capable to characterize many types of systems affected by delay effects, such as the systems represented by retarded and neutral type functional differential equations [6, 7]. The corresponding stability criteria of Liapunov's direct method for coupled differential-functional equations are also summarized mathematically.

Note that this chapter is not intended to provide a thorough and systematic study on the mathematical theories of coupled differential-functional equations or functional differential equations. For more information on the rich theories behind this topic, readers can refer to [6, 10, 61]. Also see [62] for an excellent treatise on the theoretical fundamentals of dynamical systems and mathematical control theories.

1.2.1 Systems with delays characterized by Coupled Differential-Functional Equations

The expression of a general coupled differential-functional equation [10] is presented as follows

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) := \lim_{\eta \downarrow 0} \frac{\boldsymbol{x}(t+\eta)-\boldsymbol{x}(t)}{\eta} = \boldsymbol{f}\left(t, \boldsymbol{x}(t), \mathbf{y}_t(\cdot)\right), \quad \boldsymbol{y}(t) = \boldsymbol{g}(t, \boldsymbol{x}(t), \mathbf{y}_t(\cdot)), \quad t \geq t_0 \\ &\boldsymbol{x}(t_0) = \boldsymbol{\omega} \in \mathbb{R}^n, \ \ \forall \theta \in [-r, 0), \ \ \boldsymbol{y}(t_0+\theta) = \mathbf{y}_{t_0}(\theta) = \boldsymbol{\psi}(\theta) \end{aligned} \tag{1.1}$$

where $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^n \times \widehat{\mathbb{C}}([-r, 0) \,⨾\, \mathbb{R}^n) \to \mathbb{R}^\nu$ and $\boldsymbol{g} : \mathbb{R} \times \mathbb{R}^n \times \widehat{\mathbb{C}}([-r, 0) \,⨾\, \mathbb{R}^\nu) \to \mathbb{R}^\nu$, which satisfy

$$\forall t \in \mathbb{R}, \ \mathbf{0}_n = \boldsymbol{f}\left(t, \mathbf{0}_n, \mathbf{0}_\nu\right), \quad \mathbf{0}_\nu = \boldsymbol{g}(t, \mathbf{0}_n, \mathbf{0}_\nu), \tag{1.2}$$

and $t_0 \in \mathbb{R}$ and $\boldsymbol{\psi}(\cdot) \in \widehat{\mathbb{C}}\left([-r, 0) \,⨾\, \mathbb{R}^\nu\right)$ and $\mathbf{y}_t(\cdot) \in \widehat{\mathbb{C}}\left([-r, 0) \,⨾\, \mathbb{R}^\nu\right)$ satisfying

$$\forall t \geq t_0, \ \ \forall \theta \in [-r, 0), \ \ \mathbf{y}_t(\theta) = \boldsymbol{y}(t+\theta). \tag{1.3}$$

The notation $\widehat{\mathbb{C}}([-r, 0) \,⨾\, \mathbb{R}^\nu)$ stands for the function space of bounded right piecewise continuous functions $\boldsymbol{f}(\tau) \in \mathbb{R}^\nu$ which together with a uniform norm $\|\boldsymbol{f}(\cdot)\|_\infty := \sup_{\tau \in [-r,0)} \|\boldsymbol{f}(\tau)\|_2$ constitutes a Banach space. According to what has been pointed out in Section 2 of [10], the uniqueness and existence of the solution of (1.1) can be established by using the procedures in [63] which require certain properties must be satisfied by $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^n \times \widehat{\mathbb{C}}([-r, 0) \,⨾\, \mathbb{R}^n) \to \mathbb{R}^\nu$ and $\boldsymbol{g} : \mathbb{R} \times \mathbb{R}^n \times \widehat{\mathbb{C}}([-r, 0) \,⨾\, \mathbb{R}^\nu) \to \mathbb{R}^\nu$.

Now let $\boldsymbol{g}(t, \boldsymbol{x}(t), \mathbf{y}_t(\cdot)) := \boldsymbol{x}(t)$ which satisfies (1.2), then the corresponding system (1.1) in this case becomes

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \mathbf{x}_t(\cdot)), \\ &\boldsymbol{x}(t_0) = \boldsymbol{\omega} \in \mathbb{R}^n, \ \ \forall\theta \in [-r, 0), \ \ \boldsymbol{x}(t_0+\theta) = \mathbf{x}_{t_0}(\theta) = \boldsymbol{\psi}(\theta) \end{aligned} \tag{1.4}$$

where $\mathbf{x}_t(\cdot) \in \widehat{\mathbb{C}}\left([-r, 0) \,;\, \mathbb{R}^n\right)$ satisfies

$$\forall t \geq t_0, \ \ \forall\theta \in [-r, 0), \ \ \mathbf{x}_t(\theta) = \boldsymbol{x}(t+\theta). \tag{1.5}$$

The differential equation in (1.4) can be further simplified into

$$\begin{aligned} &\dot{\boldsymbol{\xi}}(t) = \widehat{\boldsymbol{f}}\left(t, \boldsymbol{\xi}_t(\cdot)\right), \qquad t \geq t_0 \\ &\forall\theta \in [-r, 0], \ \ \boldsymbol{\xi}(t_0+\theta) = \boldsymbol{\xi}_{t_0}(\theta) = \boldsymbol{\phi}(\theta), \end{aligned} \tag{1.6}$$

where $\widehat{\boldsymbol{f}} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{C}([-r, 0] \,;\, \mathbb{R}^n) \to \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r, 0] \,;\, \mathbb{R}^n)$ and $\boldsymbol{\xi}_t(\theta)$ satisfies

$$\forall t \geq t_0, \ \ \forall\theta \in [-r, 0], \ \ \boldsymbol{\xi}_t(\theta) = \boldsymbol{\xi}(t+\theta). \tag{1.7}$$

Note that in (1.6) we have chosen a continuous function $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r, 0] \,;\, \mathbb{R}^n)$ for the initial condition in (1.6), which is simpler than the piecewise continuous initial function $\boldsymbol{\psi}(\cdot)$ in (1.4). Now (1.6) is the standard expression of general functional differential equations [6, 7]. On the other hand, consider the following neutral functional differential equation [6, 7]

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) = \acute{\boldsymbol{f}}(t, \mathbf{y}_t(\cdot)), \quad \boldsymbol{x}(t) = \boldsymbol{y}(t) - \boldsymbol{h}(t, \mathbf{y}_t(\cdot)), \quad t \geq t_0 \\ &\forall\theta \in [-r, 0], \ \mathbf{y}_{t_0}(\theta) = \boldsymbol{\phi}(\theta) \end{aligned} \tag{1.8}$$

where $t_0 \in \mathbb{R}$ and $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r, 0] \,;\, \mathbb{R}^n)$ and $\mathbf{y}_t(\cdot)$ satisfies

$$\forall t \geq t_0, \ \ \forall\theta \in [-r, 0], \ \ \mathbf{y}_t(\theta) = \boldsymbol{y}(t+\theta). \tag{1.9}$$

Moreover $\acute{\boldsymbol{f}} : \mathbb{R} \times \mathbb{C}([-r, 0] \,;\, \mathbb{R}^n) \to \mathbb{R}^\nu$ and $\boldsymbol{h} : \mathbb{R} \times \mathbb{C}([-r, 0] \,;\, \mathbb{R}^n) \to \mathbb{R}^n$ in (1.8) satisfy

$$\forall t \in \mathbb{R}, \ \ \mathbf{0}_n = \acute{\boldsymbol{f}}(t, \mathbf{0}_n), \quad \mathbf{0}_n = \boldsymbol{h}(t, \mathbf{0}_n). \tag{1.10}$$

Note that (1.8) can be reformulated into

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) = \grave{\boldsymbol{f}}\left(t, \boldsymbol{y}(t), \acute{\mathbf{y}}_t(\cdot)\right) = \grave{\boldsymbol{f}}\left(t, \Big[\boldsymbol{x}(t) + \boldsymbol{h}(t, \boldsymbol{y}(t), \acute{\mathbf{y}}_t(\cdot))\Big], \acute{\mathbf{y}}_t(\cdot)\right) = \widetilde{\boldsymbol{f}}\left(t, \boldsymbol{x}(t), \acute{\mathbf{y}}_t(\cdot)\right), \quad t \geq t_0 \\ &\boldsymbol{y}(t) = \boldsymbol{x}(t) + \boldsymbol{h}(t, \boldsymbol{y}(t), \acute{\mathbf{y}}_t(\cdot)) = \widetilde{\boldsymbol{h}}\left(t, \boldsymbol{x}(t), \acute{\mathbf{y}}_t(\cdot)\right), \\ &\boldsymbol{x}(t_0) = \boldsymbol{\phi}(0) - \boldsymbol{h}(t_0, \boldsymbol{\phi}(\cdot)), \ \ \forall\theta \in [-r, 0), \ \acute{\mathbf{y}}_{t_0}(\theta) = \boldsymbol{\phi}(\theta) \end{aligned} \tag{1.11}$$

where $\acute{\mathbf{y}}_t(\cdot)$ satisfies

$$\forall t \geq t_0, \ \ \forall\theta \in [-r, 0), \ \ \acute{\mathbf{y}}_t(\theta) = \boldsymbol{y}(t+\theta), \tag{1.12}$$

and the functionals $\widetilde{\boldsymbol{f}} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{C}([-r, 0) \,;\, \mathbb{R}^n) \to \mathbb{R}^\nu$ and $\widetilde{\boldsymbol{h}} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{C}([-r, 0) \,;\, \mathbb{R}^\nu) \to \mathbb{R}^\nu$ satisfy

$$\forall t \in \mathbb{R}, \ \ \mathbf{0} = \widetilde{\boldsymbol{h}}(t, \mathbf{0}, \mathbf{0}), \quad \mathbf{0} = \widetilde{\boldsymbol{f}}(t, \mathbf{0}, \mathbf{0}). \tag{1.13}$$

Now it is clear to see that (1.11) can be analyzed via (1.1) which illustrates that (1.1) can be applied to model certain neutral functional differential equations.

In the context of system engineering, the stability of a system is one of the fundamental properties which we want to study. In the next subsection, the corresponding Liapunov's direct approach for CDFEs is presented.

1.2.2 LKF approach for the stability analysis of CDFEs

The following Liapunov-Krasovskii stability theorem for of CDFEs is taken from Theorem 3 of [10] and paraphrased with our own notations. For further details of exact mathematical definitions of different types of stability, see Definition 1 and Definition 2 in [10].

Theorem 1.1. *Let* $\boldsymbol{f}: \mathbb{R} \times \mathbb{R}^n \times \widehat{\mathbb{C}}([-r,0) \,;\, \mathbb{R}^n) \to \mathbb{R}^\nu$ *and* $\boldsymbol{g}: \mathbb{R} \times \mathbb{R}^n \times \widehat{\mathbb{C}}([-r,0) \,;\, \mathbb{R}^\nu) \to \mathbb{R}^\nu$ *in* (1.1) *to satisfy the prerequisites of Theorem 3 of [10], and assume that* $\boldsymbol{y}(t) = \boldsymbol{g}(t, \boldsymbol{x}(t), \mathbf{y}_t(\cdot))$ *in* (1.1) *is uniformly input to state stable*[2]*. Moreover, let* $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathbb{C}(\mathbb{R}_{\geq 0} \,;\, \mathbb{R}_{\geq 0})$ *to be non-decreasing function and* $\forall \theta > 0$, $\alpha_1(\theta) > 0$, $\alpha_2(\theta) > 0$ *with* $\alpha_1(0) = \alpha_2(0) = 0$*. Then the trivial solution* $\boldsymbol{x}(t) \equiv \mathbf{0}_n$, $\boldsymbol{y}(t) \equiv \mathbf{0}_\nu$ *of* (1.1) *is uniformly stable, if there exist differentiable functionals* $v: \mathbb{R} \times \mathbb{R}^n \times \widehat{\mathbb{C}}([-r,0) \,;\, \mathbb{R}^\nu) \to \mathbb{R}_{\geq 0}$ *such that* $\forall t \in \mathbb{R}, v(t, \mathbf{0}_n, \mathbf{0}_\nu) = 0$ *and*

$$\alpha_1\left(\|\boldsymbol{\omega}\|_2\right) \leq v(t_0, \boldsymbol{\omega}, \boldsymbol{\phi}(\cdot)) \leq \alpha_2\left(\|\boldsymbol{\omega}\|_2 \vee \|\boldsymbol{\phi}(\cdot)\|_\infty\right) \tag{1.14}$$

$$\dot{v}(t_0, \boldsymbol{\omega}, \boldsymbol{\phi}(\cdot)) = \left.\frac{\mathsf{d}^+}{\mathsf{d}t} v(t, \boldsymbol{x}(t), \mathbf{y}_t(\cdot))\right|_{t=t_0, \boldsymbol{x}(t_0)=\boldsymbol{\omega}, \mathbf{y}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} \leq -\alpha_3\left(\|\boldsymbol{\omega}\|_2\right) \tag{1.15}$$

for any initial condition $\boldsymbol{\omega} \in \mathbb{R}^n$ *and* $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r,0) \,;\, \mathbb{R}^n)$ *in* (1.1)*, where* $t_0 \in \mathbb{R}$, $\|\boldsymbol{\omega}\|_2 \vee \|\boldsymbol{\phi}(\cdot)\|_\infty := \max\left(\|\boldsymbol{\omega}\|_2, \|\boldsymbol{\phi}(\cdot)\|_\infty\right)$ *and* $\frac{\mathsf{d}^+}{\mathsf{d}x} f(x) = \operatorname{limsup}_{\eta \downarrow 0} \frac{f(x+\eta) - f(x)}{\eta}$*. Furthermore, if* $\forall \theta > 0$, $\alpha_3(\theta) > 0$*, then the trivial solution* $\boldsymbol{x}(t) \equiv \mathbf{0}_n$, $\boldsymbol{y}(t) \equiv \mathbf{0}_\nu$ *of* (1.1) *is uniformly asymptotically stable. In addition, the trivial solution* $\boldsymbol{x}(t) \equiv \mathbf{0}_n$, $\boldsymbol{y}(t) \equiv \mathbf{0}_\nu$ *of* (1.1) *is globally uniformly asymptotically stable if* $\lim_{\theta \to \infty} \alpha_1(\theta) = \infty$*.*

Note that if $\widehat{\boldsymbol{x}}(t)$ and $\widehat{\boldsymbol{y}}(t)$ are any solution of (1.1), then their stability can be determined by the trivial solution $\boldsymbol{z}(t) = \mathbf{0}_n$ and $\boldsymbol{\zeta}(t) = \mathbf{0}_\nu$ of the system

$$\begin{aligned} \dot{\boldsymbol{z}}(t) &= \boldsymbol{f}\Big(t, \boldsymbol{z}(t) + \widehat{\boldsymbol{x}}(t), \boldsymbol{z}_t(\cdot) + \widehat{\mathbf{y}}_t(\cdot)\Big) - \boldsymbol{f}\left(t, \widehat{\boldsymbol{x}}(t), \widehat{\mathbf{y}}_t(\cdot)\right), \\ \boldsymbol{\zeta}(t) &= \boldsymbol{g}\Big(t, \boldsymbol{z}(t) + \widehat{\boldsymbol{x}}(t), \boldsymbol{z}_t(\cdot) + \widehat{\mathbf{y}}_t(\cdot)\Big) - \boldsymbol{g}(t, \widehat{\boldsymbol{x}}(t), \widehat{\mathbf{y}}_t(\cdot)) \end{aligned} \tag{1.16}$$

where $\boldsymbol{z}_t(\cdot)$ satisfies

$$\forall t \geq t_0, \;\; \forall \theta \in [-r, 0), \;\; \boldsymbol{z}_t(\theta) = \boldsymbol{z}(t+\theta). \tag{1.17}$$

If a system considered in (1.4)–(1.7) is concerned, then Theorem 1.1 can be modified to deal with (1.6) where Theorem 1.1 becomes Theorem 2.1 in [6] or Theorem 1.3 in [64]. Moreover, one can conclude that the stability of certain types of neutral functional differential equations can be analyzed by Theorem 1.1 via the representation of (1.1) based on what we have demonstrated in (1.8)–(1.11).

Remark 1.1. Theorem 1.1 can be considered as an extension of the direct Liapunov method for systems with finite dimensional. It provides an effective tool to verify the stability of (1.1) without explicitly knowing the analytic expressions of the solution of (1.1).

The direct Liapunov approach in Theorem 1.1 generally can only provide a sufficient condition to determine the stability of (1.1). In the context of control engineering, it is desirable to numerically construct Liapunov functions (functionals) where conditions like (1.14) and (1.15) are implied by the feasible solution of certain optimization programs. In this thesis, we focus on the synthesis and stability analysis of linear systems with distributed delay. Unlike the situation of using a quadratic Liapunov function to analyze the stability of an LTI delay-free system, sufficient and necessary stability conditions generally cannot be derived by constructing LKFs for linear systems with delays. In the following section, we review some existing methods for the stability analysis and stabilization of linear delay systems with particular emphasis on linear systems with distributed delays, where we present some recent development of methodologies and discuss the technical difficulties on this topic.

[2] For the mathematical definition of the uniformly input to state stability of $\boldsymbol{y}(t) = \boldsymbol{g}(t, \boldsymbol{x}(t), \mathbf{y}_t(\cdot))$, see Definition 2 in [10]

1.3 Literature review on the stability analysis of linear systems with distributed delay

In this section, the review of two major branches (frequency and time-domain approaches) of existing methods for the stability analysis and stabilization of linear delay systems are presented with special emphasis on linear systems with distributed delays. The scenarios proposed in this thesis are based on the construction of LKFs which belongs to the category of time-domain approaches. We also provide a brief summary concerning the development of the optimization methods via semidefinite programmings by which our proposed stability (stabilization) conditions in this thesis can be numerically solved.

To effectively describe the existing works on the stability analysis and stabilization of linear systems with distributed delays, we use the following linear distributed delay system

$$\dot{\boldsymbol{x}}(t) = A_1\boldsymbol{x}(t) + A_2\boldsymbol{x}(t-r) + \int_{-r}^{0} A_3(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau, \quad r>0 \tag{1.18}$$

as a reference in time-domain, where $A_3(\cdot) \in \mathbb{L}^2\left([-r,0] ⨟ \mathbb{R}^{\nu}\right)$, $r>0$ and $\boldsymbol{x}(t)\in\mathbb{R}^n$. The spectrum of (1.18):

$$\left\{s\in\mathbb{C} : p(s)=0\right\}, \quad p(s) = \det\left(sI_n - A_1 - A_2\mathsf{e}^{-rs} - \int_{-r}^{0} A_3(\tau)\mathsf{e}^{\tau s}\mathsf{d}\tau\right). \tag{1.19}$$

is used as a reference to discuss existing frequency-domain-based methods.

1.3.1 Frequency-domain approaches

The information of the spectrum in (1.19) is determined via the zeros of a complex-valued function $p(\cdot)$ including complex exponential functions. Unlike the case of a finite-dimensional LTI system which can only have finite numbers of characteristic roots, a linear delay system can have countably infinite number of number of characteristic roots [64, 65], hence more advanced mathematical theories may be required for the analysis of (1.19).

Essentially, using (1.19) to determine the stability of (1.18) converges to the problem of computing the spectral abscissa [5] of $\{s\in\mathbb{C} : p(s)=0\}$. Specifically, it is true that (1.18) is asymptotically (exponentially) stable if and only if (see Theorem 1.5 in [64]) $\max\{\Re(s) : p(s)=0\} < 0$[3]. Thus by using the value of $\max\{\Re(s) : p(s)=0\}$ which might be acquired by computing the zeros of $p(s)$ within a critical region in the complex plane, the stability of (1.18) can be determined. Concerning existing which may handle this type of problem, one can consider the methods developed for computing the zeros of general analytic functions [66–68], or solutions proposed for solving nonlinear eigenvalue problems [69–79]. Here we name a few frequency-domain approaches further which can handle a general[4] distributed delay term $\int_{-r}^{0} A_3(\tau)\mathsf{e}^{\tau s}\mathsf{d}\tau$ in the spectrum with available code implementation in Matlab environment,. The numerical method in [80–83] (Matlab code in `http://cdlab.uniud.it/software#eigAM-eigTMN`) allows one to analyze the stability of a retarded (renewal) linear system with multiple numbers of discrete and distributed delays, as long as the values of delays are given and explicit expressions of the distributed delay terms are provided. On the other hand, the algorithms in [84–86] provide solutions for the computation and analysis of quasi-polynomials, which can be applied to calculate the spectral abscissa of (1.19) with a given $r>0$. However, the implementation of the algorithm [86] in Matlab environment (Matlab code in `http://www.cak.fs.cvut.cz/algorithms/qpmr`) cannot be applied if the integral term $\int_{-r}^{0} A_3(\tau)\mathsf{e}^{\tau s}\mathsf{d}\tau$ in (1.19) cannot be denoted by a closed form expression.

Note that here we have no intention to make a detailed survey on the frequency-domain approaches for the analysis of stability of delay system. For more information on this topic, interested readers can refer

[3]The criteria are different for neutral type delay systems or coupled differential functional systems

[4]"General" here means that $A(\tau)$ is not a constant over $[-r,0]$

to the monographs [64, 65, 87–91]. In addition, more frequency-domain-based schemes for the stability analysis of linear delay systems can be found in [92–106]. (Apart from the approaches in [96, 103], these methods may not be able to handle the stability of systems containing distributed delays with non-constant kernels).

On the other hand, the research on the stabilization of systems with delays in frequency-domain is a rich subject which can be approached via different perspectives. Early works on this topic can be found in [107, 108] where the Smith predictor is discussed. Other frequency-domain-based methods for the stabilization of linear delay system are listed as follow:

- Stabilization of linear system including distributed delay: [109–111]
- Stabilization of linear systems with discrete delays [112–117]
- PID controller design for linear system with discrete delays [118–122]
- Stabilization of linear systems with discrete delays and performance constraints [123–126]
- Synthesis methods for single-input-single-output infinite dimensional systems [127–129]
- Synthesis methods for multiple-input-multiple-output infinite dimensional systems [11, 12, 130]

To the best of the author's knowledge, the newest trend of frequency-domain-based methods for the stabilization of linear systems with delays is represented by the results in [5, 123, 125] and [131, 132][5]. These methods are predominately nourished by the recent development of algorithms for non-smooth optimization [133–138].

1.3.2 Time-domain approaches

The LKF approach is one of the major time-domain approaches for the stability analysis of systems with delays. By exploiting the properties of LKFs with predefined structures, one may construct stability (synthesis) conditions which are denoted by the unknown parameters of the functionals. It is desirable to pose stability (synthesis) conditions as optimization constraints which can be efficiently solved by numerical algorithms. The main challenges for this approach reside in deriving non-conservative stability (synthesis) conditions which are significantly influenced by the predefined structures and mathematical handling of the functionals to be constructed. As a matter of fact, the LKF approach for delay systems can be considered as an extension of the second Liapunov approach for finite dimensional systems, where the latter one has been successfully applied to characterize the stability of $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$ via the existence of a quadratic Liapunov function $v(\boldsymbol{\xi}) = \boldsymbol{\xi}^\top P\boldsymbol{\xi}$. General LKFs like

$$v(\boldsymbol{\phi}(\cdot)) := \boldsymbol{\phi}^\top(0)P_1\boldsymbol{\phi}(0) + 2\boldsymbol{\phi}^\top(0)\int_{-r}^{0} P_2(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau + \int_{-r}^{0}\int_{-r}^{0} \boldsymbol{\phi}^\top(\theta)P_3(\tau,\theta)\boldsymbol{\phi}(\tau)\mathsf{d}\tau\mathsf{d}\theta + \int_{-r}^{0} \boldsymbol{\phi}^\top(\tau)Q(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \quad (1.20)$$

has been previously investigated in [139, 140] for analyzing the stability of a simple delay system $\dot{\boldsymbol{x}}(t) = A_1\boldsymbol{x}(t) + A_2\boldsymbol{x}(t-r)$, where $P \in \mathbb{S}^n$, $P_2(\tau) \in \mathbb{R}^{n\times n}$, $P_3(\tau,\theta) \in \mathbb{R}^{n\times n}$, $Q(\tau) \in \mathbb{S}^{n\times n}$ with $P_3(\tau,\theta) = P_3^\top(\theta,\tau)$, $\forall\tau;\theta \in [-r,0]$. It has been shown in [141] that the form of (1.20) with $Q(\tau) = Q_1 + (\tau+r)Q_2$, which can render (1.20) to admit a quadratic lower bound [141], is adequate to provide sufficient and necessary stability conditions for the asymptotic stability. Thus (1.20) with $Q(\tau) = Q_1 + (\tau+r)Q_2$ can be considered as an example of complete Liapunov-Krasovskii functional[6]. In fact, it has been demonstrated in [143] that

[5]The synthesis scheme in [131, 132] are developed for infinite dimensional linear systems, hence it may be applied to design a controller for a linear system with delays whenever it is applicable.

[6]For a thorough study on the theory of complete LKFs for linear delay system, see [142]

there exists a complete LKF which can determine the stability of a general linear delay system. Moreover, the idea of 'Complete LKF' has been extended in [10] as

$$v(\boldsymbol{\phi}(\cdot)) := \boldsymbol{\xi}^\top P_1 \boldsymbol{\xi} + 2\boldsymbol{\xi}^\top \int_{-r}^{0} P_2(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau + \int_{-r}^{0}\int_{-r}^{0} \boldsymbol{\phi}^\top(\theta) P_3(\tau,\theta)\boldsymbol{\phi}(\tau)\mathsf{d}\tau\mathsf{d}\theta + \int_{-r}^{0} \boldsymbol{\phi}^\top(\tau) Q(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \quad (1.21)$$

with $P \in \mathbb{S}^n$, $P_2(\tau) \in \mathbb{R}^{n\times\nu}$, $P_3(\tau,\theta) = P_3^\top(\theta,\tau) \in \mathbb{R}^{\nu\times\nu}$ and $Q(\tau) \in \mathbb{S}^\nu$, which can be applied to provide sufficient and necessary conditions for the stability of a coupled differential-difference equation

$$\dot{\boldsymbol{x}}(t) = A_1\boldsymbol{x}(t) + A_2\boldsymbol{y}(t-r), \quad \boldsymbol{y}(t) = A_3\boldsymbol{x}(t) + A_4\boldsymbol{y}(t-r) \quad (1.22)$$

with $r > 0$ and $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{y}(t) \in \mathbb{R}^\nu$.

Remark 1.2. Note that for a system with multiple delays such as

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \sum_{i=1}^{p} A_i\boldsymbol{x}(t-r_i), \quad r_{i+1} > r_i, \ \forall i = 1\cdots p, \quad (1.23)$$

the structure of the corresponding the complete LKF can be easily determined by adding more 'delay related' integral terms in (1.21). See the functional in Chapter 7 of [64] for details.

In contrast to the situation of finding a quadratic function $\boldsymbol{x}^\top(t)Q\boldsymbol{x}(t)$ for $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}$ where Q has finite dimensions, the dimensions of decision variables $P_2(\cdot)$, $P_3(\cdot,\cdot)$ of (1.21) are infinite dimensional which are significantly difficult to be constructed numerically. As a matter of fact, the handling of the distributed delay term $A_3(\cdot)$ in (1.18) inherits similar difficulties due to its infinite dimension. Consequently, early results based on the LKF approach generally assume simple structures for the matrix parameters in (1.21) and a constant distributed delay kernel $A_3(\tau) = D \in \mathbb{R}^{n\times n}$. For instance, with $P_2(\tau) = P_3(\tau,\theta) = \mathsf{O}_n$ and $Q(\tau) = Q_1$, one may use (1.21) to derive delay independent conditions for $\dot{\boldsymbol{x}}(t) = A_1\boldsymbol{x}(t) + A_2\boldsymbol{x}(t-r)$, which ensures that the system is asymptotically stable irrespective to the values of delay $r > 0$. However, such stability condition is too restrictive since some system may only be stable within specific ranges of delay values, thus it is imperative to derive delay-dependent stability conditions which are related to the value of delays.

Supported by the popularization of LMIs [56], there have been tremendous amounts of literature since the middle of the 90s dedicated to this topic on both stability analysis [144–157] and stabilization [158–169]. For comprehensive collections of the existing literature on this topic, see the monographs in [64, 170–173]. Although choosing $P_2(\cdot)$, $P_3(\cdot,\cdot)$ to be constant leads to stability (synthesis) conditions with finite dimensional, the induced conservatism is obvious given the fact that $P_2(\cdot)$, $P_3(\cdot,\cdot)$ in (1.21) are assumed to be general functions. Thus it is certainly reasonable to consider $P_2(\cdot)$, $P_3(\cdot,\cdot)$ with more general structure. On the other hand, unilaterally adding more terms to LKFs may not necessarily lead to less conservative stability conditions. One of the critical factors in the procedure of deriving stability conditions via the LKFs approach is the application of integral inequalities whose lower bounds may have a significant impact to the conservatism of the resulting stability (synthesis) conditions. Finally, it is important to mention that only sufficient conditions can be obtained generally via the LKF approach due to the intrinsic mathematical structures encountered in the procedure of deriving stability (synthesis) conditions which are generally denoted by matrix inequalities. Thus how to construct stability (synthesis) conditions with less conservatism and fewer variables become the paramount goal to be achieved if the LKF approach is considered.

Since $P_2(\cdot)$ and $P_3(\cdot,\cdot)$ in (1.21) are of infinite dimensional, thus discretization or approximation scheme may be applied to $P_2(\cdot)$ and $P_3(\cdot,\cdot)$ to construct variables with finite dimensions. Initiated in [174] by Gu, the variables $P_2(\cdot)$, $P_3(\cdot,\cdot)$ and $Q(\cdot)$ in (1.21) are discretized by piecewise linear functions over subregions of a delay interval. This idea has produced fruitful results over the past decade [175–179] and has been successfully extended to tackle linear coupled differential-difference system [10, 180, 181], and linear systems with distributed delays terms which are of piecewise constant integrand [182, 183]. However, the assumption of using piecewise linear functions as the basis of $P_2(\cdot)$, $P_3(\cdot,\cdot)$ and $Q(\cdot)$ can be conservative, and it is not clear how to deal with a distributed delay term with more general integrands. By using the application of full-block S-procedure [184], a novel approach is presented in [185] where a linear system with rational distributed delay kernels is considered. The method in [185] can be applied to handle general distributed terms via approximations though it is not clear about the relationship between the stability of the original and approximated system. However, the stabilization condition in [185] demands (A, B) to be stabilizable[7] where A is the delay-free state space matrix and B is the input gain matrix, thereby inferring that the induced conservatisms cannot be ignored.

On the other hand, the research on finding new integral inequalities for the construction of LKFs is increasingly becoming popular. In [156], a general form of Wirtinger inequality is derived which generalizes the integral form of Jensen inequality. A significant breakthrough was first made in [186] where Legendre-Bessel integral inequality is derived. This integral inequality generalizes the previous Jensen and Wirtinger inequalities and it is perfectly suitable to be applied to construct stability conditions via functionals like (1.21) with polynomials kernels for $P_2(\cdot)$ and $P_3(\cdot,\cdot)$. A distinct feature in [186] is that the feasibility of the stability conditions is a hierarchy with respect to the degree of the Legendre polynomials in the LKF. Moreover, the numerical examples tested in [186] and the subsequent literature [187] had clearly demonstrated the advantage of the stability conditions in [186, 187] over existing results based on less conservatism and lower numerical complexity. Following this idea, Legendre-Bessel integral inequality has been further extended in [188] to cope with linear distributed delay systems where the distributed delay kernels can be any continuous functions approximated via Legendre polynomials. Although the method of handling distributed delay is based on approximation, the stability of the original distributed delay system was analyzed in [188] and the results are not based on the stability of an approximated system. This can be reflected by the fact that the approximation error of the distributed delay function is included by the hierarchical stability condition in [188]. A potential problem of the method developed in [188] is that very large values for the degree of Legendre polynomials might be required to approximate a distributed delay kernel function if the form of the function is not "friendly" towards polynomials approximations.

Remark 1.3. Apart from the LMI-based LKF approach in time-domain, there are also other types of methods which can be applied to the stability analysis and stabilization of systems with delays. Early results can be found in [189, 190] where the ideas of both predictor controllers and transcendental matrix equations are employed to transfer the original delay system into an equivalent delay free system. However, the approaches in [189, 190] inherit obvious conservatism originated from the restriction imposed on the system's parameters with uninviting numerical solvability. For instance, necessary conditions or sufficient and necessary conditions for the stability of systems with delays have been proposed in [191–195] and [196–198], respectively, based on the delay Liapunov matrix approach. However, as mentioned in [197], the numerical solutions of the delay Liapunov matrix remains critical to the applicability of this approach, whose difficulty is possibly derived from the mathematical complexity of delay systems and general distributed-delay terms. Another representative example is the construction of predictor controllers [199] for systems with input delays [200–203], and for systems with both state and input delays [204–210]. It is worthy to mention that the idea of the classical prediction scheme has been further integrated with the constructive synthesis approaches to attain stabilization for the delay systems possessing certain structures (backstepping, forwarding) [211–213]. Finally, the semi-discretization scheme[8] in [90] can be applied for the stability

[7] For the case of stability analysis, this means that the stability condition can be feasible only if the delay-free matrix A is Hurwitz

[8] This approach might be interpreted as a mixed combination of both time and frequency-domain approaches

analysis of linear-periodic time-delay systems with distributed delays.

1.3.3 Computational Tools for the Analysis and Synthesis of Delay Systems

Applying semidefinite programmings to solve problems in the context of control engineering has proved to be fruitful in terms of stability analysis and stabilization [56]. For instance, by using a quadratic Liapunov function for an LTI system with finite dimensions, the problem of stability analysis or stabilization can be transformed into a problem characterized by optimization constraints denoted by linear matrix inequalities which can be efficiently solved by using interior point algorithms for semidefinite programming [214]. A good advantage of using LMIs approach is that the stability (stabilization) conditions can be extended (modified) to incorporate further information, such as dissipativity [215] and uncertainties [216], without necessarily introducing intractable mathematical complications. There has been a series of papers and monographs dedicated to the application of LMIs related to the subject of systems and control [184, 217–219].

Recently, three papers [220–222] have demonstrated similar methodologies to solve polynomials optimization problems rooted in the theory of algebraic geometry [223–225]. The so-called sum-of-squares (SoS) programming can impose positive constraints on polynomials decision variables which ultimately can be solved via LMIs with finite dimension. This provides an effective way to solve infinite dimensional robust LMIs which are polynomially dependent to uncertainties over compact sets. A successive series of results in [221, 226–231] contains comprehensive applications from a variety of standpoints. In terms of delay systems, the constructions of general Krasovskii functionals via SoS approach are considered in [232–235] where the functionals, with similar structures as (1.21), contains integral terms with polynomials kernels. However, the implications of the numerical burden when SoS is applied to construct LKFs still need to be addressed compared to standard LMIs approaches.

1.4 Research motivations and outline of thesis

The research motivations and the outline of this thesis are presented in this section. The first subsection includes the theoretical and practical motivations to investigate the stability and stabilization of linear systems with distributed delays. Moreover, we provide a summary in the second subsection to outline the works of this thesis for each chapter.

1.4.1 Research Motivations

The first research motivation of this thesis is the fact that there are not too many existing solutions for the stability analysis and stabilization of linear systems with general distributed-delays. Mathematically, a linear system with both discrete and general distributed-delays can represent a significantly wide class of general linear delay system.[9] Thus the contribution of the methods for the stability (stabilization) of linear systems with distributed delays is very important to the theory of general linear delay systems as a whole. The results developed in [186–188] point to a very promising direction to apply the LKF approach to handle the stability of delay systems. However, there is still wide space for the development of new approaches, and many questions concerning distributed delay still can be addressed. For instance, can we derive non-conservative stability (synthesis) conditions for linear systems with a non-trivial distributed delay term where the stability (synthesis) conditions can detect delay margins? Moreover, can a distributed delay term be tackled directly in the context of stability analysis (stabilization) even if no approximation is employed? On the other hand, can we apply other types of functions instead of only Legendre polynomials [188] to handle a general distributed-delay term in the context of stability analysis? Based on the mathematical nature of constructing stability conditions via LKFs, the previous three questions imply that new integral

[9] For instance, see Chapter 7 in [6] for the general model of autonomous linear functional differential system.

inequalities are required to be developed which must be able to handle non-polynomials kernels. Finally, the synthesis solutions for systems with delays still need to be further addressed since it may not be trivial to construct convex conditions for the solutions of synthesis problems via the LKF approach. All these questions are worthy to be considered and in fact answered by the results presented in this thesis.

The second motivation is the ubiquitous, both theoretical and practical, applications of linear systems with delays. For theoretical applications, if the transfer function of a linear distributed parameter system (linear infinite dimensional system) is identical to the transfer function of a system with delays, then its stability can be analyzed by the corresponding delay model via constructing LKFs. Indeed, it has been shown in [236–239] that coupled PDE-ODE systems, which can be applied to model drilling mechanisms [240], can be equivalently described by systems with delays. Thus if a linear distributed parameter system contains a distributed delay term, this system might be analyzed by an equivalent linear system with a distributed delay. Furthermore, one of the appealing feature of the LKF approach compared to common frequency-domain solutions is that intractable mathematical difficulties are not necessarily introduced when the method is extended to consider dissipativity [173], uncertainties [241], or uncertain bounded time-varying delays [242]. In fact, the rich existing results on the stability analysis and control of networked control [243–247] and sampled-data systems [248–252] advanced by the LKF approach can undoubtedly demonstrate the importance of developing effective solutions for linear systems with time-varying delays. On the other hand, it has been shown in [253] the digital communication channel, with stochastic packet delay and loss, of a networked control system can be modeled via distributed delays. Hence new methods on the systems with distributed delays based on the LKF approach may lead to significant advancements on the modeling and control of networked control systems, which has become one of the major subjects in the field of control engineering.

1.4.2 Outline of Thesis

The contents of the rest of the chapters in this thesis are summarized as follows.

In **Chapter 2**, we examine the problem of stabilizing a linear system with distributed delays subject to dissipativity constraints. The distributed delay terms exist in states, inputs and outputs of the system, and the distributed delay kernels can include a certain class of elementary functions such as polynomials, trigonometric and exponential functions. Sufficient conditions for the existence of a stabilizing state feedback controller under the dissipativity constraints are derived in terms of linear and bilinear matrix inequalities (BMIs) via constructing an LKF related to the distributed kernels. The construction of the functional is achieved through the application of a novel general integral inequality. To tackle the non-convexity induced by the BMI in the synthesis conditions, Projection Lemma is employed to produce convex conditions denoted by LMIs. Moreover, an iterative algorithm is constructed to further improve the feasibility of our methods. Finally, numerical examples are presented to demonstrate the strength and effectiveness of the proposed methodology.

In **Chapter 3**, the solutions of stability analysis and stabilization in Chapter 2 are further extended to deal with the problem of stabilizing an uncertain linear system with distributed delays subject to dissipativity constraints, where the uncertainties are of linear fractional forms and subject to the constraints of full block scaling structures. To handle the complex uncertainties with linear fractional structures, a lemma is derived which establishes a relation between simple LMIs and a robust LMI with linear fractional uncertainties where the well-posedness of the uncertainties can be determined by a matrix inequality condition with finite dimensions. Based on the results derived in Chapter 2 without considering the presence of system uncertainties, two theorems containing synthesis conditions can be derived based on the application of the aforementioned lemma where the conditions of the second theorem are convex. Similar to the paradigm utilized in Chapter 2, an iterative algorithm is also proposed to solve the BMI in the first theorem to further reduce conservatism, where the algorithm can be initiated by the feasible solutions of the aforementioned second theorem. A distinct feature of the result in this chapter is that the proposed method is further modified to design a non-fragile dynamical state feedback for a linear system with input delays, where both the plant and resulting controller are robust against uncertainties. More importantly, the corresponding iterative algorithm for the design of non-fragile dynamical state feedback can be initiated simply by the gain of a constructible predictor controller, without appealing to a separate theorem. A battery of numerical examples is tested to demonstrate the effectiveness of our proposed methods.

Given the importance of having optimal integral inequalities demonstrated by the results in the previous two chapters, three general classes of integral inequality are developed in **Chapter 4** which can be utilized with the LKF approach to handle stability related problems for linear systems with delay. Our inequalities exhibit very general structures in terms of the generality of weight functions and integral kernels of the lower quadratic bounds. Almost all existing inequalities in the peer-reviewed literature, including those with free matrix variables, are the special cases of our proposed inequalities. Moreover, relations are established in terms of the inequality bound gaps between our proposed inequalities. For specific applications concerning systems with delays, our inequalities are applied to construct a stability condition via the LKF approach for a linear CDDS with a distributed delay. It is shown that the resulting stability condition is invariant with respect to a parameter in the LKF and equivalent stability conditions can be derived by using different types of inequalities. Finally, it is important to stress that the proposed inequalities can be applied in more general contexts such as the stability analysis of PDE-related systems or sampled-data systems.

Chapter 5 presents a new method for the dissipativity and stability analysis of a linear CDDS with general distributed-delays at both state and output. More precisely, the distributed delay terms under consideration can contain any $\mathbb{L}^2$ functions which are approximated via a class of elementary functions which includes the option of Legendre polynomials. By using this broader class of functions compared to the existing approach in [188] where only Legendre polynomials are utilized for approximations, one can construct LKFs with more general structures as compared to the existing approach in [188] where the functional is parameterized via Legendre polynomials. Furthermore, a novel generalized integral inequality is also proposed to incorporate approximation error in our stability (dissipativity) conditions. Based on the proposed approximation scenario with the proposed integral inequality, sufficient conditions determining the dissipativity and stability of a CDDS are derived in terms of linear matrix inequalities. In addition, several hierarchies in terms of the feasibility of the proposed conditions are derived under certain constraints. Finally, several numerical examples are presented in this chapter to show the effectiveness of our proposed methodologies.

The problem of delay range stability analysis for a CDDS with distributed delays is investigated in **Chapter 6** where the system is also subject to a dissipative constraint. Polynomials distributed delay kernels are considered in this chapter so that tractable stability conditions can be obtained. A LKF with non-constant matrix parameters, which are related to the delay value polynomially, is applied in this chapter to construct sufficient conditions which guarantee range stability of a linear CDDS subject to a dissipative constraint. The proposed sufficient conditions for the stability and dissipativity of the system are denoted by sum-of-squares constraints which are constructed based on the application of a matrix relaxation technique for robust LMIs without introducing any potential conservatism. Furthermore, the proposed methods can be extended to solve delay margin estimation problems for a linear CDDS subject to prescribed dissipative constraint. Finally, numerical examples are presented to demonstrate the effectiveness of the proposed methodologies.

In **Chapter 7**, new methods are developed to stabilize a linear distributed delay system whose delay is time-varying and bounded by given constants. The distributed delay terms exist at the states, inputs and outputs of the system, and the distributed delay kernels in this chapter can be functions belonging to a class of elementary functions. Furthermore, a novel integral inequality is proposed to construct synthesis (stability) conditions via an LKF where the conditions are expressed in terms of matrix inequalities. The proposed synthesis (stability) conditions, which can determine the stability and dissipativity of the system with a supply function, are related to the values of the bounds of the time-varying delay where the information of the derivatives of the time-varying delays is absent. Given what we have presented in Chapter 2 concerning the handling of bilinear matrix inequalities, the resulting synthesis (stability) conditions in this chapter can be either solved directly by the standard solvers of SDPs if they are convex, or reshaped into LMIs, or solved by an iterative algorithm. Finally, numerical examples are presented to demonstrate the effectiveness of the proposed methodologies.

At the end of this thesis, we outline some future works and suggestions in **Chapter 8**.

Chapter 2

Dissipative Stabilization for a Linear Delay System with Distributed Delays

2.1 Introduction

Among many models of Time-Delay Systems (TDS) [6], distributed delay systems (DDS) cover a wide range of real-time applications [38, 39]. For a rigorous treatment and benchmark results on the frequency-domain approaches for linear TDS or DDS, see the monograph [5] and the references therein. As for time-domain approaches, the construction of LKFs [64] has been adopted as the most common method to undertake both stability analysis and controller synthesis. In particular, the complete Liapunov Krasovskii functional (CLKF) [64, 142], which can provide a sufficient and necessary stability condition for linear delay systems such as $\dot{\boldsymbol{x}}(t) = A_1\boldsymbol{x}(t) + A_2\boldsymbol{x}(t-r), r \geq 0$, generalizes most of the existing proposed functionals. For a thorough treatise on the fundamental theories of CKLF and its mathematical derivation, see [142] and references therein.

In contrast to constructing a quadratic function within the context of semi-definite programming, the decision variables of CLKF possess infinite dimension which leads to significant difficulties to calculate these variables numerically. In addition, similar problems have been encountered in dealing with non-constant distributed delay terms. If one assumes a linear system with constant distributed delay terms and analyze this system with a functional with constant decision variables, then finite dimension constraints denoted by LMIs can be obtained accordingly. There has been a significant series of literature on this direction to perform either stability analysis or controller synthesis for linear DDS [165, 168, 169]. For a collection of the previous works on this topic, see the monographs [172, 173].

For dealing with non-constant distributed delay terms, the results in [254] have demonstrated that certain linear DDS can be transformed into a system with only discrete delays. However, this method inherits obvious conservatism due to the presence of additional dynamics required by adding new states. An alternative synthesis approach, based on the discretization scheme proposed in [255], is presented in [183] considering linear DDS with a piecewise constant distributed delay term. Moreover, by using the application of full-block S-procedure [184], a novel synthesis scenario is presented in [185] to tackle systems with rational distributed delay kernels, which is capable of dealing with general distributed terms via approximations. However, the derived stabilization conditions require (A, B) to be stabilizable in [185] (A is the delay-free state space matrix and B is the input gain matrix), thereby inferring that the induced conservatisms cannot be ignored. Finally, a systematic way to construct controllers for linear systems with discrete and distributed delays, having forwarding or backstepping structures, has been investigated in [211].

In this chapter, we propose methods for stabilizing linear DDS with distributed delays in states, inputs and outputs. The structure of the distributed delay terms considered here can be non-constant, as the delay kernels functions can be polynomials, trigonometric and exponential functions. A quadratic supply function [173, 215] is also incorporated by our synthesis schemes, which can provide a broad characterization of controller performances. Furthermore, a new integral inequality is derived for the formulation of the synthesis conditions, which can be considered as a generalization of the recent proposed Bessel-Legendre inequality [186, 187]. By constructing a general LKF via the application this inequality, sufficient conditions for the existence of a stabilizing state feedback controller taking into account the dissipativity constraints are derived which are denoted by matrix inequalities containing a bilinear inequality. To circumvent the bi-linearity induced by the product between the parameters of the controller and functional, Projection Lemma [256] is applied so that convex synthesis conditions can be derived in terms of LMIs. To further reduce the conservatism of our methods, an iterative algorithm is derived based on the scenario proposed in [257]. Unlike existing methods, our proposed synthesis solutions neither require (A, B) to be stabilizable as in [185], nor demand forwarding or backstepping structures as in [211]. In addition, the integral kernels of our LKF are not necessarily polynomials compared to the existing results in [186, 187, 234]. With respect to the performance of stability analysis, a numerical example presented in this chapter demonstrate that our approach can outperform the method in [188] in terms of numerical complexity. This is largely due to the fact that the distributed delay terms in this chapter are handled without appealing to the use of polynomials approximations [188], as some functions are difficult to be approximated by polynomials with insufficient degrees.

The chapter is organized as follows. The synthesis problem is first formulated in Section 2.2. Secondly, we present vital mathematical results in Section 2.3 for the derivation of the synthesis solutions in this chapter. Next, the main results on controller synthesis are presented in Section 2.4. To demonstrate the capacity and effectiveness of our methodologies, numerical examples are investigated in Section 4 before the final conclusion in Section 5.

2.2 Problem formulations

The following property of the Kronecker product will be used throughout the whole work.

Lemma 2.1. *$\forall X \in \mathbb{R}^{n\times m}$, $\forall Y \in \mathbb{R}^{m\times p}$, $\forall Z \in \mathbb{R}^{q\times r}$,*

$$(X \otimes I_q)(Y \otimes Z) = (XY) \otimes (I_q Z) = (XY) \otimes Z = (XY) \otimes (Z I_r) = (X \otimes Z)(Y \otimes I_r). \tag{2.1}$$

Moreover, $\forall X \in \mathbb{R}^{n\times m}$, we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes X = \begin{bmatrix} A\otimes X & B\otimes X \\ C\otimes X & D\otimes X \end{bmatrix} \tag{2.2}$$

for any A, B, C, D with appropriate dimensions for the partition of the block matrix.

Consider the following model of linear distributed delay system

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A_1\boldsymbol{x}(t) + A_2\boldsymbol{x}(t-r) + \int_{-r}^{0} \widetilde{A}_3(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau + B_1\boldsymbol{u}(t) + B_2\boldsymbol{u}(t-r) \\
&\qquad + \int_{-r}^{0} \widetilde{B}_3(\tau)\boldsymbol{u}(t+\tau)\mathsf{d}\tau + D_1\boldsymbol{w}(t) \\
\boldsymbol{z}(t) &= C_1\boldsymbol{x}(t) + C_2\boldsymbol{x}(t-r) + \int_{-r}^{0} \widetilde{C}_3(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau + B_4\boldsymbol{u}(t) + B_5\boldsymbol{u}(t-r) \\
&\qquad + \int_{-r}^{0} \widetilde{B}_6(\tau)\boldsymbol{u}(t+\tau)\mathsf{d}\tau + D_2\boldsymbol{w}(t) \\
\dot{\boldsymbol{x}}(t) &:= \lim_{\eta\downarrow 0} \frac{\boldsymbol{x}(t+\eta)-\boldsymbol{x}(t)}{\eta}, \qquad \forall\theta\in[-r,0],\ \ \boldsymbol{x}(t_0+\theta)=\boldsymbol{\phi}(\theta), \qquad t\geq t_0
\end{aligned} \tag{2.3}$$

where $t_0 \in \mathbb{R}$ and $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r,0] ; \mathbb{R}^n)$, and $\boldsymbol{x}(t) \in \mathbb{R}^n$ satisfies the delay equation in (2.3), $\boldsymbol{u}(t) \in \mathbb{R}^p$ denotes input signals, $\boldsymbol{w}(\cdot) \in \widehat{\mathbb{L}}^2([t_0,\infty) ; \mathbb{R}^q)$ represents disturbance and $\boldsymbol{z}(t) \in \mathbb{R}^m$ is the regulated output. Note that $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r,0] ; \mathbb{R}^n)$ is the initial condition for (2.3) at $t = t_0$. The size of the state spaces matrices in (2.3) is determined by the values of $n; m; p; q \in \mathbb{N}$. Note that here we assume the delay value $r \geq 0$ is known. Finally, $\widetilde{A}_3(\tau)$, $\widetilde{B}_3(\tau)$, $\widetilde{C}_3(\tau)$ and $\widetilde{B}_6(\tau)$ satisfy the following assumption:

Assumption 2.1. There exist $\mathbf{Col}_{i=1}^{d} f_i(\tau) = \boldsymbol{f}(\cdot) \in \mathbb{C}^1([-r,0] ; \mathbb{R}^d)$ with $d \in \mathbb{N}$, and constant matrices $A_3 \in \mathbb{R}^{n\times dn}$, $B_3 \in \mathbb{R}^{n\times dp}$, $C_3 \in \mathbb{R}^{m\times dn}$, $B_6 \in \mathbb{R}^{m\times dp}$ such that $\forall\tau \in [-r,0]$, $\mathbb{R}^{n\times n} \ni \widetilde{A}_3(\tau) = A_3F(\tau)$ and $\mathbb{R}^{n\times p} \ni \widetilde{B}_3(\tau) = B_3(\boldsymbol{f}(\tau)\otimes I_p)$ and $\mathbb{R}^{m\times n} \ni \widetilde{C}_3(\tau) = C_3F(\tau)$ and $\mathbb{R}^{m\times p} \ni \widetilde{B}_6(\tau) = B_6(\boldsymbol{f}(\tau)\otimes I_p)$ where $\mathbb{R}^{dn\times n} \ni F(\tau) := \boldsymbol{f}(\tau)\otimes I_n$. In addition, $\boldsymbol{f}(\cdot)$ satisfies the following property:

$$\exists M \in \mathbb{R}^{d\times d} : \frac{\mathsf{d}\boldsymbol{f}(\tau)}{\mathsf{d}\tau} = M\boldsymbol{f}(\tau) \tag{2.4}$$

$$\int_{-r}^{0} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau) \succ 0 \tag{2.5}$$

Remark 2.1. The condition (2.4) in Assumption 2.1 indicates that the functions in $\boldsymbol{f}(\cdot)$ are the solutions of linear homogeneous differential equations with constant coefficients such as polynomials, trigonometric and exponential functions. Namely, $\boldsymbol{f}(\cdot)$ contains functions belong to the entries of a matrix exponential function $\mathsf{e}^{X\tau}$, $X \in \mathbb{R}^{d\times d}$. In addition, there is no limitation on the size of the dimension of $\boldsymbol{f}(\tau)$ as long as it is able to cover all the elements in the distributed terms in (2.3). For (2.5), it indicates that the functions in $\boldsymbol{f}(\cdot)$ are linearly independent in a Lebesgue sense based on Theorem 7.2.10 in [258]. As for the generality of $\boldsymbol{f}(\tau)$, there are many applications can be modeled by (2.3) compatible with Assumption 2.1. For example, the compartmental dynamic systems with distributed delays mentioned in [259], and the distributed delay systems with gamma distributions in [185] and [260] with a finite delay range.

Assume that all states are available for feedback in (2.3) and the open-loop system is stabilized by a state feedback controller $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ with $K \in \mathbb{R}^{p\times n}$. Substitute $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ into (2.3) and considering Assumption 2.1 with (2.1), the input distributed delay matrices become

$$\begin{aligned}
B_3F(\tau)K &= B_3(\boldsymbol{f}(\tau)\otimes I_p)K = B_3(I_d\otimes K)(\boldsymbol{f}(\tau)\otimes I_n), \\
B_6F(\tau)K &= B_6(\boldsymbol{f}(\tau)\otimes I_p)K = B_6(I_d\otimes K)(\boldsymbol{f}(\tau)\otimes I_n).
\end{aligned} \tag{2.6}$$

Now by $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ and (2.6), the closed-loop system can be written as

$$\begin{aligned}
&\dot{\boldsymbol{x}}(t) = \big(\mathbf{A} + \mathbf{B}_1\left[(I_{2+d}\otimes K)\oplus \mathsf{O}_q\right]\big)\boldsymbol{\chi}(t), \ \ \boldsymbol{z}(t) = \big(\mathbf{C} + \mathbf{B}_2\left[(I_{2+d}\otimes K)\oplus \mathsf{O}_q\right]\big)\boldsymbol{\chi}(t), \ \ t\geq t_0 \\
&\forall\theta\in[-r,0], \ \ \boldsymbol{x}(t_0+\theta)=\boldsymbol{\phi}(\theta)
\end{aligned} \tag{2.7}$$

with $t_0 \in \mathbb{R}$ and $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r,0] ; \mathbb{R}^n)$ in (2.3), where

$$\mathbf{A} = \begin{bmatrix} A_1 & A_2 & A_3 & D_1 \end{bmatrix}, \ \ \mathbf{B}_1 = \begin{bmatrix} B_1 & B_2 & B_3 & \mathsf{O}_{n\times q} \end{bmatrix} \tag{2.8}$$

$$\mathbf{C} = \begin{bmatrix} C_1 & C_2 & C_3 & D_2 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} B_4 & B_5 & B_6 & \mathsf{O}_{m\times q} \end{bmatrix} \tag{2.9}$$

$$\boldsymbol{\chi}(t) := \mathbf{Col} \begin{bmatrix} \boldsymbol{x}(t) & \boldsymbol{x}(t-r) & \int_{-r}^{0} F(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau & \boldsymbol{w}(t) \end{bmatrix}, \quad F(\tau) = \boldsymbol{f}(\tau) \otimes I_n. \tag{2.10}$$

2.3 Important mathematical tools

In this section, important theoretical instruments for the derivation of our synthesis conditions in this chapter are presented. These include the Krasovskii stability criteria to determine the stability of delay systems and the definition of dissipativity. Furthermore, a new integral inequality is proposed here which will be applied in the construction of an LKF in the next section.

Lemma 2.2. *Given $r > 0$, the closed-loop system (2.7) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is uniformly globally asymptotically stable at its origin if there exist $\epsilon_1; \epsilon_2; \epsilon_3 > 0$ and a differentiable functional $v : \mathbb{C}([-r, 0] ⨾ \mathbb{R}^n) \to \mathbb{R}$ with $v(\mathbf{0}_n) = 0$ such that*

$$\epsilon_1 \left\|\boldsymbol{\phi}(0)\right\|_2^2 \le v(\boldsymbol{\phi}(\cdot)) \le \epsilon_2 \left\|\boldsymbol{\phi}(\cdot)\right\|_\infty^2, \tag{2.11}$$

$$\left.\frac{\mathsf{d}^+}{\mathsf{d}t} v(\mathbf{x}_t(\cdot))\right|_{t=t_0, \mathbf{x}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} \le -\epsilon_3 \left\|\boldsymbol{\phi}(0)\right\|_2^2 \tag{2.12}$$

hold for any $\boldsymbol{\phi}(\cdot) \in \mathbb{C}\left([-r, 0] ⨾ \mathbb{R}^n\right)$ in (2.7), where $t_0 \in \mathbb{R}$ and $\left\|\boldsymbol{\phi}(\cdot)\right\|_\infty := \sup_{-r\le\tau\le 0} \left\|\boldsymbol{\phi}(\tau)\right\|_2$ and $\frac{\mathsf{d}^+}{\mathsf{d}x} f(x) = \operatorname{limsup}_{\eta\downarrow 0} \frac{f(x+\eta)-f(x)}{\eta}$. Moreover, $\mathbf{x}_t(\cdot)$ in (2.12) is given by $\forall t \ge t_0$, $\forall\theta \in [-r, 0]$, $\mathbf{x}_t(\theta) = \boldsymbol{x}(t + \theta)$ where $\boldsymbol{x} : [t_0 - r, \infty) \to \mathbb{R}^n$ satisfies (2.7) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$.

Proof. Let the functions $u(\cdot), v(\cdot), w(\cdot)$ in Theorem 3 of [64] to be quadratic functions with positive parameters ϵ_1, ϵ_2 and ϵ_3, then Lemma 2.2 can be obtained accordingly since (2.7) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is a particular case of the system $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \mathbf{x}_t(\cdot)))$, $\mathbf{x}_t(\cdot) \in \mathbb{C}\left([-r, 0] ⨾ \mathbb{R}^n\right)$ considered in Theorem 3 of [64]. ■

The following definition of dissipativity is presented based on the general definition of dissipativity in [261].

Definition 2.1. Given $r > 0$, the closed-loop system (2.7) with a supply function $s(\boldsymbol{z}(t), \boldsymbol{w}(t))$ is said to be dissipative if there exists a differentiable functional $v : \mathbb{C}([-r, 0] ⨾ \mathbb{R}^n) \to \mathbb{R}_{\ge 0}$ such that

$$\forall t \ge t_0, \quad \dot{v}(\mathbf{x}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \le 0 \tag{2.13}$$

with $t_0 \in \mathbb{R}$ in (2.7), where $\dot{v}(\mathbf{x}_t(\cdot))$ is well defined for all $t \ge t_0$ and $\mathbf{x}_t(\cdot)$ satisfies $\forall t \ge t_0$, $\forall\theta \in [-r, 0]$, $\mathbf{x}_t(\theta) = \boldsymbol{x}(t + \theta)$ with $\boldsymbol{x}(t)$ and $\boldsymbol{z}(t)$ in (2.7) with $\boldsymbol{w}(\cdot) \in \widehat{\mathbb{L}}^2([t_0, \infty) ⨾ \mathbb{R}^q)$.

If (2.13) holds, then we have

$$\forall t \ge t_0, \;\; v(\mathbf{x}_t(\cdot)) - v(\mathbf{x}_{t_0}(\cdot)) \le \int_{t_0}^{t} s(\boldsymbol{z}(\theta), \boldsymbol{w}(\theta))\mathsf{d}\theta \tag{2.14}$$

which now is in line with the original definition of dissipativity in [261], given $\dot{v}(\mathbf{x}_t(\cdot))$ is well defined for all $t \ge t_0$.

To characterize dissipativity, a quadratic supply function

$$s(\boldsymbol{z}(t), \boldsymbol{w}(t)) = \begin{bmatrix} \boldsymbol{z}(t) \\ \boldsymbol{w}(t) \end{bmatrix}^\top \mathbf{J} \begin{bmatrix} \boldsymbol{z}(t) \\ \boldsymbol{w}(t) \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} \widetilde{J}^\top J_1^{-1} \widetilde{J} & J_2 \\ * & J_3 \end{bmatrix} \in \mathbb{S}^{(m+q)}, \;\; \widetilde{J}^\top J_1^{-1} \widetilde{J} \preceq 0, \;\; J_1^{-1} \prec 0 \tag{2.15}$$

is applied in this chapter where the form of $\mathbf{J}$ is constructed considering the general quadratic constraints applied in [215] together with the idea of factorizing the matrix U_j in [215]. Note that (2.15) is able to characterize numerous performance criteria such as

- $\mathbb{L}^2$ gain performance: $J_1 = -\gamma I_m$, $\widetilde{J} = I_m$, $J_2 = \mathsf{O}_{m\times q}$, $J_3 = \gamma I_q$ where $\gamma > 0$.

- Passivity: $J_1 \prec 0$, $\widetilde{J} = \mathsf{O}_m$, $J_2 = I_m$, $J_3 = \mathsf{O}_m$ with $m = q$.

The following general integral inequality is derived to be applied for the construction of LKFs in the next section.

Lemma 2.3. *Given $\mathcal{K} \subseteq \mathbb{R} \cup \{\pm\infty\}$ where the Lebesgue measure of $\mathcal{K}$ is non-zero, and $U \in \mathbb{S}^n_{\succeq 0}$ and $\boldsymbol{g}(\tau) = \mathbf{Col}_{i=1}^d g_i(\tau) \in \mathbb{L}_2(\mathcal{K} ⨟ \mathbb{R}^d)$ with $d \in \mathbb{N}$ which satisfies*

$$\int_{\mathcal{K}} \boldsymbol{g}(\tau)\boldsymbol{g}^\top(\tau)\mathsf{d}\tau \succ 0, \tag{2.16}$$

then we have

$$\forall \boldsymbol{x}(\cdot) \in \mathbb{L}_2(\mathcal{K} ⨟ \mathbb{R}^n), \quad \int_{\mathcal{K}} \boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau \geq \int_{\mathcal{K}} \boldsymbol{x}^\top(\tau)G^\top(\tau)\mathsf{d}\tau\,(\mathsf{G}\otimes U)\int_{\mathcal{K}} G(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau, \tag{2.17}$$

where $\mathsf{G}^{-1} = \int_{\mathcal{K}} \boldsymbol{g}(\tau)\boldsymbol{g}^\top(\tau)\mathsf{d}\tau \succ 0$ and $G(\tau) := \boldsymbol{g}(\tau) \otimes I_n$ such that $\boldsymbol{g}(\tau) := \mathbf{Col}_{i=1}^d g_i(\tau)$.

Proof. The proof is inspired by the results in [186, 188]. To begin with, we can conclude that the matrix G is invertible given (2.16). Let $\boldsymbol{y}(\cdot) \in \mathbb{L}^2(\mathcal{K} ⨟ \mathbb{R}^n)$ and

$$\boldsymbol{y}(\tau) := \boldsymbol{x}(\tau) - G^\top(\tau)(\mathsf{G}\otimes I_n)\int_{\mathcal{K}} G(\theta)\boldsymbol{x}(\theta)\mathsf{d}\theta. \tag{2.18}$$

Substituting (2.18) into $\int_{\mathcal{K}} \boldsymbol{y}^\top(\tau)U\boldsymbol{y}(\tau)\mathsf{d}\tau$ gives

$$\begin{aligned}\int_{\mathcal{K}} \boldsymbol{y}^\top(\tau)U\boldsymbol{y}(\tau)\mathsf{d}\tau = \int_{\mathcal{K}} \boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau + \mathbf{z}^\top \int_{\mathcal{K}} (\mathsf{G}\otimes I_n)^\top G(\tau)UG^\top(\tau)(\mathsf{G}\otimes I_n)\mathsf{d}\tau\mathbf{z}\\ - 2\int_{\mathcal{K}} \boldsymbol{x}^\top(\tau)UG^\top(\tau)\mathsf{d}\tau(\mathsf{G}\otimes I_n)\mathbf{z}\end{aligned} \tag{2.19}$$

where $\mathbf{z} := \int_{\mathcal{K}} G(\theta)\boldsymbol{x}(\theta)\mathsf{d}\theta$.

Now apply (2.1) to the product term $UG^\top(\tau)$ in (2.19) and consider the fact that $G(\tau) = \boldsymbol{g}(\tau) \otimes I_n$, we have

$$U\left(\boldsymbol{g}^\top(\tau)\otimes I_n\right) = U\left(\boldsymbol{g}^\top(\tau)\otimes I_n\right)\left(\boldsymbol{g}^\top(\tau)\otimes I_n\right)(I_d\otimes U) = G^\top(\tau)(I_d\otimes U). \tag{2.20}$$

Applying (2.20) to the term $\int_{\mathcal{K}} \boldsymbol{x}^\top(\tau)UG^\top(\tau)\mathsf{d}\tau(\mathsf{G}\otimes I_n)\mathbf{z}$ in (2.19) yields

$$\int_{\mathcal{K}} \boldsymbol{x}^\top(\tau)UG^\top(\tau)\mathsf{d}\tau(\mathsf{G}\otimes I_n)\mathbf{z} = \mathbf{z}^\top(I_d\otimes U)(\mathsf{G}\otimes I_n)\mathbf{z} = \mathbf{z}^\top(\mathsf{G}\otimes U)\mathbf{z}. \tag{2.21}$$

Furthermore, by (2.20) and the fact that $\mathsf{G} = \mathsf{G}^\top$, the term $\int_{\mathcal{K}}(\mathsf{G}\otimes I_n)^\top G(\tau)UG^\top(\tau)(\mathsf{G}\otimes I_n)\mathsf{d}\tau$ in (2.19) can be reformulated into

$$\begin{aligned}&\int_{\mathcal{K}}(\mathsf{G}\otimes I_n)G(\tau)G^\top(\tau)(I_d\otimes U)(\mathsf{G}\otimes I_n)\mathsf{d}\tau = \int_{\mathcal{K}}(\mathsf{G}\otimes I_n)G(\tau)G^\top(\tau)(\mathsf{G}\otimes U)\mathsf{d}\tau\\ &= (\mathsf{G}\otimes I_n)\int_{\mathcal{K}}(\boldsymbol{g}(\tau)\otimes I_n)\left(\boldsymbol{g}^\top(\tau)\otimes I_n\right)\mathsf{d}\tau(\mathsf{G}\otimes U) = (\mathsf{G}\otimes I_n)\left(\int_{\mathcal{K}}\boldsymbol{g}(\tau)\boldsymbol{g}^\top(\tau)\mathsf{d}\tau\otimes I_n\right)(\mathsf{G}\otimes U)\\ &= \mathsf{G}\otimes U.\end{aligned} \tag{2.22}$$

By (2.22) and (2.21), (2.19) is rewritten into the form

$$\int_{\mathcal{K}} \boldsymbol{y}^\top(\tau)U\boldsymbol{y}(\tau)\mathsf{d}\tau = \int_{\mathcal{K}} \boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau - \int_{\mathcal{K}} \boldsymbol{x}^\top(\tau)G^\top(\tau)\mathsf{d}\tau\,(\mathsf{G}\otimes U)\int_{\mathcal{K}} G(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau. \tag{2.23}$$

Given $U \succeq 0$, now (2.17) can be derived from (2.23). This finishes the proof. ■

Remark 2.2. By the conclusions of Theorem 7.2.10 in [258], the condition in (2.16) indicates that the functions in $\boldsymbol{g}(\cdot)$ are linearly independent in a Lebesgue sense.

Remark 2.3. Given $\mathcal{K} = [-r, 0], r > 0$ and $\boldsymbol{g}(\tau) = \boldsymbol{\ell}_d(\tau) = \mathsf{Col}_{i=0}^{d}\, \ell_i(\tau)$ with the Legendre polynomials[1] [186–188, 262]

$$\ell_d(\tau) := \sum_{k=0}^{d} \binom{d}{k}\binom{d+k}{k}\left(\frac{\tau}{r}\right)^k, \quad \forall d \in \mathbb{N}\cup\{0\}, \quad \forall \tau \in [-r, 0], \tag{2.24}$$

then (2.17) holds with $\mathsf{G}^{-1} = \bigoplus_{i=0}^{d} \frac{r}{2i+1}$ in this case. This demonstrates the fact that (2.17) can be regarded as a generalization of the Bessel-Legendre Inequality proposed in [186, 187]. Furthermore, (2.17) can be also considered as a generalization of the results in [263]. Finally, it is worthy to emphasize that a discrete version of (2.17)

$$\sum_{k\in\mathcal{J}} \boldsymbol{x}^\top(k) U \boldsymbol{x}(k) \geq [*]\left(\mathsf{G}\otimes U\right)\left(\sum_{k\in\mathcal{J}} \left(\boldsymbol{g}(k)\otimes I_n\right)\boldsymbol{x}(k)\right), \quad \mathsf{G}^{-1} = \sum_{k\in\mathcal{J}} \boldsymbol{g}(k)\boldsymbol{g}^\top(k) \succ 0, \quad \mathcal{J}\subseteq\mathbb{Z} \tag{2.25}$$

can be easily derived given the connections between integrations and summations.

The following Projection Lemma will be applied in the derivation of convex synthesis conditions in Chapter 2,3 and 7.

Lemma 2.4 (Projection Lemma)**.** *[256] Given $n; p; q \in \mathbb{N}$, $\Pi \in \mathbb{S}^n, P \in \mathbb{R}^{q\times n}, Q \in \mathbb{R}^{p\times n}$, there exists $\Theta \in \mathbb{R}^{p\times q}$ such that the following two propositions are equivalent :*

$$\Pi + P^\top \Theta^\top Q + Q^\top \Theta P \prec 0, \tag{2.26}$$

$$P_\perp^\top \Pi P_\perp \prec 0 \;\; and \;\; Q_\perp^\top \Pi Q_\perp \prec 0, \tag{2.27}$$

where the columns of $P_\perp$ and $Q_\perp$ contain bases of null space of matrix P and Q, respectively, which means that $PP_\perp = \mathsf{O}$ and $QQ_\perp = \mathsf{O}$.

Proof. Refer to [256] and [173]. ∎

2.4 Main results on controller synthesis

In this section, the main results on controller synthesis in view of dissipativity are summarized in Theorem 2.1 and 2.2 and Algorithms 1. The proposed methods are based on the construction of

$$v(\mathbf{x}_t(\cdot)) := \begin{bmatrix} \boldsymbol{x}(t) \\ \int_{-r}^{0} F(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau \end{bmatrix}^\top \begin{bmatrix} P & Q \\ * & R \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \int_{-r}^{0} F(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau \end{bmatrix} + \int_{-r}^{0} \boldsymbol{x}^\top(t+\tau)\left(S + (\tau+r)U\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau \tag{2.28}$$

via the application of (2.17), where $\boldsymbol{x}(t)$ satisfies (2.7) and $F(\tau) = \boldsymbol{f}(\tau)\otimes I_n$ with $\boldsymbol{f}(\cdot)$ in Assumption 2.1, and the values of $P \in \mathbb{S}^n$, $Q \in \mathbb{R}^{n\times dn}$, $R \in \mathbb{S}^{dn}$, $S; U \in \mathbb{S}^n$ are to be constructed numerically. Note that (2.28) can be considered as a particular case of the Complete Liapunov Krasovskii Functional (CKLF) in [64] whose infinite dimensional variables are parameterized by Q and R here via the integral term $\int_{-r}^{0} F(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau$.

In the following theorem, sufficient conditions for the existence of a state feedback controller which stabilizes (2.7) and takes into account the supply rate (2.15) are derived in terms of matrix inequalities.

Theorem 2.1. *Given $\boldsymbol{f}(\cdot)$ and M in* (2.1)*, then the closed-loop system* (2.7) *with the supply rate function in* (2.15) *is dissipative and the trivial solution $\boldsymbol{x}(t) \equiv \mathbf{0}_n$ of* (2.7) *with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is globally asymptotically*

[1]The expression of Legendre polynomials here is obtained by setting $\alpha = \beta = 0$ for the experssion of Jacobi polynomials over $[-r, 0]$. See (4.6) for more details.

stable if there exist $K \in \mathbb{R}^{n\times p}$ *and* $P \in \mathbb{S}^n$, $Q \in \mathbb{R}^{n\times dn}$, $R \in \mathbb{S}^{dn}$ *and* $S; U \in \mathbb{S}^n$ *such that the following conditions hold,*

$$\begin{bmatrix} P & Q \\ * & R + \mathsf{F} \otimes S \end{bmatrix} \succ 0, \quad S \succeq 0, \quad U \succeq 0, \tag{2.29}$$

$$\mathbf{\Phi} + \mathsf{Sy}\left(\mathbf{P}^\top \mathbf{\Pi}\right) \prec 0 \tag{2.30}$$

where $\mathbf{P} := \begin{bmatrix} P & \mathsf{O}_n & Q & \mathsf{O}_{n\times q} & \mathsf{O}_{n\times m} \end{bmatrix}$, $\mathbf{\Pi} := \begin{bmatrix} \mathbf{A} + \mathbf{B}_1 \left[(I_{2+d} \otimes K) \oplus \mathsf{O}_q\right] & \mathsf{O}_{n\times m} \end{bmatrix}$ *and*

$$\mathbf{\Phi} := \mathsf{Sy}\left(\begin{bmatrix} Q \\ \mathsf{O}_{n\times dn} \\ R \\ \mathsf{O}_{(q+m)\times dn} \end{bmatrix} \begin{bmatrix} \mathbf{F} & \mathsf{O}_{dn\times m} \end{bmatrix}\right) + \mathsf{Sy}\left(\begin{bmatrix} \mathsf{O}_{(2n+dn)\times m} \\ -J_2^\top \\ \widetilde{J} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma} & \mathsf{O}_m \end{bmatrix}\right)$$
$$+ [S + rU] \oplus [-S] \oplus [-\mathsf{F} \otimes U] \oplus (-J_3) \oplus J_1 \tag{2.31}$$

with $\mathsf{F}^{-1} := \int_{-r}^0 \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$ *and*

$$\mathbf{F} = \begin{bmatrix} F(0) & F(-r) & -M \otimes I_n & \mathsf{O}_{dn\times q} \end{bmatrix}, \quad \mathbf{\Sigma} = \mathbf{C} + \mathbf{B}_2 \left[(I_{2+d} \otimes K) \oplus \mathsf{O}_q\right] \tag{2.32}$$

with $F(\tau)$ *given in* (2.10). *Moreover, the matrices of* $\mathbf{A}$*,* $\mathbf{B}_1$*,* $\mathbf{B}_2$ *and* $\mathbf{C}$ *which contains the parameters of the open-loop system* (2.3) *are given in* (2.8) *and* (2.9).

Proof. To begin with, note that $s(\boldsymbol{z}(t), \boldsymbol{w}(t))$ in (2.15) can be written as

$$s(\boldsymbol{z}(t), \boldsymbol{w}(t)) = \boldsymbol{z}^\top(t)\widetilde{J}^\top J_1^{-1}\widetilde{J}\boldsymbol{z}(t) + \mathsf{Sy}\left[\boldsymbol{z}^\top(t)J_2\boldsymbol{w}(t)\right] + \boldsymbol{w}^\top(t)J_3\boldsymbol{w}(t). \tag{2.33}$$

where only the term $\boldsymbol{z}^\top(t)\widetilde{J}^\top J_1^{-1}\widetilde{J}\boldsymbol{z}(t)$ introduces nonlinearity with respect to the undetermined variables in (2.7) and (2.28). Substituting $\boldsymbol{z}(t)$ in equation (2.7) into $\boldsymbol{z}^\top(t)\widetilde{J}^\top J_1^{-1}\widetilde{J}\boldsymbol{z}(t)$ produces

$$\boldsymbol{z}^\top(t)\widetilde{J}^\top J_1^{-1}\widetilde{J}\boldsymbol{z}(t) = \boldsymbol{\chi}^\top(t)\mathbf{\Sigma}^\top\widetilde{J}^\top J_1^{-1}\widetilde{J}\mathbf{\Sigma}\boldsymbol{\chi}(t) \tag{2.34}$$

where $\mathbf{\Sigma}$ is given in (2.32) and $\boldsymbol{\chi}(t)$ has been defined in (2.10).

Now differentiate the functional $v(\mathbf{x}_t(\cdot))$ in (2.28) along the trajectory of the closed-loop system (2.7) and consider (2.4), (2.33), (2.34) and the relation

$$\frac{\mathsf{d}}{\mathsf{d}t}\int_{-r}^0 F(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau = F(0)\boldsymbol{x}(t) - F(-r)\boldsymbol{x}(t-r) - (M \otimes I_n)\int_{-r}^0 F(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau = \mathbf{F}\boldsymbol{\chi}(t) \tag{2.35}$$

with $\mathbf{F}$ in (2.32). Then it yields

$$\begin{aligned}
&\forall t \geq t_0, \quad \dot{v}(\mathbf{x}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \\
&= \boldsymbol{\chi}^\top(t)\,\mathsf{Sy}\left(\begin{bmatrix} I_n & \mathsf{O}_{n\times dn} \\ \mathsf{O}_n & \mathsf{O}_{n\times dn} \\ \mathsf{O}_{dn\times n} & I_{dn} \\ \mathsf{O}_{q\times n} & \mathsf{O}_{q\times dn} \end{bmatrix} \begin{bmatrix} P & Q \\ * & R \end{bmatrix} \begin{bmatrix} \mathbf{A} + \mathbf{B}_1 \left[(I_{2+d} \otimes K) \oplus \mathsf{O}_q\right] \\ \mathbf{F} \end{bmatrix}\right)\boldsymbol{\chi}(t) \\
&+ \boldsymbol{x}^\top(t)\left(S + rU\right)\boldsymbol{x}(t) - \boldsymbol{x}^\top(t-r)S\boldsymbol{x}(t-r) - \int_{-r}^0 \boldsymbol{x}^\top(t+\tau)U\boldsymbol{x}(t+\tau)\mathsf{d}\tau - \boldsymbol{w}^\top(t)J_3\boldsymbol{w}(t) \\
&- \boldsymbol{\chi}^\top(t)\,\mathsf{Sy}\left(\mathbf{\Sigma}^\top\widetilde{J}^\top J_1^{-1}\widetilde{J}\mathbf{\Sigma} + \begin{bmatrix} \mathsf{O}_{(2n+dn)\times m} \\ J_2^\top \end{bmatrix}\mathbf{\Sigma}\right)\boldsymbol{\chi}(t).
\end{aligned} \tag{2.36}$$

Assume $U \succeq 0$, then applying (2.17) to the integral $\int_{-r}^0 \boldsymbol{x}^\top(t+\tau)U\boldsymbol{x}(t+\tau)\mathsf{d}\tau$ in (2.36) with $\boldsymbol{g}(\tau) = \boldsymbol{f}(\tau)$ produces

$$\forall t \geq t_0, \quad \int_{-r}^0 \boldsymbol{x}^\top(t+\tau)U\boldsymbol{x}(t+\tau)\mathsf{d}\tau \geq \int_{-r}^0 \boldsymbol{x}^\top(t+\tau)F^\top(\tau)\mathsf{d}\tau\,(\mathsf{F} \otimes U)\int_{-r}^0 F(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau, \tag{2.37}$$

where $\mathsf{F}^{-1} = \int_{-r}^{0} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$. Now apply (2.37) to (2.36), then we have

$$\forall t \geq t_0, \quad \dot{v}(\mathbf{x}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \leq \boldsymbol{\chi}^\top(t) \left(\boldsymbol{\Psi} - \boldsymbol{\Sigma}^\top \widetilde{J}^\top J_1^{-1} \widetilde{J} \boldsymbol{\Sigma}\right) \boldsymbol{\chi}(t) \tag{2.38}$$

with $\boldsymbol{\chi}(t)$ defined in (2.10), where

$$\begin{aligned}\boldsymbol{\Psi} := \mathbf{Sy}&\left(\begin{bmatrix} P & Q \\ \mathsf{O}_n & \mathsf{O}_{n\times nd} \\ Q^\top & R \\ \mathsf{O}_{q\times n} & \mathsf{O}_{q\times nd}\end{bmatrix} \begin{bmatrix} \mathbf{A} + \mathbf{B}_1 \left[(I_{2+d} \otimes K) \oplus \mathsf{O}_q\right] \\ \mathbf{F} \end{bmatrix}\right) \\ &- \left([-S - rU] \oplus S \oplus (\mathsf{F} \otimes U) \oplus J_3\right) - \mathbf{Sy}\left(\begin{bmatrix} \mathsf{O}_{(2n+dn)\times m} \\ J_2^\top \end{bmatrix} (\mathbf{C} + \mathbf{B}_2 \left[(I_{2+d} \otimes K) \oplus \mathsf{O}_q\right])\right)\end{aligned} \tag{2.39}$$

containing all the matrix terms induced by $\dot{v}(\mathbf{x}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t))$ in (2.36) excluding (2.34). Now based on the property of positive definite matrices, it is easy to see that if

$$\boldsymbol{\Psi} - \boldsymbol{\Sigma}^\top \widetilde{J}^\top J_1^{-1} \widetilde{J} \boldsymbol{\Sigma} \prec 0, \;\; U \succeq 0 \tag{2.40}$$

is satisfied then the dissipative inequality in (2.13) : $\forall t \geq t_0$, $\dot{v}(\mathbf{x}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \leq 0$ holds. Moreover, utilizing the Schur complement to the first inequality in (2.40) concludes that

$$\begin{bmatrix} \boldsymbol{\Psi} & \boldsymbol{\Sigma}^\top \widetilde{J}^\top \\ * & J_1 \end{bmatrix} = \mathbf{P}^\top \boldsymbol{\Pi} + \boldsymbol{\Phi} \prec 0, \;\; U \succeq 0, \tag{2.41}$$

holds if and only if (2.40) holds given $J_1^{-1} \prec 0$ in (2.15), where $\boldsymbol{\Phi}$ is defined in (2.31). As a result, it follows that the existence of the feasible solutions of (2.41) implies the existence of (2.28) satisfying (2.13). By considering the properties of positive definite matrices, it is obvious that given (2.41) holds then $\exists \epsilon_3 > 0$, $\forall t \geq t_0$ $\dot{v}(\mathbf{x}_t(\cdot)) \leq -\epsilon_3 \|\boldsymbol{x}(t)\|_2^2 = -\epsilon_3 \|\mathbf{x}_t(0)\|_2^2$ where $\boldsymbol{x}(t)$ here satisfies (2.7) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$. Let $t = t_0$ with the fact[2] that $\forall \theta \in [-r, 0]$, $\mathbf{x}_{t_0}(\theta) = \boldsymbol{x}(t_0 + \theta) = \boldsymbol{\phi}(\theta)$ in (2.7), then we have $\exists \epsilon_3 > 0$, $\dot{v}(\mathbf{x}_{t_0}(\cdot)) = \dot{v}(\boldsymbol{\phi}(\cdot)) \leq -\epsilon_3 \|\mathbf{x}_{t_0}(0)\|_2^2 = -\epsilon_3 \|\boldsymbol{\phi}(0)\|_2^2$ for any $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r, 0] ⨾ \mathbb{R}^n)$ in (2.7), which gives (2.12). Thus we can prove that if (2.41) holds then (2.28) satisfies (2.13) and (2.12).

Now we start to prove that (2.28) satisfies (2.11) if (2.29) holds. Given the fact that $\forall \theta \in [-r, 0]$, $\boldsymbol{x}(t_0 + \tau) = \boldsymbol{\phi}(\theta)$ in (2.7), let $t = t_0$ in (2.28) and $S \succeq 0$, and then applying (2.17) with $\boldsymbol{g}(\tau) = \boldsymbol{f}(\tau)$ to the integral $\int_{-r}^{0} \boldsymbol{x}^\top(t+\tau) S \boldsymbol{x}(t+\tau)\mathsf{d}\tau$ in (2.28) with $t = t_0$ yields that the inequality

$$\int_{-r}^{0} \boldsymbol{\phi}^\top(\tau) S \boldsymbol{\phi}(\tau)\mathsf{d}\tau \geq \left[\int_{-r}^{0} F(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau\right]^\top (\mathsf{F} \otimes S) \int_{-r}^{0} F(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \tag{2.42}$$

holds for any $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r, 0] ⨾ \mathbb{R}^n)$ in (2.7). Apply (2.42) to (2.28) with $t = t_0$ and $S \preceq 0$, we have

$$\begin{aligned} v(\mathbf{x}_{t_0}(\cdot)) = v(\boldsymbol{\phi}(\cdot)) \geq \begin{bmatrix} \boldsymbol{\phi}(0) \\ \int_{-r}^{0} F(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \end{bmatrix}^\top \begin{bmatrix} P & Q \\ * & R + \mathsf{F} \otimes S \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}(0) \\ \int_{-r}^{0} F(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \end{bmatrix} \\ + \int_{-r}^{0} (\tau + r)\boldsymbol{\phi}^\top(\tau) U \boldsymbol{\phi}(\tau)\mathsf{d}\tau \end{aligned} \tag{2.43}$$

for any $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r, 0] ⨾ \mathbb{R}^n)$ in (2.7). According to the property of positive definite matrices considering the structure of (2.43), it is obvious to conclude that if (2.29) are satisfied, then $\exists \epsilon_1 > 0 : v(\boldsymbol{\phi}(\cdot)) \geq \epsilon_1 \|\boldsymbol{\phi}(0)\|_2$ for any $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r, 0] ⨾ \mathbb{R}^n)$ in (2.7). Furthermore, for (2.28) with $t = t_0$ it follows that $\exists \lambda_1, \lambda_2 > 0$ such

[2] Again $\boldsymbol{x}(t)$ here satisfies (2.7) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$

that

$$
\begin{aligned}
v(\boldsymbol{\phi}(\cdot)) &\leq \begin{bmatrix} \boldsymbol{\phi}(0) \\ \int_{-r}^{0} F(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \end{bmatrix}^\top \lambda_1 \begin{bmatrix} \boldsymbol{\phi}(0) \\ \int_{-r}^{0} F(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \end{bmatrix} + \lambda_1 \int_{-r}^{0} \sup_{-r\leq\tau\leq 0} \|\boldsymbol{\phi}(\tau)\|_2^2 \mathsf{d}\tau \\
&\leq \boldsymbol{\phi}^\top(0)\lambda_1\boldsymbol{\phi}(0) + r\lambda_1 \|\boldsymbol{\phi}(\cdot)\|_\infty^2 + \int_{-r}^{0} \boldsymbol{\phi}^\top(\tau)F^\top(\tau)\left(\lambda_2 \mathsf{F}\otimes I_n\right)\int_{-r}^{0} F(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \\
&\leq \boldsymbol{\phi}^\top(0)\lambda_1\boldsymbol{\phi}(0) + \int_{-r}^{0} \boldsymbol{\phi}^\top(\tau)\lambda_2\boldsymbol{\phi}(\tau)\mathsf{d}\tau + r\|\boldsymbol{\phi}(\cdot)\|_\infty^2 \leq (\lambda_1 + r\lambda_2 + r\lambda_1)\|\boldsymbol{\phi}(\cdot)\|_\infty^2
\end{aligned}
\tag{2.44}
$$

for any $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r,0] ⨾ \mathbb{R}^n)$ in (2.7), which is derived via (2.17) with $\boldsymbol{g}(\tau) = \boldsymbol{f}(\tau)$ and the property $\forall X \in \mathbb{S}^n, \exists\lambda > 0, \lambda I_n - X \succ 0$. This shows that (2.11) can be satisfied for any functional in the form of (2.28). As a result, we have shown that if (2.29) holds then the functional in (2.28) satisfies (2.11).

Since we have proved that (2.28) satisfies (2.13) and (2.12) provided that (2.41) holds, thus one can conclude that the feasible solutions of (2.29) and (2.30) infer the existence of the corresponding (2.28) satisfying the dissipative inequality in (2.13) and the stability criteria in (2.11) and (2.12). This completes the proof. ■

Since (2.30) is non-convex B_1,B_2,B_3 are of non-zero values, thus a genuine stabilization problem may not be solved via standard semidefinite programming solvers. As a result, we specifically derive the following theorem based on the application of Projection Lemma, where the conditions for dissipative stabilization with certain given parameters.[3] are convex which can be solved via standard semidefinite programming solvers.

Theorem 2.2. *Let $\boldsymbol{f}(\cdot)$ and M in* (2.1) *and $\{\alpha_i\}_{i=1}^{2+d} \subset \mathbb{R}$, $d \in \mathbb{N}$ be given, then the closed-loop system* (2.7) *with the supply rate function in* (2.15) *is dissipative and the trivial solution $\boldsymbol{x}(t) \equiv \mathbf{0}_n$ of* (2.7) *with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is globally asymptotically stable if there exist $\acute{P} \in \mathbb{S}^n$, $X \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{p\times n}$, $\acute{Q} \in \mathbb{R}^{n\times dn}$, $\acute{R} \in \mathbb{S}^{dn}$, $\acute{S};\acute{U} \in \mathbb{S}^n$ such that*

$$
\begin{bmatrix} \acute{P} & \acute{Q} \\ * & \acute{R} + \mathsf{F}\otimes\acute{S} \end{bmatrix} \succ 0, \quad \acute{S} \succ 0, \quad \acute{U} \succ 0, \tag{2.45}
$$

$$
\acute{\boldsymbol{\Theta}} := \mathbf{Sy}\left(\begin{bmatrix} I_n \\ \mathbf{Col}_{i=1}^{2+d}\alpha_i I_n \\ \mathsf{O}_{(q+m)\times n} \end{bmatrix}\begin{bmatrix} -X & \acute{\boldsymbol{\Pi}} \end{bmatrix}\right) + \begin{bmatrix} \mathsf{O}_n & \acute{\mathbf{P}} \\ * & \acute{\boldsymbol{\Phi}} \end{bmatrix} \prec 0, \tag{2.46}
$$

hold, where $\mathsf{F}^{-1} := \int_{-r}^{0} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$ and

$$
\begin{aligned}
&\acute{\mathbf{P}} := \begin{bmatrix} \mathsf{O}_n & \acute{P} & \mathsf{O}_n & \acute{Q} & \mathsf{O}_{n\times q} & \mathsf{O}_{n\times m} \end{bmatrix}, \quad \acute{\boldsymbol{\Pi}} := \begin{bmatrix} \mathbf{A}(I_{2+d}\otimes X) + \mathbf{B}_1(I_{2+d}\otimes V) & \mathsf{O}_{n\times m} \end{bmatrix}, \\
&\acute{\boldsymbol{\Phi}} := \mathbf{Sy}\left(\begin{bmatrix} \acute{Q} \\ \mathsf{O}_{n\times dn} \\ \acute{R} \\ \mathsf{O}_{q\times dn} \end{bmatrix}\begin{bmatrix} \mathbf{F} & \mathsf{O}_{dn\times m} \end{bmatrix}\right) + \left(\acute{S} + r\acute{U}\right) \oplus \left(-\acute{S}\right) \oplus \left(-\mathsf{F}\otimes\acute{U}\right) \oplus (-J_3) \oplus J_1 \\
&\quad + \mathbf{Sy}\left(\begin{bmatrix} \mathsf{O}_{(2n+dn)\times m} \\ -J_2^\top \\ \widetilde{J} \end{bmatrix}\begin{bmatrix} \acute{\boldsymbol{\Sigma}} & \mathsf{O}_m \end{bmatrix}\right).
\end{aligned}
\tag{2.47}
$$

with $\acute{\boldsymbol{\Sigma}} = \begin{bmatrix} C_1X + B_4V & C_2X + B_5V & C_3(I_d\otimes X) + B_2(I_d\otimes V) & D_2 \end{bmatrix}$ and $\mathbf{F}$ in (2.32). *Furthermore the controller parameter is calculated via $K = VX^{-1}$ where the invertibility of X is ensured by* (2.46).

[3] It is illustrated later in Remark 2.4 that it possible to only adjust the value of one parameter with other parameters assigned to be zeros

Proof. It is easy to observe that the source of non-convexity in (2.30) is the matrix products in $\mathsf{Sy}\left(\mathbf{P}^\top\mathbf{\Pi}\right)$ concerning P and Q. Now reformulate (2.30) into the form

$$\begin{bmatrix}\mathbf{\Pi}\\ I_{2n+dn+q+m}\end{bmatrix}^\top\begin{bmatrix}\mathsf{O}_n & \mathbf{P}\\ * & \mathbf{\Phi}\end{bmatrix}\begin{bmatrix}\mathbf{\Pi}\\ I_{2n+dn+q+m}\end{bmatrix}\prec 0. \tag{2.48}$$

It is clear that the structure of (2.48) is in line with the structure of the inequalities in (2.27) as part of the statements in Lemma 2.4. However, since there are two matrix inequalities in (2.27), a new matrix inequality must be constructed accordingly. Consider the matrix inequality

$$\Upsilon^\top\begin{bmatrix}\mathsf{O}_n & \mathbf{P}\\ * & \mathbf{\Phi}\end{bmatrix}\Upsilon\prec 0 \tag{2.49}$$

with $\Upsilon^\top := \begin{bmatrix}\mathsf{O}_{(q+m)\times(3n+dn)} & I_{q+m}\end{bmatrix}$, which can be further simplified into

$$\Upsilon^\top\begin{bmatrix}\mathsf{O}_n & \mathbf{P}\\ * & \mathbf{\Phi}\end{bmatrix}\Upsilon=\begin{bmatrix}-J_3-\mathsf{Sy}(D_2^\top J_2) & D_2^\top\widetilde{J}\\ * & J_1\end{bmatrix}\prec 0. \tag{2.50}$$

As a matter of fact, (2.50) is identical to the very matrix resulted from extracting the 2×2 block matrix at the right bottom of (2.48). Consequently, one can conclude that (2.50) is automatically satisfied if condition (2.48) holds, thus it introduces no additional conservatism to the original condition.

To utilize Lemma 2.4, two matrices $\mathbf{U}$, $\mathbf{Y}$ need to be constructed based on the inequalities in (2.27) satisfying

$$\mathbf{U}_\perp=\Upsilon,\quad \mathbf{Y}_\perp=\begin{bmatrix}\mathbf{\Pi}\\ I_{2n+dn+q+m}\end{bmatrix} \tag{2.51}$$

where Υ and $\begin{bmatrix}\mathbf{\Pi}^\top & I_{2n+dn+q+m}\end{bmatrix}^\top$ contain bases of the null spaces of $\mathbf{U}$ and $\mathbf{Y}$, respectively. For $\Upsilon^\top :=$ $\begin{bmatrix}\mathsf{O}_{(q+m)\times(3n+dn)} & I_{q+m}\end{bmatrix}$, we have $\operatorname{rank}(\Upsilon)=q+m$ which gives $\operatorname{rank}(\mathbf{U})=3n+dn$ by the rank nullity theorem. Similarly, we can conclude that $\operatorname{rank}(\mathbf{Y})=n$. Without losing generality, left

$$\mathbf{Y}:=\begin{bmatrix}-I_n & \mathbf{\Pi}\end{bmatrix},\ \mathbf{U}:=\begin{bmatrix}I_{3n+dn} & \mathsf{O}_{(3n+dn)\times(q+m)}\end{bmatrix} \tag{2.52}$$

by which we have

$$\mathbf{Y}\mathbf{Y}_\perp=\mathbf{Y}\begin{bmatrix}\mathbf{\Pi}\\ I_{2n+dn+q+m}\end{bmatrix}=\mathsf{O}_{n\times(2n+dn+q+m)},\quad \mathbf{U}\mathbf{U}_\perp=\mathsf{O}_{(3n+dn)\times(q+m)}.$$

Now the choice of $\mathbf{U}$ and $\mathbf{Y}$ in (2.52) satisfies $\operatorname{rank}(\mathbf{U})=3n+dn$, $\operatorname{rank}(\mathbf{Y})=n$, and Υ, $\begin{bmatrix}\mathbf{\Pi}^\top & I_{2n+dn+q+m}\end{bmatrix}^\top$ contain bases of the null spaces of $\mathbf{U}$ and $\mathbf{Y}$, respectively. Now one can apply Lemma 2.4 to (2.48) and (2.50) yields that (2.48) and (2.50) hold if and only if

$$\exists\mathbf{W}\in\mathbb{R}^{(3n+dn)\times n}:\mathsf{Sy}\begin{bmatrix}\mathbf{U}^\top\mathbf{W}\mathbf{Y}\end{bmatrix}+\begin{bmatrix}\mathsf{O}_n & \mathbf{P}\\ * & \mathbf{\Phi}\end{bmatrix}\prec 0. \tag{2.53}$$

with $\mathbf{U}$ and $\mathbf{Y}$ defined in (2.52). To ultimately produce convex synthesis conditions, let $\{\alpha_i\}_{i=1}^{d+2}\subset\mathbb{R}$ be given and

$$\mathbf{W}:=\begin{bmatrix}W^\top & \mathsf{Col}_{i=1}^{d+2}\,\alpha_i W\end{bmatrix}^\top \tag{2.54}$$

with $W\in\mathbb{R}^{n\times n}$. Substituting both (2.54) and (2.52) into (2.53) produces

$$\mathbf{\Theta}=\mathsf{Sy}\left(\mathbf{U}^\top\begin{bmatrix}W\\ \mathsf{Col}_{i=1}^{d+2}\,\alpha_i W\end{bmatrix}\mathbf{Y}\right)+\begin{bmatrix}\mathsf{O}_n & \mathbf{P}\\ * & \mathbf{\Phi}\end{bmatrix}\prec 0. \tag{2.55}$$

where not (2.55) can be convexified via congruence transformations.

Note that because of the structural constraints in (2.54), the inequality in (2.55) is no longer an equivalent but only sufficient condition implying (2.48). It is important to stress here that the invertibility of W is implied by (2.55) since the expression $-W - W^\top$ is the only element at the first diagonal block of (2.55). Now applying congruence transformations to (2.55) and (2.29) with the fact that W^{-1} is well defined, we have

$$\begin{gathered}((I_{3+d} \otimes X) \oplus I_{q+m})^\top \boldsymbol{\Theta} ((I_{3+d} \otimes X) \oplus I_{q+m}) \prec 0, \\ \left(I_{d+1} \otimes X^\top\right) \begin{bmatrix} P & Q \\ * & R + \mathsf{F} \otimes S \end{bmatrix} (I_{d+1} \otimes X) \succ 0, \quad X^\top S X \succ 0, \quad X^\top U X \succ 0\end{gathered} \tag{2.56}$$

holds if and only if (2.55) and (2.29) hold, where $X^\top := W^{-1}$. Moreover, letting

$$\begin{bmatrix} \acute{P} & \acute{Q} \\ * & \acute{R} \end{bmatrix} := \left(I_{1+d} \otimes X^\top\right) \begin{bmatrix} P & Q \\ * & R \end{bmatrix} (I_{1+d} \otimes X), \quad \begin{bmatrix} \acute{S} & \acute{U} \end{bmatrix} := X^\top \begin{bmatrix} SX & UX \end{bmatrix} \tag{2.57}$$

and considering (2.56) with (2.1) yields (2.45) and

$$[*]\boldsymbol{\Theta} ((I_{3+d} \otimes X) \oplus I_{q+m}) = \acute{\boldsymbol{\Theta}} = \mathbf{Sy}\left(\begin{bmatrix} I_n \\ \mathbf{Col}_{i=1}^{2+d} \alpha_i I_n \\ \mathsf{O}_{(q+m)\times n} \end{bmatrix} \begin{bmatrix} -X & \acute{\boldsymbol{\Pi}} \end{bmatrix} \right) + \begin{bmatrix} \mathsf{O}_n & \acute{\mathbf{P}} \\ * & \acute{\boldsymbol{\Phi}} \end{bmatrix} \prec 0 \tag{2.58}$$

where $\acute{\mathbf{P}} = X\mathbf{P}\left[(I_{3+d} \otimes X) \oplus I_{q+m}\right] = \begin{bmatrix} \mathsf{O}_n & \acute{P} & \mathsf{O}_n & \acute{Q} & \mathsf{O}_{n\times(q+m)} \end{bmatrix}$ and

$$\begin{aligned}\acute{\boldsymbol{\Pi}} = \boldsymbol{\Pi}\left[(I_{3+d} \otimes X) \oplus I_{q+m}\right] &= \begin{bmatrix} \mathbf{A}(I_{3+d} \otimes X) + \mathbf{B}_1 (I_{3+d} \otimes KX) & \mathsf{O}_{n\times m} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}(I_{3+d} \otimes X) + \mathbf{B}_1 (I_{3+d} \otimes V) & \mathsf{O}_{n\times m} \end{bmatrix}\end{aligned} \tag{2.59}$$

with $V = KX$, and

$$\begin{aligned}\acute{\boldsymbol{\Phi}} &= \left[\left(I_{2+d} \otimes X^\top\right) \oplus I_{q+m}\right] \boldsymbol{\Phi} \left[(I_{2+d} \otimes X) \oplus I_{q+m}\right] = \\ &\mathbf{Sy}\left(\begin{bmatrix} \acute{Q} \\ \mathsf{O}_{n\times dn} \\ \acute{R} \\ \mathsf{O}_{(q+m)\times dn} \end{bmatrix} \begin{bmatrix} \mathbf{F} & \mathsf{O}_{dn\times m} \end{bmatrix} \right) + \left(\begin{bmatrix} \acute{S} + r\acute{U} \end{bmatrix} \oplus \begin{bmatrix} -\acute{S} \end{bmatrix} \oplus \begin{bmatrix} -\mathsf{F} \otimes \acute{U} \end{bmatrix} \oplus J_3 \oplus (-J_1) \right) \\ &\qquad + \mathbf{Sy}\left(\begin{bmatrix} \mathsf{O}_{(2n+dn)\times m} \\ -J_2^\top \\ \widetilde{J} \end{bmatrix} \begin{bmatrix} \acute{\boldsymbol{\Sigma}} & \mathsf{O}_{n\times m} \end{bmatrix} \right)\end{aligned} \tag{2.60}$$

with $\acute{\boldsymbol{\Sigma}} = \begin{bmatrix} C_1 X + B_4 V & C_2 X + B_5 V & C_3 (I_d \otimes X) + B_2 (I_d \otimes V) & D_2 \end{bmatrix}$, are the same given in (2.47). Note that (2.58) is the same as (2.46), and the form of $\acute{\boldsymbol{\Phi}}$ in (2.60) is derived considering the relation

$$\begin{aligned}\mathbf{F}\left[(I_{2+d} \otimes X) \oplus I_{q+m}\right] = \begin{bmatrix} \widehat{\mathbf{F}} \otimes I_n & \mathsf{O}_{dn\times q} \end{bmatrix} \left[(I_{2+d} \otimes X) \oplus I_{q+m}\right] &= \begin{bmatrix} I_d \widehat{\mathbf{F}} \otimes X I_n & \mathsf{O}_{dn\times q} \end{bmatrix} \\ &= (I_d \otimes X) \begin{bmatrix} \widehat{\mathbf{F}} \otimes I_n & \mathsf{O}_{dn\times q} \end{bmatrix}.\end{aligned} \tag{2.61}$$

Moreover, because the expression $-X - X^\top$ is the only term at the first diagonal block of $\acute{\boldsymbol{\Theta}}$ in (2.46), thus X is invertible if (2.46) holds, which is in line with the fact that a full rank W is inferred by (2.55).

Consequently, the equivalence between (2.29) and (2.45) has been shown. Furthermore, since (2.55) is equivalent to (2.46) which infers (2.30), hence one can conclude that there exist matrices such that (2.29) and (2.30) are satisfied if there exist feasible solutions of (2.45) and (2.46). Because feasible solutions of (2.29) and (2.30) infer the existence of an LKF (2.28) satisfying (2.13) and (2.11), (2.12), thus it shows that feasible solutions of (2.45) and (2.46) infers the existence of (2.28) satisfying the corresponding stability and dissipativity criteria. This finishes the proof. ■

Remark 2.4. Considering the structure of (2.46), some values of $\{\alpha_i\}_{i=1}^{2+d} \subset \mathbb{R}$ may be more crucial than others in terms of their influence on the feasibility of (2.46). For instance, α_1 might be the most crucial one since it affects the feasibility of the diagonal related to A_1 in (2.46). A simple assignment of the values for $\{\alpha_i\}_{i=1}^{2+d} \subset \mathbb{R}$ can be $\alpha_1 \in \mathbb{R}$ and $\alpha_i = 0, i = 2 \cdots 2 + d$ which allows one to only adjust the value of α_1 to use Theorem 2.2.

Remark 2.5. Even without considering dissipativity constraints, it is still possible to introduce slack variables as in (2.46) to solve a synthesis problem. However, in such situation, Projection Lemma may not be applied since it may not be able to construct two matrix inequalities as in (2.50). Instead, a particular version of Projection Lemma, called Finsler Lemma [264], which only demands one inequality similar to the structure of the inequalities in (2.27), can be applied in such situation. By using the notation of empty matrices, the corresponding synthesis condition derived via the application of Finsler Lemma can be obtained by setting $m = q = 0$ in (2.46).

Remark 2.6. It is important to stress that one can simply apply Finsler lemma at the step of (2.48) so that a similar condition with more extra variables than (2.53) can be obtained. This indicates the fact that there is advantage to apply Projection Lemma over Finsler Lemma if (2.50) can be constructed without introducing extra conservatism.

2.4.1 An inner convex approximation algorithm for Theorem 2.1

By prescribing the values of $\{\alpha_i\}_{i=1}^{2+d} \subset \mathbb{R}$, a dissipative stabilizing K can be obtained by solving the constraints in Theorem 2.2 via standard solvers for semidefinite programming. However, the structure we have utilized in (2.54) can introduce potential conservatism compared to the synthesis condition in Theorem 2.1. In this subsection, an iterative algorithm is presented based on the algorithm proposed in [257] to solve Theorem 2.1. The resulting algorithm avoids the introduction of slack variables as in Theorem 2.2 and its initial values can be supported by the feasible solutions of Theorem 2.2.

First of all, the nonconvex inequality (2.30) can be rewritten into

$$\mathfrak{D}(\mathbf{\Lambda}, K) = \mathsf{Sy}\left[\mathbf{P}^\top \mathbf{\Pi}\right] + \mathbf{\Phi} = \mathsf{Sy}\left[\mathbf{P}^\top \mathbf{B}\left[(I_{2+d} \otimes K) \oplus \mathsf{O}_{p+m}\right]\right] + \widehat{\mathbf{\Phi}} \prec 0 \tag{2.62}$$

given the structure of $\mathbf{\Pi}$ in (2.30), where $\mathbf{B} = \begin{bmatrix}\mathbf{B}_1 & \mathsf{O}_{n\times m}\end{bmatrix}$ and $\widehat{\mathbf{\Phi}} = \mathsf{Sy}\left(\mathbf{P}^\top \begin{bmatrix}\mathbf{A} & \mathsf{O}_{n\times m}\end{bmatrix}\right) + \mathbf{\Phi}$ and $\mathbf{\Lambda} = \begin{bmatrix}P & Q\end{bmatrix}$ with P and Q in Theorem 2.1. Now consider the expression

$$\Delta\left(\mathbf{\Omega}, \widetilde{\mathbf{\Omega}}, \mathbf{\Gamma}, \widetilde{\mathbf{\Gamma}}\right) := \begin{bmatrix}\mathbf{\Omega}^\top - \widetilde{\mathbf{\Omega}}^\top & \mathbf{\Gamma}^\top - \widetilde{\mathbf{\Gamma}}^\top\end{bmatrix}\left[Z \oplus (I_n - Z)\right]^{-1}\begin{bmatrix}\mathbf{\Omega} - \widetilde{\mathbf{\Omega}} \\ \mathbf{\Gamma} - \widetilde{\mathbf{\Gamma}}\end{bmatrix} + \mathsf{Sy}\left(\widetilde{\mathbf{\Omega}}^\top \mathbf{\Gamma} + \mathbf{\Omega}^\top \widetilde{\mathbf{\Gamma}} - \widetilde{\mathbf{\Omega}}^\top \widetilde{\mathbf{\Gamma}}\right) + \mathbf{T} \tag{2.63}$$

where $\mathbf{\Omega}; \widetilde{\mathbf{\Omega}} \in \mathbb{R}^{n\times\mu}$, $\mathbf{\Gamma}; \widetilde{\mathbf{\Gamma}} \in \mathbb{R}^{n\times\mu}$ and $\mathbf{T} \in \mathbb{S}^\mu$, $Z \in \{X \in \mathbb{S}^n : X \oplus (I_n - X) \succ 0\}$. By Example 3 in [257], it is obvious that (2.63) satisfies

$$\mathbf{T} + \mathsf{Sy}\left(\mathbf{\Omega}^\top \mathbf{\Gamma}\right) \preceq \Delta\left(\mathbf{\Omega}, \widetilde{\mathbf{\Omega}}, \mathbf{\Gamma}, \widetilde{\mathbf{\Gamma}}\right), \quad \mathbf{T} + \mathsf{Sy}\left(\mathbf{\Omega}^\top \mathbf{\Gamma}\right) = \Delta(\mathbf{\Omega}, \mathbf{\Omega}, \mathbf{\Gamma}, \mathbf{\Gamma}) \tag{2.64}$$

for all $\mathbf{\Omega}; \widetilde{\mathbf{\Omega}} \in \mathbb{R}^{n\times\mu}$ and for all $\mathbf{\Gamma}; \widetilde{\mathbf{\Gamma}} \in \mathbb{R}^{n\times\mu}$ with $\widetilde{\mathbf{T}} \in \mathbb{S}^\mu$ and $Z \oplus (I_n - Z) \succ 0$, which indicates that $\Delta\left(\bullet, \widetilde{\mathbf{\Omega}}, \bullet, \widetilde{\mathbf{\Gamma}}\right)$ in (2.63) is a psd-overestimate of $\acute{\Delta}(\mathbf{\Omega}, \mathbf{\Gamma}) = \mathbf{T} + \mathsf{Sy}\left[\mathbf{\Omega}^\top \mathbf{\Gamma}\right]$ with respect to the parameterization

$$\begin{bmatrix}\mathbf{vec}(\widetilde{\mathbf{\Omega}}) \\ \mathbf{vec}(\widetilde{\mathbf{\Gamma}})\end{bmatrix} = \begin{bmatrix}\mathbf{vec}(\mathbf{\Omega}) \\ \mathbf{vec}(\mathbf{\Gamma})\end{bmatrix}. \tag{2.65}$$

Now let $\mu = 2n + dn + q + m$ and $Z \oplus (I_n - Z) \succ 0$ and

$$\begin{aligned}
&\mathbf{T} = \widehat{\mathbf{\Phi}},\ \ \mathbf{\Omega} = \mathbf{P} = \begin{bmatrix} P & \mathsf{O}_n & Q & \mathsf{O}_{n\times q} & \mathsf{O}_{n\times m}\end{bmatrix},\ \ \mathbf{\Lambda} = \begin{bmatrix} P & Q \end{bmatrix} \\
&\widetilde{\mathbf{\Omega}} = \widetilde{\mathbf{P}} = \begin{bmatrix} \widetilde{P} & \mathsf{O}_n & \widetilde{Q} & \mathsf{O}_{n\times q} & \mathsf{O}_{n\times m}\end{bmatrix},\ \widetilde{\mathbf{\Lambda}} = \begin{bmatrix} \widetilde{P} & \widetilde{Q} \end{bmatrix},\ \widetilde{P} \in \mathbb{S}^n,\ \widetilde{Q} \in \mathbb{R}^{n\times dn} \\
&\mathbf{\Gamma} = \mathbf{B}\mathbf{K} = \mathbf{B}\left[(I_{2+d} \otimes K) \oplus \mathsf{O}_{p+m}\right],\ \ \widetilde{\mathbf{\Gamma}} = \mathbf{B}\widetilde{\mathbf{K}} = \mathbf{B}\left[\left(I_{2+d} \otimes \widetilde{K}\right) \oplus \mathsf{O}_{p+m}\right],\ \ \widetilde{K} \in \mathbb{R}^{p\times n}
\end{aligned} \tag{2.66}$$

with $\widehat{\mathbf{\Phi}}$, $\mathbf{B}$, $\mathbf{\Lambda}$ and K in (2.62). By (2.64) with (2.66), we have

$$\begin{aligned}
\mathfrak{D}(\mathbf{\Lambda}, K) = \widehat{\mathbf{\Phi}} + \mathsf{Sy}\left[\mathbf{P}^\top \mathbf{B}\mathbf{K}\right] \preceq \mathfrak{J}\left(\mathbf{\Lambda}, \widetilde{\mathbf{\Lambda}}, K, \widetilde{K}\right) &:= \widehat{\mathbf{\Phi}} + \mathsf{Sy}\left(\widetilde{\mathbf{P}}^\top \mathbf{B}\mathbf{K} + \mathbf{P}^\top \mathbf{B}\widetilde{\mathbf{K}} - \widetilde{\mathbf{P}}^\top \mathbf{B}\widetilde{\mathbf{K}}\right) \\
&+ \begin{bmatrix} \mathbf{P}^\top - \widetilde{\mathbf{P}}^\top & \mathbf{K}^\top \mathbf{B}^\top - \widetilde{\mathbf{K}}^\top \mathbf{B}^\top \end{bmatrix} \left[Z \oplus (I_n - Z)\right]^{-1} \begin{bmatrix} \mathbf{P} - \widetilde{\mathbf{P}} \\ \mathbf{B}\mathbf{K} - \mathbf{B}\widetilde{\mathbf{K}} \end{bmatrix}
\end{aligned} \tag{2.67}$$

where $\mathfrak{J}\left(\bullet, \widetilde{\mathbf{\Lambda}}, \bullet, \widetilde{K}\right)$ is a psd-convex overestimate of $\mathfrak{D}(\mathbf{\Lambda}, K)$ in (2.62) with respect to the parameterization

$$\begin{bmatrix} \mathbf{vec}(\widetilde{\mathbf{\Lambda}}) \\ \mathbf{vec}(\widetilde{K}) \end{bmatrix} = \begin{bmatrix} \mathbf{vec}(\mathbf{\Lambda}) \\ \mathbf{vec}(K) \end{bmatrix} \tag{2.68}$$

Now it is obvious that $\mathfrak{J}\left(\mathbf{\Lambda}, \widetilde{\mathbf{\Lambda}}, K, \widetilde{K}\right) \prec 0$ infers $\mathfrak{D}(\mathbf{\Lambda}, K)$ in (2.62). Moreover, applying the Schur complement to the inequality $\mathfrak{J}\left(\mathbf{\Lambda}, \widetilde{\mathbf{\Lambda}}, K, \widetilde{K}\right) \prec 0$ concludes that $\mathfrak{J}\left(\mathbf{\Lambda}, \widetilde{\mathbf{\Lambda}}, K, \widetilde{K}\right) \prec 0$ with $Z \in \{X \in \mathbb{S}^n : X \oplus (I_n - X) \succ 0\}$ if and only if

$$\begin{bmatrix} \widehat{\mathbf{\Phi}} + \mathsf{Sy}\left(\widetilde{\mathbf{P}}^\top \mathbf{B}\mathbf{K} + \mathbf{P}^\top \mathbf{B}\widetilde{\mathbf{K}} - \widetilde{\mathbf{P}}^\top \mathbf{B}\widetilde{\mathbf{K}}\right) & \mathbf{P}^\top - \widetilde{\mathbf{P}}^\top & \mathbf{K}^\top \mathbf{B}^\top - \widetilde{\mathbf{K}}^\top \mathbf{B}^\top \\ * & -Z & \mathsf{O}_n \\ * & * & Z - I_n \end{bmatrix} \prec 0 \tag{2.69}$$

which now can be handled by standard interior algorithms of semidefinite programmings provided that the values of $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{K}}$ are given. To apply the methods in [257], one has to determine an initial value for $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{K}}$ which must be included by the corresponding elements in the relative interior of the feasible set of (2.29)–(2.30) in Theorem 2.1. Namely, one may use $\widetilde{P} \leftarrow P$, $\widetilde{Q} \leftarrow Q$ and $\widetilde{K} \leftarrow K$ as the initial data for (2.69) where P, Q and K is a feasible solution of Theorem 2.1.

By compiling all the aforementioned procedures according to the expositions in [257], Algorithm 1 can be constructed as follows where $\mathbf{x}$ consists of all the decision variables of $R \in \mathbb{S}^{dn}$, $S; U \in \mathbb{S}^n$ in Theorem 2.1 and $Z \in \mathbb{S}^n$ in (2.69), while $\mathbf{\Lambda}$, $\widetilde{\mathbf{\Lambda}}$, K, $\widetilde{K}$ in Algorithm 1 are in line with (2.66). Furthermore, ρ_1, ρ_2 and ε are given constants for regularizations and determining error tolerance, respectively.

Remark 2.7. When a convex objective function is contained by Theorem 2.1, for instance $\mathbb{L}^2$ gain γ minimization, a termination condition might be added to Algorithm 1 concerning the values of objective function between two successive iterations [257]. Nonetheless, this condition has not been applied in our numerical examples in this chapter. Moreover, note that a termination condition in terms of the number of the iterations in the while loop can be added in Algorithm 1. Finally, note that the regularization term $\mathsf{tr}\left[\rho_1[*]\left(\mathbf{\Lambda} - \widetilde{\mathbf{\Lambda}}\right) + \rho_2[*]\left(K - \widetilde{K}\right)\right]$ in Algorithm 1 is a special case[4] of the general regularization term

$$\frac{1}{2}\begin{bmatrix} \mathbf{x} - \widetilde{\mathbf{x}} \\ \mathbf{vec}\left(\mathbf{\Lambda} - \widetilde{\mathbf{\Lambda}}\right) \\ \mathbf{vec}\left(K - \widetilde{K}\right) \end{bmatrix}^\top Q_k \begin{bmatrix} \mathbf{x} - \widetilde{\mathbf{x}} \\ \mathbf{vec}\left(\mathbf{\Lambda} - \widetilde{\mathbf{\Lambda}}\right) \\ \mathbf{vec}\left(K - \widetilde{K}\right) \end{bmatrix}, \quad Q_k \succeq 0$$

corresponding to the one proposed in CSDP [257].

[4]This can be understood by the relation $\mathsf{tr}(A^\top B) = \mathbf{vec}(A)^\top \mathbf{vec}(B)$, see the vectorization section in `http://www.ee.ic.ac.uk/hp/staff/dmb/matrix/property.html` which is part of [265].

Algorithm 1: An inner convex approximation solution for Theorem 2.1

begin

Find values for $\mathbf{\Lambda} = \begin{bmatrix} P & Q \end{bmatrix}$ and K included by the corresponding elements in the relative interior of the feasible set of Theorem 2.1. This may be attained by the feasible solution of Theorem 2.2.

update $\widetilde{\mathbf{\Lambda}} \longleftarrow \mathbf{\Lambda}, \quad \widetilde{K} \longleftarrow K,$

solve $\min_{\mathbf{x},\mathbf{\Lambda},K} \mathsf{tr}\left[\rho_1[*](\mathbf{\Lambda} - \widetilde{\mathbf{\Lambda}}) + \rho_2[*](K - \widetilde{K})\right]$ subject to (2.29) and (2.69) to obtain the values of $\mathbf{\Lambda}$ and K

while $\dfrac{\left\|\begin{bmatrix}\mathbf{vec}(\mathbf{\Lambda}) \\ \mathbf{vec}\,(K)\end{bmatrix} - \begin{bmatrix}\mathbf{vec}(\widetilde{\mathbf{\Lambda}}) \\ \mathbf{vec}(\widetilde{K})\end{bmatrix}\right\|_\infty}{\left\|\begin{bmatrix}\mathbf{vec}(\widetilde{\mathbf{\Lambda}}) \\ \mathbf{vec}(\widetilde{K})\end{bmatrix}\right\|_\infty + 1} \geq \varepsilon$ **do**

  update $\widetilde{\mathbf{\Lambda}} \longleftarrow \mathbf{\Lambda}, \quad \widetilde{K} \longleftarrow K;$

  solve $\min_{\mathbf{x},\mathbf{\Lambda},K} \mathsf{tr}\left[\rho_1[*](\mathbf{\Lambda} - \widetilde{\mathbf{\Lambda}}) + \rho_2[*](K - \widetilde{K})\right]$ subject to (2.29) and (2.69) to obtain $\mathbf{\Lambda}$ and K;

end

end

Remark 2.8. The most challenging step in using Algorithm 1 is its initialization if one only considers the synthesis condition in Theorem 2.1. Nevertheless, given what has been proposed in Theorem 2.2, one way to acquire the initial values of $\widetilde{P}$, $\widetilde{Q}$ and $\widetilde{K}$ is to find a feasible solution of (2.45) and (2.46) with given values of $\{\alpha_i\}_{i=1}^{2+d}$. [5]

2.5 Numerical examples

Two numerical examples are presented in this section to demonstrate the effectiveness of the proposed methods in Chapter 2. The examples were tested in Matlab via the optimization interface Yalmip [266]. We use SeDuMi [267] and SDPT3 [268–270] for the solvers of semidefinite programmings.

2.5.1 Stability analysis of distributed delay systems

The example in this subsection has been reported in the first numerical example in the author's journal paper [57] where semidefinite programmings are solved via SeDuMi [267].

Consider a distributed delay system

$$\dot{x}(t) = 0.395x(t) - 5\int_{-r}^{0} \cos(12\tau)x(t+\tau)\mathsf{d}\tau \tag{2.70}$$

with $t_0 \in \mathbb{R}$, which corresponds to $A_1 = 0.395$, $A_2 = 0$ and $\widetilde{A}_3(\tau) = -5\cos(12\tau)$ with $n = 1$ and $p = m = q = 0$ in (2.3) and the remaining parts of the state space matrices in (2.3) corresponding to (2.70) are empty matrices. Since $0.395 > 0$ and the distributed term contains a trigonometric function, the methodologies in [271] and [272] are not able to analyze the stability of (2.70).

[5] Note that as we have elaborated in Remark 2.4, one can apply $\alpha_1 = \alpha_2 = 0$ and $\alpha_i = 0, i = 4 \cdots 2 + d$ which allows users to only adjust the value of α_3 to apply Theorem 2.2.

In order to apply the methodology in this chapter, $\boldsymbol{f}(\cdot)$ in Assumption 2.1 is chosen to be

$$\boldsymbol{f}(\tau) = \begin{bmatrix} 1 \\ \sin(12\tau) \\ \cos(12\tau) \end{bmatrix} \quad \text{with} \quad M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 12 \\ 0 & -12 & 0 \end{bmatrix} \tag{2.71}$$

with $A_3 = \begin{bmatrix} 0 & 0 & -5 \end{bmatrix}$ which satisfies Assumption 2.1 with $d = 3, n = 1$. Furthermore, applying the spectrum methods in [80, 81] with $M = 20$ as the discretization index yields Figure 2.1 as a stability diagram, where it plots the values of $\operatorname{sign}[\Re(\lambda)]$ with λ denoting the rightmost characteristics roots of the system (2.70). Specifically, by testing sufficient large r by the code in [81], it occurs that $[0.104, 0.1578]$, $[0.6276, 0.6814]$, $[1.1512, 1.205]$, $[1.6748, 1.7286]$ and $[2.1984, 2.2522]$ are the stable delay intervals of (2.70).

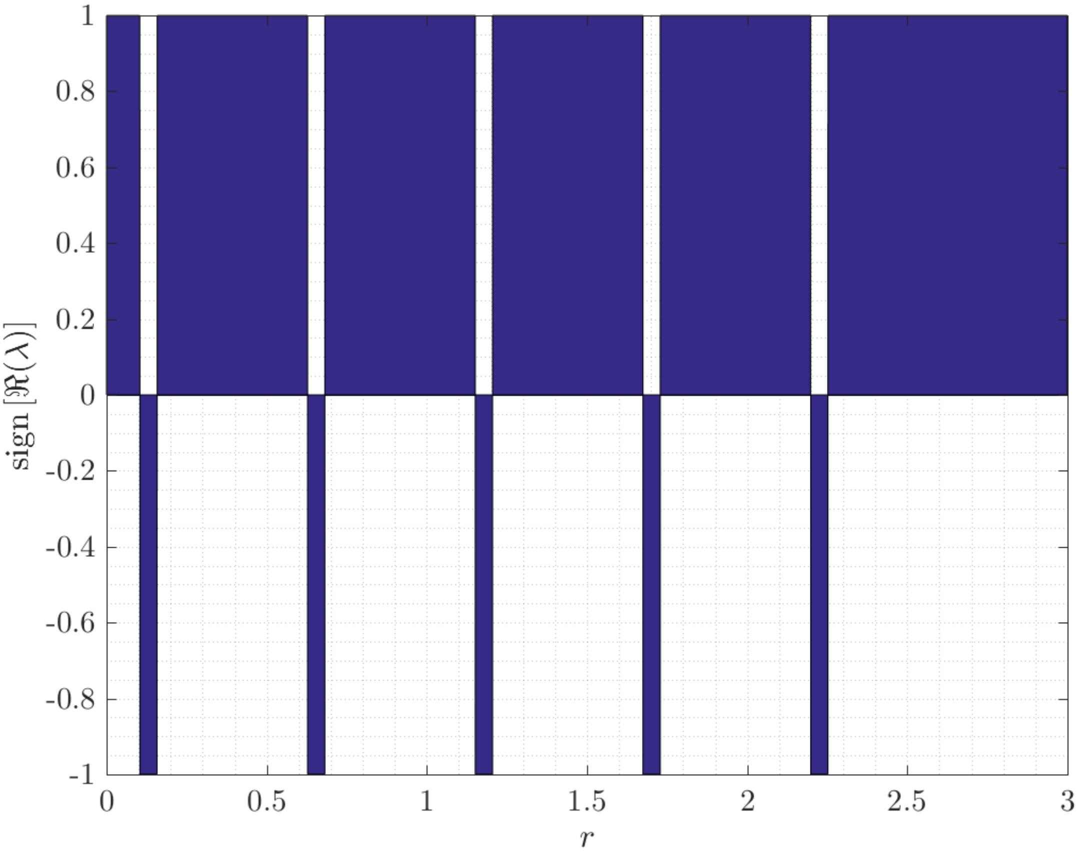

Figure 2.1: Diagram showing stability regions of (2.70)

In Table 2.1, we compare our proposed methodology against the approximation approach in [188] in terms of the ability to detect the boundaries of the stable delay intervals of (2.70). Note that all the semidefinite programs corresponding to the results in Table 2.1 are solved via SeDuMi [267]. It can be observed that it requires $d = 3$ (which is equivalent to the N in [188]) with 12 variables to detect all the boundaries of the first stability intervals. As r increases, a larger degree d of Legendre polynomials is required in order to produce feasible results. It follows that $d = 22$ with 302 variables is required to detect the upper stability boundary 2.2522. In contrast, applying Theorem 2.1 with (2.71) and $A_3 = \begin{bmatrix} 0 & 0 & -5 \end{bmatrix}$ to (2.70), we are able to detect all the boundaries of the stable intervals with only 12 decision variables.

Methodology	first interval	second interval	third interval	forth interval	fifth interval	NDVs
[272]	–	–	–	–	–	–
[185]	–	–	–	–	–	–
[188], $d = 2$	$[0.104, 0.1578]$	–	–	–	–	12
[188], $d = 8$	$[0.104, 0.1578]$	$[0.6276, 0.6814]$	–	–	–	57
[188], $d = 13$	$[0.104, 0.1578]$	$[0.6276, 0.6814]$	$[1.1512, 1.205]$	–	–	122
[188], $d = 17$	$[0.104, 0.1578]$	$[0.6276, 0.6814]$	$[1.1512, 1.205]$	$[1.6748, 1.7286]$	–	192
[188], $d = 22$	$[0.104, 0.1578]$	$[0.6276, 0.6814]$	$[1.1512, 1.205]$	$[1.6748, 1.7286]$	$[2.1984, 2.2522]$	302
Theorem 2.1, $d = 2$	$[0.104, 0.1578]$	$[0.6276, 0.6814]$	$[1.1512, 1.205]$	$[1.6748, 1.7286]$	$[2.1984, 2.2522]$	12

Table 2.1: Feasible Stability Testing Intervals (NDVs stands for the number of decision variables).

Remark 2.9. The boundaries of the stable intervals of (2.70) can be accurately detected by the approach in [188] and Theorem 2.1. This illustrates the fact that the methods of both [188] and Theorem 2.1 are consistent with the reliable calculations in [80], which is indeed not common in comparison to existing time-domain approaches. However, a clear contribution of our method is that fewer variables might be required for the distributed kernels exhibiting patterns of intensive oscillations, which is exactly the case of (2.70).

Remark 2.10. For practical systems, the proposed methods in this chapter can be applied to the models of hematopoietic cell maturation in [32] or SIR Epidemic in [19].

2.5.2 Dissipative static state feedback controller design

A portion of this subsection has been reported in subsection 4.2.1 in the author's paper [57]. All semidefinite programs in this subsection are solved via [270] except for the resulting controller (2.75) which was reported in [57] and solved via SeDuMi [267].

Consider (2.3) with $r = 1$ and the following state space matrices

$$
\begin{aligned}
&A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1 & -1 \\ 0 & 0.9 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ B_2 = B_3(\tau) = \mathbf{0}_2, \ D_1 = \begin{bmatrix} 0.1 & -0.11 \\ 0.21 & 0.1 \end{bmatrix}, \\
&\widetilde{A}_3(\tau) = \begin{bmatrix} -0.4 - 0.1\mathrm{e}^{\tau}\sin(20\tau) + 0.3\mathrm{e}^{\tau}\cos(20\tau) & 1 + 0.2\mathrm{e}^{\tau}\sin(20\tau) + 0.2\mathrm{e}^{\tau}\cos(20\tau) \\ -1 + 0.01\mathrm{e}^{\tau}\sin(20\tau) - 0.2\mathrm{e}^{\tau}\cos(20\tau) & 0.4 + 0.3\mathrm{e}^{\tau}\sin(20\tau) + 0.4\mathrm{e}^{\tau}\cos(20\tau) \end{bmatrix}, \\
&C_1 = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}, \ C_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \ B_4 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \ B_5 = B_6(\tau) = \mathbf{0}_2, \\
&\widetilde{C}_3(\tau) = \begin{bmatrix} 0.2\mathrm{e}^{\tau}\sin(20\tau) & 0.1 + 0.1\mathrm{e}^{\tau}\cos(20\tau) \\ 0.1\mathrm{e}^{\tau}\sin(20\tau) - 0.1\mathrm{e}^{\tau}\cos(20\tau) & -0.2 + 0.3\mathrm{e}^{\tau}\sin(20\tau) \end{bmatrix}, \ D_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.12 & 0.1 \end{bmatrix}
\end{aligned}
\tag{2.72}
$$

which corresponds to $n = q = m = 2$ and $p = 1$. Since (A_1, B_1) is not stabilizable, the method in [185] cannot be applied here regardless of the fact that $\widetilde{A}_3(\tau)$ and $\widetilde{C}_3(\tau)$ might be approximated via rational functions. Moreover, based on the structures of $\widetilde{A}_3(\tau)$ and $\widetilde{C}_3(\tau)$ in (2.72), the corresponding delay system does not have either forwarding or backstepping structures without having transformations. Thus, the constructive approaches in [211] may not be applicable here. Now by using the spectrum method in [80, 81] with a testing gird vector of different values of delays, one can make the estimation[6] that the delay system with (2.72) is unstable for $0 \le r \le 10$. Furthermore, the problem of $\mathbb{L}^2$ attenuation is considered here for the system with (2.72) which corresponds to $J_3 = -J_1 = \gamma I_2$, $\widetilde{J} = I_2$ and $J_2 = \mathsf{O}_2$ for the supply rate function in (2.15).

[6]It is an estimation since only a finite amount of pointwise delay values can be tested by the method in [81].

Observing the elements inside of $\widetilde{A}_3(\tau)$, $\widetilde{C}_3(\tau)$, we choose

$$\boldsymbol{f}(\tau) = \begin{bmatrix} 1 & 10\mathrm{e}^{\tau}\sin(20\tau) & 10\mathrm{e}^{\tau}\cos(20\tau) \end{bmatrix}^{\top} \tag{2.73}$$

which gives

$$\begin{gathered} M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 20 \\ 0 & -20 & 1 \end{bmatrix} \\ A_3 = 0.1\begin{bmatrix} -4 & 10 & -0.1 & 0.2 & 0.3 & 0.2 \\ -10 & 4 & 0.01 & 0.3 & -0.2 & 0.4 \end{bmatrix}, \quad C_3 = 0.1\begin{bmatrix} 0 & 1 & 0.2 & 0 & 0 & 0.1 \\ 0 & -2 & 0.1 & 0.3 & -0.1 & 0 \end{bmatrix} \end{gathered} \tag{2.74}$$

in accordance to Assumption 2.1 with $d = 3$, $n = m = q = 2$.

Remark 2.11. For $\widetilde{A}_3(\tau)$, $\widetilde{C}_3(\tau)$ in (2.72), the matrices A_3 and C_3 can be determined via direct observations given that the structure of $\boldsymbol{f}(\tau)$ in (2.73) is well ordered. On the other hand, applying the substitution $\theta_1 = 10\mathrm{e}^{\tau}\sin(20\tau)$, $\theta_2 = 10\mathrm{e}^{\tau}\cos(20\tau)$ to $\widetilde{A}_3(\tau)$, $\widetilde{C}_3(\tau)$ yields $\widetilde{A}_3(\tau) = \widehat{A}_3(\theta_1, \theta_2) = A_3 M(\theta_1, \theta_2)$, $\widetilde{C}_3(\tau) = \widehat{C}_3(\theta_1, \theta_2) = C_3 M(\theta_1, \theta_2)$, where A_3 and C_3 can be obtained by the correspondence of the coefficient with respect to θ_1, θ_2. Finally, one can use the formulas $A_3 = \begin{bmatrix} \widehat{A}_3(0,0) & \frac{\partial \widehat{A}_3(\theta_1,\theta_2)}{\partial \theta_1} & \frac{\partial \widehat{A}_3(\theta_1,\theta_2)}{\partial \theta_2} \end{bmatrix}$ and $C_3 = \begin{bmatrix} \widehat{C}_3(0,0) & \frac{\partial \widehat{C}_3(\theta_1,\theta_2)}{\partial \theta_1} & \frac{\partial \widehat{C}_3(\theta_1,\theta_2)}{\partial \theta_2} \end{bmatrix}$ to obtain the values of A_3 and C_3 based on the application of differentiation. (The differentiation approach is suggested by Kwin in `https://www.mathworks.com/matlabcentral/answers/309797-extracting-the-coefficient-of-a-polynomials-matrix#answer_241277`)

Apply Theorem 2.2 to (2.7) with the parameters in (2.72)–(2.74) and $\alpha_1 = 1$, $\{\alpha_i\}_{i=2}^{8} = 0$. Then it shows that the system (2.3) with the system parameters in (2.72) is stabilized by

$$\boldsymbol{u}(t) = \begin{bmatrix} 1.7839 & -6.3792 \end{bmatrix} \boldsymbol{x}(t) \tag{2.75}$$

with the performance $\min \gamma = 0.3468$. Note that after this step, we apply SDPT3 [270] as the solver for semidefinite programmings. Due to the simplification applied in the step (2.54), the value of $\min \gamma$ calculated by Theorem 2.2 might be more conservative compared to Theorem 2.1 with a given value of K. Consequently, we apply Theorem 2.1 with $K = \begin{bmatrix} 1.7839 & -6.3792 \end{bmatrix}$ to (2.7) with the parameters in (2.72)–(2.74), which shows that (2.75) is able to achieve $\min \gamma = 0.27077$. To verify (2.75) is a stabilizing controller for the system with (2.72), we again apply the spectrum method in [80] to the resulting closed-loop system. It yields $-0.1606 < 0$ as the real part of the rightmost characteristic root pair, which proves that the resulting closed-loop system is stable.

It is important to mention that if the tuning factors $\{\alpha_i\}_{i=1}^{d+2}$ in (2.46) are not given, then (2.46) becomes bilinear which requires the application of nonlinear solvers such as Penlab [273] to be handled numerically. In fact, one of the main motivations to apply a nonlinear solver to solve the synthesis condition in Theorem 2.1 is to find out the potential optimal values of tuning factors $\{\alpha_i\}_{i=1}^{d+2}$ so they can be substituted into (2.46) to produce convex constraints. Nevertheless, as what has been presented in this subsection, it is possible to solve Theorem 2.2 without appealing to a nonlinear solver by only adjusting one parameter as argued in Remark 2.4.

To apply Algorithm 1 to calculate controllers with further improvement concerning performance, one can first solve Theorem 2.1 with the controller gain in (2.75) to calculate a feasible solution of P and Q. Then the controller gain in (2.75) with the aforementioned P and Q can be applied as the initial values for $\widetilde{K}$, $\widetilde{P}$, $\widetilde{Q}$ in Algorithm 1. The results produced by Algorithm 1 with $\rho_1 = \rho_2 = 10^{-8}$ and $\varepsilon = 10^{-12}$ are summarized in Table 2.2 in which NoIs standards for the number of iterations executed by the while loop in Algorithm 1. Furthermore, the quantities of the spectral abscissas of the closed-loop system spectra in Table 2.2 are calculated by the method in [80, 81]. Clearly, the results in Table 2.2 demonstrate that more iterations lead to better $\min \gamma$ value at the expense of larger computational burdens.

Controller gain K	$\begin{bmatrix} 2.472 \\ -9.3914 \end{bmatrix}^\top$	$\begin{bmatrix} 2.9906 \\ -11.6878 \end{bmatrix}^\top$	$\begin{bmatrix} 3.3886 \\ -13.4488 \end{bmatrix}^\top$	$\begin{bmatrix} 3.7325 \\ -14.9736 \end{bmatrix}^\top$
$\min \gamma$	0.27041	0.27031	0.27027	0.27025
NoIs	10	20	30	40
Spectral abscissa of the closed-loop system	−0.1595	−0.1587	−0.1582	−0.1578

Table 2.2: $\min \gamma$ produced by different iterations

Remark 2.12. Some functions in the $\boldsymbol{f}(\cdot)$ in (2.74) have been scaled compared to the form in (2.71). This is due to the fact that in some situations having F with a large condition number may affect the numerical solvability of the corresponding optimization programs.

Remark 2.13. Note that Simulink is not utilized throughout the paper to simulate a closed-loop system[7] with a distributed delay calculated by our proposed methods. This is due to the consideration that there are no proper numerical solvers for delay systems in Simulink where one can only employ numerical solvers for ODEs. On the other hand, the trajectory of a system with distributed delays of constant delay values could be handled by the DDE23 solver in Matlab by discretizing the distributed delays into multiple discrete forms. However, the consequences of discretizing distributed delays may require further investigation in terms of whether it can produce accurate numerical solutions of a distributed delay system.

[7]The stability of the closed-loop systems in this thesis are verified by frequency domain approaches whenever it can be achieved.

Chapter 3

Dissipative Stabilization for Uncertain Linear Distributed Delay Systems

3.1 Introduction

To take into account modeling errors and the influence of the system's operating environment, it is more realistic to incorporate uncertainties into the state space parameters of the mathematical model of a system. Characterizing uncertainties in the models of systems has been extensively researched over the past decades [274–278] and the methodologies might be also adopted to handle problems pertaining to linear parameter varying systems when the Liapunov approaches are considered [173, 217, 279, 280].

Since the characterization of the robustness of a system is directly affected by the complexity of uncertainties, thus having uncertainties with more general structures in a system can lead to more general results in terms of system's robustness. One of the common structures of uncertainties is the norm-bounded uncertainty: [275, 281] $G\Delta H$. $\Delta^\top\Delta \preceq I$ [1] where G and H are given and Δ can be a function of time t or a function of other variables. To handle norm-bounded uncertainties in the context of solving linear matrix inequalities, Petersen Lemma [275] (See the summary in Lemma C.10.1 of [173] also) was introduced which can provide tractable conditions for an LMI term possessing norm-bounded uncertainties. The idea of Petersen Lemma was further extended in [282] to handle uncertainties which are of linear fractional form. On the other hand, the handling of norm-bounded uncertainties subject to a full block scaling constraint is elaborated by Lemma C.10.4 in [173] in which the proof is supported by the linearization procedure in [283]. Nevertheless, it is certainly more beneficial to consider using linear fractional uncertainties subject to full block scaling constraints.

In Chapter 3, the problem of stabilizing an uncertain linear DDS with distributed delays in states, inputs and outputs is investigated where the structure of the distributed delay terms follow the same class considered in Chapter 2. We propose methodologies dealing with calculating a static state feedback controller for a linear distributed delay system having uncertainties with linear fractional form. Namely, the uncertainties of state-space matrices are in the form of $G(I-\Delta F)^{-1}\Delta H$ with[2] Δ subject to the full

[1] Note that the constraints on Δ can be chosen as $\Delta^\top\Delta \preceq R$ where $R \succ 0$.

[2] Note that each state space matrix of the system in this chapter has its own uncertainty parameter (constraint), the term $G(I-\Delta F)^{-1}\Delta H$ and its constraint only provide a common characteristic of the uncertainties considered in this chapter

block scaling constraint

$$\Delta \in \left\{ \widehat{\Delta} \;\middle|\; \begin{bmatrix} I \\ \widehat{\Delta} \end{bmatrix}^{\top} \begin{bmatrix} \Xi^{-1} & \Lambda \\ * & \Gamma \end{bmatrix} \begin{bmatrix} I \\ \widehat{\Delta} \end{bmatrix} \succeq 0 \right\}, \quad \Xi^{-1} \succ 0, \quad \Gamma \preceq 0.$$

Furthermore, the coefficients of distributed delay functions are also assumed to be affected by uncertainties, hence the model of the uncertain system considered in this chapter is sufficiently general. On the other hand, the proposed scenarios on static controller design can be modified to design a non-fragile dynamic state feedback controller for an uncertain LTI system with input delays. Interestingly, compared to the case of finding a robust static state controller, the computation of a non-fragile dynamic state controller can be more "easier" due to the available results pertaining to the construction of predictor controllers for input delay systems.

The rest of the chapter is organized as follows. The plant of an open-loop uncertain system is first presented in section 3.2 where the uncertainties are of linear fractional form subject to full block scaling constraints. Since the synthesis solutions for a linear distributed delay system have been presented in Chapter 2, hence they can be adapted to handle the uncertain counterpart as long as the uncertainties therein can be coped with by a mathematical scenario. To handle uncertainties in the form of $G(I - \Delta F)^{-1}\Delta H$ in the context of semidefinite programming, Lemma 3.1 is introduced which is applied later for the derivation of the optimization constraints for robust synthesis. By utilizing this lemma, the main results on the synthesis of robust static state feedback controllers are summarized in Theorem 3.1 and 3.2 in section 3.3 by means of matrix inequalities of finite dimensions. Moreover, an iterative algorithm for solving the bilinear matrix inequality in Theorem 3.1 is also derived subsequently based on what has been presented in Chapter 2 for the derivation of Algorithm 1. Next, the whole section 3.4 is dedicated to the study of designing a non-fragile dynamic state feedback controller for an uncertain LTI system with input delays where both the models of both the plant and controller incorporate uncertainties terms. Following the same strategy proposed in the previous section, the corresponding synthesis condition for the existence of a non-fragile dynamic state feedback controller is presented in Theorem 3.3 which can be solved by the iterative algorithm outlined in Algorithm 3. Interestingly, the iterative algorithm in section 3.4 for the design of a robust dynamic state feedback can be initiated directly via the gains of constructed predictor controllers. Finally, several numerical examples are presented in section 3.5 to demonstrate the capacity and effectiveness of the proposed methodologies.

3.2 Problem formulation

Consider the following linear uncertain distributed delay system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \grave{A}_1\boldsymbol{x}(t) + \grave{A}_2\boldsymbol{x}(t-r) + \int_{-r}^{0} \grave{A}_3 F(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau + \grave{B}_1\boldsymbol{u}(t) + \grave{B}_2\boldsymbol{u}(t-r) \\ &\qquad + \int_{-r}^{0} \grave{B}_3 F(\tau)\boldsymbol{u}(t+\tau)\mathsf{d}\tau + \grave{D}_1\boldsymbol{w}(t), \quad t \geq t_0 \\ \boldsymbol{z}(t) &= \grave{C}_1\boldsymbol{x}(t) + \grave{C}_2\boldsymbol{x}(t-r) + \int_{-r}^{0} \grave{C}_3(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau + \grave{B}_4\boldsymbol{u}(t) + \grave{B}_5\boldsymbol{u}(t-r) \\ &\qquad + \int_{-r}^{0} \grave{B}_6 F(\tau)\boldsymbol{u}(t+\tau)\mathsf{d}\tau + \grave{D}_2\boldsymbol{w}(t) \\ &\forall \theta \in [-r, 0], \quad \boldsymbol{x}(t_0+\theta) = \boldsymbol{\phi}(\theta) \end{aligned} \tag{3.1}$$

to be stabilized, where $t_0 \in \mathbb{R}$, $\boldsymbol{x}(t) \in \mathbb{R}^n$ satisfies (3.1), $\boldsymbol{u}(t) \in \mathbb{R}^p$ denotes input signals, $\boldsymbol{w}(\cdot) \in \widehat{\mathbb{L}}^2([t_0, \infty) ; \mathbb{R}^q)$ represents disturbance, $\boldsymbol{z}(t) \in \mathbb{R}^m$ is the regulated output, and $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r, 0] ; \mathbb{R}^n)$ denotes initial condition. Moreover, the distributed delay function is $\mathbb{R}^{n \times dn} \ni F(\tau) = \boldsymbol{f}(\tau) \otimes I_n$ with $\boldsymbol{f}(\cdot) \in \mathbb{C}^1\left([-r, 0] ; \mathbb{R}^d\right)$

which satisfies

$$\int_{-r}^{0} \boldsymbol{f}(\tau)\boldsymbol{f}^{\top}(\tau)\mathsf{d}\tau \succ 0,$$
$$\exists M \in \mathbb{R}^{d\times d} : \frac{\mathsf{d}\boldsymbol{f}(\tau)}{\mathsf{d}\tau} = M\boldsymbol{f}(\tau). \tag{3.2}$$

The state space parameters in (3.1) are defined as

$$\begin{bmatrix}\grave{A}_1 & \grave{B}_1 & \grave{A}_2 & \grave{B}_2 & \grave{A}_3 & \grave{B}_3 & \grave{D}_1\end{bmatrix}$$
$$= \begin{bmatrix}A_1 & B_1 & A_2 & B_2 & A_3 & B_3 & D_1\end{bmatrix} + \mathop{\mathbf{Row}}_{i=1}^{7} \left(G_i(I-\Delta_i F_i)^{-1}\Delta H_i\right) \tag{3.3}$$

and

$$\begin{bmatrix}\grave{C}_1 & \grave{B}_4 & \grave{C}_2 & \grave{B}_5 & \grave{C}_3 & \grave{B}_6 & \grave{D}_2\end{bmatrix}$$
$$= \begin{bmatrix}C_1 & B_4 & C_2 & B_5 & C_3 & B_6 & D_2\end{bmatrix} + \mathop{\mathbf{Row}}_{i=8}^{14} \left[G_i(I-\Delta_i F_i)^{-1}\Delta H_i\right] \tag{3.4}$$

where the dimensions of the parameters A_i; C_i, $i = 1\cdots 3$ and B_j, $j = 1\cdots 6$ and D_1, D_2 are identical to the parameters in (3.1) and Assumption 2.1 in the previous chapter. The given matrices G_i, F_i, H_i with $i = 1\cdots 14$ determine the configuration of the uncertainty Δ_i which subject to the full block constraints [173, 216]

$$\Delta_i \in \left\{ \widehat{\Delta}_i \;\middle|\; \begin{bmatrix} I \\ \widehat{\Delta}_i \end{bmatrix}^{\top} \begin{bmatrix} \Xi_i^{-1} & \Lambda_i \\ * & \Gamma_i \end{bmatrix} \begin{bmatrix} I \\ \widehat{\Delta}_i \end{bmatrix} \succeq 0 \right\}, \quad \forall i = 1\cdots 14, \;\; \Xi_i^{-1} \succ 0, \;\; \Gamma_i \preceq 0 \tag{3.5}$$

where Ξ_i, Λ_i and Γ_i are given. Finally, all matrices in (3.3)–(3.5) are supposed to have compatible dimensions.

Remark 3.1. Note that in this chapter we skip the decomposition procedure for the distributed delay terms as in Assumption 2.1 of Chapter 1. It is a fact that any distributed delay term which contains functions as entries of a matrix exponential $\mathsf{e}^{X\tau}$, $X \in \mathbb{R}^{d\times d}$ can be written via the forms of distributed delay terms in (3.3).

The constraints of uncertainty (3.5) can be rewritten into $\widehat{\Delta}_i^{\top}\Gamma_i\widehat{\Delta}_i + \mathbf{Sy}(\Lambda_i\widehat{\Delta}_i) + \Xi_i^{-1} \succeq 0, \forall i = 1\cdots 14$, which is equivalent to a single inequality $\bigoplus_{i=1}^{14}\left(\widehat{\Delta}_i^{\top}\Gamma_i\widehat{\Delta}_i + \mathbf{Sy}(\Lambda_i\widehat{\Delta}_i) + \Xi_i^{-1}\right) \succeq 0$. Moreover, using the property of diagonal matrices $(X+Y)\oplus(X+Y) = (X\oplus X)+(Y\oplus Y)$ and $XY\oplus XY = (X\oplus X)(Y\oplus Y)$ shows that (3.5) is equivalent to

$$\bigoplus_{i=1}^{14}\Delta_i \in \mathcal{U} := \left\{ \bigoplus_{i=1}^{14}\widehat{\Delta}_i \;\middle|\; \begin{bmatrix} I \\ \bigoplus_{i=1}^{14}\widehat{\Delta}_i \end{bmatrix}^{\top} \begin{bmatrix} \bigoplus_{i=1}^{14}\Xi_i^{-1} & \bigoplus_{i=1}^{14}\Lambda_i \\ * & \bigoplus_{i=1}^{14}\Gamma_i \end{bmatrix} \begin{bmatrix} I \\ \bigoplus_{i=1}^{14}\widehat{\Delta}_i \end{bmatrix} \succeq 0 \right\}$$
$$\bigoplus_{i=1}^{14}\Xi_i^{-1} \succ 0, \quad \bigoplus_{i=1}^{14}\Gamma_i \preceq 0. \tag{3.6}$$

Having demonstrated the equivalence relation between (3.6) and (3.5), (3.6) will be applied in the next section to derive our robust synthesis conditions due to its compact structure.

Remark 3.2. The uncertainties in (3.3) with the constraints in (3.6) provide a very general characterization of uncertainties in a linear distributed delay system. Note that the robust terms $\grave{A}_3 = A_3 + G_5(I - \Delta_5F_5)^{-1}\Delta H_5$, $\grave{B}_3 = B_3 + G_6(I-\Delta_6F_6)^{-1}\Delta_6H_6$, $\grave{C}_3 = C_3 + G_{12}(I-\Delta_{12}F_{12})^{-1}\Delta_{12}H_{12}$ and $\grave{B}_6 = B_6 + G_{13}(I-\Delta_{13}F_{13})^{-1}\Delta H_{13}$ lead to the distributed terms $\widetilde{A}_3(\tau) = A_3F(\tau) + G_5(I-\Delta_5F_5)^{-1}\Delta_5H_5F(\tau)$ and $\widetilde{B}_3(\tau) = B_3F(\tau) + G_6(I-\Delta_6F_6)^{-1}\Delta H_6F(\tau)$ and $\widetilde{C}_3(\tau) = C_3F(\tau) + G_{12}(I-\Delta_{12}F_{12})^{-1}\Delta H_{12}F(\tau)$ and $\widetilde{B}_6(\tau) = B_6F(\tau) + G_{13}(I-\Delta_{13}F_{13})^{-1}\Delta H_{13}F(\tau)$, respectively. This further demonstrates the fact that the uncertainties associated with the distributed terms are sufficiently general in (3.3), as all the coefficients of the functions in $\widetilde{A}_3(\tau)$, $\widetilde{B}_3(\tau)$, $\widetilde{C}_3(\tau)$ and $\widetilde{B}_6(\tau)$ are subject to the variations of $G_5(I-\Delta_5F_5)^{-1}\Delta H_5$, $G_6(I-\Delta_6F_6)^{-1}\Delta H_6$, $G_{12}(I-\Delta_{12}F_{12})^{-1}\Delta H_{12}$ and $G_{13}(I-\Delta_{13}F_{13})^{-1}\Delta H_{12}$, respectively.

Now substitute $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ into (3.1), one can derive the expression of the closed-loop uncertain system

$$\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\ \boldsymbol{z}(t)\end{bmatrix} = \left(\begin{bmatrix}\mathbf{A}+\mathbf{B}_1\left[(I_{2+d}\otimes K)\oplus \mathsf{O}_q\right]\\ \mathbf{C}+\mathbf{B}_2\left[(I_{2+d}\otimes K)\oplus \mathsf{O}_q\right]\end{bmatrix} + \begin{bmatrix}\mathop{\mathsf{Row}}_{i=1}^{7} G_i & \mathsf{O}\\ \mathsf{O} & \mathop{\mathsf{Row}}_{i=8}^{14} G_i\end{bmatrix}\left(I-\bigoplus_{i=1}^{14}\Delta_iF_i\right)^{-1}\boldsymbol{\Delta}\begin{bmatrix}\mathbf{H}_1\\ \mathbf{H}_2\end{bmatrix}\right)\boldsymbol{\chi}(t) \tag{3.7}$$
$$\forall\theta\in[-r,0],\quad \boldsymbol{x}(t_0+\theta)=\boldsymbol{\phi}(\theta)$$

with $\boldsymbol{\chi}(t)$ in (2.10), where $\boldsymbol{\Delta} := \bigoplus_{i=1}^{14}\Delta_i$ and

$$\begin{aligned}\mathbf{H}_1 &= \left(\begin{bmatrix}H_1\\ H_2K\end{bmatrix}\oplus\begin{bmatrix}H_3\\ H_4K\end{bmatrix}\oplus\begin{bmatrix}H_5\\ H_6\,(I_d\otimes K)\end{bmatrix}\oplus H_7\right),\\ \mathbf{H}_2 &= \left(\begin{bmatrix}H_8\\ H_9K\end{bmatrix}\oplus\begin{bmatrix}H_{10}\\ H_{11}K\end{bmatrix}\oplus\begin{bmatrix}H_{12}\\ H_{13}\,(I_d\otimes K)\end{bmatrix}\oplus H_{14}\right)\end{aligned} \tag{3.8}$$

In order to handle the uncertainties structure in (3.3) and (3.6), the following lemma is derived.

3.2.1 A Lemma concerning uncertainties

Lemma 3.1. *For arbitrary* $n;m;p;q\in\mathbb{N}$, $\Theta_1\in\mathbb{S}^p_{\succ 0}$, $\Theta_3\in\mathbb{S}^m_{\preceq 0}$, $\Theta_2\in\mathbb{R}^{p\times m}$, $\Phi\in\mathbb{S}^n$, $G\in\mathbb{R}^{n\times m}$, $H\in\mathbb{R}^{p\times n}$, $F\in\mathbb{R}^{p\times m}$ *if*

$$\exists\alpha>0:\begin{bmatrix}I_m & -I_m-\alpha F^\top\Theta_2 & \alpha F^\top\\ * & I_m-\alpha\Theta_3 & \mathsf{O}_{m\times p}\\ * & * & \alpha\Theta_1\end{bmatrix}\succ 0, \tag{3.9}$$

then

$$\Phi+\mathsf{Sy}\left[G(I_m-\Delta F)^{-1}\Delta H\right]\prec 0,\ \forall\Delta\in\mathcal{F}\subseteq\mathcal{D}:=\left\{\widehat{\Delta}\in\mathbb{R}^{m\times p}\,\middle|\,[*]\begin{bmatrix}\Theta_1^{-1} & \Theta_2\\ * & \Theta_3\end{bmatrix}\begin{bmatrix}I_p\\ \widehat{\Delta}\end{bmatrix}\succeq 0\right\} \tag{3.10}$$

holds if

$$\exists\kappa>0:\begin{bmatrix}\Phi & G+\kappa H^\top\Theta_2 & \kappa H^\top\\ * & \kappa F^\top\Theta_2+\kappa\Theta_2^\top F+\kappa\Theta_3 & \kappa F^\top\\ * & * & -\kappa\Theta_1\end{bmatrix}\prec 0. \tag{3.11}$$

Moreover, for the situation when $\Theta_1^{-1}=\mathsf{O}_p$, (3.11) *and* (3.9) *become*

$$\exists\kappa>0:\begin{bmatrix}\Phi & G+\kappa H^\top\Theta_2\\ * & \kappa F^\top\Theta_2+\kappa\Theta_2^\top F+\kappa\Theta_3\end{bmatrix}\prec 0, \tag{3.12}$$

$$\exists\alpha>0:\begin{bmatrix}I_m & -I_m-\alpha F^\top\Theta_2\\ * & I_m-\alpha\Theta_3\end{bmatrix}\succ 0, \tag{3.13}$$

respectively.

Proof. See Appendix A. ∎

Remark 3.3. The assumptions $\Theta_1\succ 0$ in this lemma is motivated by the expectation to derive convex conditions (3.9) and (3.11) by the application of the Schur complement at the steps of (A.15) and (A.7).

Lemma 3.1 is able to cover a wide range of uncertainty configurations such as the common norm bounded uncertainties. Namely, let $\mathbb{S}^p \ni \Theta_1^{-1} = R \succ 0$, $\Theta_2 = \mathsf{O}_{p\times m}$, $\Theta_3 = -I_m$, $F = \mathsf{O}_{p\times m}$ and the corresponding $\mathcal{D} = \left\{\widehat{\Delta} \in \mathbb{R}^{m\times p} : \widehat{\Delta}^\top\widehat{\Delta} \preceq R\right\}$ and

$$\exists\kappa > 0 : \begin{bmatrix} \Phi & G & \kappa H^\top \\ * & -\kappa I_m & \mathsf{O}_m \\ * & * & -\kappa R^{-1} \end{bmatrix} \prec 0 \tag{3.14}$$

which corresponds to (3.11). Now it is obvious that (3.14) is equivalent to

$$\exists\kappa > 0 : \Phi - \begin{bmatrix} G & \kappa H^\top \end{bmatrix} \begin{bmatrix} -\kappa I_m & \mathsf{O}_m \\ * & -\kappa R^{-1} \end{bmatrix}^{-1} \begin{bmatrix} G^\top \\ \kappa H \end{bmatrix} = \Phi + \kappa^{-1} G G^\top + \kappa H^\top R H \tag{3.15}$$

where (3.15) with $m = p$ is equivalent to the result of Petersen Lemma (See Lemma C.10.1 in [173]). Note that in this case the well-posedness condition (3.9) does not need to be considered given $F = \mathsf{O}_{p\times m}$.

Furthermore, consider the case of $F \neq \mathsf{O}_{p\times m}$, $\Theta_1^{-1} = R \in \mathbb{S}^p_{\succ 0}$, $\Theta_2 = \mathsf{O}_{p\times m}$ and $\Theta_3 = -I_m$ with $\mathcal{D} = \left\{\widehat{\Delta} \in \mathbb{R}^{m\times p} : \widehat{\Delta}^\top\widehat{\Delta} \preceq R\right\}$, then Lemma 3.1 can handle the linear fractional uncertainties considered by the Rational versions of Petersen's Lemma in [282] and [173]. Namely, the corresponding conditions of (3.9) and (3.11) are

$$\exists\alpha > 0 : \begin{bmatrix} I_m & -I_m & \alpha F^\top \\ * & I_m + \alpha I_m & \mathsf{O}_{m\times p} \\ * & * & \alpha R^{-1} \end{bmatrix} \succ 0, \tag{3.16}$$

$$\exists\kappa > 0 : \begin{bmatrix} \Phi & G & \kappa H^\top \\ * & -\kappa I_m & \kappa F^\top \\ * & * & -\kappa R^{-1} \end{bmatrix} \prec 0. \tag{3.17}$$

For (3.16), it is equivalent to

$$I_m - \begin{bmatrix} -I_m & \alpha F^\top \end{bmatrix} \begin{bmatrix} (\alpha+1)^{-1} I_m & \mathsf{O}_{m\times p} \\ * & \alpha^{-1} R \end{bmatrix} \begin{bmatrix} -I_m \\ \alpha F \end{bmatrix} = I_m - (\alpha+1)^{-1} I_m - \alpha F^\top R F \succ 0. \tag{3.18}$$

which is equivalent to there exists $\alpha > 0$ such that

$$\alpha(\alpha+1)F^\top R F + I_m \prec (\alpha+1) I_m \iff \alpha(\alpha+1)F^\top R F - \alpha I_m \prec 0$$
$$\iff (\alpha+1)\, F^\top R F - I_m \prec 0. \tag{3.19}$$

Furthermore, for any $\alpha > 0$ we have $F^\top R F - I_m \preceq (\alpha+1)\, F^\top R F - I_m \prec 0$ since $F^\top R F \preceq (\alpha+1)\, F^\top R F$ for any $\alpha > 0$ based on the fact that $\alpha > 0$ and $F^\top R F \succeq 0$ with $R \succ 0$. Hence (3.19) infers $F^\top R F - I_m \prec 0$.[3] This shows that $F^\top R F - I_m \prec 0$ is implied by (3.16) which further shows that (3.16) implies that the well-posedness conditions in [173] and [282] corresponding to the cases of $m = p$ and $R = I_p$, respectively, are satisfied. On the other hand, we can conclude that (3.17) holds if and only if

$$\exists\kappa > 0 : \begin{bmatrix} I_n & \mathsf{O}_{n\times m} & \mathsf{O}_{n\times p} \\ \mathsf{O}_{p\times n} & \mathsf{O}_{p\times m} & I_p \\ \mathsf{O}_{m\times n} & I_m & \mathsf{O}_{m\times p} \end{bmatrix} \begin{bmatrix} \Phi & G & \kappa H^\top \\ G^\top & -\kappa I_m & \kappa F^\top \\ \kappa H & \kappa F & -\kappa R^{-1} \end{bmatrix} \begin{bmatrix} I_n & \mathsf{O}_{n\times p} & \mathsf{O}_{n\times m} \\ \mathsf{O}_{m\times n} & \mathsf{O}_{m\times p} & I_m \\ \mathsf{O}_{p\times n} & I_p & \mathsf{O}_{p\times m} \end{bmatrix}$$
$$= \begin{bmatrix} \Phi & G & \kappa H^\top \\ \kappa H & \kappa F & -\kappa R^{-1} \\ G^\top & -\kappa I_m & \kappa F^\top \end{bmatrix} \begin{bmatrix} I_n & \mathsf{O}_{n\times p} & \mathsf{O}_{n\times m} \\ \mathsf{O}_{m\times n} & \mathsf{O}_{m\times p} & I_m \\ \mathsf{O}_{p\times n} & I_p & \mathsf{O}_{p\times m} \end{bmatrix} = \begin{bmatrix} \Phi & \kappa H^\top & G \\ \kappa H & -\kappa R^{-1} & \kappa F \\ G^\top & \kappa F^\top & -\kappa I_m \end{bmatrix} \prec 0 \tag{3.20}$$

[3] It is possible to show that $F^\top R F - I_m \prec 0$ infers $(\alpha+1)F^\top R F - I_m \prec 0$ based on the application of eigendecomposition and the property of real numbers.

holds given the properties of congruence transformation. Now apply the Schur complement to (3.20), it yields that (3.20) is equivalent to

$$\exists\kappa > 0,\ \ \Phi + \begin{bmatrix}\kappa H^\top & G\end{bmatrix}\kappa^{-1}\begin{bmatrix}R^{-1} & -F\\ -F^\top & I_m\end{bmatrix}^{-1}\begin{bmatrix}\kappa H\\ G^\top\end{bmatrix}$$
$$= \Phi + \begin{bmatrix}\sqrt{\kappa}H^\top & \sqrt{\kappa^{-1}}G\end{bmatrix}\begin{bmatrix}R^{-1} & -F\\ -F^\top & I_m\end{bmatrix}^{-1}\begin{bmatrix}\sqrt{\kappa}H\\ \sqrt{\kappa^{-1}}G^\top\end{bmatrix} \prec 0 \quad (3.21)$$

Now let $m = p$, then (3.21) is equivalent to the result in (C.30) of Lemma C.10.2 in [173], which further shows the equivalence between (3.17) and (C.31) of Lemma C.10.2 with $m = p$. Moreover, with $\sqrt{\kappa} = \varepsilon^{-1} > 0$ and $R = I_q$, then (3.21) is equivalent to the inequality in (11) of [282].

3.3 Main results on controller synthesis

Now we combine the synthesis results in Theorem 2.1 with (3.3) and (3.6). By using Lemma 3.1, it results in the following theorem which provides sufficient conditions for the existence of a state feedback controller ensuring both robust dissipativity and stability.

Theorem 3.1. *Given* $\boldsymbol{f}(\cdot)$ *and* M *in* (3.2)*, then the uncertain closed-loop system* (3.7) *with the supply rate function in* (2.15) *is robustly dissipative subject to the uncertainty constraints in* (3.6)*, and the trivial solution* $\boldsymbol{x}(t) \equiv \mathbf{0}_n$ *of* (3.7) *with* $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ *is robustly globally asymptotically stable subject to the uncertainty constraints in* (3.6)*, if there exist* $P \in \mathbb{S}^n$*,* $Q \in \mathbb{R}^{n\times dn}$*,* $R \in \mathbb{S}^{dn}$ *and* $S; U \in \mathbb{S}^n$ *and* $\varkappa_1, \varkappa_2 > 0$ *such that the inequalities in* (2.29) *and the following conditions are satisfied,*

$$\begin{bmatrix} I & -I - \varkappa_1\mathbf{F}^\top\mathbf{J}_2 & \varkappa_1\mathbf{F}^\top\\ * & I - \varkappa_1\mathbf{J}_3 & \mathsf{O}\\ * & * & \varkappa_1\mathbf{J}_1\end{bmatrix} \succ 0, \tag{3.22}$$

$$\begin{bmatrix}\mathbf{P}^\top\mathbf{\Pi} + \mathbf{\Phi} & \mathbf{G} + \varkappa_2\mathbf{H}^\top\mathbf{J}_2 & \varkappa_2\mathbf{H}^\top\\ * & \varkappa_2\mathbf{F}^\top\mathbf{J}_2 + \varkappa_2\mathbf{J}_2^\top\mathbf{F} + \varkappa_2\mathbf{J}_3 & \varkappa_2\mathbf{F}^\top\\ * & * & -\varkappa_2\mathbf{J}_1\end{bmatrix} \prec 0, \tag{3.23}$$

where the structure of $\mathbf{P}^\top\mathbf{\Pi} + \mathbf{\Phi}$ *is given in Theorem 2.1 with the nominal state space parameters in* (3.1)*, and*

$$\mathbf{G} := \begin{bmatrix} P & \mathsf{O}_{n\times m}\\ \mathsf{O}_n & \mathsf{O}_{n\times m}\\ Q^\top & \mathsf{O}_{dn\times m}\\ \mathsf{O}_{q\times n} & -J_2^\top\\ \mathsf{O}_{m\times n} & \widetilde{J}\end{bmatrix}\begin{bmatrix}\underset{i=1}{\overset{7}{\textsf{Row}}}\, G_i & \mathsf{O}\\ \mathsf{O} & \underset{i=8}{\overset{14}{\textsf{Row}}}\, G_i\end{bmatrix},\quad \begin{bmatrix}\mathbf{F}\\ \mathbf{J}_1\\ \mathbf{J}_2\\ \mathbf{J}_3\end{bmatrix} := \begin{bmatrix}\bigoplus_{i=1}^{14} F_i\\ \bigoplus_{i=1}^{14}\Xi_i\\ \bigoplus_{i=1}^{14}\Lambda_i\\ \bigoplus_{i=1}^{14}\Gamma_i\end{bmatrix} \tag{3.24}$$

$$\mathbf{H} := \begin{pmatrix}\begin{bmatrix}H_1\\ H_2K\end{bmatrix}\oplus\begin{bmatrix}H_3\\ H_4K\end{bmatrix}\oplus\begin{bmatrix}H_5\\ H_6\left(I_d\otimes K\right)\end{bmatrix}\oplus H_7\\ \begin{bmatrix}H_8\\ H_9K\end{bmatrix}\oplus\begin{bmatrix}H_{10}\\ H_{11}K\end{bmatrix}\oplus\begin{bmatrix}H_{12}\\ H_{13}\left(I_d\otimes K\right)\end{bmatrix}\oplus H_{14}\end{pmatrix}\begin{bmatrix}I_{2n+dn+q} & \mathsf{O}_{m\times(2n+dn+q)}\end{bmatrix} \tag{3.25}$$

with G_i*,* F_i*,* H_i *and* Ξ_i*,* Λ_i*,* Γ_i *given in* (3.7) *and* (3.5)*, respectively.*

Proof. Substituting the expression of the uncertain closed-loop system (3.7) into (2.30), then we have the inequality

$$\forall\mathbf{\Delta}\in\mathcal{U},\ \ \textsf{Sy}\left(\mathbf{P}^\top\mathbf{\Pi}\right) + \mathbf{\Phi} + \textsf{Sy}\left[\mathbf{G}\left(\mathbf{I} - \mathbf{\Delta}\mathbf{F}\right)^{-1}\mathbf{\Delta}\mathbf{H}\right] \prec 0, \tag{3.26}$$

where $\mathbb{R}^{\nu_1\times\nu_2} \ni \mathbf{\Delta} := \bigoplus_{i=1}^{14}\Delta_i$ with ν_1, ν_2 determined by the dimensions of Δ_i, $i = 1\cdots 14$, and $\mathcal{U}$ is given in (3.6). Note that (3.26) can be derived based on the structure of (2.30) and (3.7). Now it is obvious that

$$\forall\acute{\mathbf{\Delta}} \in \mathcal{W},\ \ \mathsf{Sy}\left(\mathbf{P}^\top\mathbf{\Pi}\right) + \mathbf{\Phi} + \mathsf{Sy}\left[\mathbf{G}\left(\mathbf{I} - \acute{\mathbf{\Delta}}\mathbf{F}\right)^{-1}\acute{\mathbf{\Delta}}\mathbf{H}\right] \prec 0, \tag{3.27}$$

infers (3.27) with

$$\mathcal{W} := \left\{\widetilde{\mathbf{\Delta}} \in \mathbb{R}^{\nu_1\times\nu_2}\ \middle|\ \begin{bmatrix} I_{\nu_2} \\ \widetilde{\mathbf{\Delta}}\end{bmatrix}^\top \begin{bmatrix} \bigoplus_{i=1}^{14}\Xi_i^{-1} & \bigoplus_{i=1}^{14}\Lambda_i \\ * & \bigoplus_{i=1}^{14}\Gamma_i \end{bmatrix} \begin{bmatrix} I_{\nu_2} \\ \widetilde{\mathbf{\Delta}}\end{bmatrix} \succeq 0\right\}$$
$$\bigoplus_{i=1}^{14}\Xi_i^{-1} \succ 0,\ \ \bigoplus_{i=1}^{14}\Gamma_i \preceq 0 \tag{3.28}$$

since $\mathcal{U} \subseteq \mathcal{W}$. Now applying Lemma 3.1 to (3.27) with (3.6) yields that (3.27) is inferred by (3.22) and (3.23), which further indicates that (3.26) is inferred by (3.22) and (3.23). This shows that the existence of feasible solutions of (2.29), (3.22) and (3.23) infer the existence of (2.28) satisfying the corresponding robust version of the stability criteria in (2.11), (2.12) and the robust version of the dissipativity in (2.13), which further infer the robust stability of the trivial solution $\boldsymbol{x}(t) \equiv \mathbf{0}_n$ of (3.7) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ and its robust dissipativity with (2.15). ■

Remark 3.4. The inequality in (3.23) is non-convex if K and B_1,B_2,B_3 are of non-zero values. On the other hand, a standard robust dissipative (stability) analysis problem can be solved by the convex constraints in Theorem 3.1 with $K = \mathsf{O}_{n\times q}$ or $B_1 = B_2 = \mathsf{O}_n$ and $B_3 = \mathsf{O}_{n\times dn}$.

Similar to Theorem 2.2 in the previous chapter, we specifically derive the following theorem providing a convex optimization-based solution for a genuine robust dissipative control problem.

Theorem 3.2. *Given $\boldsymbol{f}(\cdot)$ and M in* (3.2) *and $\{\alpha_i\}_{i=1}^{2+d} \subset \mathbb{R}$ with $d \in \mathbb{N}$, then the uncertain closed-loop system* (3.7) *with the supply rate function in* (2.15) *is robustly dissipative subject to the uncertainty constraints in* (3.6)*, and the trivial solution $\boldsymbol{x}(t) \equiv \mathbf{0}_n$ of* (3.7) *with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is robustly globally asymptotically stable subject to* (3.6)*, if there exist $\acute{P} \in \mathbb{S}^n$, $X \in \mathbb{R}_{[n]}^{n\times n}$, $V \in \mathbb{R}^{p\times n}$, $\acute{Q} \in \mathbb{R}^{n\times dn}$, $\acute{R} \in \mathbb{S}^{dn}$, $\acute{S};\acute{U} \in \mathbb{S}^n$ and $\varkappa_1, \varkappa_2 > 0$ such that* (2.45) *and the following conditions are satisfied,*

$$\begin{bmatrix} I & -I - \varkappa_1\mathbf{F}^\top\mathbf{J}_2 & \varkappa_1\mathbf{F}^\top \\ * & I - \varkappa_1\mathbf{J}_3 & \mathsf{O} \\ * & * & \varkappa_1\mathbf{J}_1 \end{bmatrix} \succ 0,\ \ \begin{bmatrix} \acute{\mathbf{\Theta}} & \acute{\mathbf{G}} + \varkappa_2\mathbf{H}^\top\mathbf{J}_2 & \varkappa_2\acute{\mathbf{H}}^\top \\ * & \varkappa_2\mathbf{F}^\top\mathbf{J}_2 + \varkappa_2\mathbf{J}_2^\top\mathbf{F} + \varkappa_2\mathbf{J}_3 & \varkappa_2\mathbf{F}^\top \\ * & * & -\varkappa_2\mathbf{J}_1 \end{bmatrix} \prec 0, \tag{3.29}$$

where $\acute{\mathbf{\Theta}}$ is defined in (2.46) *and*

$$\acute{\mathbf{G}} := \begin{bmatrix} I_n & \mathsf{O}_{n\times m} \\ \mathsf{Col}_{i=1}^{d+2}\,\alpha_i I_n & \mathsf{O}_{(d+2)n\times m} \\ \mathsf{O}_{q\times n} & -J_2^\top \\ \mathsf{O}_{m\times n} & \widetilde{J} \end{bmatrix} \begin{bmatrix} \mathop{\mathsf{Row}}\limits_{i=1}^{7} G_i & \mathsf{O} \\ \mathsf{O} & \mathop{\mathsf{Row}}\limits_{i=8}^{14} G_i \end{bmatrix}, \tag{3.30}$$

$$\acute{\mathbf{H}} := \begin{pmatrix} \begin{bmatrix} H_1X \\ H_2V \end{bmatrix} \oplus \begin{bmatrix} H_3X \\ H_4V \end{bmatrix} \oplus \begin{bmatrix} H_5(I_d\otimes X) \\ H_6\,(I_d\otimes V) \end{bmatrix} \oplus H_7 \\ \begin{bmatrix} H_8X \\ H_9V \end{bmatrix} \oplus \begin{bmatrix} H_{10}X \\ H_{11}V \end{bmatrix} \oplus \begin{bmatrix} H_{12}(I_d\otimes X) \\ H_{13}(I_d\otimes V) \end{bmatrix} \oplus H_{14} \end{pmatrix} \begin{bmatrix} \mathsf{O}_{(2n+dn+q)\times n} & I_{2n+dn+q} & \mathsf{O}_{(2n+dn+q)\times m} \end{bmatrix} \tag{3.31}$$

with G_i, F_i, H_i and Ξ_i, Λ_i, Γ_i given in (3.7) *and* (3.5)*, respectively.*

Proof. Substituting (3.7) into (2.46) and considering the proof procedure of Theorem 2.2, the corresponding robust version of (2.46) can be derived as

$$\forall \boldsymbol{\Delta} \in \mathcal{U}, \quad \acute{\boldsymbol{\Theta}} := \mathbf{Sy}\left(\begin{bmatrix} I_n \\ \mathbf{Col}_{i=1}^{2+d} \alpha_i I_n \\ \mathsf{O}_{(q+m)\times n} \end{bmatrix} \begin{bmatrix} -X & \widehat{\boldsymbol{\Pi}} \end{bmatrix} \right) + \begin{bmatrix} \mathsf{O}_n & \acute{\mathbf{P}} \\ * & \widehat{\boldsymbol{\Phi}} \end{bmatrix} \prec 0. \tag{3.32}$$

where $\mathbb{R}^{\nu_1\times\nu_2} \ni \boldsymbol{\Delta} := \bigoplus_{i=1}^{14} \Delta_i$ and

$$\widehat{\boldsymbol{\Pi}} = \acute{\boldsymbol{\Pi}} + \left(\mathop{\mathbf{Row}}_{i=1}^{7} G_i \right) \left(I - \bigoplus_{i=1}^{7} \Delta_i F_i \right)^{-1} \left(\bigoplus_{i=1}^{7} \Delta_i \right)$$
$$\times \left(\begin{bmatrix} H_1 X \\ H_2 V \end{bmatrix} \oplus \begin{bmatrix} H_3 X \\ H_4 V \end{bmatrix} \oplus \begin{bmatrix} H_5 (I_d \otimes X) \\ H_6 (I_d \otimes V) \end{bmatrix} \oplus H_7 \right) \begin{bmatrix} I_{2n+dn+q} & \mathsf{O}_{(2n+dn+q)\times m} \end{bmatrix} \tag{3.33}$$

with $\acute{\boldsymbol{\Pi}}$, $\acute{\mathbf{P}}$ in line with the definitions in (2.47), and

$$\widehat{\boldsymbol{\Phi}} := \mathbf{Sy}\left(\begin{bmatrix} Q \\ \mathsf{O}_{n\times dn} \\ R \\ \mathsf{O}_{q\times dn} \end{bmatrix} \begin{bmatrix} \mathbf{F} & \mathsf{O}_{dn\times m} \end{bmatrix} \right) + (S + rU) \oplus (-S) \oplus (-\mathsf{F} \otimes U) \oplus (-J_3) \oplus J_1$$
$$+ \mathbf{Sy}\left(\begin{bmatrix} \mathsf{O}_{(2n+dn)\times m} \\ -J_2^\top \\ \widetilde{J} \end{bmatrix} \begin{bmatrix} \grave{C}_1 X + \grave{B}_4 V & \grave{C}_2 X + \grave{B}_5 V & \grave{C}_3 (I_d \otimes X) + \grave{B}_2 (I_d \otimes V) & \grave{D}_2 & \mathsf{O}_m \end{bmatrix} \right). \tag{3.34}$$

Now considering the expressions of (3.32)–(3.34), (3.32) can be reformulated into

$$\forall \boldsymbol{\Delta} \in \mathcal{U}, \ \acute{\boldsymbol{\Theta}} + \mathbf{Sy}\left[\acute{\mathbf{G}} \left(\mathbf{I}_\nu - \boldsymbol{\Delta}\mathbf{F} \right)^{-1} \boldsymbol{\Delta} \acute{\mathbf{H}} \right] \prec 0 \tag{3.35}$$

with $\mathbb{R}^{\nu_1\times\nu_2} \ni \boldsymbol{\Delta} := \bigoplus_{i=1}^{14} \Delta_i$ and $\acute{\mathbf{G}}$ in (3.30) and $\acute{\mathbf{H}}$ in (3.31) and $\acute{\boldsymbol{\Theta}}$ in (2.46). Similar to the proof of Theorem 3.1, it is obvious that

$$\forall \acute{\boldsymbol{\Delta}} \in \mathcal{W}, \ \acute{\boldsymbol{\Theta}} + \mathbf{Sy}\left[\acute{\mathbf{G}} \left(\mathbf{I}_{\nu_1} - \acute{\boldsymbol{\Delta}}\mathbf{F} \right)^{-1} \acute{\boldsymbol{\Delta}} \acute{\mathbf{H}} \right] \prec 0, \tag{3.36}$$

infers (3.35) since $\mathcal{U} \subseteq \mathcal{W}$ where $\mathcal{W}$ is given in (3.28). Note that also the well-posedness of the uncertainty in (3.36) infers the well-posedness of the uncertainties in (3.35) since $\mathcal{U} \subseteq \mathcal{W}$. Using Lemma 3.1 to (3.36) with $\mathcal{W}$ in (3.28) yields that the existence of the feasible solutions of (3.29) infers (3.36) which further infers (3.35), where (3.29) also ensure the well-posedness of the linear fractional uncertainties in (3.36) and (3.35) and ultimately (3.7). Considering what has been presented in the proof of Theorem 2.2, this shows that the existence of feasible solutions of (2.45) and (3.29) infer the existence of (2.28) satisfying the corresponding robust version of (2.11), (2.12) and (2.13), which further infers the robust stability of the origin of (3.7) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$, and its robust dissipativity with (2.15). Finally, the first inequality in (3.29) also infers that all the uncertainties in (3.7) are well-posed. This completes the proof. ■

Remark 3.5. It is worthy to mention that all the uncertainties in (3.3) may be 'pull out' into interconnection form as it has been demonstrated in [184, 216]. However, for the sake of producing a single convex condition not requiring the application of the dualization lemma [173, 184], the uncertainties have been chosen in the form of (3.3).

3.3.1 An inner convex approximation solution of Theorem 3.1

Similar to what we have presented in subsection 2.4.1, an iterative algorithm is presented in this subsection based on the algorithm proposed in [257] to solve Theorem 3.1. The resulting iterative algorithm can be initiated by the feasible solutions of Theorem 2.2.

We first present the following lemma to derive the condition for inner convex approximation.

Lemma 3.2. *Given $A \in \mathbb{S}^n$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{S}^m$, $D \in \mathbb{S}^p_{\prec 0}$ and $E \in \mathbb{R}^{n\times p}$, we have*

$$\begin{bmatrix} A - ED^{-1}E^\top & B \\ B^\top & C \end{bmatrix} \prec 0 \iff \begin{bmatrix} A & E & B \\ E^\top & D & \mathsf{O}_{p\times n} \\ B^\top & \mathsf{O}_{n\times p} & C \end{bmatrix} \tag{3.37}$$

Proof. Note that

$$\begin{bmatrix} A - ED^{-1}E^\top & B \\ B^\top & C \end{bmatrix} = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} - \begin{bmatrix} E \\ \mathsf{O}_{m\times p} \end{bmatrix} D^{-1} \begin{bmatrix} E^\top & \mathsf{O}_{p\times m} \end{bmatrix}. \tag{3.38}$$

Applying the Schur complement to (3.38) gives that (3.38) holds if and only if

$$\begin{bmatrix} A & B & E \\ B^\top & C & \mathsf{O}_{m\times p} \\ E^\top & \mathsf{O}_{p\times m} & D \end{bmatrix} \prec 0 \tag{3.39}$$

Now apply congruence transformation to (3.3.1), we have (3.3.1) holds if and only

$$\begin{aligned} &\begin{bmatrix} I_n & \mathsf{O}_{n\times m} & \mathsf{O}_{n\times p} \\ \mathsf{O}_{m\times n} & \mathsf{O}_{m\times p} & I_m \\ \mathsf{O}_{p\times n} & I_p & \mathsf{O}_{p\times m} \end{bmatrix} \begin{bmatrix} A & B & E \\ B^\top & C & \mathsf{O}_{m\times p} \\ E^\top & \mathsf{O}_{p\times m} & D \end{bmatrix} \begin{bmatrix} I_n & \mathsf{O}_{n\times m} & \mathsf{O}_{n\times p} \\ \mathsf{O}_{m\times n} & \mathsf{O}_{m\times p} & I_m \\ \mathsf{O}_{p\times n} & I_p & \mathsf{O}_{p\times m} \end{bmatrix} \\ &= \begin{bmatrix} A & B & E \\ E^\top & \mathsf{O}_{p\times m} & D \\ B^\top & C & \mathsf{O}_{m\times p} \end{bmatrix} \begin{bmatrix} I_n & \mathsf{O}_{n\times m} & \mathsf{O}_{n\times p} \\ \mathsf{O}_{m\times n} & \mathsf{O}_{m\times p} & I_m \\ \mathsf{O}_{p\times n} & I_p & \mathsf{O}_{p\times m} \end{bmatrix} = \begin{bmatrix} A & E & B \\ E^\top & D & \mathsf{O}_{p\times n} \\ B^\top & \mathsf{O}_{n\times p} & C \end{bmatrix} \end{aligned} \tag{3.40}$$

■

Similar to the structure of (2.62), realize that (3.23) can be rewritten into

$$\begin{bmatrix} \mathsf{Sy}\left[\mathbf{P}^\top \mathbf{B}\left[(I_{2+d}\otimes K)\oplus \mathsf{O}_{p+m}\right]\right] + \widehat{\mathbf{\Phi}} & \mathbf{G} + \varkappa_2 \mathbf{H}^\top \mathbf{J}_2 & \varkappa_2 \mathbf{H}^\top \\ * & \varkappa_2 \mathbf{F}^\top \mathbf{J}_2 + \varkappa_2 \mathbf{J}_2^\top \mathbf{F} + \varkappa_2 \mathbf{J}_3 & \varkappa_2 \mathbf{F}^\top \\ * & * & -\varkappa_2 \mathbf{J}_1 \end{bmatrix} \prec 0. \tag{3.41}$$

where $\mathbf{B}$ and $\widehat{\mathbf{\Phi}}$ are defined in (2.62), and other parameters have been given in Theorem 3.1. Now apply (2.64) with (2.66) and the conclusion in (3.37) to (3.41), one can conclude that

$$\begin{bmatrix} \widehat{\mathbf{\Phi}} + \mathsf{Sy}\left(\widetilde{\mathbf{P}}^\top \mathbf{B}\mathbf{K} + \mathbf{P}^\top \mathbf{B}\widetilde{\mathbf{K}} - \widetilde{\mathbf{P}}^\top \mathbf{B}\widetilde{\mathbf{K}}\right) & \mathbf{P}^\top - \widetilde{\mathbf{P}}^\top & \mathbf{K}^\top \mathbf{B}^\top - \widetilde{\mathbf{K}}^\top \mathbf{B}^\top & \mathbf{G} + \varkappa_2 \mathbf{H}^\top \mathbf{J}_2 & \varkappa_2 \mathbf{H}^\top \\ * & -Z & \mathsf{O}_n & \mathsf{O} & \mathsf{O} \\ * & * & Z - I_n & \mathsf{O} & \mathsf{O} \\ * & * & * & \varkappa_2 \mathbf{F}^\top \mathbf{J}_2 + \varkappa_2 \mathbf{J}_2^\top \mathbf{F} + \varkappa_2 \mathbf{J}_3 & \varkappa_2 \mathbf{F}^\top \\ * & * & * & * & -\varkappa_2 \mathbf{J}_1 \end{bmatrix} \prec 0 \tag{3.42}$$

infers (3.41), where all the matrices in (3.42) are in line with the definitions in Theorem 3.1 and (2.66).

By using the results in [257], the following Algorithm 2 can be constructed similar to Algorithm 1, where $\mathbf{x}$ contains all the decision variables of $R \in \mathbb{S}^{nd}$ and $S;U;Z \in \mathbb{S}^n$. Moreover, let $\mathbf{\Lambda} := \begin{bmatrix} P & Q \end{bmatrix}$ and $\widetilde{\mathbf{\Lambda}} := \begin{bmatrix} \widetilde{P} & \widetilde{Q} \end{bmatrix}$. Furthermore, ρ_1, ρ_2 and ε are given constants for regularizations and determining error tolerance, respectively.

Remark 3.6. When a convex objective function is contained by Theorem 3.1, for instance $\mathbb{L}^2$ gain γ minimization, a termination condition might be added to Algorithm 2 concerning the improvement of objective function between two successive iterations [257]. Nonetheless, this condition has not been applied in our numerical examples in this chapter.

Algorithm 2: An inner convex approximation solution for Theorem 3.1

begin

Find initial values for $\widetilde{\mathbf{\Lambda}} = \begin{bmatrix} \widetilde{P} & \widetilde{Q} \end{bmatrix}$ and $\widetilde{K}$ belonging to the relative interior of the feasible set of Theorem 3.1.

solve $\min\limits_{\mathbf{x},\mathbf{\Lambda},K} \mathsf{tr}\left[\rho_1[*](\mathbf{\Lambda} - \widetilde{\mathbf{\Lambda}}) + \rho_2[*](K - \widetilde{K})\right]$ subject to (2.29), (3.22) and (3.42) to obtain $\mathbf{\Lambda}$ and K

while $\dfrac{\left\| \begin{bmatrix} \mathbf{vec}(\mathbf{\Lambda}) \\ \mathbf{vec}\,(K) \end{bmatrix} - \begin{bmatrix} \mathbf{vec}(\widetilde{\mathbf{\Lambda}}) \\ \mathbf{vec}(\widetilde{K}) \end{bmatrix} \right\|_\infty}{\left\| \begin{bmatrix} \mathbf{vec}(\widetilde{\mathbf{\Lambda}}) \\ \mathbf{vec}(\widetilde{K}) \end{bmatrix} \right\|_\infty + 1} \geq \varepsilon$ **do**

  update $\widetilde{\mathbf{\Lambda}} \longleftarrow \mathbf{\Lambda}, \quad \widetilde{K} \longleftarrow K$;

  solve $\min\limits_{\mathbf{x},\mathbf{\Lambda},K} \mathsf{tr}\left[\rho_1[*](\mathbf{\Lambda} - \widetilde{\mathbf{\Lambda}}) + \rho_2[*](K - \widetilde{K})\right]$ subject to (2.29), (3.22) and (3.42) to obtain $\mathbf{\Lambda}$ and K;

end

end

Remark 3.7. Initial values of $\widetilde{P}$, $\widetilde{Q}$ and $\widetilde{K}$ in Algorithm 2 can be provided by the feasible solutions of (2.45) and (3.29) with given values of $\{\alpha_i\}_{i=1}^{2+d}$. Note that similar to what we have explained in Remark 2.4 in the previous chapter, one can apply $\alpha_1 = \alpha_2 = 0$ and $\alpha_i = 0, i = 4 \cdots 2 + d$ which allows users to only adjust the value of α_3 to apply Theorem 3.2.

3.4 Application to dissipative resilient stabilizations of a linear system with a discrete input delay

In this section, it is shown that the idea presented in section 3.3 can be adapted to handle the synthesis problem of stabilizing a linear input delay system by means of a dynamical state feedback controller under both general uncertainties and a dissipative constraint. Due to the mathematical structure of the closed-loop system resulted from the stabilization by a dynamical state controller, the synthesis conditions proposed in this section can calculate the gains of a resilient controller that the controller itself is robust against uncertainties. Moreover, unlike the situation of calculating the gain of a state feedback controller in Algorithm 2, Algorithm 3 proposed in this section may be initialized based on the values of the gains of explicitly constructed predictor controllers.

3.4.1 Formulation of Synthesis Problem

Consider the following system

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \grave{A}\boldsymbol{x}(t) + \grave{B}\boldsymbol{u}(t-r) + \grave{D}_1\boldsymbol{w}(t), \quad t \geq t_0 \\
\boldsymbol{z}(t) &= \grave{C}_1 \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{u}(t) \end{bmatrix} + \grave{C}_2 \begin{bmatrix} \boldsymbol{x}(t-r) \\ \boldsymbol{u}(t-r) \end{bmatrix} + \int_{-r}^{0} \grave{C}_3 \left(\sqrt{\mathsf{F}}\boldsymbol{f}(\tau) \otimes I_\nu\right) \begin{bmatrix} \boldsymbol{x}(t+\tau) \\ \boldsymbol{u}(t+\tau) \end{bmatrix} \mathsf{d}\tau + \grave{D}_2\boldsymbol{w}(t)
\end{aligned}
$$

where $r > 0$ and $t_0 \in \mathbb{R}$ and $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^p$ and $\boldsymbol{w}(t) \in \mathbb{R}^q$ and $\boldsymbol{f}(\cdot) \in \mathbb{C}^1\left([-r, 0]; \mathbb{R}^d\right)$ with $\mathsf{F}^{-1} = \int_{-r}^{0} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau \succ 0$ and $\sqrt{\mathsf{F}}$ stands for the unique[4] square root of F. The state space matrices in (3.43) contain uncertainties. Specifically, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are the nominal part of the state space

[4] For the uniqueness of the square root of a positive semidefinite matrix, see Theorem 7.2.6 in [258]

matrices $\grave{A}$ and $\grave{B}$ without uncertainties[5], and we assume that there exists $K \in \mathbb{R}^{p\times n}$ such that $A + BK$ is Hurwitz. The matrix parameters in the output are defined as

$$\begin{bmatrix}\grave{C}_1 & \grave{C}_2 & \grave{C}_3 & \grave{D}_2\end{bmatrix} = \begin{bmatrix}C_1 & C_2 & C_3 & D_2\end{bmatrix} + \mathop{\mathbf{Row}}_{i=5}^{8}\begin{bmatrix}G_i(I-\Delta_i F_i)^{-1}\Delta H_i\end{bmatrix} \tag{3.43}$$

where $C_1; C_2 \in \mathbb{R}^{m\times\nu}$, $C_3 \in \mathbb{R}^{m\times d\nu}$ with $\nu = n + p$. Furthermore, $\boldsymbol{f}(\cdot) \in \mathbb{C}^1\left(\mathbb{R} \,;\, \mathbb{R}^d\right)$ in (3.43) satisfies

$$\exists K \in \mathbb{R}^{p\times n}, \exists X \in \mathbb{R}^{p\times p},\ \exists \widehat{K} \in \mathbb{R}^{n\times dp},\ \begin{bmatrix}\mathsf{O}_{p\times n} & (KA - XK)\,\mathsf{e}^{-A\tau}B\end{bmatrix} = \widehat{K}\left(\sqrt{\mathsf{F}}\boldsymbol{f}(\tau)\otimes I_\nu\right) \tag{3.44}$$

$$\exists M \in \mathbb{R}^{d\times d}: \frac{\mathsf{d}\boldsymbol{f}(\tau)}{\mathsf{d}\tau} = M\boldsymbol{f}(\tau). \tag{3.45}$$

where the motivation of having (3.44) will be explained later in light of the structure of predictor controllers.

Now to stabilize (3.43), we consider a dynamical state feedback controller

$$\dot{\boldsymbol{u}}(t) = K_1\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{u}(t)\end{bmatrix} + K_2\begin{bmatrix}\boldsymbol{x}(t-r)\\ \boldsymbol{u}(t-r)\end{bmatrix} + \int_{-r}^{0} K_3\left(\sqrt{\mathsf{F}}\boldsymbol{f}(\tau)\otimes I_\nu\right)\begin{bmatrix}\boldsymbol{x}(t+\tau)\\ \boldsymbol{u}(t+\tau)\end{bmatrix}\mathsf{d}\tau \tag{3.46}$$

and assume that all states are measurable, where $K_1; K_2 \in \mathbb{R}^{p\times\nu}$ and $K_3 \in \mathbb{R}^{p\times d\nu}$ are controller gains with $\nu = n + p$. Due to the uncertainties caused by implementation environments and disturbances, it is more realistic to consider

$$\dot{\boldsymbol{u}}(t) = \grave{K}_1\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{u}(t)\end{bmatrix} + \grave{K}_2\begin{bmatrix}\boldsymbol{x}(t-r)\\ \boldsymbol{u}(t-r)\end{bmatrix} + \int_{-r}^{0} \grave{K}_3\left(\sqrt{\mathsf{F}}\boldsymbol{f}(\tau)\otimes I_\nu\right)\begin{bmatrix}\boldsymbol{x}(t+\tau)\\ \boldsymbol{u}(t+\tau)\end{bmatrix}\mathsf{d}\tau + \grave{D}_3\boldsymbol{w}(t), \tag{3.47}$$

as the mathematical model of (3.46) for the theoretical analysis discussed in this section, where $\grave{K}_1$,$\grave{K}_2$,$\grave{K}_3$ and $\grave{D}_3$ contains uncertainties.

Remark 3.8. It is extremely important to stress that (3.47) is only a mathematical model for (3.46) to be considered by the theoretical synthesis methods in this section. When the values of K_1,K_2 and K_3 are obtained by the proposed scenarios, the resulting controller will be implemented by the model in (3.46). Thus no uncertainties or $\boldsymbol{w}(t)$ are included by the actual implementation of the resulting controller (3.46). However, since uncertainties and disturbances are taken into account by the theoretical model (3.47), the resulting controller in (3.46) is non-fragile and able to withstand external perturbations.

Now combining (3.43) and (3.47) with (3.43) produces the following closed-loop system

$$\begin{aligned}
&\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\ \dot{\boldsymbol{u}}(t)\end{bmatrix} = \left(\begin{bmatrix}\grave{A} & \mathsf{O}_{n\times p}\\ \multicolumn{2}{c}{\grave{K}_1}\end{bmatrix}\ \begin{bmatrix}\mathsf{O}_n & \grave{B}\\ \multicolumn{2}{c}{\grave{K}_2}\end{bmatrix}\ \begin{bmatrix}\mathsf{O}_{n\times d\nu}\\ \grave{K}_3\end{bmatrix}\ \begin{bmatrix}\grave{D}_1\\ \grave{D}_3\end{bmatrix}\right)\boldsymbol{\vartheta}(t), \quad t \geq t_0\\
&\boldsymbol{z}(t) = \begin{bmatrix}\grave{C}_1 & \grave{C}_2 & \grave{C}_3 & \grave{D}_2\end{bmatrix}\boldsymbol{\vartheta}(t)\\
&\forall\theta \in [-r, 0],\ \begin{bmatrix}\boldsymbol{x}(t_0+\theta)\\ \boldsymbol{u}(t_0+\theta)\end{bmatrix} = \widehat{\boldsymbol{\phi}}(\theta)
\end{aligned} \tag{3.48}$$

where $r > 0$ and $t_0 \in \mathbb{R}$ and $\widehat{\boldsymbol{\phi}}(\cdot) \in \mathbb{C}\left([-r, 0] \,;\, \mathbb{R}^{n+p}\right)$ and

$$\boldsymbol{\vartheta}(t) = \mathbf{Col}\left(\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{u}(t)\end{bmatrix},\ \begin{bmatrix}\boldsymbol{x}(t-r)\\ \boldsymbol{u}(t-r)\end{bmatrix},\ \int_{-r}^{0}\left(\sqrt{\mathsf{F}}\boldsymbol{f}(\tau)\otimes I_\nu\right)\begin{bmatrix}\boldsymbol{x}(t+\tau)\\ \boldsymbol{u}(t+\tau)\end{bmatrix},\ \boldsymbol{w}(t)\right) \tag{3.49}$$

and the matrices in (3.48) are defined as

$$\begin{aligned}
&\left(\begin{bmatrix}\grave{A} & \mathsf{O}_{n\times p}\\ \multicolumn{2}{c}{\grave{K}_1}\end{bmatrix}\ \begin{bmatrix}\mathsf{O}_n & \grave{B}\\ \multicolumn{2}{c}{\grave{K}_2}\end{bmatrix}\ \begin{bmatrix}\mathsf{O}_{n\times d\nu}\\ \grave{K}_3\end{bmatrix}\ \begin{bmatrix}\grave{D}_1\\ \grave{D}_3\end{bmatrix}\right) = \left(\begin{bmatrix}A & \mathsf{O}_{n\times p}\\ \multicolumn{2}{c}{K_1}\end{bmatrix}\ \begin{bmatrix}\mathsf{O}_n & B\\ \multicolumn{2}{c}{K_2}\end{bmatrix}\ \begin{bmatrix}\mathsf{O}_{n\times d\nu}\\ K_3\end{bmatrix}\ \begin{bmatrix}D_1\\ D_3\end{bmatrix}\right)\\
&\qquad\qquad + \mathop{\mathbf{Row}}_{i=1}^{4}\left(G_i(I-\Delta_i F_i)^{-1}\Delta H_i\right)\\
&\begin{bmatrix}\grave{C}_1 & \grave{C}_2 & \grave{C}_3 & \grave{D}_2\end{bmatrix} = \begin{bmatrix}C_1 & C_2 & C_3 & D_2\end{bmatrix} + \mathop{\mathbf{Row}}_{i=5}^{8}\begin{bmatrix}G_i(I-\Delta_i F_i)^{-1}\Delta H_i\end{bmatrix}.
\end{aligned} \tag{3.50}$$

[5] The exact structures of the uncertainties in $\grave{A}$ and $\grave{B}$ will be specified later

The uncertainties Δ_i, $i = 1 \cdots 8$ in (3.50) and (3.43) are subject to the constraints

$$\Delta_i \in \left\{ \widehat{\Delta}_i \;\middle|\; \begin{bmatrix} I \\ \widehat{\Delta}_i \end{bmatrix}^\top \begin{bmatrix} \Xi_i^{-1} & \Lambda_i \\ * & \Gamma_i \end{bmatrix} \begin{bmatrix} I \\ \widehat{\Delta}_i \end{bmatrix} \succeq 0 \right\}, \quad \forall i = 1 \cdots 8, \;\; \Xi_i^{-1} \succ 0, \;\; \Gamma_i \preceq 0. \tag{3.51}$$

Note that the matrices G_i, H_i, F_i, Δ_i in (3.50) and Ξ_i, Λ_i, Γ_i in (3.51) are locally defined in this section, thus there are not the same as in (3.3), (3.4) and (3.5). Meanwhile, the value of $D_3 \in \mathbb{R}^{p\times q}$ in (3.50) is given. Finally, similar to (3.6), the constraints in (3.51) can be reformulated into

$$\begin{gathered} \bigoplus_{i=1}^{8} \Delta_i \in \mathcal{T} := \left\{ \bigoplus_{i=1}^{8} \widehat{\Delta}_i \;\middle|\; [*] \begin{bmatrix} \bigoplus_{i=1}^{8} \Xi_i^{-1} & \bigoplus_{i=1}^{8} \Lambda_i \\ * & \bigoplus_{i=1}^{8} \Gamma_i \end{bmatrix} \begin{bmatrix} I \\ \bigoplus_{i=1}^{8} \widehat{\Delta}_i \end{bmatrix} \succeq 0 \right\}, \\ \bigoplus_{i=1}^{8} \Xi_i^{-1} \succ 0, \;\; \bigoplus_{i=1}^{8} \Gamma_i \preceq 0. \end{gathered} \tag{3.52}$$

Remark 3.9. The constraints in (3.44) indicate that the elements in $\boldsymbol{f}(\cdot)$ cover the functions in $\mathsf{e}^{-A\tau}B$ satisfying $\frac{\mathsf{d}\mathsf{e}^{-A\tau}B}{\mathsf{d}\tau} = -A\mathsf{e}^{-A\tau}B$ where the functions in $-A\mathsf{e}^{-A\tau}B$ are naturally compatible with the property in (3.45).[6] Moreover, this also means that (3.44) does not impose any extra restrictions onto the structure of A, B as long as one can find a K to make $A + BK$ to be Hurwitz. Finally, we stress that the use of $\sqrt{\mathsf{F}}$ does not affect the existence of C_3 and Γ given that $\sqrt{\mathsf{F}}$ is a symmetrical and full rank matrix.

Remark 3.10. Unlike dealing with a standard state feedback synthesis problem in (3.7), the controller parameters in (3.50) are not directly multiplied by the system parameters in (3.43). As a result, it is possible to produce resilient dissipative stabilization results with (3.50) based on what we have derived in (3.3).

Remark 3.11. The model of (3.48) can be handled by the methodologies proposed in the previous section. In addition, (3.48) is of retarded type thus it satisfies the properties summarized by Theorem 2 in [52]. Namely, it means that if the trivial solution $\boldsymbol{x}(t) \equiv \boldsymbol{0}_n$ of (3.48) with $\boldsymbol{w}(t) \equiv 0$ is robustly uniformly asymptotically stable with the "controller model" in (3.47), then the controller (3.46) can be implemented in real-time without having potential numerical stability problems provided that the accuracy of the approximation of the distributed term in (3.46) reach certain degrees.

3.4.2 An example of dynamical state controllers

Now we can start to explain the motivation of the assumption in (3.44) in light of the expression in (3.47).

Note that for a system $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t - r)$ with $r > 0$ and $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{u}(t) \in \mathbb{R}^p$, it can be always exponentially stabilized by the predictor controller

$$\boldsymbol{u}(t) = K\boldsymbol{x}(t + r) = K\left(\mathsf{e}^{Ar}\boldsymbol{x}(t) + \int_{-r}^{0} \mathsf{e}^{-A\tau}B\boldsymbol{u}(t + \tau)\mathsf{d}\tau\right) \tag{3.53}$$

for any $r > 0$ provided that $A + BK$ is Hurwitz for some $K \in \mathbb{R}^{p\times n}$. However, the form of (3.53) may not secure a safe numerical implementation. One solution of this problem is solved in [52] where a dynamical state controller with special form is proposed in (10) of [52]. Now substitute appropriate parameters into (10) of [52], one can conclude that the dynamical state feedback controller

$$\dot{\boldsymbol{u}}(t) = (KB + X)\,\boldsymbol{u}(t) + (KA - XK)\left(\mathsf{e}^{Ar}\boldsymbol{x}(t) + \int_{-r}^{0} \mathsf{e}^{-A\tau}B\boldsymbol{u}(t + \tau)\mathsf{d}\tau\right) \tag{3.54}$$

can asymptotically stabilize $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t - r)$ for any $r > 0$, where $A + BK$ and $X \in \mathbb{R}^{p\times p}$ are Hurwitz and the form of (3.54) is a particular case of the controller structure in (10) of [52]. Note that the spectrum of $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t - r)$ under (3.54) is

$$\Big\{ s \in \mathbb{C} : \det\left(sI_n - A - BK\right)\det\left(sI_p - X\right) = 0 \Big\} \tag{3.55}$$

[6] This can be understood by the property of $\mathsf{e}^{-A\tau}$ in light of Putzer Algorithm for Matrix Exponentials

which can be exponentially stable for some $K \in \mathbb{R}^{p\times n}$ given $X \in \mathbb{R}^{p\times p}$ is Hurwitz.

By the matrices in (3.54), now it is clear that (3.44) indicates the functions in $\boldsymbol{f}(\cdot)$ in (3.43) are able to cover all the functions in $\mathsf{e}^{-A\tau}B$ and $\widetilde{C}_3(\tau)$ where A and B are in line with the definitions in (3.43). Thus (3.44) implies that it is always possible to construct a predictor controller in the form of (3.54) to stabilize the nominal system of (3.43) without considering the terms of uncertainties and the disturbance $\boldsymbol{w}(\cdot)$ therein. On the other hand, the structure of (3.46) indicates that it incorporates (3.54) as a special case. This means there always exists controller gains in (3.46) which can exponentially stabilize the nominal system of (3.43) with $\boldsymbol{w}(t) \equiv 0$. Specifically, (3.54) can be denoted in the form of (3.46) as

$$K_1 = \begin{bmatrix}(KA - XK)\,\mathsf{e}^{Ar} & KB + X\end{bmatrix}, \quad K_2 = \mathsf{O}_{p\times\nu}, \quad K_3 = \widehat{K} \tag{3.56}$$

where $\widehat{K}$ is given in (3.44) and both $A + BK$ and X are Hurwitz.

Based on Theorem 3.1, the following theorem can be derived for the closed-loop system in (3.48).

Theorem 3.3. *Let the parameters G_i, F_i, H_i and Ξ_i, Λ_i, Γ_i in (3.50) and (3.51) be given. Given $\boldsymbol{f}(\cdot)$ with $\mathsf{F}^{-1} = \int_{-r}^{0} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$ and M in (3.44)–(3.45), if there exist $P \in \mathbb{S}^\nu$, $Q \in \mathbb{R}^{\nu\times d\nu}$, $R \in \mathbb{S}^{d\nu}$, $S; U \in \mathbb{S}^\nu$ and $\begin{bmatrix}K_1 & K_2 & K_3\end{bmatrix} \in \mathbb{R}^{p\times(2\nu+d\nu)}$ and $\varkappa_1, \varkappa_2 > 0$ such that the following matrix inequalities*

$$\begin{bmatrix} P & Q \\ * & R + I_d \otimes S \end{bmatrix} \succ 0, \;\; S \succeq 0, \;\; U \succeq 0 \tag{3.57}$$

$$\begin{bmatrix} I & -I - \varkappa_1 \mathbf{F}^\top \mathbf{J}_2 & \varkappa_1 \mathbf{F}^\top \\ * & I - \varkappa_1 \mathbf{J}_3 & \mathsf{O} \\ * & * & \varkappa_1 \mathbf{J}_1 \end{bmatrix} \succ 0 \tag{3.58}$$

$$\begin{bmatrix} \mathbf{\Phi} + \mathbf{Sy}\left[\mathbf{P}^\top(\mathbf{A} + \mathbf{BK})\right] & \mathbf{G} + \varkappa_2 \mathbf{H}^\top \mathbf{J}_2 & \varkappa_2 \mathbf{H}^\top \\ * & \varkappa_2 \mathbf{F}^\top \mathbf{J}_2 + \varkappa_2 \mathbf{J}_2^\top \mathbf{F} + \varkappa_2 \mathbf{J}_3 & \varkappa_2 \mathbf{F}^\top \\ * & * & -\varkappa_2 \mathbf{J}_1 \end{bmatrix} \prec 0 \tag{3.59}$$

are satisfied with

$$\mathbf{G} := \begin{bmatrix} P & \mathsf{O}_{\nu\times m} \\ \mathsf{O}_\nu & \mathsf{O}_{\nu\times m} \\ Q^\top & \mathsf{O}_{d\nu\times m} \\ \mathsf{O}_{q\times\nu} & -J_2^\top \\ \mathsf{O}_{m\times\nu} & \widetilde{J} \end{bmatrix} \left(\begin{bmatrix} \mathbf{Row}_{i=1}^{4}\, G_i & \mathsf{O} \\ \mathsf{O} & \mathbf{Row}_{i=4}^{8}\, G_i \end{bmatrix} \right), \quad \begin{bmatrix} \mathbf{F} \\ \mathbf{J}_1 \\ \mathbf{J}_2 \\ \mathbf{J}_3 \end{bmatrix} := \begin{bmatrix} \bigoplus_{i=1}^8 F_i \\ \bigoplus_{i=1}^8 \Xi_i \\ \bigoplus_{i=1}^8 \Lambda_i \\ \bigoplus_{i=1}^8 \Gamma_i \end{bmatrix} \tag{3.60}$$

$$\mathbf{H} := \begin{bmatrix} \bigoplus_{i=1}^4 H_i \\ \bigoplus_{i=4}^8 H_i \end{bmatrix} \begin{bmatrix} I_{2\nu+d\nu+q} & \mathsf{O}_{(2\nu+d\nu+q)\times m} \end{bmatrix} \tag{3.61}$$

$$\mathbf{P} := \begin{bmatrix} P & \mathsf{O}_\nu & Q & \mathsf{O}_{\nu\times q} & \mathsf{O}_{\nu\times m} \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} A & \mathsf{O}_{n\times\nu} & B & \mathsf{O}_{n\times d\nu} & D_1 & \mathsf{O}_{n\times m} \\ \mathsf{O}_{p\times n} & \mathsf{O}_{p\times\nu} & \mathsf{O}_p & \mathsf{O}_{p\times d\nu} & D_2 & \mathsf{O}_{p\times m} \end{bmatrix}$$

$$\mathbf{B} = \mathbf{Col}\left[\mathsf{O}_{n\times p}, I_p\right], \;\; \mathbf{K} := \begin{bmatrix} K_1 & K_2 & K_3 & \mathsf{O}_{p\times(q+m)} \end{bmatrix}$$

$$\begin{aligned} \mathbf{\Phi} := \mathbf{Sy} & \left(\begin{bmatrix} Q \\ \mathsf{O}_{\nu\times d\nu} \\ R \\ \mathsf{O}_{(q+m)\times d\nu} \end{bmatrix} \begin{bmatrix} \sqrt{\mathsf{F}}\boldsymbol{f}(0) \otimes I_\nu & -\sqrt{\mathsf{F}}\boldsymbol{f}(-r) \otimes I_\nu & -\left(\sqrt{\mathsf{F}} M \sqrt{\mathsf{F}^{-1}}\right) \otimes I_\nu & \mathsf{O}_{d\nu\times(q+m)} \end{bmatrix} \right) \\ & + (S + rU) \oplus (-S) \oplus (-I_d \otimes U) \oplus J_3 \oplus J_1 \\ & + \mathbf{Sy}\left(\begin{bmatrix} \mathsf{O}_{(2\nu+d\nu)\times m} \\ -J_2^\top \\ \widetilde{J} \end{bmatrix} \begin{bmatrix} C_1 & C_2 & C_3 & D_3 & \mathsf{O}_m \end{bmatrix} \right), \end{aligned} \tag{3.62}$$

then the closed-loop system (3.48) *with the supply rate function* (2.15) *is dissipative, and the origin of the closed-loop system* (3.48) *with* $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ *is robustly globally asymptotically stable subject to the uncertainty in* (3.51)

Proof. The proof of this theorem is straightforward given the results presented in Theorem 3.1 and 2.1. ■

Similar to the handling of Theorem 3.1, one can derive an iterative algorithm to solve Theorem 3.3. Now given what we have derived in subsection 3.3.1, one can conclude that the feasible solutions of

$$\begin{bmatrix} \boldsymbol{\Phi} + \mathsf{Sy}\left(\widetilde{\mathbf{P}}^\top \mathbf{B}\mathbf{K} + \mathbf{P}^\top \mathbf{B}\widetilde{\mathbf{K}} - \widetilde{\mathbf{P}}^\top \mathbf{B}\widetilde{\mathbf{K}}\right) & \mathbf{P}^\top - \widetilde{\mathbf{P}}^\top & \mathbf{K}^\top \mathbf{B}^\top - \widetilde{\mathbf{K}}^\top \mathbf{B}^\top & \mathbf{G} + \varkappa_2 \mathbf{H}^\top \mathbf{J}_2 & \varkappa_2 \mathbf{H}^\top \\ * & -Z & \mathsf{O}_\nu & \mathsf{O} & \mathsf{O} \\ * & * & Z - I_\nu & \mathsf{O} & \mathsf{O} \\ * & * & * & \varkappa_2 \mathbf{F}^\top \mathbf{J}_2 + \varkappa_2 \mathbf{J}_2^\top \mathbf{F} + \varkappa_2 \mathbf{J}_3 & \varkappa_2 \mathbf{F}^\top \\ * & * & * & * & -\varkappa_2 \mathbf{J}_1 \end{bmatrix} \prec 0 \quad (3.63)$$

implies the existence of the feasible solutions of (3.59), where $Z \in \mathbb{S}^\nu$ and

$$\begin{aligned} \widetilde{\mathbf{P}} &:= \begin{bmatrix} \widetilde{P} & \mathsf{O}_\nu & \widetilde{Q} & \mathsf{O}_{\nu \times q} & \mathsf{O}_{\nu \times m} \end{bmatrix} \quad \text{with} \quad \widetilde{P} \in \mathbb{S}^\nu \quad \text{and} \quad \widetilde{Q} \in \mathbb{R}^{\nu \times d\nu} \\ \widetilde{\mathbf{K}} &:= \begin{bmatrix} \widetilde{K}_1 & \widetilde{K}_2 & \widetilde{K}_3 & \mathsf{O}_{p \times (q+m)} \end{bmatrix} \quad \text{with} \quad \widetilde{K} = \begin{bmatrix} \widetilde{K}_1 & \widetilde{K}_2 & \widetilde{K}_3 \end{bmatrix} \in \mathbb{R}^{p \times (2\nu + d\nu)} \end{aligned} \quad (3.64)$$

and all other matrices in (3.63) are in line with the definitions in Theorem 3.3. Given the derivations in subsection 3.3.1, Algorithm 3 can be constructed similar to Algorithm 2, where $\mathbf{x}$ contains all the decision variables of $R \in \mathbb{S}^{d\nu}$ and $S; U; Z \in \mathbb{S}^\nu$, while $\boldsymbol{\Lambda} := \begin{bmatrix} P & Q \end{bmatrix} \in \mathbb{R}^{\nu \times \nu(d+1)}$ and $\widetilde{\boldsymbol{\Lambda}} := \begin{bmatrix} \widetilde{P} & \widetilde{Q} \end{bmatrix} \in \mathbb{R}^{\nu \times \nu(d+1)}$. Furthermore, ρ_1, ρ_2 and ε in Algorithm 3 are given constants for regularizations and determining error tolerance, respectively. A distinct feature of Algorithm 3 compared to Algorithm 2 is that we may use the gains of a predictor controller in (3.56) to acquire initial values for the iterative algorithm in Algorithm 3. This is because (3.54) can asymptotically stabilize the nominal system of (3.43) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ as what has been shown in subsection 3.4.2. Moreover, one can always find K and X for (3.56) since (A, B) is stabilizable. As a result, an initial value of $\widetilde{\mathbf{K}}$ and $\widetilde{\mathbf{P}}$ for Algorithm 3 might be obtained by Theorem 3.1 via (3.56) without using a separate theorem as Theorem 3.2. As for the value of K in (3.56), we suggest that it can be obtained by solving a standard convex program corresponding to designing a static state feedback controller for the system resulted from excluding all the delay terms in (3.43).

Algorithm 3: An inner convex approximation solution for Theorem 3.3

begin

Given $K \in \mathbb{R}^{p \times n}$ such that $A + BK$ is Hurwitz

solve Theorem 3.3 with (3.56) to produce P and Q.

solve Theorem 3.3 with the previous P and Q to produce K_1, K_2 and K_3.

let $\widetilde{P} \leftarrow P$, $\widetilde{Q} \leftarrow Q$, $\widetilde{K}_1 \leftarrow K_1$, $\widetilde{K}_2 \leftarrow K_2$, $\widetilde{K}_3 \leftarrow K_3$,

solve $\min\limits_{\mathbf{x}, \boldsymbol{\Lambda}, \mathbf{K}} \mathsf{tr}\left[\rho_1 [*]\left(\boldsymbol{\Lambda} - \widetilde{\boldsymbol{\Lambda}}\right) + \rho_2 [*]\left(\mathbf{K} - \widetilde{\mathbf{K}}\right)\right]$ subject to (3.57), (3.58) and (3.63) to obtain $\boldsymbol{\Lambda}$ and $\mathbf{K}$

while $\dfrac{\left\| \begin{bmatrix} \mathbf{vec}(\boldsymbol{\Lambda}) \\ \mathbf{vec}(\mathbf{K}) \end{bmatrix} - \begin{bmatrix} \mathbf{vec}(\widetilde{\boldsymbol{\Lambda}}) \\ \mathbf{vec}(\widetilde{\mathbf{K}}) \end{bmatrix} \right\|_\infty}{\left\| \begin{bmatrix} \mathbf{vec}(\widetilde{\boldsymbol{\Lambda}}) \\ \mathbf{vec}(\widetilde{\mathbf{K}}) \end{bmatrix} \right\|_\infty + 1} \geq \varepsilon$ **do**

update $\widetilde{\boldsymbol{\Lambda}} \longleftarrow \boldsymbol{\Lambda}$, $\widetilde{\mathbf{K}} \longleftarrow \mathbf{K}$;

solve $\min\limits_{\mathbf{x}, \boldsymbol{\Lambda}, \mathbf{K}} \mathsf{tr}\left[\rho_1 [*]\left(\boldsymbol{\Lambda} - \widetilde{\boldsymbol{\Lambda}}\right) + \rho_2 [*]\left(\mathbf{K} - \widetilde{\mathbf{K}}\right)\right]$ subject to (3.57), (3.58) and (3.63) to obtain $\boldsymbol{\Lambda}$ and $\mathbf{K}$;

end

end

3.5 Numerical examples

Two numerical examples are presented in this section to demonstrate the strength of the proposed scenarios in Chapter 3. The following numerical examples were tested in Matlab with Yalmip [266] as the optimization interface, respectively. Furthermore, all the analytic properties of the delay systems considered in this section are examined by the spectral method in [80, 81] with the code in `http://cdlab.uniud.it/software#eigAM-eigTMN`.

3.5.1 Robust stabilization of an uncertain distributed delay system with dissipativity

Semidefinite programs in this subsections are solved by SDPT3 [270] except for the programs corresponding to the controller in (3.69) which is solved by SeDuMi and reported in subsection 4.2.2 in [57].

Consider a system of the form (3.1) with $r = 1$ and the state space matrices

$$\begin{aligned}
&A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & -1 \\ 0 & 0.9 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.1 & -0.11 \\ 0.21 & 0.1 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} \\
&\widetilde{A}_3(\tau) = \begin{bmatrix} -0.4 - 0.1\mathrm{e}^{\tau}\sin(20\tau) + 0.3\mathrm{e}^{\tau}\cos(20\tau) & 1 + 0.2\mathrm{e}^{\tau}\sin(20\tau) + 0.2\mathrm{e}^{\tau}\cos(20\tau) \\ -1 + 0.01\mathrm{e}^{\tau}\sin(20\tau) - 0.2\mathrm{e}^{\tau}\cos(20\tau) & 0.4 + 0.3\mathrm{e}^{\tau}\sin(20\tau) + 0.4\mathrm{e}^{\tau}\cos(20\tau) \end{bmatrix} \\
&D_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.12 & 0.1 \end{bmatrix},\ C_1 = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix},\ C_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.2 \end{bmatrix} \\
&\widetilde{C}_3(\tau) = \begin{bmatrix} 0.2\mathrm{e}^{\tau}\sin(20\tau) & 0.1 + 0.1\mathrm{e}^{\tau}\cos(20\tau) \\ 0.1\mathrm{e}^{\tau}\sin(20\tau) - 0.1\mathrm{e}^{\tau}\cos(20\tau) & -0.2 + 0.3\mathrm{e}^{\tau}\sin(20\tau) \end{bmatrix}
\end{aligned} \tag{3.65}$$

with the uncertainties parameters $\Delta_i \in \mathbb{R}^{n\times n}$ subject to (3.6) with $\Lambda_i = F_i = \mathsf{O}_n, \forall i = 1\cdots 10$ and

$$\begin{aligned}
&H_1 = \begin{bmatrix} -0.1 & -0.7 \\ -0.3 & 0.3 \end{bmatrix},\ H_2 = \begin{bmatrix} 0.1 \\ -0.3 \end{bmatrix},\ H_3 = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix},\ H_4 = \left[\begin{smallmatrix} 0.2 & 0.1 & -0.1 & 0.3 & 0.12 & -0.2 \\ 0.14 & 0.25 & 0.19 & -0.11 & -0.1 & -0.23 \end{smallmatrix}\right] \\
&H_5 = \begin{bmatrix} 0.2 & 0.2 \\ 0.21 & 0.21 \end{bmatrix},\ H_6 = \begin{bmatrix} -0.12 & -0.14 \\ 0.01 & 0.2 \end{bmatrix},\ H_7 = \begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix},\ H_8 = \begin{bmatrix} 0.12 & -0.14 \\ 0.01 & 0.2 \end{bmatrix} \\
&H_9 = \left[\begin{smallmatrix} 0.2 & 0.1 & -0.1 & -0.14 & 0.1 & -0.1 \\ -0.1 & 0.3 & 0.2 & -0.1 & -0.1 & -0.1 \end{smallmatrix}\right],\ H_{10} = \begin{bmatrix} 0.22 & 0.23 \\ 0.22 & 0.23 \end{bmatrix}, G_i = \begin{bmatrix} 0.04 & 0.04 \\ 0.11 & 0.11 \end{bmatrix}, \forall i = 1\cdots 5 \\
&G_i = \begin{bmatrix} 0.17 & 0.17 \\ 0.14 & 0.14 \end{bmatrix}, \forall i = 6\cdots 10,\ \Xi_1 = \begin{bmatrix} 2.3 & 1 \\ * & 2.4 \end{bmatrix}, \Xi_2 = \begin{bmatrix} 1.5 & -0.5 \\ * & 2.9 \end{bmatrix} \\
&\Xi_3 = \begin{bmatrix} 1.7 & 0.48 \\ * & 1.6 \end{bmatrix},\ \Xi_4 = \begin{bmatrix} 2.5 & 0.51 \\ * & 2 \end{bmatrix},\ \Xi_5 = \begin{bmatrix} 1.7 & 0.44 \\ * & 1.7 \end{bmatrix},\ \Xi_6 = \begin{bmatrix} 1.6 & 0.15 \\ * & 1.4 \end{bmatrix},\ \Xi_7 = \begin{bmatrix} 3.37 & -1.1 \\ * & 1.8 \end{bmatrix} \\
&\Xi_8 = \begin{bmatrix} 1.54 & 0.13 \\ * & 1.34 \end{bmatrix},\ \Xi_9 = \begin{bmatrix} 2.7 & -0.65 \\ * & 1.87 \end{bmatrix},\ \Xi_{10} = \begin{bmatrix} 1.7 & 0.44 \\ * & 1.7 \end{bmatrix},\ \Gamma_i = \begin{bmatrix} -1.75 & -0.58 \\ * & -1.75 \end{bmatrix}, \forall i = 1\cdots 10.
\end{aligned} \tag{3.66}$$

Note that (3.65) is identical to (2.72) without considering the presence of uncertainties. Thus the methods in [185] and [211] still cannot handle (3.65) as we have argued in the previous chapter, even without considering the presence of uncertainties in (3.65). Moreover, it is shown in Chapter 2 by using the spectrum method that (3.65) is unstable for $0 \le r \le 10$ without considering the presence of uncertainties. For the controller objective, again we choose to minimize the value of $\mathbb{L}^2$ attenuation factor γ which corresponds to $J_3 = -J_1 = \gamma I_2$, $\widetilde{J} = I_2$, $J_2 = \mathsf{O}_2$ in (2.15). Now consider the parameters

$$\boldsymbol{f}(\tau) = \begin{bmatrix} 1 \\ 10\mathrm{e}^{\tau}\sin(20\tau) \\ 10\mathrm{e}^{\tau}\cos(20\tau) \end{bmatrix} \otimes I_2, \quad M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 20 \\ 0 & -20 & 1 \end{bmatrix} \tag{3.67}$$

and

$$A_3 = 0.1 \begin{bmatrix} -4 & 10 & -0.1 & 0.2 & 0.3 & 0.2 \\ -10 & 4 & 0.01 & 0.3 & -0.2 & 0.4 \end{bmatrix}, \quad C_3 = 0.1 \begin{bmatrix} 0 & 1 & 0.2 & 0 & 0 & 0.1 \\ 0 & -2 & 0.1 & 0.3 & -0.1 & 0 \end{bmatrix} \tag{3.68}$$

which are the same as in (2.74) with $d = 3,\ n = m = q = 2$.

Now we apply Theorem 3.1 to the system with the parameters in (3.65)–(3.68). It follows that the corresponding uncertain system is robustly stabilized by the controller

$$\boldsymbol{u}(t) = \begin{bmatrix} 3.2847 & -16.7739 \end{bmatrix} \boldsymbol{x}(t) \tag{3.69}$$

for any uncertainties in the corresponding $\mathcal{D}$ with $\min \gamma = 0.62$. To reduce the potential conservatism of the value of $\min \gamma$ calculated by Theorem 3.2, we apply Theorem 3.1 to the previous resulting closed-loop system with the controller gain $K = \begin{bmatrix} 3.2847 & -16.7739 \end{bmatrix}$. It shows that $K = \begin{bmatrix} 3.2847 & -16.7739 \end{bmatrix}$ can achieve $\min \gamma = 0.45941$. Next, apply the spectrum method again to the resulting closed-loop system without considering the uncertainties therein. It produces $-0.1773 < 0$ as the real part of the rightmost characteristic root pair which shows that the nominal resulting closed-loop system is stable.

Now we can apply Algorithm 2 to calculate controller gains with better performance. The results produced by Algorithm 2 with $\rho_1 = \rho_2 = 10^{-7}$ and $\varepsilon = 10^{-12}$ are summarized in Table 3.1 where NoIs standards for the number of iterations executed by the while loop in Algorithm 2. Furthermore, SPA stands for the spectral abscissas of the sets containing the characteristic roots of the nominal closed-loop systems (without uncertainties) in Table 3.1, whose values are obtained by the method in [80, 81].

Controller gains K	$\begin{bmatrix} 3.6897 \\ -19.0393 \end{bmatrix}^\top$	$\begin{bmatrix} 4.0571 \\ -21.1008 \end{bmatrix}^\top$	$\begin{bmatrix} 4.4230 \\ -23.1575 \end{bmatrix}^\top$	$\begin{bmatrix} 4.7878 \\ -25.2108 \end{bmatrix}^\top$
$\min \gamma$	0.459357	0.459321	0.459292	0.459269
NoIs	10	20	30	40
SPA	−0.177	−0.1768	−0.1766	−0.1764

Table 3.1: $\min \gamma$ produced by different iterations

The results in Table 3.1 demonstrate that more iterations lead to better $\min \gamma$ value at the expense of larger numerical complexities. Note that by substituting the resulting K in table 3.1 into Theorem 3.1, the same values of $\min \gamma$ can be obtained by solving the corresponding convex semidefinite programmings. This indicates that the numerical results produced by the iterative Algorithm 2 in terms of $\min \gamma$ are reliable.

3.5.2 Non-fragile dynamical state feedback design for an uncertain linear system with an input delay

We apply Mosek 8.0 [284] as the numerical solver to solve semidefinite programmings in this subsection.

Consider the uncertain open loop system (3.43) with $r = 3$ and the state space matrices

$$\begin{gathered} A = \begin{bmatrix} -1 & 1 \\ 0 & 0.1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix},\ C_1 = \begin{bmatrix} -0.3 & 0.4 & 0.1 \\ -0.3 & 0.1 & -0.1 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 & 0.2 & 0 \\ -0.2 & 0.1 & 0 \end{bmatrix}, \\ \widetilde{C}_3(\tau) = \begin{bmatrix} 0.2 + 0.1\mathrm{e}^{\tau} & 0.1 & 0.12\mathrm{e}^{3\tau} \\ -0.2 & 0.3 + 0.14\mathrm{e}^{2\tau} & 0.11\mathrm{e}^{3\tau} \end{bmatrix},\ D_2 = 0.12,\ \ D_3 = \begin{bmatrix} 0.14 \\ 0.1 \end{bmatrix} \end{gathered} \tag{3.70}$$

with $\Delta_i \in \mathbb{R}^{\nu \times \nu}$ subject to (3.52) with $\Lambda_i = F_i = \mathsf{O}_\nu, \forall i = 1 \cdots 8$ and

$$H_1 = \begin{bmatrix} -0.1 & -0.7 & 0 \\ -0.3 & 0.3 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},\ H_2 = \begin{bmatrix} -0.1 & 0 & 0 \\ -0.3 & 0.3 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},\ H_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0}_9^\top \\ 0.14 & 0.25 & 0.19 & -0.11 & -0.1 & -0.23 & \mathbf{0}_9^\top \\ 0.1 & 0.14 & 0.07 & 0.12 & 0.17 & 0.15 & \mathbf{0}_9^\top \end{bmatrix} \tag{3.71}$$

$$H_4 = \begin{bmatrix} 0.01 \\ 0.02 \\ 0.01 \end{bmatrix},\ H_5 = \begin{bmatrix} 0.12 & -0.14 & 0.13 \\ 0.01 & 0.2 & 0.05 \\ 0.12 & 0.18 & 0.15 \end{bmatrix},\ H_6 = \begin{bmatrix} 0.12 & -0.14 & 0.11 \\ 0.01 & 0.2 & 0.15 \\ 0.11 & 0.12 & 0.011 \end{bmatrix} \tag{3.72}$$

$$H_7 = \begin{bmatrix} 0.2 & 0.1 & -0.1 & -0.14 & 0.1 & -0.1 & \mathbf{0}_9^\top \\ -0.1 & 0.3 & 0.2 & 0.1 & -0.1 & 0.1 & \mathbf{0}_9^\top \\ 0.2 & 0.1 & -0.1 & 0.3 & 0.12 & -0.2 & \mathbf{0}_9^\top \end{bmatrix},\ H_8 = \begin{bmatrix} 0.2 \\ 0.1 \\ 0.1 \end{bmatrix} \tag{3.73}$$

$$G_1 = \begin{bmatrix} 0.04 & 0.04 & 0 \\ 0.11 & 0.11 & 0.1 \\ 0.02 & 0.01 & 0.03 \end{bmatrix},\ G_2 = \begin{bmatrix} 0 & 0 & 0.04 \\ 0.11 & 0.11 & 0.12 \\ 0.05 & 0.07 & 0.03 \end{bmatrix},\ G_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0.11 & 0.11 & 0.1 \\ 0.02 & 0.01 & 0.03 \end{bmatrix} \tag{3.74}$$

$$G_4 = \begin{bmatrix} 0.04 & 0.04 & 0.05 \\ 0.11 & 0.11 & 0.1 \\ 0.02 & 0.01 & 0.03 \end{bmatrix},\ G_i = \begin{bmatrix} 0.11 & 0.14 & 0.12 \\ 0.1 & 0.11 & 0.12 \end{bmatrix},\ \forall i = 6 \cdots 10,\ \Xi_1 = \begin{bmatrix} 2.3 & 1 & 0 \\ 1 & 2.4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.75}$$

$$\Xi_2 = \begin{bmatrix} 1.5 & -0.5 & 0.1 \\ -0.5 & 2.9 & 0.1 \\ 0.1 & 0.1 & 0.11 \end{bmatrix},\ \Xi_3 = \begin{bmatrix} 1.7 & 0.48 & 0 \\ 0.48 & 1.6 & 0 \\ 0 & 0 & 1.2 \end{bmatrix},\ \Xi_4 = \begin{bmatrix} 2.5 & 0.51 & 0.2 \\ 0.51 & 2 & 0.2 \\ 0.2 & 0.2 & 1.5 \end{bmatrix} \tag{3.76}$$

$$\Xi_5 = \begin{bmatrix} 1.7 & 0.44 & 0 \\ 0.44 & 1.7 & 0 \\ 0 & 0 & 1.6 \end{bmatrix},\ \Xi_6 = \begin{bmatrix} 1.6 & 0.15 & 0.12 \\ 0.15 & 1.4 & 0.2 \\ 0.12 & 0.2 & 1.8 \end{bmatrix},\ \Xi_7 = \begin{bmatrix} 3.37 & -1.1 & 0 \\ -1.1 & 1.8 & 0 \\ 0 & 0 & 2.2 \end{bmatrix} \tag{3.77}$$

$$\Xi_8 = \begin{bmatrix} 1.54 & 0.13 & 0 \\ 0.13 & 1.34 & 0 \\ 0 & 0 & 1.7 \end{bmatrix},\ \Gamma_i = \begin{bmatrix} -1.75 & -0.58 & 0 \\ -0.58 & -1.75 & 0 \\ 0 & 0 & -1.75 \end{bmatrix},\ i = 1 \cdots 8. \tag{3.78}$$

Now consider the functions in

$$\mathsf{e}^{-A\tau} B = \textbf{Col}\left[(10/11)\,\mathsf{e}^{-0.1\tau} - (10/11)\,\mathsf{e}^{\tau},\ \mathsf{e}^{-0.1\tau}\right] \tag{3.79}$$

and the functions inside of $\widetilde{C}_3(\tau)$, we apply

$$\boldsymbol{f}(\tau) = \textbf{Col}\left(1, \mathsf{e}^{\tau}, \mathsf{e}^{2\tau}, \mathsf{e}^{3\tau}, \mathsf{e}^{-0.1\tau}\right),\quad \frac{\mathsf{d}\boldsymbol{f}(\tau)}{\mathsf{d}\tau} = M\boldsymbol{f}(\tau) \tag{3.80}$$

with $M = 0 \oplus 1 \oplus 2 \oplus 3 \oplus (-0.1)$ as the basis function to denote all the distributed terms in (3.79) and (3.70). Now let $K = \begin{bmatrix} -0.5249 & -0.4173 \end{bmatrix}$ and $X = -0.1$ in (3.56) with (3.80), where $A + BK$ and X are Hurwitz. Then we have

$$\begin{aligned} C_3 &= \begin{bmatrix} 0.2 & 0.1 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.12 & 0 & 0 & 0 \\ -0.2 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0.14 & 0 & 0 & 0 & 0.11 & 0 & 0 & 0 \end{bmatrix} \left(\sqrt{\mathsf{F}^{-1}} \otimes I_3\right) \\ \widehat{K} &= \begin{bmatrix} \mathbf{0}_5^\top & -0.4295 & \mathbf{0}_8^\top & -0.1789 \end{bmatrix} \left(\sqrt{\mathsf{F}^{-1}} \otimes I_3\right) \end{aligned} \tag{3.81}$$

where $\mathsf{F}^{-1} = \int_{-r}^{0} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$ with $\boldsymbol{f}(\tau)$ in (3.80), and

$$K_1 = \begin{bmatrix} 0.0235 & -0.2629 & -0.5173 \end{bmatrix},\ K_2 = \mathbf{0}_3^\top, \tag{3.82}$$

where K_1, K_2 and $\widehat{K}$ can be used to initiate the iterative algorithm in Algorithm 3.

Given the parameters in (3.70)–(3.78), applying Theorem 3.3 to (3.48) with the controller gains in (3.81) and (3.82) yields a feasible solution and shows the controller can achieve $\min \gamma = 0.99295$. Following the procedures in Algorithm 3, the resulting P and Q produced by the previous optimization problem concerning K_1, K_2 and $\widehat{K}$ in (3.81)–(3.82) are substituted into Theorem 3.3 which now can be solved as a convex optimization program to calculate a new $\mathbf{K}$. By doing so, we obtain

$$\begin{aligned}K_1 &= \begin{bmatrix}-0.0250 & -0.1808 & -0.5636\end{bmatrix}, \; K_2 = \begin{bmatrix}-0.0098 & 0.0067 & -0.0394\end{bmatrix}, \\ K_3 &= \begin{bmatrix}0.0294 & -0.0610 & -0.4612 & 0.0455 & -0.1037 & -0.2287 & -0.0011 & -0.0932 \cdots \\ & -0.1429 & -0.0181 & -0.0767 & -0.0931 & -0.0071 & -0.0328 & -0.5428\end{bmatrix}\end{aligned} \tag{3.83}$$

which can achieve the performance $\gamma = 0.92315$. To proceed, use (3.83) with the values of the associated P and Q for $\grave{\mathbf{Y}}$ and $\grave{\mathbf{K}}$ to follow the steps in Algorithm 3. The results produced by Algorithm 3 with $\rho_1 = 0.01$, $\rho_2 = 0.01$ and $\varepsilon = 10^{-12}$ are summarized in Table 3.2 where NoIs standards for the number of iterations executed by the while loop in Algorithm 3, and SPA stands for the spectral abscissas of the sets containing the characteristic roots of the resulting nominal closed-loop systems (without uncertainties). Note that the values of SPA are calculated via the methods in [80, 81].

$\min \gamma$	0.88264	0.8562911	0.831113	0.807242
NoIs	100	200	300	400
SPA	−0.2749	−0.2635	−0.2608	−0.2874

Table 3.2: $\min \gamma$ produced by Algorithm 3 with different numbers of iterations

Moreover, the controller gains corresponding to the results in Table 3.2 are presented as follows.

$$\begin{aligned}\mathrm{NoI} = 100: K_1 &= \begin{bmatrix}-0.0288 & -0.206 & -0.6271\end{bmatrix}, \; K_2 = \begin{bmatrix}-0.0132 & 0.0124 & -0.0411\end{bmatrix} \\ K_3 &= \begin{bmatrix}0.016 & -0.0383 & -0.4609 & 0.0395 & -0.11 & -0.2298 & -0.0175 & -0.1108 \cdots \\ & -0.1579 & -0.0225 & -0.0872 & -0.1107 & -0.0167 & -0.0236 & -0.5467\end{bmatrix}\end{aligned} \tag{3.84}$$

$$\begin{aligned}\mathrm{NoI} = 200: K_1 &= \begin{bmatrix}-0.0274 & -0.2203 & -0.6585\end{bmatrix}, \; K_2 = \begin{bmatrix}-0.0109 & 0.012 & -0.0351\end{bmatrix} \\ K_3 &= \begin{bmatrix}0.0016 & -0.0174 & -0.4539 & 0.0288 & -0.1194 & -0.2313 & -0.0235 & -0.1306 \cdots \\ & -0.1657 & -0.0283 & -0.1051 & -0.1186 & -0.0306 & -0.0083 & -0.5541\end{bmatrix}\end{aligned} \tag{3.85}$$

$$\begin{aligned}\mathrm{NoI} = 300: K_1 &= \begin{bmatrix}-0.0294 & -0.2350 & -0.6847\end{bmatrix}, \; K_2 = \begin{bmatrix}-0.0072 & 0.0069 & -0.0289\end{bmatrix} \\ K_3 &= \begin{bmatrix}-0.0015 & -0.0006 & -0.4439 & 0.0271 & -0.1219 & -0.2308 & -0.0286 & -0.1433 \cdots \\ & -0.1741 & -0.0325 & -0.1206 & -0.1217 & -0.0381 & -0.0019 & -0.562\end{bmatrix}\end{aligned} \tag{3.86}$$

$$\begin{aligned}\mathrm{NoI} = 400: K_1 &= \begin{bmatrix}-0.0479 & -0.2484 & -0.6754\end{bmatrix}, \; K_2 = \begin{bmatrix}0.0089 & 0.0057 & -0.0155\end{bmatrix} \\ K_3 &= \begin{bmatrix}-0.0055 & 0.0258 & -0.4477 & 0.0205 & -0.1192 & -0.2471 & -0.0431 & -0.1603 \cdots \\ & -0.1909 & -0.0404 & -0.1226 & -0.1304 & -0.0532 & 0.0069 & -0.569\end{bmatrix}\end{aligned} \tag{3.87}$$

Clearly, the results in Table 3.2 demonstrate that more iterations lead to smaller values of $\min \gamma$ at the expense of more numbers of iterations. In addition, it also shows an example of using Algorithm 3 to calculate controller gains with better performance. Finally, we emphasize here again that all the aforementioned resulting closed-loop systems are of the retarded type, thereby satisfying the properties stressed in Remark 3.11.

Chapter 4

Two General Classes of Integral Inequalities Including Weight Functions

4.1 Introduction

Many control and optimization problems involve the applications of integral inequalities. Notable examples can be found in the Liapunov stability analysis or stabilization of linear delay systems as we have demonstrated in the previous two chapters or PDE-related systems [236, 237, 285, 286]. Unlike analyzing the stability of an LTI system, functionals with integral structures are required to handle the stability analysis of infinite dimensional systems and the existing approaches may only lead to sufficient stability conditions due to the intrinsic limitations of their underlying mathematical structures.

For the stability analysis of time-delay systems, enormous efforts have been made to reduce the induced conservatisms when inequalities are applied for the construction of LKFs [142]. Two major classes of inequalities have been proposed. The first type can be called as the Bessel type inequality [57, 187, 262, 287]. The structures of the Bessel type inequalities resemble the structure of the Legendre-Bessel inequality first proposed in [186] which contains no extra variables other than the origin variable in the quadratic term. On the other hand, free matrix type inequalities with extra variables have been proposed in [288–290] motivated by their applications to the stability analysis of time-varying delay systems. Meanwhile, by considering the existing results in the literature, one can clearly see that the applicable structures of LKFs are directly affected by the availability of integral inequalities. Thus it is certainly beneficial to construct LKFs with the support of optimal inequalities.

In this chapter, we derive three general integral inequalities with a detailed analysis of their properties. The proposed inequalities might be applied to a variety of applications, these include but are not limited to delay (time-varying) related systems, PDE-related systems, and sampled-data systems, etc. We propose our first integral inequality in Section 4.2 and we show that it generalizes many existing Bessel-type inequalities in [57, 186, 187, 262, 287, 291–293]. On the other hand, the second integral inequality, which is derived based on the idea discussed in Lemma 4.1 of [294], is presented in Section 4.3 where it shows a relation concerning the inequality bound gaps between the first and second proposed inequalities. Furthermore, the third inequality, which is of the free matrix type, is derived in Section 4.4 generalizing the existing inequalities in [288–290]. We then prove an important conclusion concerning the inequality bound gaps between our proposed three inequalities, by which relations between many existing inequalities might be established. To show a concrete application of our proposed inequalities, we apply them in Section 4.5 to derive stability condition for a linear CDDS [10] with a distributed delay by constructing a parameterized complete LKFs. We show that equivalent stability conditions, whose feasibility is invariant with respect to a parameter of the LKF, can be obtained by the application of proposed inequalities. The core contributions in this chapter are rooted in the generality of the proposed inequalities supported by the nice properties concerning their inequality bound gaps. This provides great potential to apply them to tackle problems in the context of control and optimizations.

4.2 First inequality

The first integral inequality is derived in this section where the generality of the inequality is demonstrated mathematically comparing with existing results in the literature.

To present our results in this section, we define the weighted Lebesgue function space

$$\mathbb{L}^2_{\varpi}\left(\mathcal{K} \,;\, \mathbb{R}^d\right) := \left\{\boldsymbol{\phi}(\cdot) \in \mathbb{L}_f\left(\mathcal{K} \,;\, \mathbb{R}^d\right) : \|\boldsymbol{\phi}(\cdot)\|_{2,\varpi} < \infty\right\} \tag{4.1}$$

with $d \in \mathbb{N}$ and the semi-norm $\|\boldsymbol{\phi}(\cdot)\|_{2,\varpi} := \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{\phi}^\top(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau$ where $\varpi(\cdot) \in \mathbb{L}_f(\mathcal{K} \,;\, \mathbb{R}_{\geq 0})$ and the function $\varpi(\cdot)$ has only countably infinite or finite number of zero values. Furthermore, $\mathcal{K} \subseteq \mathbb{R} \cup \{\pm\infty\}$ and the Lebesgue measure of $\mathcal{K}$ is non-zero.

Theorem 4.1. *Given $\varpi(\cdot)$ in (4.1) and $U \in \mathbb{S}^n_{\succ 0}$ and $\boldsymbol{f}(\cdot) \in \mathbb{L}^2_{\varpi}\left(\mathcal{K} \,;\, \mathbb{R}^d\right)$ which satisfies*

$$\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau \succ 0, \tag{4.2}$$

then we have

$$\forall \boldsymbol{x}(\cdot) \in \mathbb{L}^2_{\varpi}(\mathcal{K} \,;\, \mathbb{R}^n), \quad \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau \geq \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)F^\top(\tau)\mathsf{d}\tau\,(\mathsf{F} \otimes U)\int_{\mathcal{K}} \varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau \tag{4.3}$$

where $F(\tau) := \boldsymbol{f}(\tau) \otimes I_n$ and $\mathsf{F}^{-1} = \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$.

Proof. See Appendix B for details. ■

The inequality (4.3) holds for any $\boldsymbol{f}(\cdot) \in \mathbb{L}^2_{\varpi}\left(\mathcal{K} \,;\, \mathbb{R}^d\right)$ satisfying (4.2) with a given $\varpi(\cdot)$. Note that the constraint (4.2) in Theorem 4.1 indicates the functions in $\boldsymbol{f}(\cdot)$ are linear independently (See the Theorem 7.2.10 in [258]) in a Lebesgue sense. Thus the flexibility of the choice of $\boldsymbol{f}(\cdot)$ is very general for (4.3). This includes the situation of $\boldsymbol{f}(\tau)$ containing orthogonal functions, elementary functions or other type of function as long as they are linearly independent in a Lebesgue sense. As a result, the structure of (4.3) is by far the most general Bessel type inequality in terms of the applicable integral kernels of the lower quadratic bound. Finally, the inequality in (4.3) still holds if $U \succeq 0$. Note that $U \succ 0$ in (4.3) is taken as the prerequisite of Theorem 4.1 to make sure that relations can be established between (4.3) and the integral inequalities proposed in later sections.

The generality of (4.3) will be demonstrated with mathematical details as follows. To do so, let us first give the standard expression of Jacobi polynomials (See 22.3.2 in [295])

$$j_d^{\alpha,\beta}(\tau)_{-1}^{1} := \frac{\gamma(d+1+\alpha)}{d!\gamma(d+1+\alpha+\beta)}\sum_{k=0}^{d}\binom{d}{k}\frac{\gamma(d+k+1+\alpha+\beta)}{\gamma(k+1+\alpha)}\left(\frac{\tau-1}{2}\right)^k, \quad \tau \in [-1,1]. \tag{4.4}$$

over $[-1,1]$, where $\gamma(\cdot)$ stands for the standard gamma function with $d \in \mathbb{N}_0$ and $\alpha > -1, \beta > -1$. The polynomials in (4.6) follows the following orthogonal property (See 22.2.1 in [295])

$$\int_{-1}^{1}(1-\tau)^\alpha(\tau+1)^\beta \boldsymbol{j}_d^{\alpha,\beta}(\tau)_{-1}^{1}\left[\boldsymbol{j}_d^{\alpha,\beta}(\tau)_{-1}^{1}\right]^\top \mathsf{d}\tau = \bigoplus_{k=0}^{d}\frac{2^{\alpha+\beta+1}\gamma(k+\alpha+1)\gamma(k+\beta+1)}{k!(2k+\alpha+\beta+1)\gamma(k+\alpha+\beta+1)} \tag{4.5}$$

with $\boldsymbol{j}_d^{\alpha,\beta}(\tau)_{-1}^1 = \mathsf{Col}_{i=0}^d j_i^{\alpha,\beta}(\tau)_{-1}^1$, where the polynomials in (4.4) are orthogonal with respect to $(1-\tau)^\alpha(\tau+1)^\beta$ over $[-1,1]$. However, it is preferable to derive a general expression for Jacobi polynomials defined over $[a,b]$ with $b>a$. Specifically, consider the affine transformation $\frac{2\tau-a-b}{b-a}\to\tau$ where the affine function $\frac{2\tau-a-b}{b-a}$ satisfies $-1\le\frac{2\tau-a-b}{b-a}\le 1$ for $\tau\in[a,b]$ with $b>a$. The shift-scaled Jacobi polynomials $j_d^{\alpha,\beta}\left(\frac{2\tau-a-b}{b-a}\right)_{-1}^1$ is expressed as

$$j_d^{\alpha,\beta}(\tau)_a^b := j_d^{\alpha,\beta}\left(\frac{2\tau-a-b}{b-a}\right)_{-1}^1 = \frac{\gamma(d+1+\alpha)}{d!\gamma(d+1+\alpha+\beta)}\sum_{k=0}^d\binom{d}{k}\frac{\gamma(d+k+1+\alpha+\beta)}{\gamma(k+1+\alpha)}\left(\frac{\tau-b}{b-a}\right)^k \tag{4.6}$$

with $\tau\in[a,b]$. Now using the affine transformation $\frac{2\tau-a-b}{b-a}\to\tau$ to (4.5) yields

$$\begin{aligned}\int_a^b\left(\frac{-2\tau+2b}{b-a}\right)^\alpha\left(\frac{2\tau-2a}{b-a}\right)^\beta\boldsymbol{j}_d^{\alpha,\beta}\left(\frac{2\tau-a-b}{b-a}\right)_{-1}^1\left[\boldsymbol{j}_d^{\alpha,\beta}\left(\frac{2\tau-a-b}{b-a}\right)_{-1}^1\right]^\top \mathsf{d}\left(\frac{2\tau-a-b}{b-a}\right)\\ =\frac{2^{\alpha+\beta+1}}{(b-a)^{\alpha+\beta+1}}\int_a^b(b-\tau)^\alpha(\tau-a)^\beta\boldsymbol{j}_d^{\alpha,\beta}(\tau)_a^b\left[\boldsymbol{j}_d^{\alpha,\beta}(\tau)_a^b\right]^\top\mathsf{d}\tau\\ =\bigoplus_{k=0}^d\frac{2^{\alpha+\beta+1}\gamma(k+\alpha+1)\gamma(k+\beta+1)}{k!(2k+\alpha+\beta+1)\gamma(k+\alpha+\beta+1)}\end{aligned} \tag{4.7}$$

where $\boldsymbol{j}_d^{\alpha,\beta}(\tau)_a^b := \mathsf{Col}_{i=0}^d j_k^{\alpha,\beta}(\tau)_a^b$ for $\tau\in[a,b]$. Moreover, the equality in (4.7) can be rewritten into

$$\int_a^b(b-\tau)^\alpha(\tau-a)^\beta\boldsymbol{j}_d^{\alpha,\beta}(\tau)_a^b\left[\boldsymbol{j}_d^{\alpha,\beta}(\tau)_a^b\right]^\top\mathsf{d}\tau=\bigoplus_{k=0}^d\frac{(b-a)^{\alpha+\beta+1}\gamma(k+\alpha+1)\gamma(k+\beta+1)}{k!(2k+\alpha+\beta+1)\gamma(k+\alpha+\beta+1)}$$

which now is the expression for the orthogonality of (4.4) with respect to $\varpi(\tau)=(b-\tau)^\alpha(\tau-a)^\beta$. Note that for $\alpha=\beta=0$, (4.6) becomes Legendre polynomials `https://www.encyclopediaofmath.org/index.php?title=Jacobi_polynomials`

$$\ell_d(\tau) := \sum_{k=0}^d\binom{d}{k}\binom{d+k}{k}\left(\frac{\tau-b}{b-a}\right)^k \tag{4.8}$$

with $d\in\mathbb{N}_0$ and $\tau\in[a,b]$, which satisfies $\int_a^b\boldsymbol{\ell}_d(\tau)\boldsymbol{\ell}_d^\top(\tau)\mathsf{d}\tau=\bigoplus_{k=0}^d\frac{b-a}{(2k+1)}$.

Remark 4.1. Let $\alpha=0$ and $\beta\in\mathbb{N}_0$, then (4.6) becomes identical to the orthogonal hyper-geometric polynomials defined in the equations (13) and (14) in [287]. By using the Cauchy formula for repeated integrations[1], it is easy to see that (4.6) with $\alpha=0,\beta\in\mathbb{N}_0$ and $\alpha\in\mathbb{N}_0,\beta=0$ are also equivalent to the polynomials defined in the equations (3) and (4) in [292], respectively. (see (10) in [296] also).

Having presented the expressions of Jacobi and Legendre polynomials, the details of the existing integral inequalities as the special cases of (4.3) are listed in Table 4.1 with the corresponding $\mathcal{K}$, $\varpi(\tau)$, $\boldsymbol{f}(\tau)$ and $\boldsymbol{x}(\cdot)$. Note that some of the results in Table 4.1 require the application of the Cauchy formula for repeated integration.

[1] see (5),(6) and (25),(26) in [287] and the Lemma 1 in [290] for concrete examples

(4.3)	$\mathcal{K}$	$\varpi(\tau)$	$\boldsymbol{f}(\tau)$	$\boldsymbol{x}(\cdot)$
(5) in [187]	$[-r,0]$	1	$\boldsymbol{j}_d^{0,0}(\tau)_{-r}^0$	$\boldsymbol{x}(\cdot)$
(6) in [187]	$[-r,0]$	1	$\boldsymbol{j}_d^{0,0}(\tau)_{-r}^0$	$\dot{\boldsymbol{x}}(\cdot)$
(5) in [263]	$[a,b]$	1	$\begin{bmatrix}1\\ \boldsymbol{p}(\tau)\end{bmatrix}$	$\boldsymbol{x}(\cdot)$
(27) in [287]	$[a,b]$	$\frac{(\tau-a)^p}{(a-b)^p}$	$\boldsymbol{j}_d^{0,p}(\tau)_a^b$	$\boldsymbol{x}(\cdot)$
(34) in [287]	$[a,b]$	$\frac{(\tau-a)^p}{(a-b)^p}$	$\boldsymbol{j}_d^{0,p}(\tau)_a^b$	$\dot{\boldsymbol{x}}(\cdot)$
(1) in [292]	$[a,b]$	$(\tau-a)^{m-1}$	$\boldsymbol{j}_d^{0,m-1}(\tau)_a^b$	$\boldsymbol{x}(\cdot)$
(2) in [292]	$[a,b]$	$(b-\tau)^{m-1}$	$\boldsymbol{j}_d^{m-1,0}(\tau)_a^b$	$\boldsymbol{x}(\cdot)$
(2) in [291]	$[a,b]$	$(\tau-a)^k$	$\boldsymbol{p}_k(\tau)$	$\boldsymbol{x}(\cdot)$
(2.17)	$\mathcal{K}$	1	$\boldsymbol{g}(\tau)$	$\boldsymbol{x}(\cdot)$
(9) in [293]	$[0,+\infty]$	$K(\tau)$	$\begin{bmatrix}1\\ g(\tau)\end{bmatrix}$	$\boldsymbol{x}(\cdot)$

Table 4.1: List of integral inequalities encompassed by (4.3)

For $\boldsymbol{p}_k(\cdot)$ in Table 4.1, it is defined as

$$\boldsymbol{p}_k(\cdot) \in \left\{ \boldsymbol{f}(\cdot) = \operatorname*{\mathsf{Col}}_{i=1}^{d} f_i(\cdot) \in \mathbb{L}^2_{(\tau-a)^k}\left([a,b] ⨾ \mathbb{R}^d\right) : \int_{\mathcal{K}} (\tau-a)\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau = \bigoplus_{i=1}^{d} \int_a^b (\tau-a)^k f_i^2(\tau)\mathsf{d}\tau \right\}$$

with $k \in \mathbb{N}_0$, which means all the functions in $\boldsymbol{p}_k(\cdot)$ are orthogonal functions with respect to the corresponding weight functions. Furthermore, $\boldsymbol{p}(\cdot)$ in Table 4.1 is defined as

$$\boldsymbol{p}(\cdot) \in \left\{ \boldsymbol{f}(\cdot) = \operatorname*{\mathsf{Col}}_{i=1}^{d-1} f_i(\cdot) \in \mathbb{L}^2\left([a,b] ⨾ \mathbb{R}^{d-1}\right) : \int_a^b \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau = \bigoplus_{i=1}^{d-1} \int_a^b f_i^2(\tau)\mathsf{d}\tau \quad \& \quad \int_a^b \boldsymbol{f}(\tau)\mathsf{d}\tau = \mathbf{0}_d \right\}$$

where it contains the auxiliary functions generated by the process in Lemma 1 of [263]. The terms $\varpi(\tau) = K(\tau)$ and $\boldsymbol{f}(\tau) = \mathsf{Col}\,[1, g(\tau)]$ are in line with the definitions in the Theorem 1 of [293]. Finally, the inclusions by (4.3) in Table 4.1 concerning the inequalities in [287] is demonstrated as follows.

Let $\alpha = 0$, $\beta = p \in \mathbb{N}_0$ with $\varpi(\tau) = (\tau-a)^p$ and $\boldsymbol{f}(\tau) = \boldsymbol{j}_d^{0,p}(\tau)_a^b$, then (4.3) becomes

$$\forall \boldsymbol{x}(\cdot) \in \mathbb{L}^2_{\varpi}\left(\mathcal{K} ⨾ \mathbb{R}^n\right), \quad \int_a^b (\tau-a)^p \boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau \geq \int_a^b (\tau-a)^p \boldsymbol{x}^\top(\tau)\left(\boldsymbol{\ell}_{d+p}^\top(\tau)\otimes I_n\right)\mathsf{d}\tau\, \Xi^\top(\mathsf{D}_d \otimes U)\,\Xi \times \int_a^b (\tau-a)^p \left(\boldsymbol{\ell}_{d+p}(\tau)\otimes I_n\right)\boldsymbol{x}(\tau)\mathsf{d}\tau \tag{4.9}$$

where

$$\mathsf{D}_d^{-1} = \bigoplus_{k=0}^{d} \frac{(b-a)^{p+1}}{2k+1+p}, \quad \Xi := \mathbf{Y}(P\otimes I_n), \quad \mathbf{Y} := \left(\mathsf{J}_{0,p}\mathsf{L}^{-1}\right)\otimes I_n, \quad \boldsymbol{j}_d^{0,p}(\tau)_a^b = \mathsf{J}_{0,p}\operatorname*{\mathsf{Col}}_{i=0}^{d}\tau^i, \quad \boldsymbol{\ell}_d(\tau) = \mathsf{L}\operatorname*{\mathsf{Col}}_{i=0}^{d}\tau^i$$

with $\mathsf{J}_{0,p} \in \mathbb{R}^{(d+1)\times(d+1)}_{[d+1]}$ and $\mathsf{L} \in \mathbb{R}^{(d+1)\times(d+1)}_{[d+1]}$. Moreover, the matrix $P \in \mathbb{R}^{(d+1)\times(d+p+1)}$ satisfies $(\tau-a)^p\boldsymbol{\ell}_d(\tau) = P\boldsymbol{\ell}_{d+p}(\tau)$. By using the multiplier $(b-a)^{-p}$, it is easy to see that (27) in [287] is equivalent to (4.9). Now considering (4.9) with the substitution $\dot{\boldsymbol{x}}(\tau) \to \boldsymbol{x}(\tau)$, we have

$$\int_a^b (\tau-a)^p \dot{\boldsymbol{x}}^\top(\tau)U\dot{\boldsymbol{x}}(\tau)\mathsf{d}\tau \geq [*]\left[\Xi^\top\left(\mathsf{D}_d\otimes U\right)\Xi\right]\int_a^b (\tau-a)^p\left(\boldsymbol{\ell}_{d+p}(\tau)\otimes I_n\right)\dot{\boldsymbol{x}}(\tau)\mathsf{d}\tau = \boldsymbol{\eta}_1^\top\Omega_1^\top\Xi^\top(\mathsf{D}_d\otimes U)\,\Xi\Omega_1\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2^\top\Omega_2^\top\Xi^\top(\mathsf{D}_d\otimes U)\,\Xi\Omega_2\boldsymbol{\eta}_2 \tag{4.10}$$

where

$$\boldsymbol{\eta}_1 := \begin{bmatrix} \boldsymbol{x}(b) \\ \boldsymbol{x}(a) \\ \int_a^b (\boldsymbol{\ell}_{d+p}(\tau) \otimes I_n)\, \boldsymbol{x}(\tau)\mathsf{d}\tau \end{bmatrix}, \quad \boldsymbol{\eta}_2 := \begin{bmatrix} \boldsymbol{x}(b) \\ \boldsymbol{x}(a) \\ \int_a^b (\boldsymbol{\ell}_{d+p-1}(\tau) \otimes I_n)\, \boldsymbol{x}(\tau)\mathsf{d}\tau \end{bmatrix}$$
$$\Omega_1 = [\boldsymbol{\ell}_{d+p}(0), \ \ \boldsymbol{\ell}_{d+p}(-r) \ \ \Lambda_1] \otimes I_n \quad \Omega_2 = [\boldsymbol{\ell}_{d+p}(0) \ \ \boldsymbol{\ell}_{d+p}(-r) \ \ \Lambda_2] \otimes I_n \tag{4.11}$$

with $\Lambda_1 \in \mathbb{R}^{(d+p)\times(d+p)}$ and $\Lambda_2 \in \mathbb{R}^{(d+p)\times(d+p-1)}$ satisfying $\dot{\boldsymbol{\ell}}_{d+p}(\tau) = \Lambda_1 \boldsymbol{\ell}_{d+p}(\tau) = \Lambda_2 \boldsymbol{\ell}_{d+p-1}(\tau)$. Again by adjusting the factor $(b-a)^{-p}$ with (4.10), one can conclude that the result in the (34) of [287] can be obtained by (4.3).

It is obvious that the structure of $\boldsymbol{f}(\cdot)$ in Theorem 4.1 may significantly affect the inequality bound gaps of (4.3). In the following corollary, we show that a substitution $G\boldsymbol{f}(\tau) \to \boldsymbol{f}(\tau)$ for (4.3) with an invertible G does not change the bound gap of (4.3). This shows that when d and $f_i(\tau)$ in $\boldsymbol{f}(\tau) = \mathsf{Col}_{i=1}^d f_i(\tau)$ are fixed, then using linear combinations of $f_i(\tau)$ does not change the inequality bound gaps of (4.3).

Corollary 4.1. *Given the same $\varpi(\cdot)$, $U \in \mathbb{S}^n_{\succ 0}$ and $\boldsymbol{f}(\cdot) \in \mathbb{L}^2_{\varpi}(\mathcal{K} ⨾ \mathbb{R}^d)$ in Theorem 4.1, we have*

$$\forall \boldsymbol{x}(\cdot) \in \mathbb{L}^2_{\varpi}(\mathcal{K} ⨾ \mathbb{R}^n), \quad \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau \geq \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)F^\top(\tau)\mathsf{d}\tau\, (\mathsf{F}\otimes U) \int_{\mathcal{K}} \varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau$$
$$= \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)\Phi^\top(\tau)\mathsf{d}\tau\, (\mathsf{\Phi}\otimes U) \int_{\mathcal{K}} \varpi(\tau)\Phi(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau \tag{4.12}$$

for all $G \in \mathbb{R}^{n\times n}_{[n]}$, where $\Phi(\tau) = \boldsymbol{\varphi}(\tau)\otimes I_n$ with $\boldsymbol{\varphi}(\tau) = G\boldsymbol{f}(\tau)$, and $\mathsf{\Phi}^{-1} := \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{\varphi}(\tau)\boldsymbol{\varphi}^\top(\tau)\mathsf{d}\tau$ and $F(\tau)$, F are the same defined in Theorem 4.1.

Proof. Note that $\mathsf{\Phi}$ is calculated by the expression

$$\mathsf{\Phi}^{-1} = \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{\varphi}(\tau)\boldsymbol{\varphi}^\top(\tau)\mathsf{d}\tau = \int_{\mathcal{K}} \varpi(\tau)G\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)G^\top\mathsf{d}\tau$$
$$= G\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau G^\top = G\mathsf{F}^{-1}G^\top \tag{4.13}$$

where $\mathsf{\Phi}^{-1}$ is well defined given the fact that $G \in \mathbb{R}^{d\times d}_{[d]}$. By (4.13) with the property of the Kronecker product in (2.1), we have

$$\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)\Phi^\top(\tau)\mathsf{d}\tau\ (\mathsf{\Phi}\otimes U)\int_{\mathcal{K}} \varpi(\tau)\Phi(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau = \int_{\mathcal{K}} \boldsymbol{x}^\top(\tau)\left(\boldsymbol{\varphi}^\top(\tau)\otimes I_n\right)\mathsf{d}\tau\left[(G^{-1})^\top\mathsf{F}G^{-1}\otimes U\right]\times$$
$$\int_{\mathcal{K}} (\boldsymbol{\varphi}(\tau)\otimes I_n)\,\boldsymbol{x}(\tau)\mathsf{d}\tau = \int_{\mathcal{K}} \boldsymbol{x}^\top(t+\tau)\left(\boldsymbol{f}^\top(\tau)G^\top\otimes I_n\right)\mathsf{d}\tau\left([*](\mathsf{F}\otimes U)\left(G^{-1}\otimes I_n\right)\right)\int_{\mathcal{K}} (G\boldsymbol{f}(\tau)\otimes I_n)\,\boldsymbol{x}(t+\tau)\mathsf{d}\tau$$
$$= \int_{\mathcal{K}} \boldsymbol{x}^\top(t+\tau)F^\top(\tau)\mathsf{d}\tau\ (\mathsf{F}\otimes U)\int_{\mathcal{K}} F(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau$$

which gives (4.12) based on the inequality in (4.3). ■

Remark 4.2. Let G in (4.12) be given by the relation $G^{-2} = \int_{\mathcal{K}} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$ which infers that $G\boldsymbol{f}(\tau)$ only contains functions which are mutually orthogonal. The conclusion of Corollary 4.1 has an important implication: using $\boldsymbol{f}(\cdot)$, which may not only contain orthogonal functions, for (4.3) does not degenerate the bound gap of (4.3) compared to using the orthogonal option $G\boldsymbol{f}(\tau)$.

4.3 Second integral inequality with a slack variable

Inspired by the result in Theorem 4.1 of [294], we derive the second integral inequality in this chapter with a slack variable, where the smallest inequality bound gap is obtained when the slack variable is chosen to make the second inequality identical to (4.3). The structure of the proposed inequality in this section implies that it might be useful for the analysis of sampled-data systems as pointed out in [294] or other potential problems which are subject to future researches. We also show that our proposed inequality in fact generalizes many existing results in the literature.

We first present the following lemma which is crucial for the derivation of the results in this subsection. The lemma is partially taken from Lemma 4.1 in [294].

Lemma 4.1. *Given matrices* $C \in \mathbb{S}^m_{\succ 0}$*,* $B \in \mathbb{R}^{m\times n}$*, then*

$$\forall M \in \mathbb{R}^{m\times n}, \;\; B^\top C^{-1} B \succeq M^\top B + B^\top M - M^\top C M \tag{4.14}$$

where $B^\top C^{-1} B = M^\top B + B^\top M - M^\top C M$ *can be obtained with* $M = C^{-1}B$*.*

It is obvious that

Proof.

$$M^\top B + B^\top M - M^\top C M = B^\top C^{-1} B - \begin{bmatrix} B^\top & M^\top C \end{bmatrix} \begin{bmatrix} C^{-1} & C^{-1} \\ C^{-1} & C^{-1} \end{bmatrix} \begin{bmatrix} B \\ CM \end{bmatrix} \preceq B^\top C^{-1} B \tag{4.15}$$

since

$$\begin{aligned} \begin{bmatrix} -B^\top & M^\top C \end{bmatrix} \begin{bmatrix} C^{-1} & C^{-1} \\ C^{-1} & C^{-1} \end{bmatrix} \begin{bmatrix} -B \\ CM \end{bmatrix} &= \begin{bmatrix} -B^\top C^{-1} + M^\top & -B^\top C^{-1} + M^\top \end{bmatrix} \begin{bmatrix} -B \\ CM \end{bmatrix} \\ &= B^\top C^{-1} B - M^\top B - B^\top M + M^\top C M \end{aligned} \tag{4.16}$$

and $\begin{bmatrix} C^{-1} & C^{-1} \\ C^{-1} & C^{-1} \end{bmatrix} \succeq 0$. Moreover, $B^\top C^{-1} B = M^\top B + B^\top M - M^\top C M$ if $M = C^{-1}B$. ■

Theorem 4.2. *Given the same* $\varpi(\cdot)$*,* U *and* $\boldsymbol{f}(\cdot)$ *defined in Theorem 4.1, then*

$$\forall \boldsymbol{x}(\cdot) \in \mathbb{L}^2_\varpi(\mathcal{K} ⨾ \mathbb{R}^n), \; \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau) U \boldsymbol{x}(\tau)\mathsf{d}\tau \geq \boldsymbol{\zeta}^\top \left[\mathbf{Sy}(H^\top\Omega) - H^\top\left(\mathsf{F}^{-1}\otimes U^{-1}\right) H\right]\boldsymbol{\zeta} \tag{4.17}$$

with $H \in \mathbb{R}^{dn\times\nu}$*, where* $\Omega\boldsymbol{\zeta} = \int_{\mathcal{K}} \varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau$ *with* $\boldsymbol{\zeta} \in \mathbb{R}^\nu$ *and* $\Omega \in \mathbb{R}^{dn\times\nu}$*. Finally, with* $H = (\mathsf{F}\otimes U)\,\Omega$*, then* (4.17) *and* (4.3) *become identical and this is the case that the smallest inequality bound gap of* (4.17) *is attained.*

Proof. Since $\Omega\boldsymbol{\zeta} = \int_{\mathcal{K}} \varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau$ with $\Omega \in \mathbb{R}^{dn\times\nu}$ and $\boldsymbol{\zeta} \in \mathbb{R}^\nu$, (4.3) can be rewritten into

$$\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau) U \boldsymbol{x}(\tau)\mathsf{d}\tau \geq \boldsymbol{\zeta}^\top\Omega^\top\left(\mathsf{F}\otimes U\right)\Omega\boldsymbol{\zeta}. \tag{4.18}$$

Now since $\mathsf{F} \succ 0$ and $U \succ 0$ with $\mathsf{F}^{-1}\otimes U^{-1} = (\mathsf{F}\otimes U)^{-1}$, applying Lemma 4.1 to the lower bound of (4.18) yields the results of Theorem 4.2. ■

Remark 4.3. Note that the definition $\Omega\boldsymbol{\zeta} = \int_{\mathcal{K}} \varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau$ does not add constraint on $\boldsymbol{f}(\cdot)$ or $\boldsymbol{x}(\cdot)$ since one can always find certain values of $\boldsymbol{\zeta}$ and Ω to make the equality holds. In the context of analyzing the stability of systems with delays, one can choose a fixed Ω with appropriate $\boldsymbol{\zeta}$ to render $\Omega\boldsymbol{\zeta} = \int_{\mathcal{K}} \varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau$ to be an identity valid for all $\boldsymbol{x}(\cdot) \in \mathbb{L}^2_\varpi(\mathcal{K} ⨾ \mathbb{R}^n)$.

Remark 4.4. Corollary 4.2 generalizes the Theorem 4.1 in [294] with $\boldsymbol{f}(\tau) = 1$ and $\varpi(\tau) = 1$. Furthermore, let $\mathcal{K} = [a,b]$ and $\varpi(\cdot) = 1$ and $\boldsymbol{f}(\tau)$ to contain Legendre polynomials over $[a,b]$, then Lemma 1 in [296] can be obtained from (4.17) with appropriate $\boldsymbol{\zeta}$ and Ω using the substitution $\dot{\boldsymbol{x}}(\cdot) \to \boldsymbol{x}(\cdot)$. Now consider the fact that the left hand of the inequality (9) in [296] can be rewritten into a one fold integral with a weight function by using the Cauchy formula for repeated integrations. Let $\mathcal{K} = [a,b]$, $\varpi(\tau) = (\tau-a)^m$ and $\boldsymbol{f}(\tau)$ to contain Jacobi polynomials associated with $(\tau-a)^m$ over $[a,b]$, hence the integral inequality in [296] can be obtained by (4.17) with appropriate $\boldsymbol{\zeta}$ and Ω using the substitution $\dot{\boldsymbol{x}}(\cdot) \to \boldsymbol{x}(\cdot)$. As a result, all inequalities in [296] are the particular examples of (4.17). Finally, since (4.17) is equivalent to (4.3), it also indicates that equivalence relations can be established between the inequalities in [292, 296].

Similar to Corollary 4.1, we show in the following corollary that using a substitution $G\boldsymbol{f}(\tau) \to \boldsymbol{f}(\tau)$ to (4.17) with an invertible G does not change the smallest achievable inequality bound gap of (4.17).

Corollary 4.2. *Given the same $\varpi(\cdot)$, U and $\boldsymbol{f}(\cdot)$ in Theorem 4.1 (Corollary 4.2), then for any $G \in \mathbb{R}^{n\times n}_{[n]}$ we have*

$$\forall \boldsymbol{x}(\cdot) \in \mathbb{L}^2_\varpi(\mathcal{K} ⨾ \mathbb{R}^n), \quad \int_\mathcal{K} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau \geq \widehat{\boldsymbol{\zeta}}^\top \left[\mathbf{Sy}\left(\widehat{H}^\top\widehat{\Omega}\right) - H^\top\left(\Phi^{-1}\otimes U^{-1}\right)H\right]\widehat{\boldsymbol{\zeta}} \tag{4.19}$$

with $H \in \mathbb{R}^{dn\times\nu}$, where $\Phi^{-1} := \int_\mathcal{K}\varpi(\tau)\boldsymbol{\varphi}(\tau)\boldsymbol{\varphi}^\top(\tau)\mathsf{d}\tau$ with $\boldsymbol{\varphi}(\tau) = G\boldsymbol{f}(\tau)$, and $\widehat{\Omega}\widehat{\boldsymbol{\zeta}} = \int_\mathcal{K}\varpi(\tau)\Phi(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau$ with $\widehat{\boldsymbol{\zeta}} \in \mathbb{R}^\nu$ and $\widehat{\Omega} \in \mathbb{R}^{dn\times\nu}$. Finally, (4.19) and (4.3) become identical with $\widehat{H} = (\mathsf{F}\otimes U)\widehat{\Omega}$ and this is the situation that the smallest inequality bound gap of (4.19) is attained which is invariant to the value of $G \in \mathbb{R}^{n\times n}_{[n]}$ and identical to the smallest achievable inequality bound gap of (4.17).

Proof. Let $G \in \mathbb{R}^{n\times n}_{[n]}$. Since $\widehat{\Omega}\widehat{\boldsymbol{\zeta}} = \int_\mathcal{K}\varpi(\tau)\Phi(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau$ with $\widehat{\boldsymbol{\zeta}} \in \mathbb{R}^\nu$ and $\widehat{\Omega} \in \mathbb{R}^{dn\times\nu}$, (4.12) can be rewritten into

$$\int_\mathcal{K} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau \geq \widehat{\boldsymbol{\zeta}}^\top\widehat{\Omega}^\top(\Phi\otimes U)\widehat{\Omega}\widehat{\boldsymbol{\zeta}}. \tag{4.20}$$

By (4.13), it is obvious that $\Phi \succ 0$ for all $G \in \mathbb{R}^{n\times n}_{[n]}$. Now since $\Phi \succ 0$ and $U \succ 0$, applying Lemma 4.1 to the lower bound of (4.18) yields the results in (4.2). ■

4.4 Third integral inequality of free matrix type

This section is devoted to the presentation of another general integral inequality with extra matrix variables, including the analysis of its inequality bound gaps and other properties. The proposed inequality can be regarded belonging to the class of free matrix type inequalities which have been previously researched in [288–290] and applied in dealing with the stability analysis of systems with time-varying delays via the LKF approach. As mentioned in Remark 7 of [289], the utilization of a free matrix type inequality can avoid appealing to the use of reciprocally convex combination in the situation of analyzing the stability of a system with a time-varying delay. See the Remark 7 in [289] for further references therein. Finally, we can prove that the smallest achievable inequality bound gap of the proposed inequality in this section is the same as (4.3) and (4.17) under the same $\varpi(\cdot)$, U and $\boldsymbol{f}(\cdot)$, and it is invariant for any $G \in \mathbb{R}^{n\times n}_{[n]}$ if $G\boldsymbol{f}(\cdot)$ is considered.

The following lemma is applied for the derivations of the integral inequality in this section, and it can be straightforwardly obtained via the definition of matrix multiplication.

Lemma 4.2. *Given a matrix $X := \mathbf{Row}_{i=1}^d X_i \in \mathbb{R}^{n\times d\rho n}$ with $n; d; \rho \in \mathbb{N}$ and a function $\boldsymbol{f}(\tau) = \mathbf{Col}_{i=1}^d f_i(\tau) \in \mathbb{R}^d$, we have*

$$X(\boldsymbol{f}(\tau)\otimes I_{\rho n}) = \sum_{i=1}^d f_i(\tau)X_i = \left(\boldsymbol{f}^\top(\tau)\otimes I_n\right)\widehat{X} \tag{4.21}$$

where $\widehat{X} := \mathbf{Col}_{i=1}^d X_i \in \mathbb{R}^{dn\times\rho n}$.

Theorem 4.3. *Let $\varpi(\cdot)$ be given as in (4.1) and $U \in \mathbb{S}^n_{\succ 0}$ and $\boldsymbol{f}(\cdot) \in \mathbb{L}^2_\varpi(\mathcal{K} ⨾ \mathbb{R}^d)$ which satisfies the inequality in (4.2). For any $Y \in \mathbb{S}^{\rho dn}$ and $X = \mathbf{Row}_{i=1}^d X_i \in \mathbb{R}^{n\times\rho dn}$ satisfying*

$$\begin{bmatrix} U & -X \\ * & Y \end{bmatrix} \succeq 0, \tag{4.22}$$

we have

$$\forall \boldsymbol{x}(\cdot) \in \mathbb{L}^2_\varpi(\mathcal{K} ⨾ \mathbb{R}^n), \quad \int_\mathcal{K} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau \geq \boldsymbol{z}^\top\left[\mathbf{Sy}\left(\Upsilon^\top\widehat{X}\right) - \mathsf{W}\right]\boldsymbol{z}, \tag{4.23}$$

where $\rho \in \mathbb{N}$ and $\mathsf{W} := \int_\mathcal{K}\varpi(\tau)(\boldsymbol{f}^\top(\tau)\otimes I_{\rho n})Y(\boldsymbol{f}(\tau)\otimes I_{\rho n})\mathsf{d}\tau \in \mathbb{S}^{\rho n}$ and $\widehat{X} = \mathbf{Col}_{i=1}^d X_i \in \mathbb{R}^{dn\times\rho n}$, and $\Upsilon\boldsymbol{z} = \int_\mathcal{K}\varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau$ with $\Upsilon \in \mathbb{R}^{dn\times\rho n}$ and $\boldsymbol{z} \in \mathbb{R}^{\rho n}$.

Proof. Given (4.22) and $\Upsilon\boldsymbol{z} = \int_{\mathcal{K}} \varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau$ with $\Upsilon \in \mathbb{R}^{dn\times\rho n}$ and $\boldsymbol{z} \in \mathbb{R}^{\rho n}$ with $\rho \in \mathbb{N}$, we have

$$\int_{\mathcal{K}} \varpi(\tau)\,[*]^\top \begin{bmatrix} U & -X \\ * & Y \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(\tau) \\ \boldsymbol{f}(\tau)\otimes\boldsymbol{z} \end{bmatrix} \mathsf{d}\tau = \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau - \mathbf{Sy}\left[\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)X\left(\boldsymbol{f}(\tau)\otimes\boldsymbol{z}\right)\mathsf{d}\tau\right] + \int_{\mathcal{K}} \varpi(\tau)\left(\boldsymbol{f}(\tau)\otimes\boldsymbol{z}\right)^\top Y\left(\boldsymbol{f}(\tau)\otimes\boldsymbol{z}\right)\mathsf{d}\tau \geq 0. \quad (4.24)$$

Now using the property of the Kronecker product in (2.1) with (4.21) to the terms in (4.24) yields

$$\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)X\left(\boldsymbol{f}(\tau)\otimes\boldsymbol{z}\right)\mathsf{d}\tau = \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)X(\boldsymbol{f}(\tau)\otimes I_{\rho n})\mathsf{d}\tau\boldsymbol{z} = \left(\sum_{i=1}^{d}\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)f_i(\tau)\mathsf{d}\tau X_i\right)\boldsymbol{z}$$
$$= \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)(\boldsymbol{f}^\top(\tau)\otimes I_n)\mathsf{d}\tau\widehat{X}\boldsymbol{z} = \boldsymbol{z}^\top\Upsilon^\top\widehat{X}\boldsymbol{z} \quad (4.25)$$

and

$$\int_{\mathcal{K}} \varpi(\tau)\left(\boldsymbol{f}(\tau)\otimes\boldsymbol{z}\right)^\top Y\left(\boldsymbol{f}(\tau)\otimes\boldsymbol{z}\right)\mathsf{d}\tau = \boldsymbol{z}^\top\int_{\mathcal{K}} \varpi(\tau)\left(\boldsymbol{f}^\top(\tau)\otimes I_{\rho n}\right)Y\left(\boldsymbol{f}(\tau)\otimes I_{\rho n}\right)\mathsf{d}\tau\boldsymbol{z} = \boldsymbol{z}^\top\mathsf{W}\boldsymbol{z} \quad (4.26)$$

where $X = \mathbf{Row}_{i=1}^{d} X_i \in \mathbb{R}^{n\times\rho dn}$ and $\widehat{X} = \mathbf{Col}_{i=1}^{d} X_i$. Substituting (4.25) and (4.26) into (4.24) gives (4.23). This finishes the proof. ■

Remark 4.5. Since $\boldsymbol{f}(\cdot)$ in Theorem 4.3 is subject to the same constraint $\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau \succ 0$ as the $\boldsymbol{f}(\cdot)$ in Theorem 4.1, hence the structure of (4.23) is more general than existing free matrix type inequalities in the literature. Moreover, let $\mathcal{K} = [a,b]$ and $\varpi(\cdot) = 1$ and $\boldsymbol{f}(\tau)$ comprising the Legendre polynomials over $[a,b]$, then one can obtain Lemma 3 in [289] by Theorem 4.3 with appropriate Υ and $\boldsymbol{z}$ and the substitution $\dot{\boldsymbol{x}}(\cdot) \to \boldsymbol{x}(\cdot)$. Note that this also means that Theorem 4.3 covers the special cases of Lemma 3 in [289] such as [288] mentioned therein.

The following theorem shows the relation between (4.3) and (4.17) and (4.23) in terms of inequality bound gaps.

Theorem 4.4. *By choosing the same $\varpi(\cdot)$, U and $\boldsymbol{f}(\cdot)$ for Theorem 4.1, 4.2 and 4.3, one can always find X and Y for* (4.22) *to render* (4.23) *to become* (4.3)*, and the smallest achievable inequality bound gap of* (4.23) *is identical to* (4.17) *which in this case is the inequality bound gap of* (4.3)*.*

Proof. See Appendix C. ■

Remark 4.6. Let $\mathcal{K} = [a,b]$, $\varpi(\cdot) = 1$ and $\boldsymbol{f}(\tau)$ to contain Legendre polynomials over $[a,b]$, then the Theorem 1 of [289] can be obtained from Theorem 4.4 with appropriate Υ and $\boldsymbol{z}$ considering the substitution $\dot{\boldsymbol{x}}(\cdot) \to \boldsymbol{x}(\cdot)$. As we have proved that (4.17) is equivalent to (4.3), thus (4.23) is equivalent to (4.17). Consequently, it is possible to show that there are comprehensive equivalence relations[2] between the inequalities in [289, 292, 296] given what we have presented in Remark 4.4.

Theorem 4.4 plays a great role in bridging the relations between (4.3), (4.17) and (4.23). Since all these three inequalities are essentially equivalent in terms of inequalities bound gaps, hence if one finds a special example of one of these three inequalities then it corresponds to two 'equivalent' inequalities.

The following Corollary 4.3 can be established for (4.23) similar to what we want to show in Corollary 4.1.

Corollary 4.3. *Given the same $\varpi(\cdot)$, U and $\boldsymbol{f}(\cdot)$ in Theorem 4.3, then for any $Y \in \mathbb{S}^{\rho dn}$ and $X = \mathbf{Row}_{i=1}^{d} X_i \in \mathbb{R}^{n\times\rho dn}$ satisfying*

$$\begin{bmatrix} U & -X \\ * & Y \end{bmatrix} \succeq 0, \quad (4.27)$$

[2]The equivalence relations here are understood by considering the structure of inequalities irrespective of using $\boldsymbol{x}(\cdot)$ or $\dot{\boldsymbol{x}}(\cdot)$.

we have

$$\forall \boldsymbol{x}(\cdot) \in \mathbb{L}^2_{\varpi}(\mathcal{K} ⨾ \mathbb{R}^n), \quad \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau \geq \widehat{\boldsymbol{z}}^\top \left[\mathbf{Sy}\left(\Pi^\top \widehat{X}\right) - \mathsf{V}\right]\widehat{\boldsymbol{z}} \tag{4.28}$$

for all $G \in \mathbb{R}^{n\times n}_{[n]}$*, where* $\widehat{X} = \mathbf{Col}_{i=1}^d X_i \in \mathbb{R}^{dn\times \rho n}$ *and*

$$\begin{aligned} &\mathbb{S}^{\rho n} \ni \mathsf{V} := \int_{\mathcal{K}} \varpi(\tau)(\boldsymbol{\varphi}^\top(\tau)\otimes I_{\rho n})Y(\boldsymbol{\varphi}(\tau)\otimes I_{\rho n})\mathsf{d}\tau, \;\; \boldsymbol{\varphi}(\tau) = G\boldsymbol{f}(\tau) \\ &\Pi\widehat{\boldsymbol{z}} = \int_{\mathcal{K}} \varpi(\tau)\Phi(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau, \;\; \Pi \in \mathbb{R}^{dn\times\rho n}, \;\; \widehat{\boldsymbol{z}} \in \mathbb{R}^{\rho n}, \;\; \Phi(\tau) = \boldsymbol{\varphi}(\tau)\otimes I_n. \end{aligned} \tag{4.29}$$

Finally, under the same $\varpi(\cdot)$***,*** U ***and*** $\boldsymbol{f}(\cdot)$***,*** *(4.28) has the same smallest achievable bound gap as (4.23) which is the inequality bound gap of (4.3) and it is invariant to the value of* $G \in \mathbb{R}^{n\times n}_{[n]}$.

Proof. Let $\varpi(\cdot)$, U and $\boldsymbol{f}(\cdot)$ in Theorem 4.3 be given throughout the entire proof. The inequality in (4.28) can be obtained based on the substitution $G\boldsymbol{f}(\cdot) = \boldsymbol{\varphi}(\cdot) \to \boldsymbol{f}(\cdot)$ in (4.23). Now consider the inequality

$$\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau \geq \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)\Phi^\top(\tau)\mathsf{d}\tau\,(\mathsf{\Phi}\otimes U)\int_{\mathcal{K}} \varpi(\tau)\Phi(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau \tag{4.30}$$

in (4.12). By using the conclusion of Theorem 4.4 with the fact that $\Pi\widehat{\boldsymbol{z}} = \int_{\mathcal{K}} \varpi(\tau)\Phi(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau$, we know that the smallest achievable bound gap of (4.28) is identical to the inequality bound gap of (4.30). Since both (4.3) and (4.30) are part of (4.12), hence one can conclude that the smallest achievable bound gap of (4.28) is equal to the inequality bound gap of (4.3) which is invariant to the values of $G \in \mathbb{R}^{n\times n}_{[n]}$. Since in Theorem 4.4 we have shown that the inequality bound gap of (4.3) is identical to the smallest achievable bound gap of (4.23), then it proves the results in Corollary 4.3. ■

Remark 4.7. Together with all the results we have presented, it is possible to establish a chain of relations among the inequalities in [297]–[293] (See Table 4.1) and their "slack variables" counterpart obtained from the inequalities we have presented with appropriate $\Omega, \boldsymbol{\zeta}$ in (4.17), and $\widehat{\Omega}, \widehat{\boldsymbol{\zeta}}$ in (4.19), and $\Upsilon, \boldsymbol{z}$ in (4.23), and $\Pi, \widehat{\boldsymbol{z}}$ in (4.28).

4.5 Applications of integral inequalities to the stability analysis of a system with delays

To demonstrate the usefulness of the results we had derived, we derive a stability condition in this section for a linear CDDS with a distributed delay via constructing a parameterized version of the complete LKF [10] based on the application of (4.3). We show that the resulting stability condition is invariant with respect to a matrix parameter in the LKF. In addition, it is also shown that equivalent stability conditions which preserve the invariance property can be also derived by the application of (4.17) and (4.23).

Consider a linear coupled differential-difference system of the form

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) = A_1\boldsymbol{x}(t) + A_2\boldsymbol{y}(t-r) + \int_{-r}^0 \widetilde{A}_3(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau \\ &\boldsymbol{y}(t) = A_4\boldsymbol{x}(t) + A_5\boldsymbol{y}(t-r) \\ &\boldsymbol{x}(t_0) = \boldsymbol{\xi}, \quad \forall\theta \in [-r,0], \;\; \boldsymbol{y}(t_0+\theta) = \boldsymbol{\phi}(\theta) \end{aligned} \tag{4.31}$$

where $t_0 \in \mathbb{R}$ and $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r,0) ⨾ \mathbb{R}^\nu)$, and the notation $\widehat{\mathbb{C}}([-r,0) ⨾ \mathbb{R}^n)$ stands for the space of bounded right piecewise continuous functions endowed with the norm $\|\boldsymbol{\phi}(\cdot)\|_\infty = \sup_{\tau\in\mathcal{X}} \|\boldsymbol{\phi}(\tau)\|_2$. Furthermore, $\boldsymbol{x}(t) \in \mathbb{R}^n$ with $\boldsymbol{y}(t) \in \mathbb{R}^\nu$ is the solution of (4.31) and the size of the state space parameters in (4.31) are determined by $n, \nu \in \mathbb{N}$. We also assume that $\rho(A_5) < 1$ which ensures the input to state stability of $\boldsymbol{y}(t) = A_4\boldsymbol{x}(t) + A_5\boldsymbol{y}(t-r)$ [10], where $\rho(A_5)$ is the spectral radius of A_5. Since $\rho(A_5) < 1$ is independent of r, thus this condition ensures the input to state stability of $\boldsymbol{y}(t) = A_4\boldsymbol{x}(t) + A_5\boldsymbol{y}(t-r)$ for all $r > 0$. Finally, $\widetilde{A}_3(\tau)$ satisfies the following assumption.

Assumption 4.1. There exist $\mathbf{Col}_{i=1}^{d} f_i(\tau) = \boldsymbol{f}(\cdot) \in \mathbb{C}^1(\mathbb{R} \,⨾\, \mathbb{R}^d)$ with $d \in \mathbb{N}$, and $A_3 \in \mathbb{R}^{n\times\nu d}$ such that for all $\tau \in [-r, 0]$ we have $\widetilde{A}_3(\tau) = A_3 F(\tau) \in \mathbb{R}^{n\times\nu}$ where $F(\tau) := \boldsymbol{f}(\tau) \otimes I_\nu \in \mathbb{R}^{\nu d\times\nu}$. In addition, we assume that $\boldsymbol{f}(\cdot)$ here satisfies $\int_{-r}^{0} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathrm{d}\tau \succ 0$ and the following property:

$$\int_{-r}^{0} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathrm{d}\tau \succ 0 \tag{4.32}$$

$$\exists M \in \mathbb{R}^{d\times d}, \quad \frac{\mathrm{d}\boldsymbol{f}(\tau)}{\mathrm{d}\tau} = M\boldsymbol{f}(\tau), \tag{4.33}$$

$$\exists N_1 \in \mathbb{R}^{\delta\times d}_{[\delta]}, \ \exists N_2 \in \mathbb{R}^{\delta\times d}_{[\delta]}, \ (\tau + r)N_1\boldsymbol{f}(\tau) = N_2\boldsymbol{f}(\tau) \tag{4.34}$$

where $0 \le \delta \le d \in \mathbb{N}$.

Remark 4.8. Many models of delay systems are encompassed by (4.31), which is the main reason why (4.31) is chosen as the foundation of the analysis in this chapter. Specifically, see the examples in [10, 57] and the references therein.

Examples of $\boldsymbol{f}(\cdot)$ in Assumption 4.1 can be the solutions of homogeneous differential equations. For instance let $\boldsymbol{f}(\tau) = \mathbf{Col}\,(1, \mathrm{e}^\tau, \tau, \tau\mathrm{e}^\tau)$ and $N_1\boldsymbol{f}(\tau) = \mathbf{Col}\,(1, \mathrm{e}^\tau)$. Then we have

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \ N_1 = \begin{bmatrix} I_2 & \mathsf{O}_2 \end{bmatrix}, \ N_2 = \begin{bmatrix} r & 1 & 0 & 0 \\ 0 & r & 0 & 1 \end{bmatrix} \tag{4.35}$$

Note that for any $\boldsymbol{f}(\cdot)$ satisfying (4.33), one can always enlarge the dimension of $\boldsymbol{f}(\cdot)$ with new added functions to render it satisfying (4.34). On the other hand, (4.34) is satisfied for any $\boldsymbol{f}(\cdot) \in \mathbb{C}^1(\mathbb{R} \,⨾\, \mathbb{R}^d)$ if N_1 and N_2 are empty matrices which implies that the constraint in (4.34) can be omitted based on appropriate situations. Note that the rank constraint on N_1 in (4.34) ensures that $N_1 \int_{-r}^{0}(\tau + r)\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathrm{d}\tau N_1^\top \succ 0$.

To prove the results in this section, we present the following lemma which contains Liapunov-Krasovskii stability criteria for (4.31).

Lemma 4.3. *Given $r > 0$, the system in (4.31) is globally uniformly asymptotically (exponentially)[3] stable at its origin, if there exist $\epsilon_1; \epsilon_2; \epsilon_3 > 0$ and a differentiable functional $v : \mathbb{R}^n \times \widehat{\mathbb{C}}([-r, 0) \,⨾\, \mathbb{R}^\nu) \to \mathbb{R}_{\ge 0}$ such that $v(\mathbf{0}_n, \mathbf{0}_\nu) = 0$ and*

$$\epsilon_1 \|\boldsymbol{\xi}\|_2^2 \le v(\boldsymbol{\xi}, \boldsymbol{\phi}(\cdot)) \le \epsilon_2 \left(\|\boldsymbol{\xi}\|_2 \vee \|\boldsymbol{\phi}(\cdot)\|_\infty\right)^2 \tag{4.36}$$

$$\dot{v}(\boldsymbol{\xi}, \boldsymbol{\phi}(\cdot)) := \left.\frac{\mathrm{d}^+}{\mathrm{d}t} v(\boldsymbol{x}(t), \mathbf{y}_t(\cdot))\right|_{t=t_0, \boldsymbol{x}(t_0)=\boldsymbol{\xi}, \mathbf{y}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} \le -\epsilon_3 \|\boldsymbol{\xi}\|_2^2 \tag{4.37}$$

for any $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r, 0) \,⨾\, \mathbb{R}^\nu)$ in (4.31), where $t_0 \in \mathbb{R}$ and $\frac{\mathrm{d}^+}{\mathrm{d}x} f(x) = \operatorname{limsup}_{\eta\downarrow 0} \frac{f(x+\eta)-f(x)}{\eta}$. Furthermore, $\mathbf{y}_t(\cdot)$ in (4.37) is defined by the equality $\forall t \ge t_0$, $\forall\theta \in [-r, 0)$, $\mathbf{y}_t(\theta) = \boldsymbol{y}(t+\theta)$ where $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ satisfying (4.31).

Proof. Let $u(\cdot), v(\cdot), w(\cdot)$ in Theorem 3 of [10] to be quadratic functions with the multiplier factors $\epsilon_1; \epsilon_2; \epsilon_3 > 0$. Since (4.31) is a particular case of the general system considered in Theorem 3 of [10], then Lemma 4.3 is obtained. ■

To analyze the stability of the origin of (4.31), consider the following parameterized LKF

$$v(\boldsymbol{\xi}, \boldsymbol{\phi}(\cdot)) := \begin{bmatrix} \boldsymbol{\xi} \\ \int_{-r}^{0} \widehat{G}(\tau)\boldsymbol{\phi}(\tau)\mathrm{d}\tau \end{bmatrix}^\top \widehat{P} \begin{bmatrix} \boldsymbol{\xi} \\ \int_{-r}^{0} \widehat{G}(\tau)\boldsymbol{\phi}(\tau)\mathrm{d}\tau \end{bmatrix} + \int_{-r}^{0} \boldsymbol{\phi}^\top(\tau)\left[S + (\tau + r)U\right]\boldsymbol{\phi}(\tau)\mathrm{d}\tau \tag{4.38}$$

[3] See [10] for the explanation on the equivalence bewteen uniform asymptotic and exponential stability for a linear coupled differential functional system

with

$$\widehat{G}(\tau) = \boldsymbol{g}(\tau) \otimes I_\nu, \ \ \boldsymbol{g}(\tau) = G\boldsymbol{f}(\tau), \quad G \in \mathbb{R}^{d\times d}_{[d]}$$

and $\boldsymbol{f}(\cdot)$ in Assumption 4.1, where $\boldsymbol{\xi} \in \mathbb{R}^n$, $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r,0) ⨾ \mathbb{R}^\nu)$ in (4.38) are the initial conditions in (4.31), and $\widehat{P} \in \mathbb{S}^{n+d\nu}$ and $S; U \in \mathbb{S}^\nu$ are unknown parameters to be determined. Note that $\widehat{G}(\tau)$ can be rewritten into $\widehat{G}(\tau) = \boldsymbol{g}(\tau)\otimes I_\nu = G\boldsymbol{f}(\tau)\otimes I_\nu = (G\otimes I_\nu)F(\tau)$ with $F(\tau) := \boldsymbol{f}(\tau)\otimes I_\nu$ based on the property of the Kronecker product in (2.1). Note that also (4.38) can be regarded as a parameterized version of the complete LKF proposed in [10].

We will show in the following theorem that the feasibility of the resulting stability condition therein remains unchanged for any $G \in \mathbb{R}^{d\times d}_{[d]}$ in (4.38) regardless of whether (4.3) or (4.23) is applied for the derivation.

Theorem 4.5. *Given $G \in \mathbb{R}^{d\times d}_{[d]}$ and $\boldsymbol{f}(\cdot)$ with M, N_1 and N_2 in Assumption 4.1, then* (4.31) *under Assumption 4.1 is globally uniformly asymptotically stable at its origin if there exists $\widehat{P} \in \mathbb{S}^{n+d\nu}$ and $S; U \in \mathbb{S}^\nu$ such that the following conditions*

$$\widehat{P} + \left[\mathsf{O}_n \oplus \left(G^{-1\top}\mathsf{F}G^{-1}\otimes S + \left(G^{-1\top}N_2^\top\widetilde{\mathsf{F}}N_2G^{-1}\right)\otimes U\right)\right] \succ 0 \tag{4.39}$$

$$S \succ 0, \ \ U \succ 0, \ \ \boldsymbol{\Phi} \prec 0 \tag{4.40}$$

hold, where $\mathsf{F}^{-1} = \int_{-r}^0 \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$ and $\widetilde{\mathsf{F}}^{-1} = \int_{-r}^0 (\tau+r)N_1\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)N_1^\top\mathsf{d}\tau$

$$\boldsymbol{\Phi} := \mathbf{Sy}\left(H\widehat{P}\begin{bmatrix}\mathbf{A}\\ \mathbf{G}\end{bmatrix}\right) + \Gamma^\top(S+rU)\Gamma - \left(\mathsf{O}_n \oplus S \oplus \left(G^{-1\top}\mathsf{F}G^{-1}\otimes U\right)\right) \tag{4.41}$$

with

$$H = \begin{bmatrix} I_n & \mathsf{O}_{n\times d\nu} \\ \mathsf{O}_{\nu\times n} & \mathsf{O}_{\nu\times d\nu} \\ \mathsf{O}_{d\nu\times n} & I_{d\nu}\end{bmatrix}, \ \ \Gamma := \begin{bmatrix}A_4 & A_5 & \mathsf{O}_{\nu\times d\nu}\end{bmatrix}, \tag{4.42}$$

$$\mathbf{A} = \begin{bmatrix}A_1 & A_2 & A_3(G^{-1}\otimes I_\nu)\end{bmatrix}, \tag{4.43}$$

$$\mathbf{G} = \begin{bmatrix}\widehat{G}(0)A_4 & \widehat{G}(0)A_5 - \widehat{G}(-r) & -\widehat{M}\end{bmatrix} \tag{4.44}$$

in which $\widehat{G}(0) = (G\otimes I_\nu)F(0)$ and $\widehat{G}(-r) = (G\otimes I_\nu)F(-r)$ and $\widehat{M} = (G\otimes I_\nu)(M\otimes I_\nu)(G^{-1}\otimes I_\nu)$. Moreover, the feasibility of (4.39) *and* (4.40) *is invariant for any $G \in \mathbb{R}^{d\times d}_{[d]}$. Note that* (4.39) *and* (4.40) *are derived by the application of* (4.3)*. On the other hand, if* (4.17) *or* (4.23) *are applied instead of* (4.3) *for the derivation of stability conditions, then the corresponding stability conditions are equivalent to* (4.39) *and* (4.40)*, respectively, and the feasibility of the resulting conditions is also invariant for any $G \in \mathbb{R}^{d\times d}_{[d]}$.*

Proof. Let $G \in \mathbb{R}^{d\times d}_{[d]}$ and $\boldsymbol{f}(\cdot)$ with M, N_1 and N_2 in Assumption 4.1 be given. Given the fact that the eigenvalues of $S + (\tau+r)U$, $\tau \in [-r,0]$ are bounded and $\widehat{G}(\tau) = (G\otimes I_n)F(\tau)$, it is obvious to see that (4.38) satisfies

$$\begin{aligned}
\exists\lambda>0,\ \exists\eta>0,\ \ v(\boldsymbol{\xi},\boldsymbol{\phi}(\cdot)) \le &\begin{bmatrix}\boldsymbol{\xi}\\ \int_{-r}^0 F(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau\end{bmatrix}^\top \lambda \begin{bmatrix}\boldsymbol{\xi}\\ \int_{-r}^0 F(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau\end{bmatrix} + \int_{-r}^0 \boldsymbol{\phi}^\top(\tau)\lambda\boldsymbol{\phi}(\tau)\mathsf{d}\tau \\
&\le \lambda\left\|\boldsymbol{\xi}\right\|_2^2 + \int_{-r}^0 \boldsymbol{\phi}^\top(\tau)F^\top(\tau)\mathsf{d}\tau\lambda\int_{-r}^0 F(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau + \lambda\left\|\boldsymbol{\phi}(\cdot)\right\|_\infty^2 \le \lambda\left\|\boldsymbol{\xi}\right\|_2^2 + \lambda\left\|\boldsymbol{\phi}(\cdot)\right\|_\infty^2 \\
&+ \int_{-r}^0 \boldsymbol{\phi}^\top(\tau)F^\top(\tau)\mathsf{d}\tau\left(\eta\mathsf{F}\otimes I_n\right)\int_{-r}^0 F(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \le \lambda\left\|\boldsymbol{\xi}\right\|_2^2 + \lambda\left\|\boldsymbol{\phi}(\cdot)\right\|_\infty^2 + \int_{-r}^0 \boldsymbol{\phi}^\top(\tau)\eta\boldsymbol{\phi}(\tau)\mathsf{d}\tau \\
&\le \lambda\left\|\boldsymbol{\xi}\right\|_2^2 + (\lambda+\eta r)\left\|\boldsymbol{\phi}(\cdot)\right\|_\infty^2 \le (\lambda+\eta r)\left\|\boldsymbol{\xi}\right\|_2^2 + (\lambda+\eta r)\left\|\boldsymbol{\phi}(\cdot)\right\|_\infty^2 \\
&\le 2\left(\lambda+\eta r\right)\left[\max\left(\left\|\boldsymbol{\xi}\right\|_2, \left\|\boldsymbol{\phi}(\cdot)\right\|_\infty\right)\right]^2
\end{aligned} \tag{4.45}$$

for any $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r_2, 0) ⨾ \mathbb{R}^\nu)$ in (4.31), where (4.45) is derived via the property of quadratic forms: $\forall X \in \mathbb{S}^n$, $\exists \lambda > 0 : \forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\mathbf{x}^\top (\lambda I_n - X)\mathbf{x} > 0$ together with the application of (4.3) with $\boldsymbol{f}(\cdot)$ in (4.1). Consequently, (4.45) shows that (4.38) satisfies the upper bound property in (4.36).

Now apply (4.3) with $\varpi(\tau) = 1$ to the integral term $\int_{-r}^{0} \boldsymbol{\phi}^\top(\tau) S \boldsymbol{\phi}(\tau) \mathsf{d}\tau$ in (4.38) given $S \succ 0$ and $\boldsymbol{f}(\cdot)$ in Assumption 4.1 and the fact that $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r, 0) ⨾ \mathbb{R}^\nu) \subset \mathbb{L}^2([-r, 0) ⨾ \mathbb{R}^\nu)$. It yields

$$\int_{-r}^{0} \boldsymbol{\phi}^\top(\tau) S \boldsymbol{\phi}(\tau) \mathsf{d}\tau \geq \int_{-r}^{0} \boldsymbol{\phi}^\top(\tau) \widehat{G}^\top(\tau) \mathsf{d}\tau \left[(G^{-1})^\top \mathsf{F} G^{-1} \otimes S\right] \int_{-r}^{0} \widehat{G}(\tau) \boldsymbol{\phi}(\tau) \mathsf{d}\tau \tag{4.46}$$

for any initial condition $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r_2, 0) ⨾ \mathbb{R}^\nu)$ in (4.31). On the other hand, apply (4.3) with $\varpi(\tau) = \tau + r$ to the term $\int_{-r}^{0} (\tau + r) \boldsymbol{\phi}^\top(\tau) U \boldsymbol{\phi}(\tau) \mathsf{d}\tau$ in (4.38) with $U \succ 0$ and $\boldsymbol{f}(\cdot)$ in Assumption 4.1. It yields

$$\begin{aligned}
\int_{-r}^{0} (\tau + r) \boldsymbol{\phi}^\top(\tau) S \boldsymbol{\phi}(\tau) \mathsf{d}\tau &\geq [*] \left(\widetilde{\mathsf{F}} \otimes U\right) \left[\int_{-r}^{0} (\tau + r) \left(N_1 \boldsymbol{f}(\tau) \otimes I_n\right) \boldsymbol{\phi}(\tau) \mathsf{d}\tau\right] \\
&= \int_{-r}^{0} \boldsymbol{\phi}^\top(\tau) \left(\boldsymbol{f}^\top(\tau) N_2^\top \otimes I_n\right) \mathsf{d}\tau \left(\widetilde{\mathsf{F}} \otimes U\right) \int_{-r}^{0} \left(N_2 \boldsymbol{f}(\tau) \otimes I_n\right) \boldsymbol{\phi}(\tau) \mathsf{d}\tau \\
&= \int_{-r}^{0} \boldsymbol{\phi}^\top(\tau) F^\top(\tau) \mathsf{d}\tau \left[\left(N_2^\top \widetilde{\mathsf{F}} N_2\right) \otimes U\right] \int_{-r}^{0} F(\tau) \boldsymbol{\phi}(\tau) \mathsf{d}\tau \\
&= \int_{-r}^{0} \boldsymbol{\phi}^\top(\tau) \widehat{G}^\top(\tau) \mathsf{d}\tau \left[\left(G^{-1\top} N_2^\top \widetilde{\mathsf{F}} N_2 G^{-1}\right) \otimes U\right] \int_{-r}^{0} \widehat{G}(\tau) \boldsymbol{\phi}(\tau) \mathsf{d}\tau
\end{aligned} \tag{4.47}$$

for any $\boldsymbol{\xi}$ and $\boldsymbol{\phi}(\cdot)$ in (4.31), where $\widetilde{\mathsf{F}}^{-1} = \int_{-r}^{0} (\tau + r) N_1 \boldsymbol{f}(\tau) \boldsymbol{f}^\top(\tau) N_1^\top \mathsf{d}\tau$ with N_1, N_2 in (4.34).

Now by using (4.46) and (4.47) to (4.38), we can conclude that if (4.39) is feasible, then it infers the existence of (4.38) satisfying (4.36) given what we have shown in (4.45). On the other hand, given the property of congruence transformations with the fact that $G \in \mathbb{R}_{[d]}^{d \times d}$, one can conclude that (4.39) holds if and only if

$$\begin{aligned}
&\left[I_n \oplus \left(G^\top \otimes I_\nu\right)\right] \widehat{P} \left[I_n \oplus (G \otimes I_\nu)\right] \\
&\quad + \left[I_n \oplus \left(G^\top \otimes I_\nu\right)\right] \left[\mathsf{O}_n \oplus \left(G^{-1\top} \mathsf{F} G^{-1} \otimes S + [*] \widetilde{\mathsf{F}} \left(N_2 G^{-1}\right) \otimes U\right)\right] \left[I_n \oplus (G \otimes I_\nu)\right] \\
&\qquad = P + \left[\mathsf{O}_n \oplus \left(\mathsf{F} \otimes S + N_2^\top \widetilde{\mathsf{F}} N_2 \otimes U\right)\right] \succ 0,
\end{aligned} \tag{4.48}$$

where $P = \left[I_n \oplus \left(G^\top \otimes I_\nu\right)\right] \widehat{P} \left[I_n \oplus (G \otimes I_\nu)\right]$. By viewing the matrix P as a new variable, it occurs that the feasibility of the last matrix inequality in (4.48) is invariant from $G \in \mathbb{R}_{[n]}^{n \times n}$. As a result, we have shown that (4.39) has the same feasibility for any invertible G.

Now we use (4.38) to start to construct conditions inferring (4.37). Differentiate $v(\boldsymbol{x}(t), \mathbf{y}_t(\cdot))$ along the trajectory of (4.31) at $t = t_0$ and consider the relation

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}t} \int_{-r}^{0} \widehat{G}(\tau) \boldsymbol{\phi}(\tau) \mathsf{d}\tau &= \frac{\mathsf{d}}{\mathsf{d}t} \int_{-r}^{0} (G \otimes I_\nu) F(\tau) \boldsymbol{\phi}(\tau) \mathsf{d}\tau = (G \otimes I_\nu) F(0) \boldsymbol{\phi}(0) - (G \otimes I_\nu) F(-r) \boldsymbol{\phi}(-r) \\
&- \widehat{M} \int_{-r}^{0} \left(G \boldsymbol{f}(\tau) \otimes I_\nu\right) \boldsymbol{\phi}(\tau) \mathsf{d}\tau = \widehat{G}(0) A_4 \boldsymbol{\xi} + \left[\widehat{G}(0) A_5 - \widehat{G}(-r)\right] \boldsymbol{\phi}(-r) - \widehat{M} \int_{-r}^{0} \widehat{G}(\tau) \boldsymbol{\phi}(\tau) \mathsf{d}\tau
\end{aligned} \tag{4.49}$$

where $\widehat{M} = (G \otimes I_\nu)(M \otimes I_\nu)(G^{-1} \otimes I_\nu)$ and (4.49) can be obtained by the relation in (4.31). Then we have

$$\begin{aligned}
\frac{\mathsf{d}^+}{\mathsf{d}t} v(\boldsymbol{x}(t), \mathbf{y}_t(\cdot)) \Big|_{t = t_0, \boldsymbol{x}(t_0) = \boldsymbol{\xi}, \mathbf{y}_{t_0}(\cdot) = \boldsymbol{\phi}(\cdot)} &= \boldsymbol{\chi}^\top \mathbf{Sy}\left(H \widehat{P} \begin{bmatrix} \mathbf{A} \\ \mathbf{G} \end{bmatrix}\right) \boldsymbol{\chi} \\
&+ \boldsymbol{\chi}^\top \left[\Gamma^\top (S + rU) \Gamma - (\mathsf{O}_n \oplus S \oplus \mathsf{O}_{d\nu})\right] \boldsymbol{\chi} - \int_{-r}^{0} \boldsymbol{\phi}^\top(\tau) U \boldsymbol{\phi}(\tau) \mathsf{d}\tau,
\end{aligned} \tag{4.50}$$

where H, $\mathbf{A}$, $\mathbf{G}$ and Γ have been defined in the statement of Theorem 4.5 and

$$\boldsymbol{\chi} := \mathbf{Col}\left(\boldsymbol{\xi},\ \boldsymbol{\phi}(-r),\ \int_{-r}^{0} \widehat{G}(\tau) \boldsymbol{\phi}(\tau) \mathsf{d}\tau\right). \tag{4.51}$$

Given $U \succ 0$ in (4.39) and apply (4.3) with $\varpi(\tau) = 1$ to the integral $\int_{-r}^{0} \boldsymbol{\phi}^\top(\tau) U \boldsymbol{\phi}(\tau) \mathsf{d}\tau$ in (4.50) similar to the procedure in (4.46). It produces

$$\int_{-r}^{0} \boldsymbol{\phi}^\top(\tau) U \boldsymbol{\phi}(\tau) \mathsf{d}\tau \geq \int_{-r}^{0} \widehat{G}(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \left[(G^{-1})^\top \mathsf{F} G^{-1} \otimes U\right] \int_{-r}^{0} \widehat{G}(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \tag{4.52}$$

for any $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}\left([-r_2, 0) ; \mathbb{R}^\nu\right)$ in (4.31). By using (4.52) to (4.50), we have

$$\left.\frac{\mathsf{d}^+}{\mathsf{d}t} v(\boldsymbol{x}(t), \mathbf{y}_t(\cdot))\right|_{t=t_0, \boldsymbol{x}(t_0)=\boldsymbol{\xi}, \mathbf{y}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} \leq \boldsymbol{\chi}^\top \boldsymbol{\Phi} \boldsymbol{\chi} \tag{4.53}$$

given $U \succ 0$ in (4.39), where $\boldsymbol{\Phi}$ is defined in (4.41). By (4.53) and (4.51), it is easy to see that the feasible solutions of (4.40) infer the existence of (4.38) satisfying (4.37).

Considering the property of congruence transformations with the fact that $G \in \mathbb{R}_{[d]}^{d \times d}$, we know that

$$\boldsymbol{\Phi} \prec 0 \iff \left[I_{n+\nu} \oplus (G^\top \otimes I_\nu)\right] \boldsymbol{\Phi} \left[I_{n+\nu} \oplus (G \otimes I_\nu)\right] = \boldsymbol{\Theta} \prec 0 \tag{4.54}$$

where

$$\boldsymbol{\Theta} := \mathbf{Sy}\left(HP\Psi\right) + \Gamma^\top (S + rU)\,\Gamma - \left[\mathsf{O}_n \oplus S \oplus (\mathsf{F} \otimes U)\right] \tag{4.55}$$

with $P = \left[I_n \oplus \left(G^\top \otimes I_\nu\right)\right] \widehat{P} \left[I_n \oplus (G \otimes I_\nu)\right]$ and H defined in (4.42) and

$$\Psi = \begin{bmatrix} A_1 & A_2 & A_3 \\ F(0)A_4 & F(0)A_5 - F(-r) & -M \otimes I_\nu \end{bmatrix} \tag{4.56}$$

which can be derived via (2.1) and (4.49). By treating the matrix P as a new variable, it is clear to see that the feasibility of (4.54) is invariant from $G \in \mathbb{R}_{[n]}^{n \times n}$, which indicates the feasibility of (4.40) remains unchanged for any invertible G.

Finally, given the conclusion in Theorem 4.4, we know that if (4.17) or (4.23) are applied for the steps at (4.46) or (4.52) instead of (4.3), then the resulting conditions with the extra constraints induced by (4.22) has the same feasibility as (4.39) and (4.40) for any $G \in \mathbb{R}_{[d]}^{d \times d}$. Since the feasibility of (4.39) and (4.40) is invariant with respect to $G \in \mathbb{R}_{[n]}^{n \times n}$, thus the feasibility of the conditions derived via (4.17) and (4.23) will be invariant with respect to $G \in \mathbb{R}_{[n]}^{n \times n}$ G as well. This finishes the proof of this theorem. ■

By Theorem 4.5, we know that applying a linear transformation to $\boldsymbol{f}(\cdot)$, namely $\boldsymbol{g}(\tau) = G\boldsymbol{f}(\tau)$ in (4.38), cannot change the feasibility of the stability conditions derived via (4.3) or (4.17) or (4.23). In fact, if $\boldsymbol{f}(\cdot)$ contains only orthogonal functions in (4.38) with $G = I_d$, such option cannot render the corresponding stability condition to be more feasible compared to the case of $G \neq I_d$ which gives a non-orthogonal structure for $\boldsymbol{g}(\cdot)$. However, although choosing orthogonal functions for $\boldsymbol{g}(\cdot)$ cannot lead to less conservative stability conditions as we have proved, it may still be beneficial to do so. Specifically, the matrix $\left(\int_{-r}^{0} \boldsymbol{g}(\tau)\boldsymbol{g}^\top(\tau)\mathsf{d}\tau\right)^{-1} = (G^{-1})^\top \mathsf{F} G^{-1}$ in (4.41) is always diagonal if $\{g_i(\cdot)\}_{i=1}^{d}$ only contains mutually orthogonal functions, which might be a positive factor towards numerical calculations.

Chapter 5

Stability and Dissipativity Analysis of Linear Coupled Differential-Difference Systems with Distributed Delays

5.1 Introduction

Coupled differential-functional equations (CDFEs), which are mathematically related to time-delay systems [173], can characterize a broad class of models concerning delay or propagation effects [298]. CDFEs are able to model systems such as standard or neutral time-delay systems or certain singular delay systems [299]. For more information on the topic of CDFEs, see [10, 61] and the references therein.

Over the past decades, a series of significant results on the stability of CDESs [300, 301] has been proposed based on the approach of constructing LKFs. In particular, the idea of the complete LKF of linear time-delay systems [173] has been extended in to formulate a complete functional for a linear coupled differential-difference system (CDDS)[1] [10], which may be constructed numerically [302] via semidefinite programming. To the best of our knowledge, however, no results have been proposed in the reviewed publications on linear CDDSs with non-trivial (non-constant) distributed delays. Generally speaking, analyzing distributed delays may require much more efforts due to the complexities induced by different types of distributed delay kernels. For the latest existing time-domain-based results in connection with distributed delays, see [48, 57, 142, 183, 185, 188].

In [188], an approximation scheme is proposed to deal with $\mathbb{L}^2$ continuous distributed delay terms based on the application of Legendre polynomials. Although only the situation of having one or two distributed delay kernels are considered in [188], the stability conditions derived in [188] are highly competent and exhibit a pattern of hierarchical feasibility enhancement with respect to the degree of the approximating Legendre polynomials. In this chapter, we propose a new approach generalizing the results in [188]. Unlike the approximation scheme in [188] where approximations are solely attained by the application of Legendre orthogonal polynomials, our proposed approximation solution is based on a class of elementary functions (this including the case of Legendre polynomials or trigonometric functions). The proposed methodology provides a unified solution which can handle the situations that multiple distributed matrix kernels are approximated individually over two different integration intervals with general matrix structures. Furthermore, unified measures concerning approximation errors are formulated via a matrix framework and these measures are included by our proposed stability and dissipativity condition.

In this chapter, we propose solutions for the dissipativity and stability analysis of a linear CDDS with

[1] A CDDS can be considered as a special case of the systems characterized by CDFEs

distributed delays at both the states and output equation. Specifically, the distributed delay kernels considered can be any $\mathbb{L}^2$ function and the kernel functions are approximated by a class of elementary functions. Many existing models with delays, such as the ones in [10, 57, 185, 188, 302] are the special cases of the considered system model in this chapter. Meanwhile, analysis of the behavior of the approximation errors is presented by using matrix representations which generalize the existing results in [188]. Furthermore, a quadratic supply function is also considered for the dissipative analysis. To incorporate the approximation errors into the optimization constraints for dissipativity and stability analysis, a general integral inequality is derived which introduces error related terms into its lower bound. By constructing an LKF with the assistance of this inequality, sufficient conditions which ensure dissipativity and asymptotic (exponential) stability can be derived in terms of linear matrix inequalities. The proposed conditions are further proved to have a hierarchical feasibility enlargement if only orthogonal functions are chosen to approximate the distributed delay kernels, which can be considered as a generalization of the result in [187]. Finally, several numerical examples are given to demonstrate the effectiveness and capacity of the proposed methodologies.

5.2 Problem formulations

The following linear CDDS

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A_1\boldsymbol{x}(t) + A_2\boldsymbol{y}(t-r_1) + A_3\boldsymbol{y}(t-r_2) + \int_{-r_1}^{0}\widetilde{A}_4(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau + \int_{-r_2}^{-r_1}\widetilde{A}_5(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau \\
&\qquad + D_1\boldsymbol{w}(t) \\
\boldsymbol{y}(t) &= A_6\boldsymbol{x}(t) + A_7\boldsymbol{y}(t-r_1) + A_8\boldsymbol{y}(t-r_2), \qquad t \ge t_0 \\
\boldsymbol{z}(t) &= C_1\boldsymbol{x}(t) + C_2\boldsymbol{y}(t-r_1) + C_3\boldsymbol{y}(t-r_2) + \int_{-r_1}^{0}\widetilde{C}_4(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau + \int_{-r_2}^{-r_1}\widetilde{C}_5(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau \\
&\qquad + C_6\dot{\boldsymbol{y}}(t-r_1) + C_7\dot{\boldsymbol{y}}(t-r_2) + D_2\boldsymbol{w}(t) \\
\boldsymbol{x}(t_0) &= \boldsymbol{\xi} \in \mathbb{R}^n, \qquad \forall\theta \in [-r_2, 0),\ \boldsymbol{y}(t_0+\theta) = \boldsymbol{\psi}(\theta), \quad \boldsymbol{\psi}(\cdot) \in \mathcal{A}\left([-r_2,0) ; \mathbb{R}^\nu\right)
\end{aligned} \tag{5.1}
$$

with distributed delays is considered in this chapter, where $r_2 > r_1 > 0$ and $t_0 \in \mathbb{R}$. The notation $\mathcal{A}\left([-r_2,0) ; \mathbb{R}^\nu\right)$ in (5.1) stands for

$$
\mathcal{A}\left([-r_2,0) ; \mathbb{R}^\nu\right) := \left\{\boldsymbol{\psi}(\cdot) \in \mathbb{C}\left([-r_2,0) ; \mathbb{R}^\nu\right) :\ \dot{\boldsymbol{\psi}}(\cdot) \in \mathbb{L}^2\left([-r_2,0) ; \mathbb{R}^\nu\right)\ \&\ \ \|\boldsymbol{\psi}(\cdot)\|_\infty + \|\dot{\boldsymbol{\psi}}(\cdot)\|_2 < +\infty\right\}
$$

where $\|\boldsymbol{\psi}(\cdot)\|_\infty := \sup_{\tau\in\mathcal{X}} \|\boldsymbol{\psi}(\tau)\|_2$ and $\dot{\boldsymbol{\psi}}(\cdot)$ stands for the weak derivatives of $\boldsymbol{\psi}(\cdot)$. Furthermore, $\boldsymbol{x}(t) \in \mathbb{R}^n, \boldsymbol{y}(t) \in \mathbb{R}^\nu$ satisfy (5.1), and $\boldsymbol{w}(\cdot) \in \widehat{\mathbb{L}}^2([t_0,\infty) ; \mathbb{R}^q)$, $\boldsymbol{z}(t) \in \mathbb{R}^m$ are the disturbance and output of (5.1), respectively. The size of the state space matrices in (5.1) are determined by the given dimensions $n; \nu \in \mathbb{N}$ and $m; q \in \mathbb{N}_0$. All the functions in the entries of the matrix-valued distributed delay terms $\widetilde{A}_4(\cdot), \widetilde{C}_4(\cdot)$ and $\widetilde{A}_5(\cdot), \widetilde{C}_5(\cdot)$ are the elements of $\mathbb{L}^2\left([-r_1,0] ; \mathbb{R}\right)$ and $\mathbb{L}^2\left([-r_2,-r_1] ; \mathbb{R}\right)$, respectively. Finally, A_7 and A_8 satisfy

$$
\sup\left\{s \in \mathbb{C} : \det\left(I_\nu - A_7\mathsf{e}^{-r_1 s} - A_8\mathsf{e}^{-r_2 s}\right) = 0\right\} < 0, \tag{5.2}
$$

which ensures input to state stability for the associated difference equation [180] of (5.1).

In order to deal with the distributed delay terms in (5.1), we first define $\acute{\boldsymbol{f}}(\cdot) \in \mathbb{C}^1\left([-r_1,0] ; \mathbb{R}^d\right)$ and $\grave{\boldsymbol{f}}(\cdot) \in \mathbb{C}^1\left([-r_2,-r_1] ; \mathbb{R}^\delta\right)$ which satisfy the conditions:

$$
\exists! M_1 \in \mathbb{R}^{d\times d},\ \exists! M_2 \in \mathbb{R}^{\delta\times\delta} : \frac{\mathsf{d}\acute{\boldsymbol{f}}(\tau)}{\mathsf{d}\tau} = M_1\acute{\boldsymbol{f}}(\tau) \ \text{ and } \ \frac{\mathsf{d}\grave{\boldsymbol{f}}(\tau)}{\mathsf{d}\tau} = M_2\grave{\boldsymbol{f}}(\tau) \tag{5.3}
$$

$$
\begin{aligned}
&\exists\acute{\boldsymbol{\phi}}(\cdot) \in \mathbb{C}^1([-r_1,0] ; \mathbb{R}^{\kappa_1}),\ \exists\grave{\boldsymbol{\phi}}(\cdot) \in \mathbb{C}^1([-r_2,-r_1] ; \mathbb{R}^{\kappa_2}),\ \exists! M_3 \in \mathbb{R}^{\kappa_1\times d},\ \exists! M_4 \in \mathbb{R}^{\kappa_2\times\delta} : \\
&\qquad\qquad \frac{\mathsf{d}\acute{\boldsymbol{\phi}}(\tau)}{\mathsf{d}\tau} = M_3\acute{\boldsymbol{f}}(\tau) \ \text{ and } \ \frac{\mathsf{d}\grave{\boldsymbol{\phi}}(\tau)}{\mathsf{d}\tau} = M_4\grave{\boldsymbol{f}}(\tau)
\end{aligned} \tag{5.4}
$$

$$\mathbb{S}^d \ni \acute{\mathsf{F}}_d^{-1} = \int_{-r_2}^{-r_1} \acute{\boldsymbol{f}}(\tau)\acute{\boldsymbol{f}}^\top(\tau)\mathsf{d}\tau \succ 0, \quad \mathbb{S}^\delta \ni \grave{\mathsf{F}}_\delta^{-1} = \int_{-r_2}^{-r_1} \grave{\boldsymbol{f}}(\tau)\grave{\boldsymbol{f}}^\top(\tau)\mathsf{d}\tau \succ 0 \tag{5.5}$$

$$\mathbb{S}^{\kappa_1} \ni \acute{\Phi}_{\kappa_1}^{-1} = \int_{-r_1}^{0} \acute{\boldsymbol{\phi}}(\tau)\acute{\boldsymbol{\phi}}^\top(\tau)\mathsf{d}\tau \succ 0, \quad \mathbb{S}^{\kappa_2} \ni \grave{\Phi}_{\kappa_2}^{-1} = \int_{-r_2}^{-r_1} \grave{\boldsymbol{\phi}}(\tau)\grave{\boldsymbol{\phi}}^\top(\tau)\mathsf{d}\tau \succ 0 \tag{5.6}$$

where $d;\delta \in \mathbb{N}$, and (5.6) indicates that the functions in $\acute{\boldsymbol{f}}(\cdot)$, $\grave{\boldsymbol{f}}(\cdot)$, $\acute{\boldsymbol{\phi}}(\cdot)$ and $\grave{\boldsymbol{\phi}}(\cdot)$ are linearly independent in a Lebesgue sense, respectively. See Theorem 7.2.10 in [258] for the explanation of the meaning of (5.6).

Remark 5.1. The constraint in (5.3) indicates that the functions in $\acute{\boldsymbol{f}}(\cdot)$, $\grave{\boldsymbol{f}}(\cdot)$ are the solutions of homogeneous differential equations with constant coefficients. (polynomials, exponential, trigonometric functions, etc) Note that the conditions in (5.4) do not put extra constraints on $\acute{\boldsymbol{f}}(\cdot)$, $\grave{\boldsymbol{f}}(\cdot)$. This is because for any given $\acute{\boldsymbol{f}}(\cdot)$, $\grave{\boldsymbol{f}}(\cdot)$ satisfying (5.3), the one can always to make the choice of $\acute{\boldsymbol{\phi}}(\tau) = \acute{\boldsymbol{f}}(\tau)$ and $\grave{\boldsymbol{\phi}}(\tau) = \grave{\boldsymbol{f}}(\tau)$ with $M_3 = M_1$ and $M_4 = M_2$ which can satisfy (5.4).

Now given $\acute{\boldsymbol{f}}(\cdot) \in \mathbb{C}^1\big([-r_1,0] \,;\, \mathbb{R}^d\big)$ and $\grave{\boldsymbol{f}}(\cdot) \in \mathbb{C}^1\big([-r_2,-r_1] \,;\, \mathbb{R}^\delta\big)$ satisfying (5.3), one can conclude that for any $\widetilde{A}_4(\cdot);\widetilde{A}_5(\cdot)$ and $\widetilde{C}_4(\cdot);\widetilde{C}_5(\cdot)$ in (5.1), there exist constant matrices $A_4 \in \mathbb{R}^{n\times(d+\mu_1)\nu}$, $A_5 \in \mathbb{R}^{n\times(\delta+\mu_2)\nu}$, $C_4 \in \mathbb{R}^{m\times(d+\mu_1)\nu}$, $C_5 \in \mathbb{R}^{m\times(\delta+\mu_2)\nu}$ and the functions $\boldsymbol{\varphi}_1(\cdot) \in \mathbb{L}^2([-r_1,0] \,;\, \mathbb{R}^{\mu_1})$, $\boldsymbol{\varphi}_2(\cdot) \in \mathbb{L}^2([-r_2,-r_1] \,;\, \mathbb{R}^{\mu_2})$ such that

$$\begin{aligned}
\widetilde{A}_4(\tau) &= A_4\left(\begin{bmatrix}\boldsymbol{\varphi}_1(\tau)\\ \acute{\boldsymbol{f}}(\tau)\end{bmatrix}\otimes I_\nu\right), \qquad \widetilde{A}_5(\tau) = A_5\left(\begin{bmatrix}\boldsymbol{\varphi}_2(\tau)\\ \grave{\boldsymbol{f}}(\tau)\end{bmatrix}\otimes I_\nu\right)\\
\widetilde{C}_4(\tau) &= C_4\left(\begin{bmatrix}\boldsymbol{\varphi}_1(\tau)\\ \acute{\boldsymbol{f}}(\tau)\end{bmatrix}\otimes I_\nu\right), \qquad \widetilde{C}_5(\tau) = C_5\left(\begin{bmatrix}\boldsymbol{\varphi}_2(\tau)\\ \grave{\boldsymbol{f}}(\tau)\end{bmatrix}\otimes I_\nu\right)
\end{aligned} \tag{5.7}$$

$$\int_{-r_1}^{0}\begin{bmatrix}\boldsymbol{\varphi}_1(\tau)\\ \acute{\boldsymbol{f}}(\tau)\end{bmatrix}\begin{bmatrix}\boldsymbol{\varphi}_1^\top(\tau) & \acute{\boldsymbol{f}}^\top(\tau)\end{bmatrix}\mathsf{d}\tau \succ 0, \qquad \int_{-r_2}^{-r_1}\begin{bmatrix}\boldsymbol{\varphi}_2(\tau)\\ \grave{\boldsymbol{f}}(\tau)\end{bmatrix}\begin{bmatrix}\boldsymbol{\varphi}_2^\top(\tau) & \grave{\boldsymbol{f}}^\top(\tau)\end{bmatrix}\mathsf{d}\tau \succ 0 \tag{5.8}$$

where $\mu_1,\mu_2 \in \mathbb{N}_0$ and (5.8) indicates that the functions in $\textbf{Col}\big[\boldsymbol{\varphi}_1(\tau),\acute{\boldsymbol{f}}(\tau)\big]$ and $\textbf{Col}\big[\boldsymbol{\varphi}_2(\tau),\grave{\boldsymbol{f}}(\tau)\big]$ are linearly independent in a Lebesgue sense, respectively. Thus (5.7) can be applied to equivalently describe the distributed delay terms in (5.1). Finally, note that (5.5) is satisfied if (5.8) holds.

Remark 5.2. The elements in $\acute{\boldsymbol{f}}(\cdot)$ and $\grave{\boldsymbol{f}}(\cdot)$ in (5.7) are chosen in view of the functions in $\widetilde{A}_4(\cdot)$, $\widetilde{A}_5(\cdot)$, $\widetilde{C}_4(\cdot)$ and $\widetilde{C}_5(\cdot)$. Note that one can always let $\acute{\boldsymbol{f}}(\cdot)$ and $\grave{\boldsymbol{f}}(\cdot)$ to only contain orthogonal functions since one can always adjust the elements in $\boldsymbol{\varphi}_1(\cdot) \in \mathbb{L}^2([-r_1,0] \,;\, \mathbb{R}^{\mu_1})$ and $\boldsymbol{\varphi}_2(\cdot) \in \mathbb{L}^2([-r_2,-r_1] \,;\, \mathbb{R}^{\mu_2})$ to satisfy (5.7). Note that $\boldsymbol{\varphi}_1(\cdot)$ and $\boldsymbol{\varphi}_2(\cdot)$ can become a 0×1 empty vector if $\mu_1 = \mu_2 = 0$. Finally, the matrix inequalities in (5.8) can be verified via numerical calculations[2] with given $\acute{\boldsymbol{f}}(\cdot)$, $\grave{\boldsymbol{f}}(\cdot)$ and $\boldsymbol{\varphi}_1(\cdot)$, $\boldsymbol{\varphi}_2(\cdot)$.

Remark 5.3. The decomposition in (5.7) is employed in this chapter to handle the distributed delay terms in (5.1) so that a well-posed dissipativity and stability condition can be derived later. This will be illustrated later in light of the results in Lemma 5.2 and Theorem 5.1. It is worthy to stress that (5.1) generalizes all the models in considered in [57, 183, 188] without considering uncertainties.

Remark 5.4. A neutral delay system

$$\frac{\mathsf{d}}{\mathsf{d}t}\left(\boldsymbol{y}(t) - A_4\boldsymbol{y}(t-r)\right) = A_1\boldsymbol{y}(t) + A_2\boldsymbol{y}(t-r) + \int_{-r}^{0} A_3(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau$$

can be equivalently expressed by a CDDS:

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A_1\boldsymbol{x}(t) + (A_2 + A_1A_4)\boldsymbol{y}(t-r) + \int_{-r}^{0} A_3(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\\
\boldsymbol{y}(t) &= \boldsymbol{x}(t) + A_4\boldsymbol{y}(t-r).
\end{aligned}$$

[2] One option is to use `vpaintegral` in Matlab which performs high-precision numerical integration.

On the other hand, if there is rank redundancy in the delay matrices, namely,

$$\frac{\mathsf{d}}{\mathsf{d}t}\left(\boldsymbol{y}(t) - A_4 N\boldsymbol{y}(t-r)\right) = A_1\boldsymbol{y}(t) + A_2 N\boldsymbol{y}(t-r) + \int_{-r}^{0} A_3(\tau)N\boldsymbol{y}(t+\tau)\mathsf{d}\tau, \tag{5.9}$$

then one can first change (5.9) into

$$\frac{\mathsf{d}}{\mathsf{d}t}\left(\boldsymbol{y}(t) - A_4\boldsymbol{z}(t-r)\right) = A_1\boldsymbol{y}(t) + A_2\boldsymbol{z}(t-r) + \int_{-r}^{0} A_3(\tau)\boldsymbol{z}(t+\tau)\mathsf{d}\tau, \quad \boldsymbol{z}(t) = N\boldsymbol{y}(t). \tag{5.10}$$

Furthermore, let $\boldsymbol{x}(t) = \boldsymbol{y}(t) - A_4\boldsymbol{z}(t-r)$ considering (5.10), one can obtain the equivalent CDDS representation

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A_1\boldsymbol{x}(t) + (A_1A_4 + A_2)\,\boldsymbol{z}(t-r) + \int_{-r}^{0} A_3(\tau)\boldsymbol{z}(t+\tau)\mathsf{d}\tau \\
\boldsymbol{z}(t) &= N\boldsymbol{x}(t) + NA_4\boldsymbol{z}(t-r)
\end{aligned}$$

which now is clearly advantageous in terms of reducing the scale of dimensionality if $\dim[\boldsymbol{z}(t)] \ll \dim[\boldsymbol{y}(t)]$. Finally, for the exploitation the rank redundancies among the state space variables of the retarded cases, see [10] for details.

In this chapter, the functions $\acute{\boldsymbol{f}}(\cdot)$ and $\grave{\boldsymbol{f}}(\cdot)$ in (5.7) are applied to approximate the functions $\boldsymbol{\varphi}_1(\cdot) \in \mathbb{L}^2([-r_1, 0]\,\mathring{;}\,\mathbb{R}^{\mu_1})$ and $\boldsymbol{\varphi}_2(\cdot) \in \mathbb{L}^2([-r_2, -r_1]\,\mathring{;}\,\mathbb{R}^{\mu_2})$ in (5.7), respectively, where $\boldsymbol{\varphi}_1(\cdot)$ and $\boldsymbol{\varphi}_2(\cdot)$ might not satisfy (5.3). Specifically, the approximations are denoted by the decomposition:

$$\boldsymbol{\varphi}_1(\tau) = \acute{\Gamma}_d\acute{\boldsymbol{f}}(\tau) + \acute{\boldsymbol{\varepsilon}}_d(\tau), \quad \boldsymbol{\varphi}_2(\tau) = \grave{\Gamma}_\delta\grave{\boldsymbol{f}}(\tau) + \grave{\boldsymbol{\varepsilon}}_\delta(\tau) \tag{5.11}$$

where $\acute{\Gamma}_d$ and $\grave{\Gamma}_\delta$ are given coefficient. Furthermore, $\acute{\boldsymbol{\varepsilon}}_d(\tau) = \boldsymbol{\varphi}_1(\tau) - \acute{\Gamma}_d\acute{\boldsymbol{f}}(\tau)$ and $\grave{\boldsymbol{\varepsilon}}_\delta(\tau) = \boldsymbol{\varphi}_2(\tau) - \grave{\Gamma}_\delta\grave{\boldsymbol{f}}(\tau)$ contain the errors of approximations. In addition, we define matrices

$$\mathbb{S}^{\mu_1\times\mu_1} \ni \acute{\mathsf{E}}_d := \int_{-r_1}^{0} \acute{\boldsymbol{\varepsilon}}_d(\tau)\acute{\boldsymbol{\varepsilon}}_d^\top(\tau)\mathsf{d}\tau, \quad \mathbb{S}^{\mu_2\times\mu_2} \ni \grave{\mathsf{E}}_\delta := \int_{-r_2}^{-r_1} \grave{\boldsymbol{\varepsilon}}_\delta(\tau)\grave{\boldsymbol{\varepsilon}}_\delta^\top(\tau)\mathsf{d}\tau \tag{5.12}$$

to measure the error residues of (5.11). Inspired by the idea of orthogonal approximation in Hilbert space [303], one option for the values of $\acute{\Gamma}_d$ and $\grave{\Gamma}_\delta$ in (5.11) is

$$\begin{aligned}
\mathbb{R}^{\mu_1\times d} \ni \acute{\Gamma}_d &:= \int_{-r_1}^{0} \boldsymbol{\varphi}_1(\tau)\acute{\boldsymbol{f}}^\top(\tau)\mathsf{d}\tau\acute{\mathsf{F}}_d,\ \acute{\mathsf{F}}_d^{-1} = \int_{-r_2}^{-r_1} \grave{\boldsymbol{f}}(\tau)\grave{\boldsymbol{f}}^\top(\tau)\mathsf{d}\tau \\
\mathbb{R}^{\mu_2\times\delta} \ni \grave{\Gamma}_\delta &:= \int_{-r_2}^{-r_1} \boldsymbol{\varphi}_2(\tau)\grave{\boldsymbol{f}}^\top(\tau)\mathsf{d}\tau\grave{\mathsf{F}}_\delta,\ \grave{\mathsf{F}}_\delta^{-1} = \int_{-r_2}^{-r_1} \grave{\boldsymbol{f}}(\tau)\grave{\boldsymbol{f}}^\top(\tau)\mathsf{d}\tau.
\end{aligned} \tag{5.13}$$

Remark 5.5. (5.13) might be interpreted as a vector form of the standard approximations (Least Squares) in Hilbert space. (See section 10.2 in [303]) If $\acute{\boldsymbol{f}}(\cdot)$ and $\grave{\boldsymbol{f}}(\cdot)$ in (5.13) contains only Legendre polynomials, then (5.11)–(5.13) generalizes the polynomials approximation scheme proposed in [188] via a matrix framework. Finally, it is very crucial to emphasize that (5.11) does not restrict one only to apply (5.13) for the values of $\acute{\Gamma}_d$ and $\grave{\Gamma}_\delta$. Other appropriate options for $\acute{\Gamma}_d$ and $\grave{\Gamma}_\delta$ can be considered as well based on specific contexts.

The system (5.1) can be re-expressed as

$$\begin{aligned}
&\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{\vartheta}(t), \quad \boldsymbol{y}(t) = \begin{bmatrix} \mathsf{O}_{\nu\times(2\nu+q)} & \Xi & \mathsf{O}_{\nu\times\nu\mu} \end{bmatrix}\boldsymbol{\vartheta}(t), \quad \boldsymbol{z}(t) = \Sigma\boldsymbol{\vartheta}(t) \\
&\boldsymbol{x}(t_0) = \boldsymbol{\xi} \in \mathbb{R}^n, \quad \forall\theta \in [-r_2, 0],\ \boldsymbol{y}(t_0+\theta) = \boldsymbol{\psi}(\theta)
\end{aligned} \tag{5.14}$$

with

$$\begin{aligned}
\mathbf{A} = \Bigg[\mathsf{O}_{n\times 2\nu} \quad D_1 \quad A_1 \quad A_2 \quad A_3 \quad & A_4\left(\begin{bmatrix}\acute{\Gamma}_d \\ I_d\end{bmatrix}\otimes I_\nu\right) \quad A_5\left(\begin{bmatrix}\grave{\Gamma}_\delta \\ I_\delta\end{bmatrix}\otimes I_\nu\right)\cdots \\
&\cdots A_4\left(\begin{bmatrix}\acute{\mathsf{E}}_d \\ \mathsf{O}_{d\times\mu_1}\end{bmatrix}\otimes I_\nu\right) \quad A_5\left(\begin{bmatrix}\grave{\mathsf{E}}_\delta \\ \mathsf{O}_{\delta\times\mu_2}\end{bmatrix}\otimes I_\nu\right)\Bigg]
\end{aligned} \tag{5.15}$$

$$\Xi = \begin{bmatrix} A_6 & A_7 & A_8 & \mathsf{O}_{\nu\times\varrho\nu} \end{bmatrix} \tag{5.16}$$

$$\begin{aligned}\Sigma = \Bigg[C_6 \quad C_7 \quad D_2 \quad C_1 \quad C_2 \quad C_3 \quad C_4\left(\begin{bmatrix}\acute{\Gamma}_d\\ I_d\end{bmatrix}\otimes I_\nu\right) \quad C_5\left(\begin{bmatrix}\grave{\Gamma}_\delta\\ I_\delta\end{bmatrix}\otimes I_\nu\right)\cdots \\ \cdots C_4\left(\begin{bmatrix}\acute{\mathsf{E}}_d\\ \mathsf{O}_{d\times\mu_1}\end{bmatrix}\otimes I_\nu\right) \quad C_5\left(\begin{bmatrix}\grave{\mathsf{E}}_\delta\\ \mathsf{O}_{\delta\times\mu_2}\end{bmatrix}\otimes I_\nu\right)\Bigg]\end{aligned} \tag{5.17}$$

$$\boldsymbol{\vartheta}(t) := \mathbf{Col}\left(\begin{bmatrix}\dot{\boldsymbol{y}}(t-r_1)\\ \dot{\boldsymbol{y}}(t-r_2)\end{bmatrix}, \begin{bmatrix}\boldsymbol{w}(t)\\ \boldsymbol{x}(t)\end{bmatrix}, \begin{bmatrix}\boldsymbol{y}(t-r_1)\\ \boldsymbol{y}(t-r_2)\end{bmatrix}, \begin{bmatrix}\int_{-r_1}^{0}\acute{F}_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\\ \int_{-r_2}^{-r_1}\grave{F}_\delta(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\end{bmatrix}, \begin{bmatrix}\int_{-r_1}^{0}\acute{E}_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\\ \int_{-r_2}^{-r_1}\grave{E}_\delta(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\end{bmatrix}\right), \tag{5.18}$$

where $\mathbb{R}^{d\nu\times\nu} \ni \acute{F}_d(\tau) := \acute{\boldsymbol{f}}(\tau)\otimes I_\nu$ and $\mathbb{R}^{\delta\nu\times\nu} \ni \grave{F}_\delta(\tau) := \grave{\boldsymbol{f}}(\tau)\otimes I_\nu$ and $\acute{E}_d(\tau) := \acute{\mathsf{E}}_d^{-1}\acute{\boldsymbol{\varepsilon}}_d(\tau)\otimes I_\nu$ and $\grave{E}_\delta(\tau) := \grave{\mathsf{E}}_\delta^{-1}\grave{\boldsymbol{\varepsilon}}_\delta(\tau)\otimes I_\nu$ with $\acute{\mathsf{E}}_d$ and $\grave{\mathsf{E}}_\delta$ in (5.12). Note that $\acute{\mathsf{E}}_d$ and $\grave{\mathsf{E}}_\delta$ in (5.12) are invertible according to what will be explained in Remark 5.8 based on what will be presented in (5.24) and (D.1). Note that also the distributed delay terms in (5.14) are derived based on the identities

$$\begin{aligned}
&\left(\begin{bmatrix}\boldsymbol{\varphi}_1(\tau)\\ \acute{\boldsymbol{f}}(\tau)\end{bmatrix}\otimes I_\nu\right)\boldsymbol{y}(t+\tau) = \left(\begin{bmatrix}\acute{\Gamma}_d\acute{\boldsymbol{f}}(\tau)+\acute{\boldsymbol{\varepsilon}}_d(\tau)\\ \acute{\boldsymbol{f}}(\tau)\end{bmatrix}\otimes I_\nu\right)\boldsymbol{y}(t+\tau) = \left(\begin{bmatrix}\acute{\Gamma}_d\\ I_d\end{bmatrix}\acute{\boldsymbol{f}}(\tau)\otimes I_\nu\right)\boldsymbol{y}(t+\tau)\\
&\quad + \left(\begin{bmatrix}I_{\mu_1}\\ \mathsf{O}_{d\times\mu_1}\end{bmatrix}\acute{\boldsymbol{\varepsilon}}_d(\tau)\otimes I_\nu\right)\boldsymbol{y}(t+\tau) = \left(\begin{bmatrix}\acute{\Gamma}_d\\ I_d\end{bmatrix}\otimes I_\nu\right)\acute{F}_d(\tau)\boldsymbol{y}(t+\tau) + \left(\begin{bmatrix}\acute{\mathsf{E}}_d\\ \mathsf{O}_{d\times\mu_1}\end{bmatrix}\otimes I_\nu\right)\acute{E}_d(\tau)\boldsymbol{y}(t+\tau)\\
&\left(\begin{bmatrix}\boldsymbol{\varphi}_2(\tau)\\ \grave{\boldsymbol{f}}(\tau)\end{bmatrix}\otimes I_\nu\right)\boldsymbol{y}(t+\tau) = \left(\begin{bmatrix}\grave{\Gamma}_\delta\grave{\boldsymbol{f}}(\tau)+\grave{\boldsymbol{\varepsilon}}_\delta(\tau)\\ \grave{\boldsymbol{f}}(\tau)\end{bmatrix}\otimes I_\nu\right)\boldsymbol{y}(t+\tau) = \left(\begin{bmatrix}\grave{\Gamma}_\delta\\ I_d\end{bmatrix}\grave{\boldsymbol{f}}(\tau)\otimes I_\nu\right)\boldsymbol{y}(t+\tau)\\
&\quad + \left(\begin{bmatrix}I_{\mu_2}\\ \mathsf{O}_{\delta\times\mu_2}\end{bmatrix}\grave{\boldsymbol{\varepsilon}}_\delta(\tau)\otimes I_\nu\right)\boldsymbol{y}(t+\tau) = \left(\begin{bmatrix}\grave{\Gamma}_\delta\\ I_\delta\end{bmatrix}\otimes I_\nu\right)\grave{F}_\delta(\tau)\boldsymbol{y}(t+\tau) + \left(\begin{bmatrix}\grave{\mathsf{E}}_\delta\\ \mathsf{O}_{\delta\times\mu_2}\end{bmatrix}\otimes I_\nu\right)\grave{E}_\delta(\tau)\boldsymbol{y}(t+\tau)
\end{aligned}$$

which themselves are obtained via the property of the Kronecker product in (2.1).

5.3 Mathematical preliminaries

In this section some important lemmas and definition are present. This includes a novel integral inequality which will be applied later for the derivation of our dissipative stability condition.

The following lemma provides sufficient conditions for the stability of (5.1). It can be interpreted as a particular case of Theorem 3 in [10] with certain modifications.

Lemma 5.1. *Given $r_2 \ge r_1 > 0$, the system* (5.1) *with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is globally uniformly asymptotically stable at its origin if there exist $\epsilon_1; \epsilon_2; \epsilon_3 > 0$ and a differentiable functional $v : \mathbb{R}^n \times \mathcal{A}([-r_2, 0) ⨾ \mathbb{R}^\nu) \to \mathbb{R}_{\ge 0}$ such that $v(\mathbf{0}_n, \mathbf{0}_\nu) = 0$ and*

$$\epsilon_1 \|\boldsymbol{\xi}\|_2^2 \le v(\boldsymbol{\xi}, \boldsymbol{\psi}(\cdot)) \le \epsilon_2 \left[\|\boldsymbol{\xi}\|_2 \vee \left(\|\boldsymbol{\psi}(\cdot)\|_\infty + \|\dot{\boldsymbol{\psi}}(\cdot)\|_2\right)\right]^2, \tag{5.19}$$

$$\dot{v}(r, \boldsymbol{\xi}, \boldsymbol{\psi}(\cdot)) := \left.\frac{\mathsf{d}^+}{\mathsf{d}t} v(\boldsymbol{x}(t), \mathbf{y}_t(\cdot))\right|_{t=t_0, \boldsymbol{x}(t_0)=\boldsymbol{\xi}, \mathbf{y}_{t_0}(\cdot)=\boldsymbol{\psi}(\cdot)} \le -\epsilon_3 \|\boldsymbol{\xi}\|_2^2 \tag{5.20}$$

for any $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\psi}(\cdot) \in \mathcal{A}([-r_2, 0) ⨾ \mathbb{R}^\nu)$ in (5.1)*, where $t_0 \in \mathbb{R}$ and $\frac{\mathsf{d}^+}{\mathsf{d}x} f(x) = \operatorname{limsup}_{\eta\downarrow 0} \frac{f(x+\eta)-f(x)}{\eta}$. Furthermore, $\mathbf{y}_t(\cdot)$ in* (5.20) *is defined by the equality $\forall t \ge t_0$, $\forall \theta \in [-r, 0)$, $\mathbf{y}_t(\theta) = \boldsymbol{y}(t+\theta)$ where $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ here satisfying* (5.1) *with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$.*

Definition 5.1 (Dissipativity)**.** Given $r_2 \ge r_1 > 0$, the coupled differential functional system (5.1) with a supply rate function $s(\boldsymbol{z}(t), \boldsymbol{w}(t))$ is said to be dissipative if there exists a differentiable functional $v : \mathbb{R}^n \times \mathcal{A}([-r_2, 0) ⨾ \mathbb{R}^\nu) \to \mathbb{R}$ such that

$$\forall t \ge t_0 : \; \dot{v}(\boldsymbol{x}(t), \mathbf{y}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \le 0 \tag{5.21}$$

with $t_0 \in \mathbb{R}$ in (5.1), where $\mathbf{y}_t(\cdot)$ is defined by the equality $\forall t \ge t_0$, $\forall \theta \in [-r_2, 0)$, $\mathbf{y}_t(\theta) = \boldsymbol{y}(t+\theta)$, and $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$ and $\boldsymbol{z}(t)$ satisfy the equialities in (5.1) with $\boldsymbol{w}(\cdot) \in \widehat{\mathbb{L}}^2([t_0, \infty) ⨾ \mathbb{R}^q)$.

To incorporate performance objectives (dissipativity) into the analysis of (5.14), a quadratic form

$$s(\boldsymbol{z}(t),\boldsymbol{w}(t)) = \begin{bmatrix}\boldsymbol{z}(t)\\ \boldsymbol{w}(t)\end{bmatrix}^{\top} \mathbf{J} \begin{bmatrix}\boldsymbol{z}(t)\\ \boldsymbol{w}(t)\end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} J_1 & J_2 \\ * & J_3 \end{bmatrix} \in \mathbb{S}^{(m+q)}, \ \widetilde{J}^{\top} J_1^{-1} \widetilde{J} \preceq 0, \ J_1^{-1} \prec 0, \ \widetilde{J} \in \mathbb{R}^{m\times m} \tag{5.22}$$

is considered for the supply function.

The following generalized new integral inequality is proposed which will be employed for the derivation of our major results on the dissipativity and stability analysis in this section. Firstly, we define the weighted Lebesgue function space

$$\mathbb{L}^2_{\varpi}\left(\mathcal{K}\,;\mathbb{R}^d\right) := \left\{\boldsymbol{\psi}(\cdot) \in \mathbb{L}_f\left(\mathcal{K}\,;\mathbb{R}^d\right) : \|\boldsymbol{\psi}(\cdot)\|_{2,\varpi} < \infty\right\} \tag{5.23}$$

with $d \in \mathbb{N}$ and $\|\boldsymbol{\psi}(\cdot)\|_{2,\varpi} := \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{\psi}^{\top}(\tau)\boldsymbol{\psi}(\tau)\mathsf{d}\tau$, where $\varpi(\cdot) \in \mathbb{L}_f(\mathcal{K}\,;\mathbb{R}_{\geq 0})$ and the function $\varpi(\cdot)$ has only countably infinite or finite number of zero values. Furthermore, $\mathcal{K} \subseteq \mathbb{R} \cup \{\pm\infty\}$ and $\int_{\mathcal{K}} \mathsf{d}\tau \neq 0$.

Lemma 5.2. *Given $\mathcal{K}$ and $\varpi(\cdot)$ in (5.23) and $U \in \mathbb{S}^n_{\succeq 0} := \{X \in \mathbb{S}^n : X \succeq 0\}$ with $n \in \mathbb{N}$. Let $\mathsf{f}(\cdot) := \mathbf{Col}_{i=1}^{d}\, \mathsf{f}_i(\cdot) \in \mathbb{L}^2_{\varpi}\left(\mathcal{K}\,;\mathbb{R}^d\right)$ and $\mathsf{g}(\cdot) := \mathbf{Col}_{i=1}^{\delta}\, \mathsf{g}_i(\cdot) \in \mathbb{L}^2_{\varpi}\left(\mathcal{K}\,;\mathbb{R}^\delta\right)$ with $d \in \mathbb{N}$ and $\delta \in \mathbb{N}_0$, in which the functions $\mathsf{f}(\cdot)$ and $\mathsf{g}(\cdot)$ satisfy*

$$\int_{\mathcal{K}} \varpi(\tau) \begin{bmatrix}\mathsf{g}(\tau)\\ \mathsf{f}(\tau)\end{bmatrix} \begin{bmatrix}\mathsf{g}^{\top}(\tau) & \mathsf{f}^{\top}(\tau)\end{bmatrix} \mathsf{d}\tau \succ 0. \tag{5.24}$$

Then we have,

$$\begin{aligned}\forall \boldsymbol{x}(\cdot) \in \mathbb{L}^2_{\varpi}(\mathcal{K}\,;\mathbb{R}^n),\ \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^{\top}(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau &\geq \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^{\top}(\tau)\mathsf{F}^{\top}(\tau)\mathsf{d}\tau\,(\mathcal{F}_d \otimes U)\int_{\mathcal{K}} \varpi(\tau)\mathsf{F}(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau \\ &+ \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^{\top}(\tau)\mathsf{E}^{\top}(\tau)\mathsf{d}\tau\left(\mathcal{E}_d^{-1} \otimes U\right)\int_{\mathcal{K}} \varpi(\tau)\mathsf{E}(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau \end{aligned} \tag{5.25}$$

where

$$\begin{aligned} &\mathsf{F}(\tau) = \mathsf{f}(\tau) \otimes I_n \in \mathbb{R}^{dn\times n}, \quad \mathcal{F}_d^{-1} = \int_{\mathcal{K}} \varpi(\tau)\mathsf{f}(\tau)\mathsf{f}^{\top}(\tau)\mathsf{d}\tau \in \mathbb{S}^d_{\succ 0} \\ &\mathsf{E}(\tau) = \mathsf{e}(\tau) \otimes I_n \in \mathbb{R}^{\delta n\times n}, \quad \mathcal{E}_d = \int_{\mathcal{K}} \varpi(\tau)\mathsf{e}(\tau)\mathsf{e}^{\top}(\tau)\mathsf{d}\tau \in \mathbb{S}^{\delta} \\ &\mathsf{e}(\tau) = \mathsf{g}(\tau) - \mathsf{A}\mathsf{f}(\tau) \in \mathbb{R}^{\delta}, \quad \mathsf{A} = \int_{\mathcal{K}} \varpi(\tau)\mathsf{g}(\tau)\mathsf{f}^{\top}(\tau)\mathsf{d}\tau\mathcal{F}_d \in \mathbb{R}^{\delta\times d}. \end{aligned} \tag{5.26}$$

Proof. See Appendix D for details. ■

Remark 5.6. By Theorem 7.2.10 in [258] and considering the fact that

$$\left(\mathbb{L}^2_{\varpi}\left(\mathcal{K}\,;\mathbb{R}^d\right) / \operatorname{Ker}\left(\|\bullet\|_{2,\varpi}\right),\ \int_{\mathcal{K}} \varpi(\tau)\bullet_1^{\top}(\tau)\bullet_2(\tau)\mathsf{d}\tau\right)$$

is an inner product space[3], we know (5.24) indicates that the functions in $\mathsf{f}(\cdot)$ and $\mathsf{g}(\cdot)$ are linearly independent in a Lebesgue sense.

The following inequality can be obtained by setting $\delta = 0$ in Lemma 5.2 based on the notion of empty matrices. Moreover, the following corollary is equivalent to Theorem 4.1 in Chapter 4.

Corollary 5.1. *Given $\mathcal{K}$ and $\varpi(\cdot)$ in (5.23) and $U \in \mathbb{S}^n_{\succeq 0} := \{X \in \mathbb{S}^n : X \succeq 0\}$ with $n \in \mathbb{N}$. Let $\mathsf{f}(\cdot) := \mathbf{Col}_{i=1}^{d}\, \mathsf{f}_i(\cdot) \in \mathbb{L}^2_{\varpi}\left(\mathcal{K}\,;\mathbb{R}^d\right)$ with $d \in \mathbb{N}$ where $\mathsf{f}(\cdot)$ satisfies*

$$\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}(\tau)\mathsf{f}^{\top}(\tau)\mathsf{d}\tau \succ 0. \tag{5.27}$$

Then the inequality

$$\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^{\top}(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau \geq \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^{\top}(\tau)\mathsf{F}^{\top}(\tau)\mathsf{d}\tau\,(\mathcal{F}_d \otimes U)\int_{\mathcal{K}} \varpi(\tau)\mathsf{F}(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau \tag{5.28}$$

holds for all $\boldsymbol{x}(\cdot) \in \mathbb{L}^2_{\varpi}(\mathcal{K}\,;\mathbb{R}^n)$, where $\mathsf{F}(\tau) = \mathsf{f}(\tau) \otimes I_n \in \mathbb{R}^{dn\times n}$ and $\mathcal{F}_d^{-1} = \int_{\mathcal{K}} \varpi(\tau)\mathsf{f}(\tau)\mathsf{f}^{\top}(\tau)\mathsf{d}\tau \in \mathbb{S}^d_{\succ 0}$.

[3] $\operatorname{Ker}\left(\|\bullet\|_{2,\varpi}\right) := \left\{\boldsymbol{f}(\cdot) \in \mathbb{L}^2_{\varpi}\left(\mathcal{K}\,;\mathbb{R}^d\right) : \|\boldsymbol{f}(\cdot)\|_{2,\varpi} = \mathbf{0}_d\right\}$

Remark 5.7. The inequality in (5.25) reduces to Lemma 1 in [188] if $\mathsf{f}(\cdot)$ contains only Legendre polynomials. Moreover, all the particular cases of (4.3) mentioned in Chapter 4 are the special cases of (5.25) since Corollary 5.1 is equivalent to Theorem 4.1.

Remark 5.8. In (5.25), $\mathsf{f}(\cdot)$ can be interpreted as to approximate $\mathbf{g}(\cdot)$. By letting $\mathsf{f}(\tau) = \acute{\boldsymbol{f}}(\tau)$ and $\mathbf{g}(\tau) = \boldsymbol{\varphi}_1(\tau)$ with $\varpi(\tau) = 1$ in Lemma 5.2, then we have $\mathcal{E}_d = \acute{\mathsf{E}}_d$ where the matrix $\acute{\mathsf{E}}_d$ is defined in (5.12). Similar procedures can be applied with $\mathsf{f}(\tau) = \grave{\boldsymbol{f}}(\tau)$ and $\mathbf{g}(\tau) = \boldsymbol{\varphi}_2(\tau)$ and $\varpi(\tau) = 1$. Furthermore, if $\mathsf{f}(\cdot)$ contains only functions which are orthogonal with respect to $\varpi(\cdot)$, then the behavior of $\mathcal{E}_d$ can be quantitatively characterized with respect to d, which will be elaborated in the following corollary. Note that (D.1) holds for any $\mathsf{A} \in \mathbb{R}^{\delta\times d}$ as long as (5.24) is satisfied, even if A is not defined as $\mathsf{A} = \int_{\mathcal{K}} \varpi(\tau)\mathbf{g}(\tau)\mathsf{f}^\top(\tau)\mathsf{d}\tau\, \mathcal{F}_d \in \mathbb{R}^{\delta\times d}$. This is an important conclusion as it infers that the error matrices $\acute{\mathsf{E}}_d$ and $\grave{\mathsf{E}}_\delta$ in (5.12) are invertible since (5.8) hold.

An interesting corollary of Lemma 5.2 is presented as follows which can be interpreted as a generalization of Lemma 1 in [188].

Corollary 5.2. *Given all the parameters defined in Lemma 5.2 with* $\{\mathsf{f}_i(\tau)\}_{i=1}^{\infty}$ *and* $\mathsf{f}(\cdot) = \mathbf{Col}_{i=1}^{d}\, \mathsf{f}_i(\cdot)$ *satisfying*

$$\forall d \in \mathbb{N},\ \ \mathcal{F}_d^{-1} = \int_{\mathcal{K}} \mathsf{f}(\tau)\mathsf{f}^\top(\tau)\mathsf{d}\tau = \bigoplus_{i=1}^{d} \left(\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_i^2(\tau)\mathsf{d}\tau\right), \tag{5.29}$$

then we have that

$$\forall d \in \mathbb{N},\ \ 0 \prec \mathcal{E}_{d+1} = \mathcal{E}_d - \left(\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}^2(\tau)\mathsf{d}\tau\right) \mathbf{a}_{d+1}\mathbf{a}_{d+1}^\top \preceq \mathcal{E}_d \tag{5.30}$$

where $\mathcal{E}_d$ *is given in Lemma 5.2 and* $\mathbf{a}_{d+1} := \left(\int_{\mathcal{K}} \varpi(\tau)\mathbf{g}(\tau)\mathsf{f}_{d+1}(\tau)\mathsf{d}\tau\right)\left(\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}^2(\tau)\mathsf{d}\tau\right)^{-1} \in \mathbb{R}^\delta$ *and* $\mathsf{f}_{d+1}(\cdot) \in \mathbb{L}_{\varpi}^2(\mathcal{K} ; \mathbb{R})$.

Proof. Note that only the dimension of $\mathsf{f}(\cdot)$ is related to d, whereas δ as the dimension of $\mathbf{g}(\cdot)$ is independent of d. It is obvious to see that given $\mathsf{f}(\cdot)$ satisfying (5.29), we have $\mathcal{F}_{d+1} = \mathcal{F}_d \oplus \left(\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}^2(\tau)\mathsf{d}\tau\right)^{-1}$ (See the Definition 1 in [262]). By using this property, it follows that for all $d \in \mathbb{N}$

$$\begin{aligned}\mathbf{e}_{d+1}(\tau) = \mathbf{g}(\tau) - \left(\int_{\mathcal{K}} \varpi(\tau)\mathbf{g}(\tau)\begin{bmatrix}\mathsf{f}^\top(\tau) & \mathsf{f}_{d+1}(\tau)\end{bmatrix}\mathsf{d}\tau\right)\left[\mathcal{F}_d \oplus \left(\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}^2(\tau)\mathsf{d}\tau\right)^{-1}\right]\begin{bmatrix}\mathsf{f}(\tau)\\ \mathsf{f}_{d+1}(\tau)\end{bmatrix} = \mathbf{g}(\tau)\\ - \begin{bmatrix}\mathsf{A}_d & \mathbf{a}_{d+1}\end{bmatrix}\begin{bmatrix}\mathsf{f}(\tau)\\ \mathsf{f}_{d+1}(\tau)\end{bmatrix} = \mathbf{e}_d(\tau) - \mathsf{f}_{d+1}(\tau)\mathbf{a}_{d+1}\end{aligned} \tag{5.31}$$

where $\mathbf{a}_{d+1}$ has been defined in (5.30) and $\mathbf{e}_d(\tau) = \mathbf{g}(\tau) - \mathsf{A}_d\mathsf{f}(\tau)$. Note that the index d is added to the symbols A and $\mathbf{e}(\tau)$ in Lemma 5.2 without causing ambiguity. By (5.31) and (D.1), we have

$$\begin{aligned}0 \prec \mathcal{E}_{d+1} = \int_{\mathcal{K}} \varpi(\tau)\mathbf{e}_{d+1}(\tau)\mathbf{e}_{d+1}^\top(\tau)\mathsf{d}\tau = \mathcal{E}_d - \mathbf{Sy}\left(\mathbf{a}_{d+1}\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}(\tau)\mathbf{e}_d^\top(\tau)\mathsf{d}\tau\right)\\ + \left(\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}^2(\tau)\mathsf{d}\tau\right)\mathbf{a}_{d+1}\mathbf{a}_{d+1}^\top.\end{aligned} \tag{5.32}$$

By (D.2) and the fact that $\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}(\tau)\mathsf{f}(\tau)\mathsf{d}\tau = \mathbf{0}_d$ due to (5.29), we have

$$\begin{aligned}\mathsf{O}_{\delta\times(d+1)} &= \int_{\mathcal{K}} \varpi(\tau)\mathbf{e}_{d+1}(\tau)\begin{bmatrix}\mathsf{f}^\top(\tau) & \mathsf{f}_{d+1}(\tau)\end{bmatrix}\mathsf{d}\tau = \int_{\mathcal{K}} \varpi(\tau)\left(\mathbf{e}_d(\tau) - \mathbf{a}_{d+1}\mathsf{f}_{d+1}(\tau)\right)\begin{bmatrix}\mathsf{f}^\top(\tau) & \mathsf{f}_{d+1}(\tau)\end{bmatrix}\mathsf{d}\tau\\ &= \int_{\mathcal{K}} \varpi(\tau)\begin{bmatrix}\mathbf{e}_d(\tau)\mathsf{f}^\top(\tau) & \mathsf{f}_{d+1}(\tau)\mathbf{e}_d(\tau)\end{bmatrix}\mathsf{d}\tau - \mathbf{a}_{d+1}\int_{\mathcal{K}} \varpi(\tau)\begin{bmatrix}\mathsf{f}_{d+1}(\tau)\mathsf{f}^\top(\tau) & \mathsf{f}_{d+1}^2(\tau)\end{bmatrix}\mathsf{d}\tau\\ &= \begin{bmatrix}\mathsf{O}_{\delta\times d} & \int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}(\tau)\mathbf{e}_d(\tau)\mathsf{d}\tau\end{bmatrix} - \begin{bmatrix}\mathsf{O}_{\delta\times d} & \int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}^2(\tau)\mathsf{d}\tau\,\mathbf{a}_{d+1}\end{bmatrix} = \mathsf{O}_{\delta\times(d+1)}.\end{aligned} \tag{5.33}$$

Now (5.33) leads to the equality $\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}(\tau)\mathbf{e}_d(\tau)\mathsf{d}\tau = \int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}^2(\tau)\mathsf{d}\tau\,\mathbf{a}_{d+1}$. Substituting this equality into (5.32) yields (5.30) given $\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}_{d+1}^2(\tau)\mathsf{d}\tau > 0$ and $\mathbf{a}_{d+1}\mathbf{a}_{d+1}^\top \succeq 0$. ■

Remark 5.9. The result of Lemma 1 in [188] is generalized by Corollary 5.2 as $\mathbf{f}(\cdot)$ can be chosen to only have Legendre polynomials with $\varpi(\tau) = 1$. Moreover, let $\acute{\boldsymbol{f}}(\cdot)$, $\grave{\boldsymbol{f}}(\cdot)$ in (5.7) to contain only orthogonal functions over $[-r_1, 0]$ and $[-r_2, -r_1]$, respectively, then $\acute{\mathsf{E}}_d$ and $\grave{\mathsf{E}}_\delta$ in (5.12) follows the property in (5.30) with $\varpi(\tau) = 1$.

5.4 Main results on dissipativity and stability analysis

The main result on the dissipativity and stability analysis of (5.1) is presented in Theorem 5.1 where the condition for the dissipativity and stability analysis of (5.1) is denoted in terms of LMIs. Moreover, we will also show in Corollary 5.3, 5.4 that the resulting condition in Theorem 5.1 can exhibit a hierarchical pattern if certain perquisites are satisfied.

Theorem 5.1. *Suppose that all functions and the parameters in* (5.3)–(5.12) *are given with* $\mu_1; \mu_2 \in \mathbb{N}_0$ *and* $d; \delta \in \mathbb{N}$. *Assume also that there exist* $\acute{\boldsymbol{g}}(\cdot) \in \mathbb{C}^1(\mathbb{R} \,; \mathbb{R}^{p_1})$, $\grave{\boldsymbol{g}}(\cdot) \in \mathbb{C}^1(\mathbb{R} \,; \mathbb{R}^{p_2})$ *and* $N_1 \in \mathbb{R}^{p_1 \times d}$, $N_2 \in \mathbb{R}^{p_2 \times \delta}$ *such that*

$$
\begin{aligned}
&(\tau + r_1)\frac{\mathsf{d}\acute{\boldsymbol{g}}(\tau)}{\mathsf{d}\tau} = N_1 \acute{\boldsymbol{f}}(\tau) && (\tau + r_2)\frac{\mathsf{d}\grave{\boldsymbol{g}}(\tau)}{\mathsf{d}\tau} = N_2 \grave{\boldsymbol{f}}(\tau) \\
&\mathbb{S}^{p_1} \ni \acute{\mathsf{G}}_{p_1}^{-1} = \int_{-r_1}^{0} (\tau + r_1)\acute{\boldsymbol{g}}(\tau)\acute{\boldsymbol{g}}^\top(\tau)\mathsf{d}\tau \succ 0 && \mathbb{S}^{p_2} \ni \grave{\mathsf{G}}_{p_2}^{-1} = \int_{-r_2}^{-r_1} (\tau + r_2)\grave{\boldsymbol{g}}(\tau)\grave{\boldsymbol{g}}^\top(\tau)\mathsf{d}\tau \succ 0.
\end{aligned}
\tag{5.34}
$$

Given $r_2 > r_1 > 0$, *then the delay system* (5.14) *with the supply rate function* (5.22) *is dissipative and the origin of* (5.14) *is globally asymptotically stable with* $\boldsymbol{w}(t) \equiv \mathbf{0}_q$, *if there exist* $P \in \mathbb{S}^l$ *and* $Q_1; Q_2; R_1; R_2; S_1; S_2; U_1; U_2 \in \mathbb{S}^\nu$ *such that the inequalities*

$$
\begin{aligned}
\mathbf{P} := P + \left(\mathsf{O}_{n+2\nu} \oplus \left[\acute{\mathsf{F}}_d \otimes Q_1\right] \oplus \left[\grave{\mathsf{F}}_\delta \otimes Q_2\right]\right) + \Pi^\top \left(G_1^\top \acute{\Phi}_{\kappa_1} G_1 \otimes S_1 + G_2^\top \grave{\Phi}_{\kappa_2} G_2 \otimes S_2\right)\Pi \\
+ \Pi^\top \left(H_1^\top \acute{\mathsf{G}}_{p_1} H_1 \otimes U_1 + H_2^\top \grave{\mathsf{G}}_{p_2} H_2 \otimes U_2\right)\Pi \succ 0,
\end{aligned}
\tag{5.35}
$$

$$
Q_1 \succeq 0,\ \ Q_2 \succeq 0,\ \ R_1 \succeq 0,\ \ R_2 \succeq 0,\ \ S_1 \succeq 0,\ \ S_2 \succeq 0,\ \ U_1 \succeq 0,\ \ U_2 \succeq 0, \tag{5.36}
$$

$$
\widetilde{\mathbf{\Omega}} = \begin{bmatrix} J_1 & \mathsf{O}_{m\times\nu} & \widetilde{J}\Sigma \\ * & -S_1 - r_1 U_1 & (S_1 + r_1 U_1)(A_6 \mathbf{A} + Y) \\ * & * & \mathbf{\Omega} \end{bmatrix} \prec 0 \tag{5.37}
$$

hold, where the positive definite matrices $\acute{\mathsf{F}}_d$, $\grave{\mathsf{F}}_\delta$, $\acute{\Phi}_{\kappa_1}$ *and* $\acute{\Phi}_{\kappa_2}$ *are given in* (5.5) *and* (5.6), *and the parameters* $\mathbf{A}$ *and* Σ *have been defined in* (5.15)–(5.17). *Moreover,*

$$
\Pi := \begin{bmatrix} \Xi & \\ \mathsf{O}_{(2\nu+\varrho\nu)\times n} & I_{2\nu+\varrho\nu} \end{bmatrix},\ \ Y := \begin{bmatrix} A_7 & A_8 & \mathsf{O}_{\nu\times(q+l+\mu\nu)} \end{bmatrix} \tag{5.38}
$$

with $\varrho = d + \delta$ *and* $l = n + 2\nu + \varrho\nu$ *and* $\mu = \mu_1 + \mu_2$, *and*

$$
\begin{aligned}
\mathbf{\Omega} := \mathbf{Sy}\left(\Theta_2^\top P \Theta_1 - \left[\mathsf{O}_{(2\nu+q+l+\mu\nu)\times 2\nu} \ \ \Sigma^\top J_2 \ \ \mathsf{O}_{(2\nu+q+l+\mu\nu)\times(l+\mu\nu)}\right]\right) \\
- \left(\mathsf{O}_{q+2\nu} \oplus \left[\Pi^\top \left(G_1^\top \acute{\Phi}_{\kappa_1} G_1 \otimes U_1 + G_2^\top \grave{\Phi}_{\kappa_2} G_2 \otimes U_2\right)\Pi\right] \oplus \mathsf{O}_{\mu\nu}\right) \\
- \Bigg([S_1 - S_2 - r_3 U_2] \oplus S_2 \oplus J_3 \oplus \mathsf{O}_n \oplus [Q_1 - Q_2 - r_3 R_2] \oplus Q_2 \oplus \left[\acute{\mathsf{F}}_d \otimes R_1\right] \oplus \left[\grave{\mathsf{F}}_\delta \otimes R_2\right] \\
\oplus \left[\acute{\mathsf{E}}_d \otimes R_1\right] \oplus \left[\grave{\mathsf{E}}_\delta \otimes R_2\right] \Bigg) + \left[\mathsf{O}_{\nu\times(2\nu+q)} \ \ \Xi \ \ \mathsf{O}_{\nu\times\nu\mu}\right]^\top (Q_1 + r_1 R_1) \left[\mathsf{O}_{\nu\times(2\nu+q)} \ \ \Xi \ \ \mathsf{O}_{\nu\times\nu\mu}\right]
\end{aligned}
\tag{5.39}
$$

where

$$
G_1 = \begin{bmatrix} \acute{\boldsymbol{\phi}}(0) & -\acute{\boldsymbol{\phi}}(-r_1) & \mathbf{0}_{\kappa_1} & -M_3 & \mathsf{O}_{\kappa_1\times d} \end{bmatrix} \quad G_2 = \begin{bmatrix} \mathbf{0}_{\kappa_2} & \grave{\boldsymbol{\phi}}(-r_1) & -\grave{\boldsymbol{\phi}}(-r_2) & \mathsf{O}_{\kappa_2\times\delta} & -M_4 \end{bmatrix} \tag{5.40}
$$

$$G_3 = \begin{bmatrix} \acute{\boldsymbol{f}}(0) & -\acute{\boldsymbol{f}}(-r_1) & \mathbf{0}_d & -M_1 & \mathsf{O}_d \end{bmatrix} \qquad G_4 = \begin{bmatrix} \mathbf{0}_\delta & \grave{\boldsymbol{f}}(-r_1) & -\grave{\boldsymbol{f}}(-r_2) & \mathsf{O}_\delta & -M_2 \end{bmatrix}, \tag{5.41}$$

$$H_1 = \begin{bmatrix} r_1\acute{\boldsymbol{g}}(0) & \mathbf{0}_{p_1} & \mathbf{0}_{p_1} & -N_1 & \mathsf{O}_{p_1} \end{bmatrix} \qquad H_2 = \begin{bmatrix} \mathbf{0}_{p_2} & (r_2-r_1)\grave{\boldsymbol{g}}(-r_1) & \mathbf{0}_{p_2} & \mathsf{O}_{p_2} & -N_2 \end{bmatrix} \tag{5.42}$$

$$\Theta_1 := \begin{bmatrix} & \mathbf{A} & \\ & I_{2\nu} \quad \mathsf{O}_{2\nu\times(q+l+\mu\nu)} & \\ \mathsf{O}_{d\nu\times(2\nu+q)} & (G_3\otimes I_\nu)\,\Pi & \mathsf{O}_{d\nu\times\nu\mu} \\ \mathsf{O}_{\delta\nu\times(2\nu+q)} & (G_4\otimes I_\nu)\,\Pi & \mathsf{O}_{\delta\nu\times\nu\mu} \end{bmatrix} \qquad \Theta_2 := \begin{bmatrix} \mathsf{O}_{l\times(2\nu+q)} & I_l & \mathsf{O}_{l\times\mu\nu} \end{bmatrix}. \tag{5.43}$$

Proof. Given $r_2 > r_1 > 0$, we consider the following LKF

$$\begin{aligned} v(\boldsymbol{\xi},\boldsymbol{\psi}(\cdot)) &= \boldsymbol{\eta}^\top(t_0)P\boldsymbol{\eta}(t_0) + \int_{-r_1}^{0} \boldsymbol{y}^\top(t+\tau)\big[Q_1+(\tau+r_1)R_1\big]\boldsymbol{y}(t+\tau)\mathsf{d}\tau \\ &+ \int_{-r_2}^{-r_1} \boldsymbol{y}^\top(t+\tau)\big[Q_2+(\tau+r_2)R_2\big]\boldsymbol{y}(t+\tau)\mathsf{d}\tau + \int_{-r_1}^{0} \dot{\boldsymbol{y}}^\top(t+\tau)\left[S_1+(\tau+r_1)U_1\right]\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau \\ &+ \int_{-r_2}^{-r_1} \dot{\boldsymbol{y}}^\top(t+\tau)\left[S_2+(\tau+r_2)U_2\right]\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau \end{aligned} \tag{5.44}$$

to be constructed to prove the statements in Theorem 5.1, where $\boldsymbol{x}(t)$ and $\mathbf{y}_t(\cdot)$ here follow the same definition in (5.21). Moreover,

$$\boldsymbol{\eta}(t) := \mathbf{Col}\left[\boldsymbol{x}(t),\ \boldsymbol{y}(t-r_1),\ \boldsymbol{y}(t-r_2),\ \int_{-r_1}^{0}\acute{F}_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau,\ \int_{-r_2}^{-r_1}\grave{F}_\delta(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\right] \tag{5.45}$$

with $\acute{F}_d(\tau)$ and $\grave{F}_\delta(\tau)$ defined in (5.18), and the matrix parameters in (5.44) are defined as $P \in \mathbb{S}^l$ and $Q_1; Q_2; R_1; R_2; S_1; S_2; U_1; U_2 \in \mathbb{S}^\nu$ with $l := n+2\nu+\varrho\nu$ and $\varrho := d+\delta$. Note that since the eigenvalues of all the matrix terms $Q_1+(\tau+r_1)R_1$, $Q_2+(\tau+r_2)R_2$ $S_1+(\tau+r_1)U_1$ and $S_2+(\tau+r_2)U_2$ in (5.44) are bounded, thus all the quadratic integrals associated with these terms are well defined since $\mathbf{y}_t(\cdot) \in \mathcal{A}\left([-r_2,0) ⨟ \mathbb{R}^\nu\right)$. On the other hand, since $\mathbf{y}_t(\cdot)$, $\acute{\boldsymbol{f}}(\tau)$ and $\grave{\boldsymbol{f}}(\tau)$ are bounded, thus the integrals in (5.45) are well defined as well.

Firstly, we prove that the existence of the feasible solutions of (5.36) and (5.37) infers that (5.44) satisfies both (5.20) and (5.21). Subsequently, we show that the existence of the feasible solutions of (5.35) and (5.36) infers that (5.44) satisfies (5.19). The existence of the upper bound of $v(\boldsymbol{x}(t),\mathbf{y}_t(\cdot))$ can be independently proved without considering the inequalities (5.35)–(5.37).

Let $t_0 \in \mathbb{R}$, differentiate $v(\boldsymbol{x}(t),\mathbf{y}_t(\cdot))$ along the trajectory of (5.14) and consider (5.22), it produces

$$\begin{aligned} \forall t \geq t_0, \quad & \dot{v}(\boldsymbol{x}(t),\mathbf{y}_t(\cdot)) - s(\boldsymbol{z}(t),\boldsymbol{w}(t)) = \boldsymbol{\vartheta}^\top(t)\,\mathbf{Sy}\left(\Theta_2^\top P\Theta_1\right)\boldsymbol{\vartheta}(t) + \boldsymbol{y}^\top(t)\left(Q_1+r_1R_1\right)\boldsymbol{y}(t) \\ &+ \boldsymbol{y}^\top(t-r_1)\left(Q_2+r_3R_2-Q_1\right)\boldsymbol{y}(t-r_1) - \boldsymbol{y}^\top(t-r_2)Q_2\boldsymbol{y}(t-r_2) + \dot{\boldsymbol{y}}^\top(t)\left(S_1+r_1U_1\right)\dot{\boldsymbol{y}}(t) \\ &+ \dot{\boldsymbol{y}}^\top(t-r_1)\left(S_2+r_3U_2-S_1\right)\dot{\boldsymbol{y}}(t-r_1) - \dot{\boldsymbol{y}}^\top(t-r_2)S_2\dot{\boldsymbol{y}}(t-r_2) - \int_{-r_1}^{0}\boldsymbol{y}^\top(t+\tau)R_1\boldsymbol{y}(t+\tau)\mathsf{d}\tau \\ &- \int_{-r_2}^{-r_1}\boldsymbol{y}^\top(t+\tau)R_2\boldsymbol{y}(t+\tau)\mathsf{d}\tau - \int_{-r_1}^{0}\dot{\boldsymbol{y}}^\top(t+\tau)U_1\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau - \int_{-r_2}^{-r_1}\dot{\boldsymbol{y}}^\top(t+\tau)U_2\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau \\ &- \boldsymbol{w}^\top(t)J_3\boldsymbol{w}(t) - \boldsymbol{\vartheta}^\top(t)\left[\Sigma^\top\widetilde{J}^\top J_1^{-1}\widetilde{J}\Sigma + \mathbf{Sy}\left(\begin{bmatrix}\mathsf{O}_{(2\nu+q+l+\mu\nu)\times 2\nu} & \Sigma^\top J_2 & \mathsf{O}_{(2\nu+q+l+\mu\nu)\times(l+\mu\nu)}\end{bmatrix}\right)\right]\boldsymbol{\vartheta}(t) \end{aligned} \tag{5.46}$$

where $\boldsymbol{\vartheta}(t)$ and Θ_1;Θ_2 have been defined in (5.18) and (5.43), respectively, and the matrices G_3 and G_4 in (5.41) are obtained via the relations

$$\begin{aligned} \int_{-r_1}^{0}\acute{F}_d(\tau)\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau &= \acute{F}_d(0)\boldsymbol{y}(t) - \acute{F}_d(-r_1)\boldsymbol{y}(t-r_1) - (M_1\otimes I_\nu)\int_{-r_1}^{0}\acute{F}_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau = \\ &= \begin{bmatrix}\mathsf{O}_{d\nu\times(q+2\nu)} & (G_3\otimes I_\nu)\,\Pi & \mathsf{O}_{d\nu\times\nu\mu}\end{bmatrix}\boldsymbol{\vartheta}(t) \end{aligned} \tag{5.47}$$

$$\begin{aligned} \int_{-r_2}^{-r_1}\grave{F}_\delta(\tau)\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau &= \grave{F}_\delta(-r_1)\boldsymbol{y}(t-r_1) - \grave{F}_\delta(-r_2)\boldsymbol{y}(t-r_2) - (M_2\otimes I_\nu)\int_{-r_2}^{-r_1}\grave{F}_\delta(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau \\ &= \begin{bmatrix}\mathsf{O}_{\delta\nu\times(q+2\nu)} & (G_4\otimes I_\nu)\,\Pi & \mathsf{O}_{\delta\nu\times\nu\mu}\end{bmatrix}\boldsymbol{\vartheta}(t) \end{aligned} \tag{5.48}$$

which themselves can be derived by using (2.2), (2.1) with (5.3).

To obtain a upper bound for (5.46), let $R_1 \succeq 0$, $R_2 \succeq 0$ so that the inequalities

$$\begin{aligned}\int_{-r_1}^{0} \boldsymbol{y}^\top(t+\tau)R_1\boldsymbol{y}(t+\tau)\mathsf{d}\tau \geq \int_{-r_1}^{0} \boldsymbol{y}^\top(t+\tau)\acute{F}_d^\top(\tau)\mathsf{d}\tau\left(\acute{\mathsf{F}}_d\otimes R_1\right)\int_{-r_1}^{0}\acute{F}_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau \\ +\int_{-r_1}^{0} \boldsymbol{y}^\top(t+\tau)\acute{E}_d^\top(\tau)\mathsf{d}\tau\left(\acute{\mathsf{E}}_d\otimes R_1\right)\int_{-r_1}^{0}\acute{E}_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau \\ \int_{-r_2}^{-r_1} \boldsymbol{y}^\top(t+\tau)R_2\boldsymbol{y}(t+\tau)\mathsf{d}\tau \geq \int_{-r_2}^{-r_1} \boldsymbol{y}^\top(t+\tau)\grave{F}_\delta^\top(\tau)\mathsf{d}\tau\left(\grave{\mathsf{F}}_\delta\otimes R_2\right)\int_{-r_2}^{-r_1}\grave{F}_\delta(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau \\ +\int_{-r_2}^{-r_1} \boldsymbol{y}^\top(t+\tau)\grave{E}_\delta^\top(\tau)\mathsf{d}\tau\left(\grave{\mathsf{E}}_\delta\otimes R_2\right)\int_{-r_2}^{-r_1}\grave{E}_\delta(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\end{aligned} \tag{5.49}$$

can be derived from (5.25) with $\mathbf{f}(\tau) = \acute{\boldsymbol{f}}(\tau)$; $\mathbf{g}(\tau) = \boldsymbol{\varphi}_1(\tau)$ and $\mathbf{f}(\tau) = \grave{\boldsymbol{f}}(\tau)$; $\mathbf{g}(\tau) = \boldsymbol{\varphi}_2(\tau)$, respectively, which matches $\acute{F}_d(\tau)$; $\grave{F}_\delta(\tau)$ in (5.18) and the expressions in (5.11). Furthermore, let $U_1 \succeq 0$ and $U_2 \succeq 0$ and apply (5.28) to the integral terms $\int_{-r_1}^{0} \dot{\boldsymbol{y}}^\top(t+\tau)U_1\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau$ and $\int_{-r_2}^{-r_1} \dot{\boldsymbol{y}}^\top(t+\tau)U_2\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau$ with $\mathbf{f}(\tau) = \acute{\boldsymbol{\phi}}(\tau)$ and $\mathbf{f}(\tau) = \grave{\boldsymbol{\phi}}(\tau)$ in (5.6), respectively, and consider the expression $\boldsymbol{y}(t) = \begin{bmatrix}\mathsf{O}_{\nu\times(2\nu+q)} & \Xi & \mathsf{O}_{\nu\times\nu\mu}\end{bmatrix}\boldsymbol{\vartheta}(t)$ in (5.16) with (2.1) and (2.2). It produces

$$\begin{aligned}\int_{-r_1}^{0} \dot{\boldsymbol{y}}^\top(t+\tau)U_1\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau \geq \int_{-r_1}^{0} \dot{\boldsymbol{y}}^\top(t+\tau)\left(\acute{\boldsymbol{\phi}}^\top(\tau)\otimes I_\nu\right)\mathsf{d}\tau\left(\acute{\Phi}_{\kappa_1}\otimes U_1\right)\int_{-r_1}^{0}\left(\acute{\boldsymbol{\phi}}(\tau)\otimes I_\nu\right)\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau \\ = \boldsymbol{\vartheta}^\top(\tau)\left[\mathsf{O}_{2\nu+q}\oplus\Pi^\top\left(G_1^\top\acute{\Phi}_{\kappa_1}G_1\otimes U_1\right)\Pi\oplus\mathsf{O}_{\nu\mu}\right]\boldsymbol{\vartheta}(\tau),\end{aligned} \tag{5.50}$$

$$\begin{aligned}\int_{-r_2}^{-r_1} \dot{\boldsymbol{y}}^\top(t+\tau)U_2\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau \geq \int_{-r_2}^{-r_1} \dot{\boldsymbol{y}}^\top(t+\tau)\left(\grave{\boldsymbol{\phi}}^\top(\tau)\otimes I_\nu\right)\mathsf{d}\tau\left(\grave{\Phi}_{\kappa_2}\otimes U_2\right)\int_{-r_2}^{-r_1}\left(\grave{\boldsymbol{\phi}}(\tau)\otimes I_\nu\right)\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau \\ = \boldsymbol{\vartheta}^\top(\tau)\left[\mathsf{O}_{2\nu+q}\oplus\Pi^\top\left(G_2^\top\grave{\Phi}_{\kappa_2}G_2\otimes U_2\right)\Pi\oplus\mathsf{O}_{\nu\mu}\right]\boldsymbol{\vartheta}(\tau)\end{aligned} \tag{5.51}$$

where G_1 and G_2 are given in (5.40) which are derived by the relations

$$\begin{aligned}\int_{-r_1}^{0}\left(\acute{\boldsymbol{\phi}}(\tau)\otimes I_\nu\right)\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau = \left(\acute{\boldsymbol{\phi}}(0)\otimes I_\nu\right)\boldsymbol{y}(t) - \left(\acute{\boldsymbol{\phi}}(-r_1)\otimes I_\nu\right)\boldsymbol{y}(t-r_1) \\ -\left(M_3\otimes I_\nu\right)\int_{-r_1}^{0}\left(\acute{\boldsymbol{\phi}}(\tau)\otimes I_\nu\right)\boldsymbol{y}(t+\tau)\mathsf{d}\tau = \left(G_1\otimes I_\nu\right)\Pi\boldsymbol{\eta}(t) \\ = \begin{bmatrix}\mathsf{O}_{\kappa_1\nu\times(q+2\nu)} & \left(G_1\otimes I_\nu\right)\Pi & \mathsf{O}_{\kappa_1\nu\times\nu\mu}\end{bmatrix}\boldsymbol{\vartheta}(t)\end{aligned} \tag{5.52}$$

$$\begin{aligned}\int_{-r_2}^{-r_1}\left(\grave{\boldsymbol{\phi}}(\tau)\otimes I_\nu\right)\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau = \left(\grave{\boldsymbol{\phi}}(-r_1)\otimes I_\nu\right)\boldsymbol{y}(t-r_1) - \left(\grave{\boldsymbol{\phi}}(-r_2)\otimes I_\nu\right)\boldsymbol{y}(t-r_2) \\ -\left(M_4\otimes I_\nu\right)\int_{-r_2}^{-r_1}\left(\grave{\boldsymbol{\phi}}(\tau)\otimes I_\nu\right)\boldsymbol{y}(t+\tau)\mathsf{d}\tau = \left(G_2\otimes I_\nu\right)\Pi\boldsymbol{\eta}(t) \\ = \begin{bmatrix}\mathsf{O}_{\kappa_2\nu\times(q+2\nu)} & \left(G_2\otimes I_\nu\right)\Pi & \mathsf{O}_{\kappa_2\nu\times\nu\mu}\end{bmatrix}\boldsymbol{\vartheta}(t).\end{aligned} \tag{5.53}$$

Now applying (5.49)–(5.51) with (5.36) to (5.46) yields

$$\begin{aligned}\forall t\geq t_0,\ \ \dot{v}(\boldsymbol{x}(t),\mathbf{y}_t(\cdot)) - s(\boldsymbol{z}(t),\boldsymbol{w}(t)) \leq \\ \boldsymbol{\vartheta}^\top(t)\left[\boldsymbol{\Omega} + \left(A_6\mathbf{A}+Y\right)^\top\left(S_1+r_1U_1\right)\left(A_6\mathbf{A}+Y\right) - \Sigma^\top\widetilde{J}^\top J_1^{-1}\widetilde{J}\Sigma\right]\boldsymbol{\vartheta}(t)\end{aligned} \tag{5.54}$$

where $\boldsymbol{\Omega}$ has been defined in (5.39). It is obvious that if $\boldsymbol{\Omega} + \left(A_6\mathbf{A}+Y\right)^\top\left(S_1+r_1U_1\right)\left(A_6\mathbf{A}+Y\right) - \Sigma^\top\widetilde{J}^\top J_1^{-1}\widetilde{J}\Sigma \prec 0$ holds with (5.36), we have

$$\exists\epsilon_3>0,\ \forall t\geq t_0,\ \ \dot{v}(\boldsymbol{x}(t),\mathbf{y}_t(\cdot)) - s(\boldsymbol{z}(t),\boldsymbol{w}(t)) \leq -\epsilon_3\left\|\boldsymbol{x}(t)\right\|_2. \tag{5.55}$$

Moreover, let $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ and consider the structure of the quadratic term in (5.54) together with the properties of negative definite matrices. One can conclude that if $\boldsymbol{\Omega} + \left(A_6\mathbf{A}+Y\right)^\top\left(S_1+r_1U_1\right)\left(A_6\mathbf{A}+Y\right) -$

$\Sigma^\top \widetilde{J}^\top J_1^{-1}\widetilde{J}\Sigma \prec 0$ and (5.36) are satisfied, it infers that

$$\exists \epsilon_3 > 0, \quad \frac{\mathsf{d}^+}{\mathsf{d}t} v(\boldsymbol{x}(t), \mathbf{y}_t(\cdot))\bigg|_{t=t_0, \boldsymbol{x}(t_0)=\boldsymbol{\xi}, \mathbf{y}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} \leq -\epsilon_3 \|\boldsymbol{\xi}\|_2 \tag{5.56}$$

where $\boldsymbol{x}(t)$ and $\mathbf{y}_t(\cdot)$ here follow the same definition in Lemma 5.1. As a result, it is obvious that (5.36) with $\boldsymbol{\Omega} + (A_6\mathbf{A} + Y)^\top (S_1 + r_1 U_1)(A_6\mathbf{A} + Y) - \Sigma^\top \widetilde{J}^\top J_1^{-1}\widetilde{J}\Sigma \prec 0$ infers (5.20) and (5.21). Finally, applying the Schur complement to $\boldsymbol{\Omega} + (A_6\mathbf{A} + Y)^\top (S_1 + r_1 U_1)(A_6\mathbf{A} + Y) - \Sigma^\top \widetilde{J}^\top J_1^{-1}\widetilde{J}\Sigma \prec 0$ with (5.36) and $J_1^{-1} \prec 0$ gives (5.37). Hence we have proved that the feasible solutions of (5.36) and (5.37) infers that (5.44) satisfies (5.20) and (5.21).

Now we start to prove that if (5.35) and (5.36) hold then (5.44) satisfies (5.19). Let $\|\boldsymbol{\psi}(\cdot)\|_\infty := \sup_{-r_2 \leq r \leq 0} \|\boldsymbol{\psi}(\tau)\|_2$ and $\|\boldsymbol{\psi}(\cdot)\|_2^2 := \int_{-r_2}^0 \boldsymbol{\psi}^\top(\tau)\boldsymbol{\psi}(\tau)\mathsf{d}\tau$. Given the structure of (5.44) with $t = t_0$, it follows that $\exists \lambda; \eta > 0$:

$$\begin{aligned}
v(\boldsymbol{x}(t_0), \mathbf{y}_{t_0}(\cdot)) &= v(\boldsymbol{\xi}, \boldsymbol{\psi}(\cdot)) \leq \boldsymbol{\eta}^\top(t_0)\lambda\boldsymbol{\eta}(t_0) + \lambda \int_{-r_2}^0 \begin{bmatrix} \boldsymbol{\psi}^\top(\tau) & \dot{\boldsymbol{\psi}}^\top(\tau) \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}(\tau) \\ \dot{\boldsymbol{\psi}}(\tau) \end{bmatrix} \mathsf{d}\tau \leq \lambda \|\boldsymbol{\xi}\|_2^2 \\
&+ (2\lambda + \lambda r_2)\|\boldsymbol{\psi}(\cdot)\|_\infty^2 + \lambda r_2 \|\dot{\boldsymbol{\psi}}(\cdot)\|_2^2 + \lambda \int_{-r_1}^0 \boldsymbol{\psi}^\top(\tau)\acute{F}_d^\top(\tau)\mathsf{d}\tau \int_{-r_1}^0 \acute{F}_d(\tau)\boldsymbol{\psi}(\tau)\mathsf{d}\tau \\
&+ \lambda \int_{-r_2}^{-r_1} \boldsymbol{\psi}^\top(\tau)\grave{F}_\delta^\top(\tau)\mathsf{d}\tau \int_{-r_2}^{-r_1} \grave{F}_\delta(\tau)\boldsymbol{\psi}(\tau)\mathsf{d}\tau \leq \lambda \|\boldsymbol{\xi}\|_2^2 + (2\lambda + \lambda r_2)\|\boldsymbol{\psi}(\cdot)\|_\infty^2 \\
&+ \lambda r_2 \|\dot{\boldsymbol{\psi}}(\cdot)\|_2^2 + \int_{-r_1}^0 \boldsymbol{\psi}^\top(\tau)\acute{F}_d^\top(\tau)\mathsf{d}\tau \left(\eta\acute{\mathsf{F}}_d \otimes I_\nu\right) \int_{-r_1}^0 \acute{F}_d(\tau)\boldsymbol{\psi}(\tau)\mathsf{d}\tau \\
&+ \int_{-r_2}^{-r_1} \boldsymbol{\psi}^\top(\tau)\grave{F}_\delta^\top(\tau)\mathsf{d}\tau \left(\eta\grave{\mathsf{F}}_\delta \otimes I_\nu\right) \int_{-r_2}^{-r_1} \grave{F}_\delta(\tau)\boldsymbol{\psi}(\tau)\mathsf{d}\tau \leq \lambda \|\boldsymbol{\xi}\|_2^2 + (2\lambda + \lambda r_2)\|\boldsymbol{\psi}(\cdot)\|_\infty^2 \\
&+ \lambda r_2 \|\dot{\boldsymbol{\psi}}(\cdot)\|_2^2 + \eta \int_{-r_1}^0 \boldsymbol{\psi}^\top(\tau)\boldsymbol{\psi}(\tau)\mathsf{d}\tau + \eta \int_{-r_2}^{-r_1} \boldsymbol{\psi}^\top(\tau)\boldsymbol{\psi}(\tau)\mathsf{d}\tau = \lambda \|\boldsymbol{\xi}\|_2^2 \\
&+ (2\lambda + \lambda r_2 + \eta r_2)\|\boldsymbol{\psi}(\cdot)\|_\infty^2 + \lambda r_2 \|\dot{\boldsymbol{\psi}}(\cdot)\|_2^2 \leq (2\lambda + \lambda r_2 + \eta r_2)\left(\|\boldsymbol{\xi}\|_2^2 + \|\boldsymbol{\psi}(\cdot)\|_\infty^2 + \|\dot{\boldsymbol{\psi}}(\cdot)\|_2^2\right) \\
&\leq (2\lambda + \lambda r_2 + \eta r_2)\left[\|\boldsymbol{\xi}\|_2^2 + \left(\|\boldsymbol{\psi}(\cdot)\|_\infty + \|\dot{\boldsymbol{\psi}}(\cdot)\|_2\right)^2\right] \\
&\leq (4\lambda + 2\lambda r_2 + 2\eta r_2)\left[\|\boldsymbol{\xi}\|_2 \vee \left(\|\boldsymbol{\psi}(\cdot)\|_\infty + \|\dot{\boldsymbol{\psi}}(\cdot)\|_2\right)\right]^2
\end{aligned} \tag{5.57}$$

for any initial condition $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\psi}(\cdot) \in \mathcal{A}([-r_2, 0) ⨾ \mathbb{R}^\nu)$ in (5.1), which is derived via (5.28) and the property of quadratic forms: $\forall X \in \mathbb{S}^n, \exists \lambda > 0 : \forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbf{x}^\top(\lambda I_n - X)\mathbf{x} > 0$. Then (5.57) shows that (5.44) satisfies the rightmost inequality in (5.19).

Now assume the inequalities in (5.36) are satisfied. Apply (5.28) with appropriate $\mathbf{f}(\cdot)$ with $\varpi(\tau) = 1$ to the integrals in (5.44) at $t = t_0$ and consider the initial conditions in (5.1), we have

$$\begin{aligned}
\int_{-r_1}^0 \boldsymbol{\psi}^\top(\tau)Q_1\boldsymbol{\psi}(\tau)\mathsf{d}\tau &\geq \int_{-r_1}^0 \boldsymbol{\psi}^\top(\tau)\acute{F}_d^\top(\tau)\mathsf{d}\tau \left(\acute{\mathsf{F}}_d \otimes Q_1\right) \int_{-r_1}^0 \acute{F}_d(\tau)\boldsymbol{\psi}(\tau)\mathsf{d}\tau, \\
\int_{-r_2}^{-r_1} \boldsymbol{\psi}^\top(\tau)Q_2\boldsymbol{\psi}(\tau)\mathsf{d}\tau &\geq \int_{-r_2}^{-r_1} \boldsymbol{\psi}^\top(\tau)\grave{F}_\delta^\top(\tau)\mathsf{d}\tau \left(\grave{\mathsf{F}}_\delta \otimes Q_2\right) \int_{-r_2}^{-r_1} \grave{F}_\delta(\tau)\boldsymbol{\psi}(\tau)\mathsf{d}\tau
\end{aligned} \tag{5.58}$$

and

$$\begin{aligned}
\int_{-r_1}^0 \dot{\boldsymbol{\psi}}^\top(\tau)S_1\dot{\boldsymbol{\psi}}(\tau)\mathsf{d}\tau &\geq \int_{-r_1}^0 \dot{\boldsymbol{\psi}}^\top(\tau)\left(\acute{\boldsymbol{\phi}}^\top(\tau) \otimes I_\nu\right)\mathsf{d}\tau \left(\acute{\Phi}_{\kappa_1} \otimes S_1\right) \int_{-r_1}^0 \left(\acute{\boldsymbol{\phi}}(\tau) \otimes I_\nu\right)\dot{\boldsymbol{\psi}}(\tau)\mathsf{d}\tau \\
&= \boldsymbol{\eta}^\top(t_0)\Pi^\top\left(G_1^\top\acute{\Phi}_{\kappa_1}G_1 \otimes S_1\right)\Pi\boldsymbol{\eta}(t_0),
\end{aligned} \tag{5.59}$$

$$\begin{aligned}
\int_{-r_2}^{-r_1} \dot{\boldsymbol{\psi}}^\top(\tau)S_2\dot{\boldsymbol{\psi}}(\tau)\mathsf{d}\tau &\geq \int_{-r_2}^{-r_1} \dot{\boldsymbol{\psi}}^\top(\tau)\left(\grave{\boldsymbol{\phi}}^\top(\tau) \otimes I_\nu\right)\mathsf{d}\tau \left(\grave{\Phi}_{\kappa_2} \otimes S_2\right) \int_{-r_2}^{-r_1} \left(\grave{\boldsymbol{\phi}}(\tau) \otimes I_\nu\right)\dot{\boldsymbol{\psi}}(\tau)\mathsf{d}\tau \\
&= \boldsymbol{\eta}^\top(t_0)\Pi^\top\left(G_2^\top\grave{\Phi}_{\kappa_2}G_2 \otimes S_2\right)\Pi\boldsymbol{\eta}(t_0)
\end{aligned} \tag{5.60}$$

which are derived via the relations in (5.52) and (5.53). Furthermore, apply (5.25) again with appropriate weight functions to the integrals $\int_{-r_1}^{0}(r_1+\tau)\dot{\boldsymbol{y}}^\top(t+\tau)U_1\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau$ and $\int_{-r_2}^{-r_1}(r_2+\tau)\dot{\boldsymbol{y}}^\top(t+\tau)U_2\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau$ for $t=t_0$ in (5.44) with $\mathbf{f}(\tau)=\acute{\boldsymbol{g}}(\tau)$, $\mathbf{f}(\tau)=\grave{\boldsymbol{g}}(\tau)$, respectively. Then it yields

$$\begin{aligned}\int_{-r_1}^{0}(r_1+\tau)\dot{\boldsymbol{\psi}}^\top(\tau)U_1\dot{\boldsymbol{\psi}}(\tau)\mathsf{d}\tau &\geq [*]\left(\acute{\mathsf{G}}_{p_1}\otimes U_1\right)\int_{-r_1}^{0}(\tau+r_1)\left(\acute{\boldsymbol{g}}(\tau)\otimes I_\nu\right)\dot{\boldsymbol{\psi}}(\tau)\mathsf{d}\tau\\ &= \boldsymbol{\eta}^\top(t_0)\Pi^\top\left(H_1^\top\acute{\mathsf{G}}_{p_1}H_1\otimes U_1\right)\Pi\boldsymbol{\eta}(t_0)\\ \int_{-r_2}^{-r_1}(r_2+\tau)\dot{\boldsymbol{\psi}}^\top(\tau)U_2\dot{\boldsymbol{\psi}}(\tau)\mathsf{d}\tau &\geq [*]\left(\grave{\mathsf{G}}_{p_2}\otimes U_2\right)\int_{-r_2}^{-r_1}(\tau+r_2)\left(\grave{\boldsymbol{g}}(\tau)\otimes I_\nu\right)\dot{\boldsymbol{\psi}}(\tau)\mathsf{d}\tau\\ &= \boldsymbol{\eta}^\top(t_0)\Pi^\top\left(H_2^\top\grave{\mathsf{G}}_{p_2}H_2\otimes U_2\right)\Pi\boldsymbol{\eta}(t_0)\end{aligned} \tag{5.61}$$

for any initial condition $\boldsymbol{\xi}\in\mathbb{R}^n$ and $\boldsymbol{\psi}(\cdot)\in\mathcal{A}\left([-r_2,0)\,⨾\,\mathbb{R}^\nu\right)$ in (5.1), where H_1 and H_2 are given in (5.42) and obtained by the relations

$$\begin{aligned}\int_{-r_1}^{0}(\tau+r_1)\left(\acute{\boldsymbol{g}}(\tau)\otimes I_\nu\right)\dot{\boldsymbol{\psi}}(\tau)\mathsf{d}\tau &= r_1\left(\acute{\boldsymbol{g}}(0)\otimes I_\nu\right)\boldsymbol{\psi}(0)-\left(N_1\otimes I_\nu\right)\int_{-r_1}^{0}\left(\acute{\boldsymbol{f}}(\tau)\otimes I_\nu\right)\boldsymbol{\psi}(\tau)\mathsf{d}\tau\\ &= \left(H_1\otimes I_\nu\right)\boldsymbol{\eta}(t_0)\end{aligned} \tag{5.62}$$

$$\begin{aligned}\int_{-r_2}^{-r_1}(\tau+r_2)\left(\grave{\boldsymbol{g}}(\tau)\otimes I_\nu\right)\dot{\boldsymbol{\psi}}(\tau)\mathsf{d}\tau &= (r_2-r_1)\left(\grave{\boldsymbol{g}}(-r_1)\otimes I_\nu\right)\boldsymbol{\psi}(-r_1)\\ &-\left(N_2\otimes I_\nu\right)\int_{-r_2}^{-r_1}\left(\grave{\boldsymbol{f}}(\tau)\otimes I_\nu\right)\boldsymbol{\psi}(\tau)\mathsf{d}\tau = \left(H_2\otimes I_\nu\right)\boldsymbol{\eta}(t_0)\end{aligned} \tag{5.63}$$

via (5.34) and the properties of the Kronecker product in (2.1) and (2.2).

With (5.36), utilizing (5.58)–(5.61) to (5.44) with $t=t_0$ and considering the initial conditions in (5.1) can conclude that (5.19) is satisfied if (5.35) and (5.36) hold. This shows that feasible solutions of (5.35)–(5.37) infers the existence of the functional in (5.44) satisfying (5.19)–(5.21). This finishes the proof. ■

Remark 5.10. By allowing m,q to be zero, Theorem 5.1 can cope with the problem of conducting stability analysis without performance requirements. Moreover, if $\acute{\boldsymbol{f}}(\cdot)$ and $\grave{\boldsymbol{f}}(\cdot)$ contain only Legendre polynomials, then Theorem 5.1 with (5.13) generalizes the two delay channel version of the stability results in [188]. (Note that the method in [188] only deals with systems with a single delay channel)

Remark 5.11. If one wants to increase the values of d and δ in (5.44) to incorporate more functions in the distributed delay terms in (5.45), then extra zeros need to be introduced to the coefficient matrices A_4, A_5 and C_4, C_5 in (5.7) in order to make (5.44) consistent with (5.7). In conclusion, there are no upper bound on the values of d and δ. Finally, (5.44) generalizes the LKF in [188] which only consider Legendre polynomials for the integral terms in (5.45).

Remark 5.12. If the condition in (5.34) is not imposed on $\acute{\boldsymbol{f}}(\cdot)$ and $\grave{\boldsymbol{f}}(\cdot)$ then dissipative conditions can still be derived but the inequalities in (5.61) can no longer be considered. In that case, the constraints (5.36) and (5.37) remain the same, and (5.35) is changed into

$$P+\left(\mathsf{O}_{n+2\nu}\oplus\left[\acute{\mathsf{F}}_d\otimes Q_1\right]\oplus\left[\grave{\mathsf{F}}_\delta\otimes Q_2\right]\right)+\Pi^\top\left(G_1^\top\acute{\Phi}_{\kappa_1}G_1\otimes S_1+G_2^\top\grave{\Phi}_{\kappa_2}G_2\otimes S_2\right)\Pi\succ 0. \tag{5.64}$$

Remark 5.13. Note that the position of the error matrices $\acute{\mathsf{E}}_d$ and $\grave{\mathsf{E}}_\delta$ in $\widetilde{\boldsymbol{\Omega}}\prec 0$ in (5.37) may cause numerical problem if the eigenvalues of $\acute{\mathsf{E}}_d$ and $\grave{\mathsf{E}}_\delta$ are too small. To circumvent this potential issue, we can apply congruence transformations to $\widetilde{\boldsymbol{\Omega}}\prec 0$ which concludes that $\widetilde{\boldsymbol{\Omega}}\prec 0$ holds if and only if

$$[*]\widetilde{\boldsymbol{\Omega}}\left[I_{m+q+n+5\nu+\varrho\nu}\oplus\left(\eta_1\acute{\mathsf{E}}_d^{-\frac{1}{2}}\otimes I_\nu\right)\oplus\left(\eta_2\grave{\mathsf{E}}_\delta^{-\frac{1}{2}}\otimes I_\nu\right)\right]\prec 0 \tag{5.65}$$

holds where $\eta_1;\eta_2\in\mathbb{R}$ are given values. Note that the diagonal elements of the transformed matrix in (5.65) are no longer associated with the error terms appear at off-diagonal elements, hence one can use the inequality (5.65) instead of (5.37).

Remark 5.14. The assumption of $r_2 > r_1 > 0$ in Theorem 5.1 indicates that there are no obvious redundant matrix parameters in (5.44) since two genuine delay channels are considered therein and (5.45) and (5.18) contain no zeros vectors. With $r_1 = 0$ or $r_2 = r_1$, one only need to consider one delay channel thus the corresponding (5.45), (5.18) and (5.44) can be simplified. Note that we do not present the corresponding dissipativity and stability condition for $r_1 = 0$ or $r_2 = r_1$ in this chapter since it can be easily derived based on the proof of Theorem 5.1 with a simplified (5.44).

In the following corollary, we show that a hierarchy of the stability condition in Theorem 5.1 can be established with respect to $\grave{\boldsymbol{\phi}}(\cdot)$ and its dimension under certain conditions.

Corollary 5.3. *Let all the functions and the parameters in* (5.3)–(5.12) *be given where* $\grave{\boldsymbol{\phi}}(\tau) := \mathbf{Col}_{i=1}^{\kappa_2} \grave{\phi}_i(\tau)$ *with* $\{\grave{\phi}_i(\cdot)\}_{i=1}^{\kappa_2} \subset \{\grave{\phi}_i(\cdot)\}_{i=1}^{\infty} \subset \mathbb{C}^1([-r_2, -r_1] ⨾ \mathbb{R})$ *satisfying*

$$\exists \varkappa \in \mathbb{N},\ \ \forall \kappa_2 \in \{j \in \mathbb{N} : j \leq \varkappa\},\ \exists! M_4 \in \mathbb{R}^{\kappa_2 \times \delta},\ \ \frac{\mathsf{d}}{\mathsf{d}\tau} \mathop{\mathbf{Col}}_{i=1}^{\kappa_2} \grave{\phi}_i(\tau) = M_4 \grave{\boldsymbol{f}}(\tau) \tag{5.66}$$

$$\forall \kappa_2 \in \mathbb{N},\ \ \grave{\Phi}_{\kappa_2} = \int_{-r_2}^{-r_1} \mathop{\mathbf{Col}}_{i=1}^{\kappa_2} \grave{\phi}_i(\tau) \mathop{\mathbf{Row}}_{i=1}^{\kappa_2} \grave{\phi}_i(\tau) \mathsf{d}\tau = \bigoplus_{j=1}^{\kappa_2} \grave{\varphi}_j,\quad \grave{\varphi}_j^{-1} = \int_{-r_2}^{-r_1} \grave{\phi}_j^2(\tau) \mathsf{d}\tau. \tag{5.67}$$

Now given $\acute{\boldsymbol{g}}(\cdot)$, $\grave{\boldsymbol{g}}(\cdot)$ *and* N_1, N_2 *in Theorem 5.1, we have*

$$\forall \kappa_2 \in \{j \in \mathbb{N} : j \leq \varkappa\},\ \ \mathcal{G}_{\kappa_2} \subseteq \mathcal{G}_{\kappa_2+1} \tag{5.68}$$

where $\varkappa \in \mathbb{N}$ *is given and*

$$\mathcal{G}_{\kappa_2} := \left\{ (r_1, r_2) \;\middle|\; r_1 > 0, r_2 > r_1 \;\;\&\;\; (5.35)\text{–}(5.37) \text{ hold } \;\&\;\; P \in \mathbb{S}^l, Q_1; Q_2; R_1; R_2; S_1; S_2; U_1; U_2 \in \mathbb{S}^\nu \right\}$$

with $l := n + 2\nu + (d + \delta)\nu$.

Proof. Given $r_2 > r_1 > 0$ and all the parameters in (5.3)–(5.7) and (5.11), (5.12), let $\mathbf{Col}(r_1, r_2) \in \mathcal{G}_{\kappa_2}$ with $\mathcal{G}_{\kappa_2} \neq \varnothing$ which infers that there exist feasible solutions for (5.35)–(5.37). Consider the situation when the dimensions and elements of $\acute{\boldsymbol{f}}(\tau)$, $\grave{\boldsymbol{f}}(\tau)$, $\acute{\boldsymbol{\phi}}(\tau)$, $\acute{\boldsymbol{g}}(\tau)$ and $\grave{\boldsymbol{g}}(\tau)$ are all fixed, and let $P \in \mathbb{S}^l$ and $Q_1; Q_2; R_1; R_2; S_1; S_2; U_1; U_2 \in \mathbb{S}^\nu$ to be a given feasible solution for $\mathbf{P}_{\kappa_2} \succ 0$, (5.36) and $\widetilde{\boldsymbol{\Omega}}_{\kappa_2} \prec 0$ at κ_2. Note that the matrix G_2 and Φ_{κ_2} in (5.37) are indexed by the value of κ_2. Given (5.36), We will show that holds the corresponding feasible solutions of (5.35) and (5.37) at $\kappa_2 + 1$ exist if feasible solutions of (5.35) and (5.37) at κ_2 exist, which proves (5.68).

The conditions in (5.67) indicate that $\grave{\phi}_i(\cdot)$ are orthogonal functions with respect to the weight function $\varpi(\tau) = 1$ over $[-r_2, -r_1]$, Assume $\kappa_2 + 1 \leq \varkappa$ and by the structure of G_2 in (5.40) with (5.66) and (5.67), we have

$$G_{2,\kappa_2+1}^\top \grave{\Phi}_{\kappa_2+1} G_{2,\kappa_2+1} = [*] \begin{bmatrix} \grave{\Phi}_{\kappa_2} & \mathbf{0}_{\kappa_2+1} \\ * & \grave{\varphi}_{\kappa_2+1} \end{bmatrix} \begin{bmatrix} G_{2,\kappa_2} \\ \mathbf{g}_{\kappa_2+1}^\top \end{bmatrix} = G_{2,\kappa_2}^\top \grave{\Phi}_{\kappa_2} G_{2,\kappa_2} + \grave{\varphi}_{\kappa_2+1} \mathbf{g}_{\kappa_2+1} \mathbf{g}_{\kappa_2+1}^\top, \tag{5.69}$$

where $\mathbf{g}_{\kappa_2+1} \in \mathbb{R}^{3+d+\delta}$ can be easily determined by the structure of G_2 with (5.53) and (5.66), and G_{2,κ_2+1} denotes the corresponding G_2 at $\kappa_2 + 1$. Note that here that no increase of the dimension indexes d, δ, p_1 and p_2 occurs. By (5.69) and considering the structure of the matrix inequalities in (5.35) and (5.37), we have

$$\begin{gathered} \mathbf{P}_{\kappa_2+1} = \mathbf{P}_{\kappa_2} + \Pi^\top \left(\grave{\varphi}_{\kappa_2+1} \mathbf{g}_{\kappa_2+1} \mathbf{g}_{\kappa_2+1}^\top \otimes S_2 \right) \Pi \\ \widetilde{\boldsymbol{\Omega}}_{\kappa_2+1} = \widetilde{\boldsymbol{\Omega}}_{\kappa_2} + \left(\mathsf{O}_{q+2\nu} \oplus \Pi^\top \left(\grave{\varphi}_{\kappa_2+1} \mathbf{g}_{\kappa_2+1} \mathbf{g}_{\kappa_2+1}^\top \otimes U_2 \right) \Pi \oplus \mathsf{O}_{\mu\nu} \right). \end{gathered} \tag{5.70}$$

Since $\grave{\varphi}_{\kappa_2+1} > 0$, $\mathbf{g}_{\kappa_2+1} \mathbf{g}_{\kappa_2+1}^\top \succeq 0$ and $S_2 \succeq 0$, $U_2 \succeq 0$ in (5.36), it is clearly to see that the feasible solutions of $\mathbf{P}_{\kappa_2} \succ 0$, $\widetilde{\boldsymbol{\Omega}}_{\kappa_2} \succ 0$ infer the existence of a feasible solution of $\mathbf{P}_{\kappa_2+1} \succ 0$, $\widetilde{\boldsymbol{\Omega}}_{\kappa_2+1} \succ 0$ given the prerequisites of Corollary 5.3. This finishes the proof. ■

Remark 5.15. A hierarchical pattern of the LMIs in Theorem 5.1 can be also established for the situation when $\acute{\boldsymbol{\phi}}(\cdot)$ contains orthogonal functions which satisfy appropriate constraints resembling (5.66) and (5.67). Note that the corresponding hierarchy result can be derived without using congruence transformations, since the dimensions of $\mathbf{P}$ in (5.35) and $\widetilde{\boldsymbol{\Omega}}$ in (5.37) are not related to the dimensions of $\acute{\boldsymbol{\phi}}(\tau) \in \mathbb{R}^{\kappa_1}$.

On the other hand, a hierarchy of the stability condition in Theorem 5.1 can be also established with respect to $\grave{\boldsymbol{g}}(\cdot)$ and its dimensions.

Corollary 5.4. *Given the functions with the parameters in* (5.3)–(5.12)*, let* $\acute{\boldsymbol{g}}(\cdot)$*,* $\grave{\boldsymbol{g}}(\cdot)$ *and* N_1*,* N_2 *in Theorem 5.1 be given where* $\grave{\boldsymbol{g}}(\tau) = \mathsf{Col}_{i=1}^{p_2} \grave{g}_i(\tau)$ *with* $\{\grave{g}_i(\cdot)\}_{i=1}^{p_2} \subset \{\grave{g}_i(\cdot)\}_{i=1}^{\infty} \subset \mathbb{C}^1([-r_2, -r_1] ; \mathbb{R})$ *satisfying*

$$\exists \alpha \in \mathbb{N},\ \ \forall p_2 \in \{j \in \mathbb{N} : j \leq \alpha\},\ \exists! N_2 \in \mathbb{R}^{p_2 \times \delta},\ \ (r_2 + \tau)\frac{\mathsf{d}}{\mathsf{d}\tau} \mathop{\mathsf{Col}}_{i=1}^{p_2} \grave{g}_i(\tau) = N_2 \grave{\boldsymbol{g}}(\tau) \tag{5.71}$$

$$\forall p_2 \in \mathbb{N},\ \ \grave{\mathsf{G}}_{p_2} = \int_{-r_2}^{-r_1} \mathop{\mathsf{Col}}_{i=1}^{p_2} \grave{g}_i(\tau) \mathop{\mathsf{Row}}_{i=1}^{p_2} \grave{g}_i(\tau)\mathsf{d}\tau = \bigoplus_{j=1}^{p_2} \grave{\mathsf{g}}_j,\quad \grave{\mathsf{g}}_j^{-1} = \int_{-r_2}^{-r_1} (\tau + r_2)\grave{g}_j^2(\tau)\mathsf{d}\tau. \tag{5.72}$$

Then we have

$$\forall p_2 \in \{j \in \mathbb{N} : j \leq \alpha\},\ \ \mathcal{H}_{p_2} \subseteq \mathcal{H}_{p_2+1} \tag{5.73}$$

where $\alpha \in \mathbb{N}$ *is given and*

$$\mathcal{H}_{p_2} := \left\{(r_1, r_2)\ \middle|\ r_1 > 0, r_2 > r_1\ \ \&\ \ (5.35)\text{–}(5.37)\ \textit{hold}\ \ \&\ \ P \in \mathbb{S}^l, Q_1; Q_2; R_1; R_2; S_1; S_2; U_1; U_2 \in \mathbb{S}^\nu\right\}$$

with $l := n + 2\nu + (d + \delta)\nu$*.*

Proof. The proof is similar to the proof of Corollary 5.3 apart from the fact that for Corollary 5.4 one only needs to consider the increase of the value of p_2 instead of κ_2 in Corollary 5.3. Given $r_2 > r_1 > 0$ with all the parameters in (5.3)–(5.7) and (5.11) and (5.12), let $\mathsf{Col}(r_1, r_2) \in \mathcal{H}_{p_2}$ with $\mathcal{H}_{p_2} \neq \varnothing$ which infers that there exist feasible solutions for (5.35)–(5.37). Let the dimensions and elements of $\acute{\boldsymbol{f}}(\tau)$, $\grave{\boldsymbol{f}}(\tau)$, $\acute{\boldsymbol{\phi}}(\tau)$, $\grave{\boldsymbol{\phi}}(\tau)$ and $\acute{\boldsymbol{g}}(\tau)$ to be all fixed, and let $P \in \mathbb{S}^l$ and $Q_1; Q_2; R_1; R_2; S_1; S_2; U_1; U_2$ to be a given feasible solution for $\mathbf{P}_{p_2} \succ 0$, (5.36) and $\widetilde{\boldsymbol{\Omega}} \prec 0$ at p_2. Note that the matrix H_2 and G_{p_2} in (5.37) are indexed by the value of κ_2 whereas $\widetilde{\boldsymbol{\Omega}} \prec 0$ is not related to $\acute{\boldsymbol{g}}(\tau)$ and $\grave{\boldsymbol{g}}(\tau)$ or their dimensions p_1, p_2. Given (5.36), we will show that the corresponding feasible solutions of (5.35) and (5.37) at $p_2 + 1$ exist if feasible solutions of (5.35) and (5.37) at p_2 exist, which leads to (5.68).

The constraints in (5.72) show that $\grave{g}_i(\cdot)$ contains functions which are orthogonal with respect to the weight function $\varpi(\tau) = (\tau + r_2)$ over $[-r_2, -r_1]$. Suppose $p_2 + 1 \leq \alpha$. Now by the structure of H_2 in (5.42) and (5.71) and (5.72), we have

$$H_{2,p_2+1}^\top \grave{\mathsf{G}}_{p_2+1} H_{2,p_2+1} = [*]\begin{bmatrix} \grave{\mathsf{G}}_{p_2} & \mathbf{0}_{p_2} \\ * & \grave{\mathsf{g}}_{p_2+1} \end{bmatrix}\begin{bmatrix} H_{2,p_2} \\ \mathbf{h}_{p_2+1}^\top \end{bmatrix} = H_{2,p_2}^\top \grave{\mathsf{G}}_{p_2} H_{2,p_2} + \grave{\mathsf{g}}_{p_2+1}\mathbf{h}_{p_2+1}\mathbf{h}_{p_2+1}^\top \tag{5.74}$$

where $\mathbf{h}_{p_2+1} \in \mathbb{R}^{3+d+\delta}$ can be easily determined by the structure of H_2 with (5.63) and (5.71), and H_{2,p_2+1} denotes the corresponding H_2 at $p_2 + 1$. Note that here the values of the dimension indexes d, δ, κ_1, κ_2 and p_1 remain unchanged.

By (5.74) and considering the structure of $\mathbf{P} \succ 0$ in (5.35), it yields

$$\mathbf{P}_{p_2+1} = \mathbf{P}_{p_2} + \Pi^\top \left(\grave{\mathsf{g}}_{p_2+1}\mathbf{h}_{p_2+1}\mathbf{h}_{p_2+1}^\top \otimes U_2\right)\Pi. \tag{5.75}$$

Since $\grave{\mathsf{g}}_{p_2+1} > 0$, $\mathbf{h}_{p_2+1}\mathbf{h}_{p_2+1}^\top \succeq 0$ with $U_2 \succeq 0$ in (5.36), one can conclude that the feasible solutions of $\mathbf{P}_{p_2} \succ 0$ infer the existence of the feasible solution of $\mathbf{P}_{p_2+1} \succ 0$ given the prerequisites in Corollary 5.4. On the other hand, since the inequality in (5.37) is not related to $\grave{\boldsymbol{g}}(\tau)$, thus $\widetilde{\boldsymbol{\Omega}} \prec 0$ remains unchanged at $p_2 + 1$. This finishes the proof. ■

Remark 5.16. Following the strategy in proving Corollary 5.4, a hierarchy of conditions in Theorem 5.1 can be also established when $\acute{\boldsymbol{g}}(\cdot)$ contains orthogonal functions satisfying appropriate constraints resembling (5.66) and (5.67). Note that the dimensions of $\mathbf{P}$ in (5.35) and $\widetilde{\boldsymbol{\Omega}}$ in (5.37) are not related to the dimensions of $\acute{\boldsymbol{g}}(\tau) \in \mathbb{R}^{p_1}$.

5.5 Numerical examples

In this section, numerical examples are presented to demonstrate the effectiveness of our proposed methods. All examples were tested in Matlab environment using Yalmip [266] with SDPT3 [270] as the numerical solver.

5.5.1 Stability analysis of a distributed delay system

Consider the following distributed delay system

$$\dot{x}(t) = 0.33x(t) - 5\int_{-r}^{0} \sin(\cos(12\tau))x(t+\tau)\mathsf{d}\tau$$
$$= 0.33x(t) - \begin{bmatrix} 5 & \mathbf{0}^\top \end{bmatrix} \int_{-r}^{0} \begin{bmatrix} \varphi_1(\tau) \\ \boldsymbol{f}(\tau) \end{bmatrix} x(t+\tau)\mathsf{d}\tau, \quad t \geq t_0 \quad (5.76)$$

with any $t_0 \in \mathbb{R}$, where $\varphi_1(\tau) = \sin(\cos(12\tau))$. The corresponding state space matrices of (5.1) for (5.76) and (5.7) are $A_1 = 0.33$ and $A_3 = -\begin{bmatrix} 5 & \mathbf{0}^\top \end{bmatrix}$ and the rest of the state space matrices in (5.1) is zero with $m = q = 0$.

Here we consider two cases for $\boldsymbol{f}(\cdot)$. The first one is $\boldsymbol{f}(\tau) = \boldsymbol{\ell}_d(\tau) = \mathsf{Col}_{i=0}^{d} \ell_i(\tau)$ with

$$\ell_d(\tau) := \sum_{k=0}^{d} \binom{d}{k}\binom{d+k}{k}\left(\frac{\tau}{r}\right)^k \quad (5.77)$$

containing Legendre polynomials with $\mathsf{F}_1^{-1} = \int_{-r}^{0} \boldsymbol{\ell}_d(\tau)\boldsymbol{\ell}_d^\top(\tau)\mathsf{d}\tau = r^{-1}\bigoplus_{i=0}^{d} 2i+1$ and the corresponding M_1 in (5.3) can be easily determined. The second one $\boldsymbol{f}(\tau) = \boldsymbol{h}_d(\tau) = \mathsf{Col}\left[1,\ \mathsf{Col}_{i=1}^{d/2} \sin 12i\tau,\ \mathsf{Col}_{i=1}^{d/2} \cos 12i\tau\right]$ contains trigonometric functions which corresponds to $M_1 = 0 \oplus \begin{bmatrix} \mathsf{O}_{d/2} & \bigoplus_{i=1}^{d/2} 12i \\ -\bigoplus_{i=1}^{d/2} 12i & \mathsf{O}_{d/2} \end{bmatrix}$ satisfying the first relation in (5.3). Note that d in $\boldsymbol{h}_d(\tau)$ must be positive even numbers and the functions in $\boldsymbol{h}_d(\tau)$ are not orthogonal over $[-r, 0]$ thus the associated F for $\boldsymbol{h}_d(\tau)$ is not a diagonal matrix. Since $0.33 > 0$, thus the method in [271] cannot be applied. Furthermore, since $\varphi_1(\tau) = \sin(\cos(12\tau))$ does not satisfy the "differentiation closure" property as in (5.3), the method in [57] cannot handle (5.76).

Now apply the spectrum methods in [80] to (5.76) with $M = 200$. The resulting information of the spectrum of (5.76) shows that the system is stable in the following intervals: $[0.093, 0.169]$, $[0.617, 0.692]$, $[1.14, 1.216]$, $[1.664, 1.739]$, $[2.188, 2.263]$ and $[2.711, 2.787]$.

In this section we apply a single delay version of Theorem 5.1 to (5.76), which is derived via the LKF

$$v(\boldsymbol{x}(t), \boldsymbol{\psi}(\cdot)) = \boldsymbol{\eta}^\top(t)P\boldsymbol{\eta}(t) + \int_{-r}^{0} \boldsymbol{y}^\top(t+\tau)\left[Q + (\tau+r)R\right]\boldsymbol{y}(t+\tau)\mathsf{d}\tau \quad (5.78)$$

as a simplified version of (5.44), where $P \in \mathbb{S}^{n+(d+1)\nu}$, $Q; R \in \mathbb{S}^\nu$ and $\boldsymbol{\eta}(t) := \mathsf{Col}\left[\boldsymbol{x}(t),\ \int_{-r}^{0} F_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\right]$ with $F_d(\tau) = \boldsymbol{f}(\tau) \otimes I_\nu$. Furthermore, the corresponding $\boldsymbol{\vartheta}(t)$ in (5.18) and (5.54) is defined as $\boldsymbol{\vartheta}(t) := \mathsf{Col}\left[\boldsymbol{x}(t),\ \boldsymbol{y}(t-r),\ \int_{-r}^{0} F_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\right]$. Now apply the corresponding stability condition derived by (5.78) with an one delay version congruence transformation (5.65) with $\eta_1 = 1$ to (5.76) with $\boldsymbol{f}(\tau) = \boldsymbol{\ell}_d(\tau)$ and $\boldsymbol{f}(\tau) = \boldsymbol{h}_d(\tau)$, respectively. The results concerning detectable delay margins are summarized in Table 5.1 and 5.2. Note that the values of N and d in these tables are presented when the margins of the stable delay intervals can be determined by the numerical results produced by Theorem 1 or the method in [188]. Note that also the results in Table 5.1 and 5.2 associated with [80, 81] are calculated with $M = 200$. Finally, NoDVs in Table 5.1 and 5.2 stands for the number of decision variables.

[80, 81]	$[0.093, 0.169]$	$[0.617, 0.692]$	$[1.14, 1.216]$
Theorem 1	$d = 3$ (NoDVs: 17)	$d = 6$ (NoDVs: 38)	$d = 10$ (NoDVs: 80)
$\boldsymbol{f}(\tau)$	$\boldsymbol{\ell}_d(\tau)$	$\boldsymbol{h}_d(\tau)$	$\boldsymbol{h}_d(\tau)$
[188]	$N = 3$ (NoDVs: 17)	$N = 11$ (NoDVs: 93)	$N = 23$ (NoDVs: 327)

Table 5.1: Testing of stable delay margins

[80]	$[1.664, 1.739]$	$[2.188, 2.263]$	$[2.711, 2.787]$
Theorem 1	$d = 10$ (NoDVs: 80)	$d = 10$ (NoDVs: 80)	$d = 10$ (NoDVs: 80)
$\boldsymbol{f}(\tau)$	$\boldsymbol{h}_d(\tau)$	$\boldsymbol{h}_d(\tau)$	$\boldsymbol{h}_d(\tau)$
[188]	–	–	–

Table 5.2: Testing of stable delay margins

Note that in Table 5.1 and 5.2 the results correspond to [188] are produced by our Theorem 1 via (5.78) with $\boldsymbol{f}(\tau) = \boldsymbol{\ell}_d(\tau)$ and $d = N$ which is essentially equivalent to the method in [188]. With $N = 25$, the margins of the stable delay intervals $[1.664, 1.739]$, $[2.188, 2.263]$ and $[2.711, 2.787]$ still cannot be detected by polynomials approximation approach proposed in [188]. For $N > 25$, our experiments show that the computational time becomes too long to accurately obtain the values of the approximation coefficient and error term via the function `vpaintegral` in Matlab. On the other hand, the function `integral` in Matlab is not an alternative option to calculate the approximation coefficient and error term in this case due to its limited capacity of numerical accuracy. The results in Tables 5.1 and 5.2 can be explained by the fact that $\varphi_1(\tau) = \sin(\cos(12\tau))$, $\tau \in [a, b]$ is not "easy" to be approximated by polynomials when the length of $[a, b]$ become relatively large. Consequently, we have shown the advantage of our method over the one in [188] when it comes to the stability analysis of (5.76).

5.5.2 Stability and dissipativity analysis with distributed delays

Consider a system of the form (5.1) with $r_1 = 2$, $r_2 = 4.05$ and the state space matrices

$$\begin{aligned}
&A_1 = \begin{bmatrix} 0.01 & 0 \\ 0 & -3 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0.1 \\ 0.2 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}, A_6 = I_2, A_7 = A_8 = \mathsf{O}_2,\ D_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} \\
&A_4\left(\begin{bmatrix} \boldsymbol{\varphi}_1(\tau) \\ \acute{\boldsymbol{f}}(\tau) \end{bmatrix} \otimes I_2\right) = \begin{bmatrix} 3\sin(18\tau) & -0.3\mathsf{e}^{\cos(18\tau)} \\ 0 & 3\sin(18\tau) \end{bmatrix},\ A_5\left(\begin{bmatrix} \boldsymbol{\varphi}_2(\tau) \\ \grave{\boldsymbol{f}}(\tau) \end{bmatrix} \otimes I_2\right) = \begin{bmatrix} -10\cos(18\tau) & 0 \\ 0.5\mathsf{e}^{\sin(18\tau)} & -10\cos(18\tau) \end{bmatrix} \\
&C_1 = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix},\ C_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.2 \end{bmatrix},\ C_3 = \begin{bmatrix} 0 & 0.1 \\ -0.1 & 0 \end{bmatrix},\ D_2 = \begin{bmatrix} 0.12 \\ 0.1 \end{bmatrix} \\
&C_4\left(\begin{bmatrix} \boldsymbol{\varphi}_1(\tau) \\ \acute{\boldsymbol{f}}(\tau) \end{bmatrix} \otimes I_\nu\right) = 0.1 \oplus 0,\ C_5\left(\begin{bmatrix} \boldsymbol{\varphi}_2(\tau) \\ \grave{\boldsymbol{f}}(\tau) \end{bmatrix} \otimes I_\nu\right) = 0.2 \oplus 0.1,\ C_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, C_7 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{5.79}$$

with $\boldsymbol{\varphi}_1(\tau) = \boldsymbol{\varphi}_2(\tau) = \begin{bmatrix} \mathsf{e}^{\sin(18\tau)} \\ \mathsf{e}^{\cos(18\tau)} \end{bmatrix}$ and $n = m = 2$, $q = 1$. We find out that the system with (5.79) is stable by applying the Matlab toolbox of the spectral method in [81]. Moreover, the minimization of $\mathbb{L}^2$ gain γ is applied as the performance criterion for the system, which corresponds to

$$\gamma > 0,\ \ J_1 = -\gamma I_2,\ \ \widetilde{J} = I_2,\ \ J_2 = \mathbf{0}_2,\ \ J_3 = \gamma \tag{5.80}$$

in (5.22).

Even one assumes the method in [185] can be extended to handle systems with multiple delay channels, it still cannot be applied here given that A_1 is not a Hurwitz matrix. In addition, since $\boldsymbol{\varphi}_1(\tau) = \boldsymbol{\varphi}_2(\tau)$ does not satisfy the "differentiation closure" property in (5.3), thus the problem of dissipativity and stability analysis may not be solved by a simple extension of the corresponding conditions in [57] for a linear CDDS, even a multiple distinct delays version of the method in [57] is derivable.

Let

$$\acute{\boldsymbol{f}}(\tau) = \acute{\boldsymbol{\phi}}(\tau) = \begin{bmatrix} 1 \\ \mathsf{Col}_{i=1}^{d} \sin 18i\tau \\ \mathsf{Col}_{i=1}^{d} \cos 18i\tau \end{bmatrix}, \quad \grave{\boldsymbol{f}}(\tau) = \grave{\boldsymbol{\phi}}(\tau) = \begin{bmatrix} 1 \\ \mathsf{Col}_{i=1}^{\delta} \sin 18i\tau \\ \mathsf{Col}_{i=1}^{\delta} \cos 18i\tau \end{bmatrix} \tag{5.81}$$

in (5.79) and (5.3), which correspond to

$$M_1 = M_3 = 0 \oplus \begin{bmatrix} \mathsf{O}_d & \bigoplus_{i=1}^{d} 18i \\ -\bigoplus_{i=1}^{d} 18i & \mathsf{O}_d \end{bmatrix}, \quad M_2 = M_4 = 0 \oplus \begin{bmatrix} \mathsf{O}_\delta & \bigoplus_{i=1}^{\delta} 18i \\ -\bigoplus_{i=1}^{\delta} 18i & \mathsf{O}_\delta \end{bmatrix} \tag{5.82}$$

in (5.3). Considering $\acute{\boldsymbol{f}}(\cdot)$, $\grave{\boldsymbol{f}}(\cdot)$ in (5.81) and $\boldsymbol{\varphi}_1(\tau) = \boldsymbol{\varphi}_2(\tau) = \begin{bmatrix} \mathsf{e}^{\sin(18\tau)} \\ \mathsf{e}^{\cos(18\tau)} \end{bmatrix}$ with (2.1) and (5.11), we obtain

$$\begin{aligned} A_4 &= \begin{bmatrix} \mathsf{O}_2 & \begin{matrix} 0 & -0.3 \\ 0 & 0 \end{matrix} & \mathsf{O}_2 & \begin{matrix} 3 & 0 \\ 0 & 3 \end{matrix} & \mathsf{O}_{2\times(4d-2)} \end{bmatrix} \\ A_5 &= \begin{bmatrix} \begin{matrix} 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \end{matrix} & \mathsf{O}_{2\times 2\delta+2} & \begin{matrix} -10 & 0 \\ 0 & -10 \end{matrix} & \mathsf{O}_{2\times 2\delta-2} \end{bmatrix} \\ C_4 &= \begin{bmatrix} \mathsf{O}_{2\times 4} & 0.1 \oplus 0 & \mathsf{O}_{2\times 4d} \end{bmatrix}, \quad C_5 = \begin{bmatrix} \mathsf{O}_{2\times 4} & 0.2 \oplus 0.1 & \mathsf{O}_{2\times 4\delta} \end{bmatrix} \end{aligned} \tag{5.83}$$

which corresponds to the distributed delay terms in (5.79).

Now apply the conditions (5.36),(5.37) and (5.65)[4] with $\eta_1 = \eta_2 = 1$ and the system's parameters in (5.79)–(5.83) where $\acute{\Gamma}_d$, $\grave{\Gamma}_\delta$ are in line with the structure in (5.13) and the matrices $\acute{\Gamma}_d$, $\grave{\Gamma}_\delta$, $\acute{\mathsf{E}}_d$, $\grave{\mathsf{E}}_\delta$ and $\acute{\mathsf{F}}_d$, $\grave{\mathsf{F}}_\delta$ are calculated computationally via the function `vpaintegral` in Matlab which can produce results with high-numerical precisions. With $d = \delta = 1$ a feasible result can be produced with $\min \gamma = 0.64655$ which requires 196 decision variables. With $d = \delta = 2$, we obtain feasible solutions with $\min \gamma = 0.32346$ requiring 376 variables. Finally, with $d = \delta = 10$ our method can produce feasible solutions with $\min \gamma = 0.31265$ with 4120 variables. It is worthy to mention that even with $d = \delta = 10$ which is a relatively large value, the duration of the calculations of $\acute{\Gamma}_d$, $\grave{\Gamma}_\delta$, $\acute{\mathsf{E}}_d$, $\grave{\mathsf{E}}_\delta$ and $\acute{\mathsf{F}}_d$, $\grave{\mathsf{F}}_\delta$ by `vpaintegral` is still acceptable (about a minute).

On the other hand, let $\acute{\boldsymbol{f}}(\tau) = \boldsymbol{j}_d^{0,0}(\tau)_{-r_1}^{0} = \mathsf{Col}_{i=0}^{d}\, j_k^{0,0}(\tau)_{-r_1}^{0}$ and $\grave{\boldsymbol{f}}(\tau) = \boldsymbol{j}_d^{0,0}(\tau)_{-r_2}^{-r_1} = \mathsf{Col}_{i=0}^{d}\, j_k^{0,0}(\tau)_{-r_2}^{-r_1}$ which are Legendre polynomials associated with $\acute{\mathsf{F}}_d = r_1^{-1}\mathsf{D}_d$ and $\grave{\mathsf{F}}_\delta = r_3^{-1}\mathsf{D}_\delta$. (See (4.6) for the definition of orthogonal polynomials) The characteristics of the functions in $\boldsymbol{\varphi}_1(\tau) = \boldsymbol{\varphi}_2(\tau)$ indicate that they might be very difficult to be approximated by polynomials. Indeed, let $d = \delta = 15$ with the corresponding A_4,A_5 and C_4,C_5. In this case, Theorem 1 yields no feasible solutions.

[4]Note that here we do not apply (5.35) in Theorem 5.1, see Remark 5.12 for further details.

Chapter 6

Dissipative Delay Range Analysis of Coupled Differential-Difference Delay Systems with Distributed Delays

6.1 Introduction

Functional differential equations [6] are able to characterize dynamical processes whose behavior is affected by its past values, i.e. dynamical systems conditioned by delay effects. Analyzing the stability property of such systems, however, is non-trivial due to its infinite dimensional nature. Two major directions, which are based on either time [64] or frequency-domain [5], have been investigated to provide solutions to characterize how delays affect the stability of systems.

For a linear delay system, the information of its stability can be obtained by analyzing its corresponding spectrum. Many different approaches [64, 96] have been developed in frequency-domain, which can provide almost a complete stability characterization when the delay systems possess certain structures. For more complex delay structures such as distributed delays with general kernels, the numerical schemes in [80, 81, 86] can produce reliable results verifying system's stability with given point-wise delay values, which suffer almost no conservatism if numerical complexities are ignored. Furthermore, the method in [123] allows one to calculate the value of $\mathcal{H}^\infty$ norm of a delay system with known point-wise delay values. However, to the best of our knowledge, none of the existing spectral based approaches can handle the problem of delay range stability analysis subject to performance objectives [215] for linear delay systems. Namely, to test whether a delay system is stable and simultaneously dissipative with a supply function [173] for all $r \in [r_1, r_2]$, where the exact delay value r is unknown but bounded by $r_1 \leq r \leq r_2$ with known values $r_2 > r_1 > 0$.

On the other hand, constructing LKFs [64, 173] has been applied as a standard approach in time-domain to analyze the stability of delay systems. Many different functionals (see [64, 172, 173] and the references therein) have been proposed among existing literature [57, 182] to analyze the problem of point-wise delay stability. Compared to its frequency-domain counterparts, time-domain approaches may be more adaptable to handle the problem of range stability analysis with performance objectives, though only sufficient conditions may be derived. In [304, 305], the results concerning the range stability of a linear discrete delay system are presented based on the principle of quadratic separation. On the other hand, a solution to the same problem has been proposed in [187, 287] based on constructing LKFs. However, no results based on the LKF approach with respect to range stability analysis have been proposed when distributed delays are concerned.[1] On the other hand, almost all existing LKFs in the literature are based

[1] The methods proposed in [260] can handle polynomials distributed delay kernels. However, the approaches in [260] are

on constant matrix parameters, which may be a very conservative approach when it comes to range stability analysis. This motives one to propose new functionals to specifically tackle the problem of range stability analysis with performance objectives or even further potential constraints.

In this chapter, we propose methodologies which allow one to conduct delay range dissipativity and stability analysis for a linear CDDS [10]; [302, 306] where the delay value is unknown but bounded. The linear CDDS model considered in this chapter contains distributed delay terms with polynomials kernels, which is able to characterize many models of time-delay systems. A novel LKF with delay-dependent matrix parameters is applied to be constructed together with a quadratic supply function to derive our dissipativity and stability condition. The resulting sufficient conditions expressed in terms of sum-of-squares constraints [307, 308] are the result of equivalently transferring some robust LMIs into SoS conditions via the relaxation technique in [309] where the transformation itself does not introduce any potential conservatism theoretically. Furthermore, the proposed scenario is extended to handle the problem of estimating the margins of a stable delay interval under a given dissipative constraint. Finally, we also prove that a hierarchy of the feasibility of our proposed dissipativity and stability condition can be established similar to the one in [187].

The chapter is organized as follows. In section 2 we formulate the linear CDDS model to be analyzed in this chapter. Subsequently, theoretical preliminaries are presented in section 3 which provide the necessary tools to derive the main results in the following section. In section 4, the main results on range stability analysis under a dissipative constraint are presented, including remarks and detailed explanations. Finally, we present several numerical examples in section 5 to demonstrate the advantage of our proposed schemes.

6.2 Problem formulation

In this chapter, the stability of the following linear coupled differential-difference system

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A_1\boldsymbol{x}(t) + A_2\boldsymbol{y}(t-r) + \int_{-r}^{0} A_3(r)L_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau + D_1\boldsymbol{w}(t), \quad t \geq t_0 \\
\boldsymbol{y}(t) &= A_4\boldsymbol{x}(t) + A_5\boldsymbol{y}(t-r) \\
\boldsymbol{z}(t) &= C_1\boldsymbol{x}(t) + C_2\boldsymbol{y}(t-r) + \int_{-r}^{0} C_3(r)L_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau + D_2\boldsymbol{w}(t) \\
\boldsymbol{x}(t_0) &= \boldsymbol{\xi}, \quad \forall\theta \in [-r, 0], \ \ \boldsymbol{y}(t_0+\theta) = \boldsymbol{\phi}(\theta)
\end{aligned} \tag{6.1}$$

with distributed delays is considered, where $t_0 \in \mathbb{R}$ and $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r, 0) ⨟ \mathbb{R}^\nu)$. Moreover, $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{y}(t) \in \mathbb{R}^\nu$ satisfy the equations in (6.1), $\boldsymbol{w}(\cdot) \in \mathbb{L}^2([t_0, \infty) ⨟ \mathbb{R}^q)$ represents disturbance, $\boldsymbol{z}(t) \in \mathbb{R}^m$ is the regulated output. Note that $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r, 0) ⨟ \mathbb{R}^n)$ are the initial conditions for the system at $t = t_0$. $\widehat{\mathbb{C}}(\mathcal{X} ⨟ \mathbb{R}^n)$ stands for the Banach space of bounded right piecewise continuous functions with a uniform norm $\|\boldsymbol{f}(\cdot)\|_\infty := \sup_{\tau\in\mathcal{X}} \|\boldsymbol{f}(\tau)\|_2$. The dimensions of the state space matrices in (6.1) are determined by the indexes $n; \nu \in \mathbb{N}$ and $m; q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, $L_d(\tau) := \boldsymbol{\ell}_d(\tau) \otimes I_\nu$ with $\boldsymbol{\ell}_d(\tau) \in \mathbb{R}^{d+1}$ contains polynomials at each row up to degree $d \in \mathbb{N}_0$. $A_3(r) \in \mathbb{R}^{n\times\varrho}$ and $C_3(r) \in \mathbb{R}^{m\times\varrho}$ are functions of r which satisfies that $rA_3(r) \in \mathbb{R}^{n\times\varrho}$ and $rC_3(r) \in \mathbb{R}^{m\times\varrho}$ are polynomials matrices of r with $\varrho = (d+1)\nu$. r is a constant but with unknown and bounded values as $r \in [r_1, r_2]$, where the values of $r_2 > r_1 > 0$ are known. Finally, it is assumed $\rho(A_5) < 1$ which ensures the input to state stability of $\boldsymbol{y}(t) = A_4\boldsymbol{x}(t) + A_5\boldsymbol{y}(t-r)$ [10] where $\rho(A_5)$ stands for the spectral radius of A_5. Since $\rho(A_5) < 1$ is independent of r, thus this condition ensures the input to state stability of $\boldsymbol{y}(t) = A_4\boldsymbol{x}(t) + A_5\boldsymbol{y}(t-r)$ for all $r > 0$.

Remark 6.1. Many delay related systems can be modeled by (6.1). See [10, 57] and the references therein. In comparison with the CDDS model in [10], (6.1) takes disturbances into account and contains distributed

derived not based on LKFs, but the principle of robust control (Quadratic Separation).

delay terms with polynomials kernels at both the state and output. In terms of real-time applications, the structures of $A_3(r)$ and $C_3(r)$ can be justified by the fact that the distributed delay gain matrices can be related to the numerical values of r.[2]

6.3 Mathematical preliminaries

We present in this section some important mathematical preliminaries for the derivation of the stability condition in the later section. This includes an integral inequality and the foundation of matrix polynomials optimization [307–309]. Without losing generalities, we assume in this chapter that $\boldsymbol{\ell}_d(\tau) = \mathsf{Col}_{i=0}^d \ell_i(r,\tau)$ in (6.1) consists of Legendre polynomials [186–188, 262]

$$\ell_d(r,\tau) := \sum_{k=0}^{d} \binom{d}{k}\binom{d+k}{k}\left(\frac{\tau}{r}\right)^k = \sum_{k=0}^{d} \binom{d}{k}\binom{d+k}{k}\tau^k r^{-k},\ \forall d \in \mathbb{N}_0,\ \forall \tau \in [-r,0], \tag{6.2}$$

with $\int_{-r}^{0} \boldsymbol{\ell}_d(\tau)\boldsymbol{\ell}_d^\top(\tau)\mathsf{d}\tau = \bigoplus_{k=0}^{d} \frac{r}{(2k+1)}$. Note that the form of (6.2) is derived from the structure of Jacobi polynomials (4.6) with $\alpha = \beta = 0$ and $a = -r$, $b = 0$.

Some properties of Legendre polynomials are summarized as follows.

Property 6.1. *Given $d \in \mathbb{N}_0$ and $\boldsymbol{m}_d(\tau) := \mathsf{Col}_{i=0}^d \tau^i$, then the following three properties hold for all $r > 0$.*

- $$\exists! \mathsf{L}_d(\cdot) \in \left(\mathbb{R}_{[d+1]}^{(d+1)\times(d+1)}\right)^{\mathbb{R}_+}, \exists! \mathsf{\Lambda}_d \in \mathbb{R}_{[d+1]}^{(d+1)\times(d+1)} : \forall \tau \in \mathbb{R},\ \boldsymbol{\ell}_d(\tau) = \mathsf{L}_d(r)\boldsymbol{m}_d(\tau) = \mathsf{\Lambda}_d\left(\bigoplus_{i=0}^{d} r^i\right)^{-1}\boldsymbol{m}_d(\tau) \tag{6.3}$$
- $$\mathsf{L}_d^{-1}(r) = \left(\bigoplus_{i=0}^{d} r^i\right)\mathsf{\Lambda}_d^{-1} \tag{6.4}$$
- $$\exists! \acute{\mathsf{L}}_d \in \mathbb{R}^{(d+1)\times(d+1)},\ \forall \tau \in \mathbb{R},\ \frac{\mathsf{d}\boldsymbol{\ell}_d(\tau)}{\mathsf{d}\tau} = r^{-1}\acute{\mathsf{L}}_d\boldsymbol{\ell}_d(\tau) \tag{6.5}$$

where $\mathbb{R}_{[n]}^{n\times n} := \{X \in \mathbb{R}^{n\times n} : \operatorname{rank}(X) = n\}$. and $\exists!$ stands for the symbol of unique existential quantification.

Proof. Since $\boldsymbol{\ell}_d(\tau)$ contains polynomials with $\int_{-r}^{0} \boldsymbol{\ell}_d(\tau)\boldsymbol{\ell}_d^\top(\tau)\mathsf{d}\tau = \bigoplus_{k=0}^{d} \frac{r}{(2k+1)}$ which is of full rank, thus (6.3) can be easily derived based on the form of (6.2) together with property of positive definite matrices. By (6.3) and $r > 0$, (6.4) can be obtained. Finally, by (6.3), we have $\frac{\mathsf{d}\boldsymbol{\ell}_d(\tau)}{\mathsf{d}\tau} = \mathsf{\Lambda}_d\left(\bigoplus_{i=0}^{d} r^i\right)^{-1}\frac{\mathsf{d}\boldsymbol{m}_d(\tau)}{\mathsf{d}\tau}$. Now it is obvious that, $\frac{\mathsf{d}\boldsymbol{m}_d(\tau)}{\mathsf{d}\tau} = \begin{bmatrix} \mathbf{0}_d^\top & 0 \\ \bigoplus_{i=1}^{d} i & \mathbf{0}_d \end{bmatrix}\boldsymbol{m}_d(\tau)$ for all $d \in \mathbb{N}_0$ if we define $\mathbf{0}_0$ and $\bigoplus_{i=1}^{0} i$ to be 0×1 and 0×0 empty matrices, respectively. Using this relation with (6.3) and (6.4) we can obtain that

$$\begin{aligned}\frac{\mathsf{d}\boldsymbol{\ell}_d(\tau)}{\mathsf{d}\tau} &= \mathsf{\Lambda}_d\left(\bigoplus_{i=0}^{d} r^i\right)^{-1}\frac{\mathsf{d}\boldsymbol{m}_d(\tau)}{\mathsf{d}\tau} = \mathsf{\Lambda}_d\left(\bigoplus_{i=0}^{d} r^i\right)^{-1}\begin{bmatrix} \mathbf{0}_d^\top & 0 \\ \bigoplus_{i=1}^{d} i & \mathbf{0}_d \end{bmatrix}\boldsymbol{m}_d(\tau) \\ &= \mathsf{\Lambda}_d\left(\bigoplus_{i=0}^{d} r^i\right)^{-1}\begin{bmatrix} \mathbf{0}_d^\top & 0 \\ \bigoplus_{i=1}^{d} i & \mathbf{0}_d \end{bmatrix}\left(\bigoplus_{i=0}^{d} r^i\right)\mathsf{\Lambda}_d^{-1}\boldsymbol{\ell}_d(\tau)\end{aligned} \tag{6.6}$$

Note that based on the final term in (6.6), (6.6) can be rewritten into

$$\begin{aligned}\frac{\mathsf{d}\boldsymbol{\ell}_d(\tau)}{\mathsf{d}\tau} &= \mathsf{\Lambda}_d\begin{bmatrix} 1 & \mathbf{0}_d^\top \\ \mathbf{0}_d & \bigoplus_{i=1}^{d} r^{-i} \end{bmatrix}\begin{bmatrix} \mathbf{0}_d^\top & 0 \\ \bigoplus_{i=1}^{d} i & \mathbf{0}_d \end{bmatrix}\begin{bmatrix} \bigoplus_{i=0}^{d-1} r^i & \mathbf{0}_d \\ \mathbf{0}_d^\top & r^d \end{bmatrix}\mathsf{\Lambda}_d^{-1}\boldsymbol{\ell}_d(\tau) \\ &= \mathsf{\Lambda}_d\begin{bmatrix} \mathbf{0}_d^\top & 0 \\ r^{-1}\left(\bigoplus_{i=1}^{d} i\right) & \mathbf{0}_d \end{bmatrix}\mathsf{\Lambda}_d^{-1}\boldsymbol{\ell}_d(\tau)\end{aligned} \tag{6.7}$$

which gives (6.5). ∎

[2] See a representative example by Example 2 in [182].

Remark 6.2. Consider distributed delay terms with standard polynomials kernels such as $\mathbb{R}^{n\times\nu} \ni \widehat{A}(\tau) = AM_d(\tau) = A\left(\boldsymbol{m}_d(\tau)\otimes I_\nu\right)$ and $\mathbb{R}^{m\times\nu} \ni \widehat{C}(\tau) = CM_d(\tau) = C\left(\boldsymbol{m}_d(\tau)\otimes I_\nu\right)$, where $\boldsymbol{m}_d(\tau) := \mathsf{Col}_{i=0}^d \tau^i$ and the matrices $A \in \mathbb{R}^{n\times\varrho}$, $C \in \mathbb{R}^{m\times\varrho}$ can be easily determined by the structure of $M_d(\tau)$. By the definition of $\boldsymbol{\ell}_d(\tau)$ in (6.2) with (6.3) and (6.4), we have $AM_d(\tau) = A\left(\mathsf{L}_d^{-1}(r)\otimes I_\nu\right)L_d(\tau)$ and $CM_d(\tau) = C\left(\mathsf{L}_d^{-1}(r)\otimes I_\nu\right)L_d(\tau)$, where $A\left(\mathsf{L}_d^{-1}(r)\otimes I_\nu\right)$ and $C\left(\mathsf{L}_d^{-1}(r)\otimes I_\nu\right)$ are polynomials matrices with respect to r corresponding to $A_3(r)$ and $C_3(r)$ in (6.1). This demonstrates that the choice of Legendre polynomials $\boldsymbol{\ell}_d(\tau)$ in (6.2) together with polynomial matrices $A_3(r)$ and $C_3(r)$ in (6.1) can handle distributed delay terms with polynomials kernels.

The following inequality has been first derived in [186, 187] with different notations.

Lemma 6.1. *Given $U(r) \in \mathbb{S}^n_{\succeq 0}$ for all $r > 0$, then the inequality*

$$\int_{-r}^{0} \boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau \geq [*]\left(r^{-1}\mathsf{D}_d\otimes U(r)\right)\left[\int_{-r}^{0}\left(\boldsymbol{\ell}_d(\tau)\otimes I_n\right)\boldsymbol{x}(\tau)\mathsf{d}\tau\right] \tag{6.8}$$

holds for all $\boldsymbol{x}(\cdot) \in \mathbb{L}^2([-r,0] ⨾ \mathbb{R}^n)$ and for all $r > 0$, where $\boldsymbol{\ell}_d(\tau)$ has been defined in (6.2) *and $\mathsf{D}_d := \bigoplus_{i=0}^d 2i+1$.*

Proof. Given $U(r) \in \mathbb{S}^n_{\succeq 0}$ for all $r > 0$. Let $\mathcal{K} = [-r,0]$ and $\boldsymbol{f}(\tau) = \boldsymbol{\ell}_d(\tau)$ in Lemma 2.16, then it gives the form of the inequality (6.8) with a known r since $\int_{-r}^0 \boldsymbol{\ell}_d(\tau)\boldsymbol{\ell}_d^\top(\tau)\mathsf{d}\tau = r\mathsf{D}_d^{-1}$. Note that the result is naturally valid for all $r > 0$ which gives this lemma. ■

Remark 6.3. Note that since $\boldsymbol{x}(\cdot) \in \mathbb{L}^2([-r,0] ⨾ \mathbb{R}^n)$ in (6.8) with the fact that all functions in $\boldsymbol{\ell}_d(\tau)$ are bounded, therefore all the integrals in (6.8) are well defined.

In the following definition, we define the space of univariate polynomials matrices. For the expression of multivariate polynomials matrices, see [307].

Definition 6.1. The space containing polynomials matrices between $\mathbb{R}^n$ to $\mathbb{R}^{p\times q}$ is defined as

$$\mathbb{R}^{p\times q}[\mathbb{R}] := \left\{ F(\cdot) \in (\mathbb{R}^{p\times q})^{\mathbb{R}} \;\middle|\; \begin{array}{c} F(x) = \sum_{i=0}^p Q_i x^p \;\;\&\;\; p \in \mathbb{N}_0 \\ \&\;\; Q_i \in \mathbb{R}^{p\times q} \end{array} \right\}. \tag{6.9}$$

Furthermore, the degree of a polynomial matrix is defined as

$$\deg\left(\sum_{i=0}^{p} Q_i x^p\right) = \max_{i=0\cdots p}\left(\left[\mathbb{1}_{\mathbb{R}^{p\times q}\setminus\{\mathsf{O}_{p\times q}\}}(Q_i)\right]i\right) \tag{6.10}$$

where $\mathbb{1}_{\mathcal{X}}(\cdot)$ is the standard indicator function. This also allows us to define

$$\mathbb{R}^{p\times q}[\mathbb{R}]_d := \left\{F(\cdot) \in \mathbb{R}^{p\times q}[\mathbb{R}] : \deg\left(F(\cdot)\right) = d\right\}, \text{ with } d \in \mathbb{N}_0 \tag{6.11}$$

which contains polynomials with degree d.

The following definition gives the space of univariate sum-of-squares polynomials matrix. For the definition of the structure of multivariate sum-of-squares polynomials matrix, see [309] for details.

Definition 6.2. A polynomial in $\mathbb{S}^m[\mathbb{R}]$ is classified as a sum-of-squares polynomial if and only if it belongs to the space

$$\boldsymbol{\Sigma}\left(\mathbb{R} ⨾ \mathbb{S}^m_{\succeq 0}\right) := \left\{ F(\cdot) \in \mathbb{S}^m[\mathbb{R}] \;\middle|\; \begin{array}{c} F(x) = \Phi(x)^\top\Phi(x) \\ \exists\Phi(\cdot) \in \mathbb{R}^{p\times m}[\mathbb{R}] \;\;\&\;\; p \in \mathbb{N} \end{array} \right\}. \tag{6.12}$$

We also define $\boldsymbol{\Sigma}_d\left(\mathbb{R} ⨾ \mathbb{S}^m_{\succeq 0}\right) := \left\{F(\cdot) \in \boldsymbol{\Sigma}\left(\mathbb{R} ⨾ \mathbb{S}^m_{\succeq 0}\right) : \deg\left(F(\cdot)\right) = 2d\right\}$ with $d \in \mathbb{N}_0$. Finally, it is obvious to see that $\boldsymbol{\Sigma}_0\left(\mathbb{R} ⨾ \mathbb{S}^m_{\succeq 0}\right) = \mathbb{S}^m_{\succeq 0}$.

The following lemma allows one to solve SoS constraints numerically via semidefinite programmings. Unlike the original Lemma 1 in [309], we only need to consider the univariate case.

Lemma 6.2. *$P(\cdot) \in \boldsymbol{\Sigma}\left(\mathbb{R} \,⨾\, \mathbb{S}^m_{\succeq 0}\right)$ if and only if there exists $Q \in \mathbb{S}^{(d+1)m}_{\succeq 0}$ such that*

$$\forall x \in \mathbb{R}, \;\; P(x) = (\boldsymbol{m}(x) \otimes I_m)^\top Q(\boldsymbol{m}(x) \otimes I_m), \tag{6.13}$$

where $\boldsymbol{m}(x) := \mathsf{Col}_{i=0}^{d}\, x^i$ with $d \in \mathbb{N}_0$.

Proof. Let $u(\cdot) = \mathsf{Col}_{i=0}^{d}\, x^i$ with $d \in \mathbb{N}_0$ in the Lemma 1 of [309], then Lemma 6.2 with $\boldsymbol{m}(x) := \mathsf{Col}_{i=0}^{d}\, x^i$ is obtained. ∎

Remark 6.4. When it comes to real-time calculations, one can only obtain a numerical result $Q \succ 0$ instead of $Q \succeq 0$. Consequently, the membership certificate produced by numerical calculations in reality is $P(\cdot) \in \boldsymbol{\Sigma}\left(\mathbb{R} \,⨾\, \mathbb{S}^m_{\succ 0}\right) \subset \boldsymbol{\Sigma}\left(\mathbb{R} \,⨾\, \mathbb{S}^m_{\succeq 0}\right)$.

6.4 Main results of dissipativity and stability analysis

In this section, the main results on range dissipativity and stability analysis are presented. The section is divided into five subsections and we first present the criteria for determining range delay stability and dissipativity for (6.1).

6.4.1 Criteria for range delay stability and dissipativity

The following range stability criteria for (6.1) can be obtained by modifying the Theorem 3 of [10].

Lemma 6.3. *Given $r_2 > r_1 > 0$, the origin of the system (6.1) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is globally uniformly asymptotically (exponentially) stable for all $r \in [r_1, r_2]$, if there exist $\epsilon_1; \epsilon_2; \epsilon_3 > 0$ and a differentiable functional $v : \mathbb{R}_+ \times \mathbb{R}^n \times \widehat{\mathbb{C}}([-r, 0) \,⨾\, \mathbb{R}^\nu) \to \mathbb{R}_{\geq 0}$ such that $\forall r \in [r_1, r_2]$, $v(r, \mathbf{0}_n, \mathbf{0}_\nu) = 0$ and*

$$\epsilon_1 \left\|\boldsymbol{\xi}\right\|_2^2 \leq v(r, \boldsymbol{\xi}, \boldsymbol{\phi}(\cdot)) \leq \epsilon_2 \left(\left\|\boldsymbol{\xi}\right\|_2 \vee \left\|\boldsymbol{\phi}(\cdot)\right\|_\infty\right)^2 \tag{6.14}$$

$$\left.\frac{\mathsf{d}^+}{\mathsf{d}t} v(r, \boldsymbol{x}(t), \mathbf{y}_t(\cdot))\right|_{t=t_0, \boldsymbol{x}(t_0)=\boldsymbol{\xi}, \mathbf{y}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} \leq -\epsilon_3 \left\|\boldsymbol{\xi}\right\|_2^2 \tag{6.15}$$

hold for all $r \in [r_1, r_2]$ and for any $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r, 0) \,⨾\, \mathbb{R}^\nu)$ in (6.1), where $t_0 \in \mathbb{R}$ and $\frac{\mathsf{d}^+}{\mathsf{d}x} f(x) = \limsup_{\eta \downarrow 0} \frac{f(x+\eta) - f(x)}{\eta}$. Furthermore, $\mathbf{y}_t(\cdot)$ in (6.15) is defined by $\forall t \geq t_0$, $\forall \theta \in [-r, 0)$, $\mathbf{y}_t(\theta) = \boldsymbol{y}(t + \theta)$ where $\boldsymbol{y}(t)$ here and $\boldsymbol{x}(t)$ in (6.15) satisfying (6.1) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$.

Proof. The Theorem 3 of [10] is for a given $r > 0$ where r is a variable of the system equation. However, it can be easily extended point-wisely by treating r in the system as an uncertain parameter belonging to an interval $[r_1, r_2]$ with $r_2 > r_1 > 0$. Moreover, the functions $V(\cdot)$,$u(\cdot)$,$v(\cdot)$ and $w(\cdot)$ in the Theorem 3 of [10] should be parameterized by r in this case. Thus a corresponding range stability criteria can be obtained which can be applied to (6.1) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$. Following the aforementioned steps and letting the functions $u(r, \cdot), v(r, \cdot), w(r, \cdot)$ to be the quadratic functions $\epsilon_i x^2$, $i = 1, 2, 3$, Lemma 6.3 can be obtained accordingly given the fact that (6.1) is a special case of the general system considered in Theorem 3 of [10]. ∎

Definition 6.3 (Dissipativity)**.** Given $r_2 > r_1 > 0$, the system in (6.1) with a supply rate function $s(\boldsymbol{z}(t), \boldsymbol{w}(t))$ is said to be dissipative for all $r \in [r_1, r_2]$, if there exists a differentiable functional $v : \mathbb{R}_+ \times \mathbb{R}^n \times \widehat{\mathbb{C}}([-r, 0) \,⨾\, \mathbb{R}^\nu) \to \mathbb{R}$ such that

$$\forall r \in [r_1, r_2], \; \forall t \geq t_0 : \; \dot{v}(r, \boldsymbol{x}(t), \mathbf{y}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \leq 0 \tag{6.16}$$

with $t_0 \in \mathbb{R}$, where $\mathbf{y}_t(\cdot)$ is defined by the equality $\forall t \geq t_0$, $\forall \theta \in [-r, 0)$, $\mathbf{y}_t(\theta) = \boldsymbol{y}(t + \theta)$, and $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$ and $\boldsymbol{z}(t)$ satisfy the equalities in (6.1) with $\boldsymbol{w}(\cdot) \in \widehat{\mathbb{L}}^2\left([t_0, \infty) \,⨾\, \mathbb{R}^q\right)$.

To incorporate dissipitivity into the analysis of (6.1), a quadratic supply function

$$s(\boldsymbol{z}(t),\boldsymbol{w}(t))=\begin{bmatrix}\boldsymbol{z}(t)\\ \boldsymbol{w}(t)\end{bmatrix}^\top \mathbf{J}\begin{bmatrix}\boldsymbol{z}(t)\\ \boldsymbol{w}(t)\end{bmatrix} \quad \text{with} \quad \mathbf{J}=\begin{bmatrix}\widetilde{J}^\top J_1^{-1}\widetilde{J} & J_2\\ * & J_3\end{bmatrix}\in\mathbb{S}^{(m+q)},\ \ \widetilde{J}^\top J_1^{-1}\widetilde{J}\preceq 0,\ \ J_1^{-1}\prec 0 \quad (6.17)$$

is applied in this chapter. For specific optimization objectives included by (6.17) such as $\mathbb{L}^2$ gain performance $J_1=-\gamma I_m,\ \widetilde{J}=I_m,\ J_2=\mathsf{O}_{m\times q},\ J_3=\gamma I_q$ where $\gamma>0$ is the value to be minimized, see more details in [215].

6.4.2 Conditions for range dissipativity and stability analysis

In this subsection, the main results on dissipativity and stability analysis are derived in the following theorem where the optimization constraints can be solved via the method of sum-of-squares programming.

Theorem 6.1. *Given $\lambda_1;\lambda_2;\lambda_3\in\mathbb{N}_0$ and $\boldsymbol{\ell}_d(\tau)$ consisting of the Legendre polynomials in* (6.2) *with $d\in\mathbb{N}_0$, the system* (6.1) *with the supply rate function* (6.17) *is dissipative for all $r\in[r_1,r_2]$, and the origin of* (6.1) *with $\boldsymbol{w}(t)\equiv\mathbf{0}_q$ is globally uniformly asymptotically stable for all $r\in[r_1,r_2]$, if there exist matrix polynomials*

$$P(\cdot)\in\mathbb{S}^{n+\varrho}\left[\mathbb{R}\right]_{\lambda_1},\quad S(\cdot)\in\mathbb{S}^{\nu}[\mathbb{R}]_{\lambda_2}\quad U(\cdot)\in\mathbb{S}^{\nu}[\mathbb{R}]_{\lambda_3},\ \widehat{P}(\cdot)\in\mathbb{S}^{n+\varrho}\left[\mathbb{R}\right]\quad \widehat{S}(\cdot);\widehat{U}(\cdot)\in\mathbb{S}^{\nu}[\mathbb{R}]$$

and $\delta_i\in\mathbb{N}_0,\ i=1\cdots 8$ with $\delta_7\neq 0$ such that

$$P(\cdot)+\left[\mathsf{O}_n\oplus(\mathsf{D}_d\otimes S(\cdot))\right]+g(\cdot)\widehat{P}(\cdot)\in\boldsymbol{\Sigma}_{\delta_1}\left(\mathbb{R}\,;\mathbb{S}^{n+\varrho}_{\succ 0}\right)\ \ \widehat{P}(\cdot)\in\boldsymbol{\Sigma}_{\delta_2}\left(\mathbb{R}\,;\mathbb{S}^{n+\varrho}_{\succeq 0}\right) \quad (6.18)$$

$$S(\cdot)+g(\cdot)\widehat{S}(\cdot)\in\boldsymbol{\Sigma}_{\delta_3}\left(\mathbb{R}\,;\mathbb{S}^{n}_{\succeq 0}\right),\ \widehat{S}(\cdot)\in\boldsymbol{\Sigma}_{\delta_4}\left(\mathbb{R}\,;\mathbb{S}^{n}_{\succeq 0}\right) \quad (6.19)$$

$$U(\cdot)+g(\cdot)\widehat{U}(\cdot)\in\boldsymbol{\Sigma}_{\delta_5}\left(\mathbb{R}\,;\mathbb{S}^{n}_{\succeq 0}\right),\ \widehat{U}(\cdot)\in\boldsymbol{\Sigma}_{\delta_6}\left(\mathbb{R}\,;\mathbb{S}^{n}_{\succeq 0}\right) \quad (6.20)$$

$$-\begin{bmatrix}J_1 & \widetilde{J}\Sigma(\cdot)\\ * & \boldsymbol{\Phi}_d(\cdot)\end{bmatrix}+g(\cdot)\boldsymbol{\Psi}(\cdot)\in\boldsymbol{\Sigma}_{\delta_7}\left(\mathbb{R}\,;\mathbb{S}^{m+q+2n+\varrho}_{\succ 0}\right),\ \ \boldsymbol{\Psi}(\cdot)\in\boldsymbol{\Sigma}_{\delta_8}\left(\mathbb{R}\,;\mathbb{S}^{m+q+2n+\varrho}_{\succeq 0}\right) \quad (6.21)$$

where $\varrho=(d+1)\nu$ and $g(r)=(r-r_1)(r-r_2)$ and

$$\boldsymbol{\Phi}_d(r):=\mathsf{Sy}\left(\begin{bmatrix}\mathsf{O}_{q\times n} & \mathsf{O}_{q\times\varrho}\\ I_n & \mathsf{O}_{n\times\varrho}\\ \mathsf{O}_n & \mathsf{O}_{n\times\varrho}\\ \mathsf{O}_{\varrho\times n} & rI_\varrho\end{bmatrix}P(r)\begin{bmatrix}D_1 & A_1 & A_2 & rA_3(r)\\ \mathsf{O}_{\varrho\times q} & L_d(0)A_4 & L_d(0)A_5-L_d(-r) & -\widehat{\mathsf{L}}_d\end{bmatrix}\right)$$
$$+\Gamma^\top(rS(r)+rU(r))\Gamma-\left[J_3\oplus\mathsf{O}_n\oplus rS(r)\oplus(r\mathsf{D}_d\otimes U(r))\right]-\mathsf{Sy}\left(\left[\Sigma^\top J_2\quad \mathsf{O}_{(n+\nu+\varrho+q)\times(n+\nu+\varrho)}\right]\right), \quad (6.22)$$

with

$$\Gamma:=\left[\mathsf{O}_{\nu\times q}\quad A_4\quad A_5\quad \mathsf{O}_{\nu\times\varrho}\right],\quad \Sigma(r):=\left[D_2\quad C_1\quad C_2\quad rC_3(r)\right]. \quad (6.23)$$

and $\widehat{\mathsf{L}}_d:=\acute{\mathsf{L}}_d\otimes I_\nu$ in which $\acute{\mathsf{L}}_d$ is given in (6.5).

Proof. The proof of this Theorem is based on the construction of the parameterized functional

$$v(r,\boldsymbol{x}(t),\mathfrak{y}_t(\cdot)):=\begin{bmatrix}\boldsymbol{x}(t)\\ \int_{-r}^0 L_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\end{bmatrix}^\top P(r)\begin{bmatrix}\boldsymbol{x}(t)\\ \int_{-r}^0 L_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\end{bmatrix}$$
$$+\int_{-r}^0\boldsymbol{y}^\top(t+\tau)\left[rS(r)+(\tau+r)U(r)\right]\boldsymbol{y}(t+\tau)\mathsf{d}\tau, \quad (6.24)$$

where $\mathfrak{y}_t(\cdot)$ follows the definition in (6.16) and the functional satisfies $v(r,\mathbf{0}_n,\mathbf{0}_\nu)=0$ for all $r\in[r_1,r_2]$ with given $r_2>r_1>0$. Furthermore, $L_d(\tau)$ in (6.24) is defined as $L_d(\tau)=\boldsymbol{\ell}_d(\tau)\otimes I_\nu$ with $\boldsymbol{\ell}_d(\tau)$ in (6.2), and the matrix parameters in (6.24) are $P(\cdot)\in\mathbb{S}^{n+\varrho}\left[\mathbb{R}\right]_{\lambda_1}$, $S(\cdot)\in\mathbb{S}^{\nu}[\mathbb{R}]_{\lambda_2}$ and $U(\cdot)\in\mathbb{S}^{\nu}[\mathbb{R}]_{\lambda_3}$ with the degree indexes $\lambda_1;\lambda_2;\lambda_3\in\mathbb{N}_0$.[3]

[3]Note that $\mathbb{S}^\nu[\mathbb{R}]_0=\mathbb{S}^\nu$

First of all, we will demonstrate that the feasible solutions of (6.20)–(6.21) *infer the existence of* (6.24) *satisfying* (6.16) *and* (6.15). Differentiating $v(r, \boldsymbol{x}(t), \mathbf{y}_t(\cdot))$ along the trajectory of (6.1) and considering the relation

$$\begin{aligned}\int_{-r}^{0} L_d(\tau)\dot{\boldsymbol{y}}(t+\tau)\mathsf{d}\tau &= L_d(0)\boldsymbol{y}(t) - L_d(-r)\boldsymbol{y}(t-r) - \widehat{\mathsf{L}}_d\frac{1}{r}\int_{-r}^{0} L_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau \\ &= L_d(0)A_4\boldsymbol{x}(t) + \left(L_d(0)A_5 - L_d(-r)\right)\boldsymbol{y}(t-r) - \widehat{\mathsf{L}}_d\frac{1}{r}\int_{-r}^{0} L_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau \end{aligned} \quad (6.25)$$

produces

$$\begin{aligned}&\forall t \geq t_0, \quad \dot{v}(r, \boldsymbol{x}(t), \mathbf{y}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \\ &= \boldsymbol{\chi}_d^\top(t)\,\mathbf{Sy}\left(\begin{bmatrix}\mathsf{O}_{q\times n} & \mathsf{O}_{q\times\varrho}\\ I_n & \mathsf{O}_{n\times\varrho}\\ \mathsf{O}_{\nu\times n} & \mathsf{O}_{\nu\times\varrho}\\ \mathsf{O}_{\varrho\times n} & rI_\varrho\end{bmatrix} P(r)\begin{bmatrix}D_1 & A_1 & A_2 & rA_3(r)\\ \mathsf{O}_{\varrho\times q} & L_d(0)A_4 & L_d(0)A_5 - L_d(-r) & -\widehat{\mathsf{L}}_d\end{bmatrix}\right)\boldsymbol{\chi}_d(t) \\ &+ \boldsymbol{\chi}_d^\top(t)\left[\Gamma^\top(rS(r) + rU(r))\,\Gamma - \left(J_3 \oplus \mathsf{O}_n \oplus rS(r) \oplus \mathsf{O}_\varrho\right)\right]\boldsymbol{\chi}_d(t) \\ &- \boldsymbol{\chi}_d^\top(t)\Big(\Sigma^\top(r)\widetilde{J}^\top J_1^{-1}\widetilde{J}\Sigma(r) + \mathbf{Sy}\left(\begin{bmatrix}\Sigma^\top(r)J_2 & \mathsf{O}_{(n+\nu+\varrho+q)\times(n+\nu+\varrho)}\end{bmatrix}\right)\Big]\boldsymbol{\chi}_d(t), \\ &- \int_{-r}^{0}\boldsymbol{y}^\top(t+\tau)U(r)\boldsymbol{y}(t+\tau)\mathsf{d}\tau,\end{aligned} \quad (6.26)$$

where

$$\boldsymbol{\chi}_d(t) := \mathbf{Col}\left(\boldsymbol{w}(t),\ \boldsymbol{x}(t),\ \boldsymbol{y}(t-r),\ \frac{1}{r}\int_{-r}^{0} L_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\right) \quad (6.27)$$

and Γ, $\Sigma(r)$ have been defined in (6.23), and $\widehat{\mathsf{L}}_d := \acute{\mathsf{L}}_d \otimes I_\nu$ in (6.25) can be obtained by (6.5) with (2.1). Assume $U(r) \succeq 0$, $\forall r \in [r_1, r_2]$. Considering the fact that $\mathbf{y}_t(\cdot) \in \widehat{\mathbb{C}}([-r, 0) ; \mathbb{R}^\nu) \subset \mathbb{L}^2([-r, 0) ; \mathbb{R}^\nu)$, now apply (6.8) to the integral $\int_{-r}^{0}\boldsymbol{y}^\top(t+\tau)U(r)\boldsymbol{y}(t+\tau)\mathsf{d}\tau$ in (6.26). It produces

$$\forall r \in [r_1, r_2], \quad \int_{-r}^{0}\boldsymbol{y}^\top(t+\tau)U(r)\boldsymbol{y}(t+\tau)\mathsf{d}\tau \geq [*]\left(r\mathsf{D}_d \otimes U(r)\right)\left[\int_{-r}^{0} r^{-1}L_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau\right] \quad (6.28)$$

with $\mathsf{D}_d = \bigoplus_{i=0}^{d} 2i+1$. Moreover, applying (6.28) to (6.26) yields

$$\forall r \in [r_1, r_2],\ \forall t \geq t_0,\ \dot{v}(r, \boldsymbol{x}(t), \mathbf{y}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \leq \boldsymbol{\chi}_d^\top(t)\left(\boldsymbol{\Phi}_d(r) - \Sigma^\top(r)\widetilde{J}^\top J_1^{-1}\widetilde{J}\Sigma(r)\right)\boldsymbol{\chi}_d(t) \quad (6.29)$$

where $\boldsymbol{\Phi}_d(r)$ and $\boldsymbol{\chi}_d(t)$ have been defined in (6.22) and (6.27), respectively. Based on the structure of (6.29), it is easy to see that if

$$\forall r \in [r_1, r_2]: \quad \boldsymbol{\Phi}_d(r) - \Sigma^\top(r)\widetilde{J}^\top J_1^{-1}\widetilde{J}\Sigma(r) \prec 0, \ \ U(r) \succeq 0 \quad (6.30)$$

is satisfied then the dissipative inequality in (6.16) : $\dot{v}(r, \boldsymbol{x}(t), \mathbf{y}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \leq 0$ holds $\forall r \in [r_1, r_2]$ and $\forall t \geq t_0$.

Furthermore, by considering the fact that $J_1^{-1} \prec 0$ and the structure of $\boldsymbol{\Phi}_d(r) - \Sigma^\top(r)\widetilde{J}^\top J_1^{-1}\widetilde{J}\Sigma(r) \prec 0$, $\forall r \in [r_1, r_2]$ with the properties of negative definite matrices, it is obvious that given (6.30) holds then there exists (6.24) and $\epsilon_3 > 0$ satisfying $\forall r \in [r_1, r_2]$, $\forall t \geq t_0$, $\dot{v}(r, \boldsymbol{x}(t), \mathbf{y}_t(\cdot)) \leq -\epsilon_3 \left\|\boldsymbol{x}(t)\right\|_2^2$ along with the trajectory of (6.1) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$. Now consider the case of $t = t_0$ for the previous inequality with the initial conditions in (6.1), it shows that the feasible solutions of (6.30) infer the existence of ϵ_3 and (6.24) satisfying (6.15). On the other hand, given $J_1^{-1} \prec 0$, applying the Schur complement to (6.30) enables one to conclude that (6.30) holds if and only if

$$\forall r \in \mathcal{G}: \quad \boldsymbol{\Theta}_d(r) = \begin{bmatrix}J_1 & \widetilde{J}\Sigma(r)\\ * & \boldsymbol{\Phi}_d(r)\end{bmatrix} \prec 0, \ \ U(r) \succeq 0 \quad (6.31)$$

with $\mathcal{G} := \left\{\rho \in \mathbb{R} : g(\rho) := (\rho - r_1)(\rho - r_2) \leq 0\right\} = [r_1, r_2]$. Now apply the matrix sum-of-squares relaxation technique proposed in [309] to (6.31), given the fact that $g(\cdot)$ naturally satisfies the qualification constraint in the Theorem 1 of [309]. Then we can conclude that (6.31) holds if and only if[4] (6.20) and (6.21) hold for some δ_i, $i = 5 \cdots 8$. This shows that the feasible solutions of (6.20)–(6.21) infer the existence of $\epsilon_3 > 0$ and (6.24) satisfying (6.16) and (6.15).

Now we will start to prove that (6.18)–(6.20) *infer that* (6.24) *satisfies* (6.14) *with* $\epsilon_1 > 0$ *and* $\epsilon_2 > 0$. Given the structure of (6.24) and consider the situation of $t = t_0$ with the initial conditions in (6.1), it follows that there exists $\lambda > 0$ such that for all $r \in [r_1, r_2]$

$$
\begin{aligned}
v(r, \boldsymbol{\xi}, \boldsymbol{\phi}(\cdot)) \leq & \begin{bmatrix} \boldsymbol{\xi} \\ \int_{-r}^{0} L_d(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \end{bmatrix}^{\top} \lambda \begin{bmatrix} \boldsymbol{\xi} \\ \int_{-r}^{0} L_d(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \end{bmatrix} + \int_{-r}^{0} \boldsymbol{\phi}^{\top}(\tau)\lambda\boldsymbol{\phi}(\tau)\mathsf{d}\tau \\
& \leq \lambda \|\boldsymbol{\xi}\|_2^2 + \int_{-r}^{0} \boldsymbol{\phi}^{\top}(\tau)L_d^{\top}(\tau)\mathsf{d}\tau\lambda \int_{-r}^{0} L_d(\tau)\boldsymbol{\phi}(\cdot)\mathsf{d}\tau + \lambda r \|\boldsymbol{\phi}(\tau)\|_{\infty}^2 \leq \lambda \|\boldsymbol{\xi}\|_2^2 + \lambda r \|\boldsymbol{\phi}(\cdot)\|_{\infty}^2 \\
& + [*] \left(\lambda \mathsf{D}_d \otimes I_n\right) \int_{-r}^{0} L_d(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \leq \lambda \|\boldsymbol{\xi}\|_2^2 + \lambda r \|\boldsymbol{\phi}(\cdot)\|_{\infty}^2 + r \int_{-r}^{0} \boldsymbol{\phi}^{\top}(\tau)\lambda\boldsymbol{\phi}(\tau)\mathsf{d}\tau \\
& \leq \lambda \|\boldsymbol{\xi}\|_2^2 + \left(\lambda r + \lambda r^2\right) \|\boldsymbol{\phi}(\cdot)\|_{\infty}^2 \leq \left(\lambda + \lambda r^2\right) \|\boldsymbol{\xi}\|_2^2 + \left(\lambda r + \lambda r^2\right) \|\boldsymbol{\phi}(\cdot)\|_{\infty}^2 \\
& \leq 2\left(\lambda r + \lambda r^2\right) \left(\|\boldsymbol{\xi}\|_2 \vee \|\boldsymbol{\phi}(\cdot)\|_{\infty}\right)^2 \leq 2\left(\lambda r_2 + \lambda r_2^2\right) \left(\|\boldsymbol{\xi}\|_2 \vee \|\boldsymbol{\phi}(\cdot)\|_{\infty}\right)^2 \quad (6.32)
\end{aligned}
$$

holds for any $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r, 0) ; \mathbb{R}^{\nu})$ in (6.1). Consequently, it shows that there exist $\epsilon_2 > 0$ such that (6.24) satisfies the upper bound property in (6.14).

Now assume $S(r) \succeq 0, \forall r \in [r_1, r_2]$. Then applying (6.8) to the integral $\int_{-r}^{0} \boldsymbol{y}^{\top}(t + \tau)S(r)\boldsymbol{y}(t + \tau)\mathsf{d}\tau$ in (6.24) at $t = t_0$ yields

$$
\forall r \in [r_1, r_2], \ \ r \int_{-r}^{0} \boldsymbol{\phi}^{\top}(\tau)S(r)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \geq \int_{-r}^{0} \boldsymbol{\phi}^{\top}(\tau)L_d^{\top}(\tau)\mathsf{d}\tau \left(\mathsf{D}_d \otimes S(r)\right) \int_{-r}^{0} L_d(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau, \qquad (6.33)
$$

with $\mathsf{D}_d = \bigoplus_{i=0}^{d} 2i + 1$. Applying (6.33) to (6.24) at $t = t_0$ produces that for all $r \in [r_1, r_2]$

$$
\begin{aligned}
v(r, \boldsymbol{\xi}, \boldsymbol{\phi}(\cdot)) \geq & \begin{bmatrix} \boldsymbol{\xi} \\ \int_{-r}^{0} L_d(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \end{bmatrix}^{\top} \left(P(r) + \left[\mathsf{O}_n \oplus (\mathsf{D}_d \otimes S(r))\right]\right) \begin{bmatrix} \boldsymbol{\xi} \\ \int_{-r}^{0} L_d(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \end{bmatrix} \\
& + \int_{-r}^{0} (\tau + r)\boldsymbol{\phi}^{\top}(\tau)U(r)\boldsymbol{\phi}(\tau)\mathsf{d}\tau. \quad (6.34)
\end{aligned}
$$

holds for any $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r, 0) ; \mathbb{R}^{\nu})$ in (6.1). Considering the structure of (6.34), it is obvious to see that if

$$
\forall r \in \mathcal{G} : \ \ \Pi_d(r) := P(r) + \left[\mathsf{O}_n \oplus (\mathsf{D}_d \otimes S(r))\right] \succ 0, \quad S(r) \succeq 0, \ \ U(r) \succeq 0 \qquad (6.35)
$$

is satisfied, then there exists $\epsilon_1 > 0$ such that for all $r \in [r_1, r_2]$ we have $\epsilon_1 \|\boldsymbol{\xi}\|_2 \leq v(r, \boldsymbol{\xi}, \boldsymbol{\phi}(\cdot))$ for any $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\phi}(\cdot) \in \widehat{\mathbb{C}}([-r, 0) ; \mathbb{R}^{\nu})$ in (6.1), where $\mathcal{G} := \left\{\rho \in \mathbb{R} : g(\rho) := (\rho - r_1)(\rho - r_2) \leq 0\right\} = [r_1, r_2]$. Now apply the matrix relaxation technique in [309] to the conditions in (6.35). Then one can conclude that (6.35) holds if and only if (6.18)–(6.20) hold for some δ_i, $i = 1 \cdots 6$. Considering the upper bound result which has been derived in (6.32), one can see the feasible solutions of (6.18)–(6.20) infer the existence of (6.24) and $\epsilon_1; \epsilon_2 > 0$ satisfying (6.14).

In conclusion, we have demonstrated that the feasible solutions of (6.18)–(6.21) infer the existence of $\epsilon_1; \epsilon_2; \epsilon_3 > 0$ and (6.24) satisfying(6.14)–(6.16). This finishes the proof. ■

Remark 6.5. Note that the structure of (6.24) is inspired by the complete LKF proposed in [10]. Because all the matrix parameters in (6.24) are related to r polynomially, thus it might be anticipated that less conservative results, when range delay stability analysis is concerned, can be produced by (6.24) in comparison to constructing an LKF with only constant matrix parameters.

[4] See the results related to the equations (1) and (6) in [309]

Remark 6.6. All SoS constraints in Theorem 6.1 can be solved numerically via the relation in (6.13). The dimension of the corresponding certificate variable Q in (6.13) is determined by the values of δ_i, $i = 1\cdots 8$ with λ_1, λ_2 and λ_3 in (6.24).

Remark 6.7. One may use different forms of $g(r)$ to characterize the set $\mathcal{G} = [r_1, r_2] = \{r \in \mathbb{R} : g(r) \leq 0\}$ as long as $g(r) \leq 0$ can equivalently characterize the interval $[r_1, r_2]$ and satisfy the qualification constraints in [309]. This also infers that a valid $g(r)$ with different form does not bring changes to the feasibility of the corresponding SoS constraints since they ultimately are equivalent to (6.31) and (6.35). Nevertheless, the form $g(r) = (r - r_1)(r - r_2)$ might be the best option to solve (6.31) and (6.35) considering its low degree, which alleviates the computational burden to solve (6.18)–(6.21).

Remark 6.8. Point-wise delay stability analysis at $r = r_0 > 0$ can be tested by solving

$$S \succeq 0,\ U \succeq 0,\ \ \Pi \succ 0,\ \ \mathbf{\Theta}_d(r_0) \prec 0 \tag{6.36}$$

in which the value of r_0 is given with $\lambda_1 = \lambda_2 = \lambda_3 = 0$ in (6.24). Since r_0 here is of fixed values, there is no need to consider non-constant polynomials matrix variables for (6.24). We emphasize that every time when (6.36) is referenced in this chapter, it assumes that $\lambda_1 = \lambda_2 = \lambda_3 = 0$ in (6.24).

6.4.3 Reducing the computational burden of Theorem 1 for certain cases

The SoS constraints in (6.18)–(6.21) can be applied to handle any form of (6.1) with given values of λ_1, λ_2, λ_3 in (6.24), supported by proper choices of δ_i, $i = 1\cdots 8$. However, if any inequality in (6.31) or (6.35) is affine with respect to r, then it can be solved equivalently via the property of convex hull to reduce numerical complexities compared to solving the equivalent SoS constraints in (6.18)–(6.21). Nevertheless, this can only happen to very special cases as what will be summarized as follows.

Case 1. Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$ in (6.24) and $rA_3(r)$ to be a constant matrix and $rC_3(r)$ to be affine in r, then the corresponding $\mathbf{\Theta}_d(r) \prec 0$ in (6.37) is affine in r and (6.18)–(6.20) become standard LMIs

$$S \succeq 0,\ U \succeq 0,\ \ \Pi \succ 0, \tag{6.37}$$

with $\delta_i = 0$, $i = 1\cdots 6$ and $\widehat{P}(r) = \mathsf{O}_{\nu+\varrho}$, $\widehat{S}(r) = \widehat{U}(r) = \mathsf{O}_{\nu}$, where (6.37) can be obtained directly from (6.35) with $\lambda_1 = \lambda_2 = \lambda_3 = 0$ also. Since $\mathbf{\Theta}_d(r) \prec 0$ in (6.37) is affine in r, hence $\forall r \in [r_1, r_2]$, $\mathbf{\Theta}_d(r) \prec 0$ can be solved by the property of convex hull instead of using (6.21). Meanwhile, $\forall r \in [r_1, r_2]$, $\mathbf{\Theta}_d(r) \prec 0$ here can still be solved via the SoS condition (6.21), with the degrees $\delta_7 = 1$ and $\delta_8 = 0$ for example. However, since using SoS does not add any extra feasibility, it is preferable in this case to solve $\forall r \in [r_1, r_2]$, $\mathbf{\Theta}_d(r) \prec 0$ by the property of convex hull instead of SoS to reduce the number of decision variables.

Case 2. If any inequality in (6.35) is affine (convex)[5] with respect to r, then it can be solved directly via the property of convex hull. Meanwhile, if unstructured matrix variables are considered in (6.24) without predefined sparsities, the only possibility for $\mathbf{\Theta}_d(r) \prec 0$ in (6.31) to be an affine (convex) matrix inequality in r is the situation when $\lambda_1 = \lambda_2 = \lambda_3 = 0$ in (6.24) with $rA_3(r)$ being a constant[6] and $rC_3(r)$ being affine in r, based on the structure of $\mathbf{\Theta}_d(r) \prec 0$. Therefore, the property of convex hull cannot be applied to solve the corresponding $\forall r \in [r_1, r_2]$, $\mathbf{\Theta}_d(r) \prec 0$ if $rA_3(r)$ is non-constant and $rC_3(r)$ is not affine in r.

We have demonstrated that for certain situations, one can solve the parameter dependent LMIs in (6.35) and (6.31) via the property of convex hull with less number of decision variables compared to solving the

[5]This may include the situation such as $S(r) = S_1 + r^4 S_2$. However, the handling of $\forall r \in [r_1, r_2]$, $S(r) \succ 0$ is identical to an affine example. In addition, the variable structure such as $S(r) = r^3 S_1$ will not be considered in this chapter, since always it can be equivalently transferred into a constant parameter.

[6]See also in Remark 5 of [287] for a relevant discussion of range stability without considering output and disturbance

equivalent SoS constraints in (6.18)–(6.21). However, based on the discussion we have made in Case 1 and Case 2, the SoS constraint in (6.21) does need to be solved with our proposed method if $rA_3(r)$ is non-constant and $rC_3(r)$ is not affine in r, which is still true even if (6.24) is parameterized only via constant matrix parameters ($\lambda_1 = \lambda_2 = \lambda_3 = 0$).

6.4.4 Estimating delay margins subject to prescribed performance objectives

Given an initial r_0 together with a supply function (6.17) (assume no decision variables in (6.17)) which renders (6.36) to be feasible, we are interested in the following problem.

Problem 1. Finding the minimum $\grave{r}$ or maximum $\acute{r}$ which render (6.1) to be stable and dissipative with (6.17) over $[\grave{r}, r_0]$ or $[r_0, \acute{r}]$, where the matrices in (6.17) contain no decision variables.

The control interpretation of this problem is straightforward: Given a specific performance objective, we want to obtain the largest stable delay interval of a delay system over which the system can always satisfy the given performance objective.

Problem 1 can be solved by the following optimization programs

$$\min \rho \quad \text{subject to} \quad (6.18) - (6.21) \quad \text{with} \quad g(r) = (r-\rho)(r-r_0) \tag{6.38}$$

or

$$\max \rho \quad \text{subject to} \quad (6.18) - (6.21) \quad \text{with} \quad g(r) = (r-r_0)(r-\rho), \tag{6.39}$$

with given λ_1, λ_2, λ_3 and δ_i, $i = 1 \cdots 8$. Specifically, we may easily handle (6.38) and (6.39) via an iterative one dimensional search scheme together with the idea of bisections [214]. Since both (6.38) and (6.39) are delay range dissipativity and stability conditions, thus the use of bisections will not produce false feasible solutions even (6.38) and (6.39) are not necessarily quasi-convex. Furthermore, as what has been elaborated in subsection 6.4.3, if any inequality in (6.35) and (6.31) is affine (convex) with respect to r, then it can be solved directly via the property of convex hull to replace the corresponding SoS conditions in (6.38) and (6.39). Finally, It is very important to emphasize here that the result of dissipativity over $[\grave{r}, r_0]$ or $[r_0, \acute{r}]$, which is produced individually by (6.38) and (6.39), cannot be automatically merged together due to the mathematical nature of range dissipativity analysis.

6.4.5 A hierarchy of the conditions in Theorem 6.1

Here we show that the feasibility of the range dissipativity and stability condition in Theorem 6.1 exhibits a hierarchy with respect to d.

Theorem 6.2. *Given $\boldsymbol{\ell}_d(\tau)$ consisting of the Legendre polynomials in* (6.2)*, we have*

$$\forall d \in \mathbb{N}_0, \ \ \mathcal{F}_d \subseteq \mathcal{F}_{d+1} \tag{6.40}$$

where

$$\begin{aligned}\mathcal{F}_d &:= \Big\{(r_1, r_2) \ \Big| \ r_2 > r_1 > 0 \ \ \& \ \ (6.35) \text{ and } (6.31) \text{ hold}\Big\} \\ &= \Big\{(r_1, r_2) \ \Big| \ r_2 > r_1 > 0 \ \ \& \ \ (6.18)\text{–}(6.21) \text{ hold } \ \& \ \ \delta_7 \in \mathbb{N} \ \ \& \ \ \delta_i; \delta_8 \in \mathbb{N}_0, i = 1 \cdots 6\Big\}.\end{aligned} \tag{6.41}$$

Proof. Let $d \in \mathbb{N}_0$ and $(r_1, r_2) \in \mathcal{F}_d$ with $\mathcal{F}_d \neq \varnothing$ and (6.35) to be satisfied by $S(r)$, $U(r)$ and $P_d(r)$ at d. Assume that $P_{d+1}(r) = P_d(r) \oplus \mathsf{O}_n$ for $P(r)$ in (6.24) at $d+1$ and consider the structure of (6.35), we have

$$\begin{gathered}\forall r \in [r_1, r_2], \ \ \Pi_{d+1}(r) = P_{d+1}(r) + \Big[\mathsf{O}_n \oplus (\mathsf{D}_{d+1} \otimes S(r))\Big] = \Pi_d(r) \oplus [(2d+3) \otimes S(r)] \succ 0, \\ \forall r \in [r_1, r_2], \ \ S(r) \succeq 0, \quad U(r) \succeq 0.\end{gathered} \tag{6.42}$$

Since $S(r) \succeq 0$ and $2d+3>0$, thus one can conclude from (6.42) that the existence of feasible solutions of $\forall r \in [r_1, r_2]$, $\Pi_d(r) \succ 0$ infers the existence of a feasible solution of $\forall r \in [r_1, r_2]$, $\Pi_{d+1}(r) \succ 0$, given $\forall r \in [r_1, r_2]$, $S(r) \succeq 0$, $U(r) \succeq 0$.

Given $P_{d+1}(r) = P_d(r) \oplus \mathsf{O}_n$ and consider (6.31) with the structure of $\mathbf{\Theta}_d(r)$ and (6.27) and $\mathbf{\Phi}_d(r)$ in (6.22), it is obvious to see that

$$\forall r \in [r_1, r_2], \quad \mathbf{\Theta}_{d+1}(r) = \begin{bmatrix} J_1 & \begin{bmatrix}\Sigma(r) & \mathsf{O}_{m\times n}\end{bmatrix} \\ * & \mathbf{\Phi}_{d+1}(r) \end{bmatrix} = \begin{bmatrix} J_1 & \begin{bmatrix}\Sigma(r) & \mathsf{O}_{m\times n}\end{bmatrix} \\ * & \mathbf{\Phi}_d(r) \oplus [-r(2d+3)U(r)] \end{bmatrix} \prec 0. \tag{6.43}$$

Since $\forall r \in [r_1, r_2]$, $U(r) \succeq 0$, we can conclude that the existence of the feasible solutions of $\forall r \in [r_1, r_2]$, $\mathbf{\Theta}_d(r) \prec 0$ infers the existence of a feasible solution of $\forall r \in [r_1, r_2]$, $\mathbf{\Theta}_{d+1}(r) \prec 0$. Consequently, we have shown that the existence of feasible solutions of (6.35) and (6.31) at d infers the existence of a feasible solution of (6.35) and (6.31) at $d+1$. Finally, since (6.42) and (6.43) at $d+1$ can be equivalently verified by the SoS conditions (6.18)–(6.21) with $d+1$ for some $\delta_7 \in \mathbb{N}$ and $\delta_i; \delta_8 \in \mathbb{N}_0$, $i = 1 \cdots 6$, thus the results in Theorem 6.2 are proved. ■

6.5 Numerical examples

We utilize several numerical examples in this section to demonstrate the strength of the proposed methods in Chapter 6. All numerical examples in Section 6.5 are calculated in Matlab environment using Yalmip [266] as the optimization interface. In addition, Mosek 8 [284] is applied as the SDP numerical solvers. Moreover, all SoS constraints are implemented via the function `coefficient` in Yalmip.

6.5.1 Delay range stability analysis

In this subsection, we consider analyzing the delay range stability of

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A_1\boldsymbol{x}(t) + A_2\boldsymbol{y}(t-r) + \int_{-r}^{0} A_3(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau \\ \boldsymbol{y}(t) &= A_4\boldsymbol{x}(t) + A_5\boldsymbol{y}(t-r) \end{aligned} \tag{6.44}$$

with different state space parameters presented in Table 6.1, in which the delay margins $r_{\min}$ and $r_{\max}$ are calculated via the software package [81] with reference to the spectral method in [80].

Parameters	A_1	A_2	$A_3(r)L_d(\tau)$	A_4	A_5	$r_{\min}$	$r_{\max}$
Example 1	$\begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \end{bmatrix}$	0	0.10016827	1.71785
Example 2	$\begin{bmatrix} 0.2 & 0.01 \\ 0 & -2 \end{bmatrix}$	O_2	$\begin{bmatrix} -1+0.3\tau & 0.1 \\ 0 & -0.1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	O_2	0.1944	1.7145

Table 6.1: Numerical Examples of (6.44)

The examples in Table 6.1 are taken from [305] and [260], respectively, which are denoted via equivalent CDDS representations. Note that Example 2 cannot be analyzed by the range stability results in [187, 287, 304].

Note that the matrix $A_3(r)$ of Example 1 and Example 2 in Table 6.1 are $A_3(r) = \mathsf{O}_{2\times(d+1)}$ and

$$A_3(r) = \begin{bmatrix} -1 & 0.1 & 0.3 & 0 & \mathsf{O}_{2\times(2d-2)} \\ 0 & -0.1 & 0 & 0 & \end{bmatrix} \left(\mathsf{L}_d^{-1}(r) \otimes I_2\right) = \begin{bmatrix} -1-0.15r & 0.1 & 0.15r & 0 & \mathsf{O}_{2\times(2d-2)} \\ 0 & -0.1 & 0 & 0 & \end{bmatrix}, \tag{6.45}$$

respectively. In the following Table 6.2–6.3, the results of detectable stable delay interval with the largest length calculated by our method are presented compared to the results in [305] and [260], respectively, where NoDVs stands for the number of decision variables. Note that the values of δ_7 and δ_8 therein are the degrees of the SoS constraints in (6.21). In addition, as what we have stated in subsection 6.4.4 concerning the reduction of the numerical complexity of Theorem 6.1, if any inequality in (6.35) is affine, then it is solved via the property of convex hull with our method to reduce computational burdens. Finally, a numerical solution of Example 1 with $r = 1$ produced by DDE23 [310] in Matlab is presented in Figure 6.1.[7]

Solutions for Delay Range Stability	$[r_1, r_2]$	NoDVs
[305] ($N = 5$)	$[0.10016829, 1.7178]$	294
Theorem 6.1 ($\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$, $d = 4$, $\delta_7 = 1$, $\delta_8 = 0$)	$[0.10016828, 1.71785]$	231
Theorem 6.1 ($\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$, $d = 5$, $\delta_7 = 1$, $\delta_8 = 0$)	$[0.10016827, 1.71785]$	291

Table 6.2: Detectable stable interval with the largest length of Example 1 in Table 6.1.

Solutions for Delay Range Stability	$[r_1, r_2]$	NoDVs
[260] ($l = 1$, $r = 3$)	$[0.2, 1.29]$	5973
[260] ($l = 2$, $r = 3$)	$[0.2, 1.3]$	14034
Theorem 6.1 ($\lambda_1 = \lambda_2 = \lambda_3 = 0$, $d = 4$, $\delta_7 = 2$, $\delta_8 = 1$)	$[0.27, 1.629]$	1394
Theorem 6.1 ($\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$, $d = 4$, $\delta_7 = 2$, $\delta_8 = 1$)	$[0.1944, 1.7145]$	1472

Table 6.3: Largest detectable stable interval of Example 2 in Table 6.1

From the outcomes summarized in Table 6.2-6.3, one can clearly observe the advantage of our proposed methods given the fact that the stable intervals[8] of Example 1 and 2 can be detected with fewer variables compared to the existing results in [260] and [305]. In addition, one can clearly see from Table 6.3 concerning the benefits of constructing a functional (6.24) with delay-dependent matrix parameters to deal with delay range stability problems. Finally, note that Theorem 6.1 does not require the constraint $A_1 + A_2$ being nonsingular as the Theorem 4 of [305] does.

Remark 6.9. Note that the number of decision variables of Theorem 6.1 in Table 6.2–6.3 might be further reduced by simplifying the SoS certificate variable in (6.13) for each case when a SoS condition needs to be solved.

6.5.2 Range dissipativity and stability analysis

Consider the following neutral delay system

$$\begin{aligned} &\frac{\mathsf{d}}{\mathsf{d}t}\left(\boldsymbol{y}(t) - A_4\boldsymbol{y}(t-r)\right) = A_1\boldsymbol{y}(t) + A_2\boldsymbol{y}(t-r) + \int_{-r}^{0} A_3(r)L_d(\tau)\boldsymbol{y}(t+\tau)\mathsf{d}\tau + D_1\boldsymbol{w}(t) \\ &\boldsymbol{z}(t) = C_1\boldsymbol{y}(t) + C_2\boldsymbol{y}(t-r) + \int_{-r}^{0} C_3(r)L_d(\tau)\boldsymbol{y}(t+\tau) + D_2\boldsymbol{w}(t) \end{aligned} \tag{6.46}$$

[7]Figure 6.1 is generated via `matlab2tikz v1.1.0` by the original figure produced in Matlab

[8]Here the stable intervals refer to the ones whose delay margins are calculated by the method in [81] as listed in Table 6.1

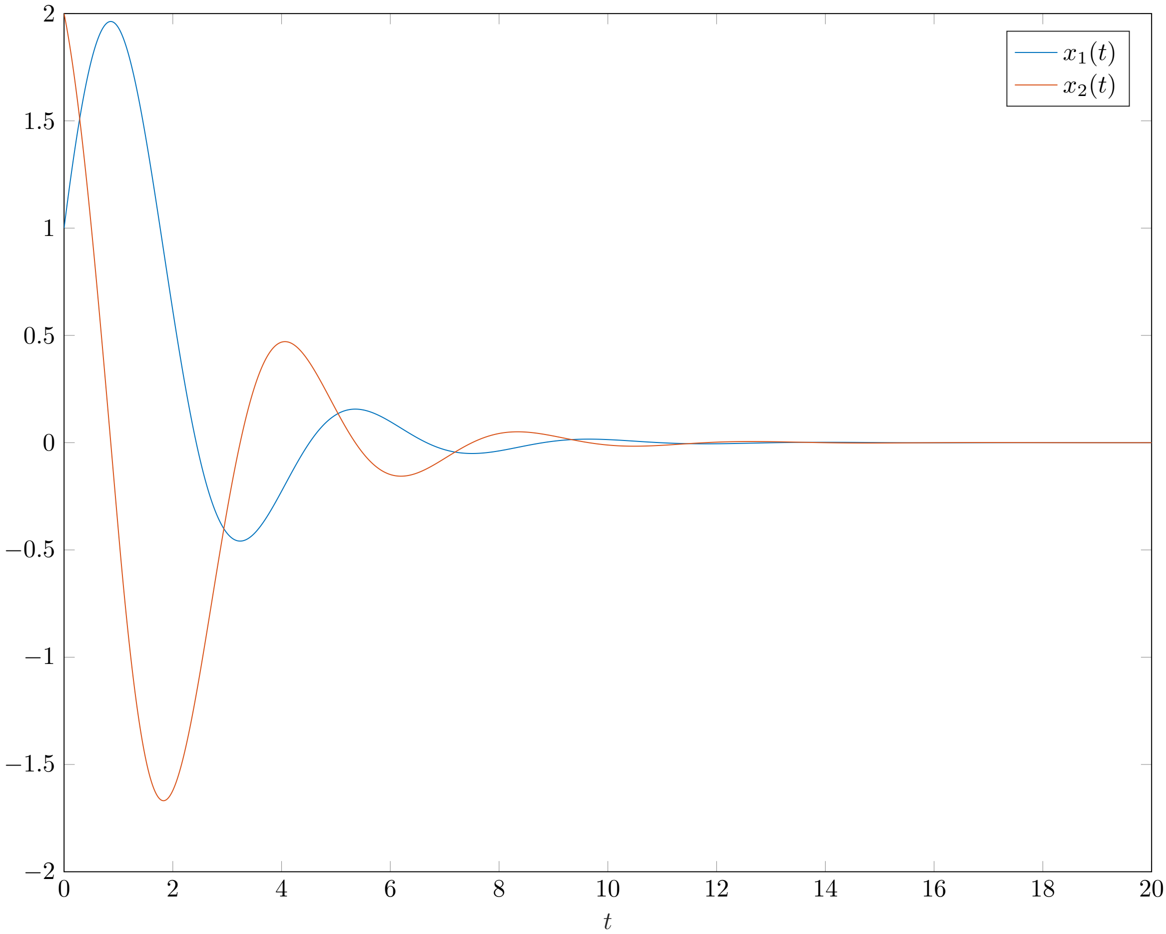

Figure 6.1: A numerical solution of Example 1 in Table 6.1

with distributed delays terms at the state and output. Let $\boldsymbol{x}(t) = \boldsymbol{y}(t) - A_4\boldsymbol{y}(t-r)$, then (6.46) can be reformulated into

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A_1\boldsymbol{x}(t) + (A_1A_4 + A_2)\boldsymbol{y}(t-r) + \int_{-r}^{0} A_3(r)L_d(\tau)\boldsymbol{y}(t+\tau) + D_1\boldsymbol{w}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{x}(t) + A_4\boldsymbol{y}(t-r), \\ \boldsymbol{z}(t) &= C_1\boldsymbol{x}(t) + (C_1A_4 + C_2)\boldsymbol{y}(t-r) + \int_{-r}^{0} C_3(r)L_d(\tau)\boldsymbol{y}(t+\tau) + D_2\boldsymbol{w}(t). \end{aligned} \tag{6.47}$$

which is now in line with the CDDS form in (6.1). Now consider a linear neutral delay system (6.46) with the parameters $A_3 = C_3 = \mathsf{O}_{3\times 3(d+1)}$ and

$$\begin{aligned} &A_1 = 100\begin{bmatrix} -2.103 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}, \quad A_2 = 100\begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix}, \quad A_4 = \frac{1}{72}\begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix} \\ &D_1 = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -0.1 & 0.1 & 0.2 \\ 0.4 & 0.01 & 0 \\ 0.1 & 0.21 & 0.1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.1 & 0 & 0.2 \\ 0.4 & 0 & -0.1 \\ 0 & -0.5 & 0.3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} \end{aligned} \tag{6.48}$$

which is modified based on the circuit model in [311]. We consider $\mathbb{L}^2$ gain γ corresponding to

$$J_1 = -\gamma I_m, \quad \widetilde{J} = I_m, \quad J_2 = \mathsf{O}_{m\times q}, \quad J_3 = \gamma I_q \tag{6.49}$$

in (6.17) as the performance objective to be minimized for (6.48).

Now assume $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$ in (6.24) with a delay range $[r_1, r_2] = [0.1, 0.5]$ and apply Theorem 6.1 to (6.47) with (6.49) and the parameters in (6.48) and $A_3 = C_3 = \mathsf{O}_{3\times 3(d+1)}$. Since all inequalities in (6.35) in this case are either affine with respect to r or simple LMIs, hence they are solved directly via the property of convex hull instead of solving the SoS conditions in (6.18)–(6.20). The results of $\min \gamma$ over $r \in [0.1, 0.5]$ are summarized in Table 6.4. Note that δ_7 and δ_8 are the degrees of the SoS constraints in (6.21). Finally, a numerical solution of the system in this case at $r = 0.1$ is produced by DDENSD [312] in Matlab presented in Figure 6.2.[9]

Theorem 6.1	$\delta_7 = 1$, $\delta_8 = 0$	$\delta_7 = 2$, $\delta_8 = 1$	$\delta_7 = 3$, $\delta_8 = 2$
$d = 1$	0.441	0.441	0.441
$d = 2$	0.364	0.364	0.364
$d = 3$	0.361	0.361	0.361

Table 6.4: Values of $\min \gamma$ valid over $r \in [0.1, 0.5]$

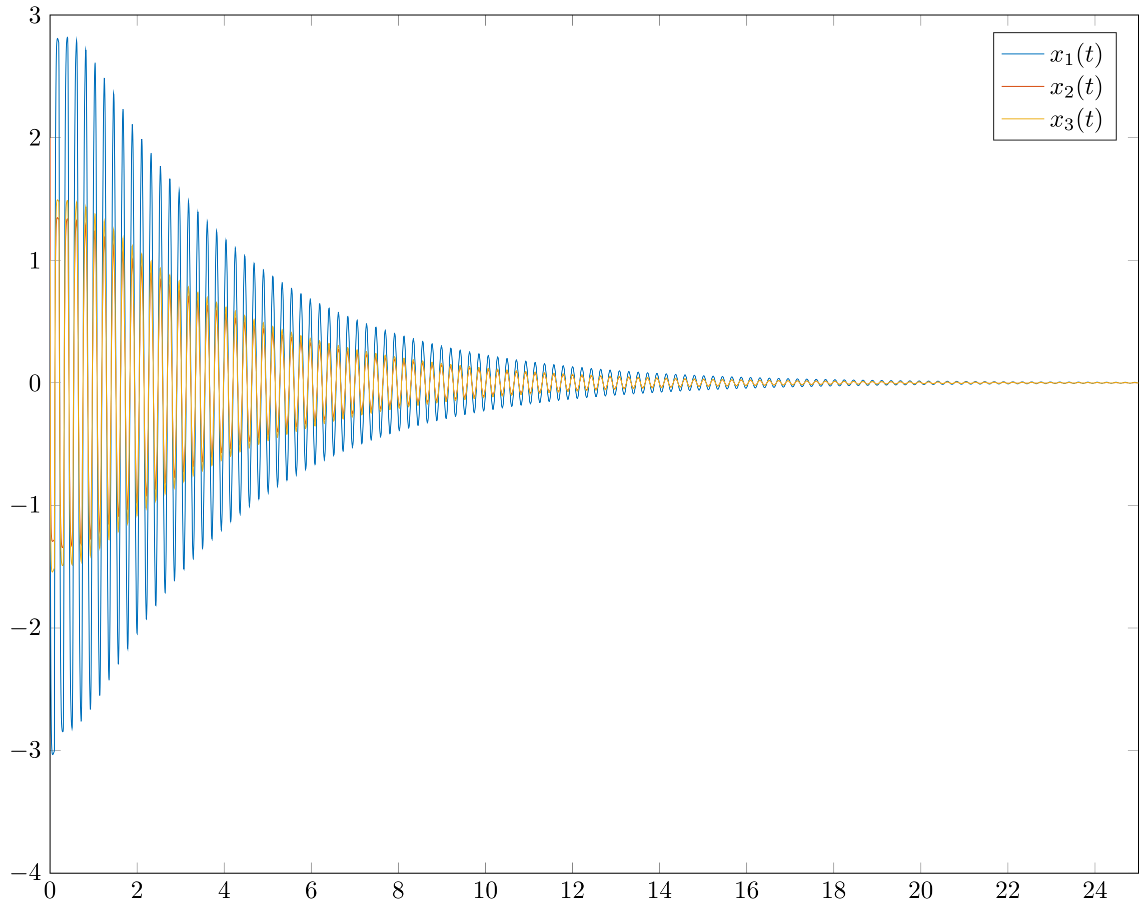

Figure 6.2: A numerical solution of (6.47) with (6.48) and $A_3 = C_3 = \mathsf{O}_{3\times 3(d+1)}$

Now apply Theorem 6.1 with constant matrix parameters $\lambda_1 = \lambda_2 = \lambda_3 = 0$ in (6.24) to the same aforementioned model with $[r_1, r_2] = [0.1, 0.5]$. The conditions in (6.35) are simple LMIs and $\mathbf{\Theta}_d(r) \prec 0$ in (6.31) can be solved by the property of convex hull in this case. However, even with $d = 10$, the corresponding range dissipativity and stability condition with $\lambda_1 = \lambda_2 = \lambda_3 = 0$ still cannot yield feasible solutions. This can demonstrate the advantage to apply an LKF with delay-dependent parameters when a

[9]Figure 6.2 is generated via `matlab2tikz v1.1.0` by the original figure produced in Matlab

functional with constant parameters is simply not strong enough to derive conditions capable of detecting a stable delay interval.

To partially verify the results in Table 6.4, we apply the `sigma` function[10] in Matlab, which can calculate the singular values ($\min\gamma$) of a dynamical system over a fixed frequency range. By extracting the peak value produced by `sigma`, it yields that the system (6.47) with (6.48) and $A_3 = C_3 = \mathbf{O}_{3\times 3(d+1)}$ exhibits $\min\gamma = 0.101074$ and $\min\gamma = 0.36064$ at $r = 0.1$ and $r = 0.5$, respectively. This shows that the values of $\min\gamma$ in Table 6.4, which are valid over $r \in [0.1, 0.5]$, are compatible with the point-wise $\min\gamma$ values obtained via `sigma` function. In addition, the best value $\min\gamma = 0.361$ in Table 6.4 is quite close to the point-wise result $\min\gamma = 0.36064$ at $r = 0.5$.

Now consider new $A_3 L_d(\tau)$ and $C_3 L_d(\tau)$ with the parameters

$$A_3(r)L_d(\tau) = \begin{bmatrix} 0.1\tau & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.3-0.1\tau \\ -0.1 & -0.2+0.1\tau & 0.2 \end{bmatrix}, \quad C_3(r)L_d(\tau) = \begin{bmatrix} 0.1 & 0 & 0.2 \\ 0.4 & 0 & -1 \\ 0 & -0.5 & 0.3 \end{bmatrix} \tag{6.50}$$

which together with (6.48) constitute a linear neutral system with distributed delays. Note that we can easily obtain the corresponding matrix coefficients as

$$\begin{gathered} A_3(r) = \begin{bmatrix} -0.05r & 0.1 & 0.3 & 0.05r & 0 & 0 & \\ 0.2 & 0.1 & 0.3+0.05r & 0 & 0 & -0.05r & \mathbf{O}_{3\times(3d-3)} \\ -0.1 & -0.2-0.05r & 0.2 & 0 & 0.05r & 0 & \end{bmatrix}, \\ C_3(r) = \begin{bmatrix} 0.1 & 0 & 0.2 & \\ 0.4 & 0 & -1 & \mathbf{O}_{3\times 3d} \\ 0 & -0.5 & 0.3 & \end{bmatrix}. \end{gathered} \tag{6.51}$$

To the best of our knowledge, there are no existing results on delay range dissipative analysis concerning linear neutral systems with non-constant distributed delay kernels. As a result, we might claim that no existing schemes can handle the problem we are dealing with here.

Assume $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$ in (6.24) with the delay range $[r_1, r_2] = [0.1, 0.5]$, now apply Theorem 6.1 to the system (6.47) with (6.48) and (6.51). Once more, since all the corresponding inequalities in (6.35) are affine with respect to r, then they are directly solved via the property of convex hull instead of solving (6.18)–(6.20). The values of the resulting $\min\gamma$ over $r \in [0.1, 0.5]$ are summarized in Table 6.5, where δ_7 and δ_8 are the degrees of the SoS constraint (6.21).

Theorem 6.1	$\delta_7 = 2$, $\delta_8 = 1$	$\delta_7 = 3$, $\delta_8 = 2$
$d = 1$	0.47	0.47
$d = 2$	0.382	0.382
$d = 3$	0.37822	0.37822

Table 6.5: values of $\min\gamma$ valid over $[0.1, 0.5]$

Unfortunately, even for the case of point-wise delays, the `sigma` function in Matlab cannot handle a distributed delay system at the current stage[11]. Thus we suggest here to use (6.36) to partially verify the results in Table 6.5. Specifically, apply (6.36) with $d = 3$ to the system (6.47) with (6.48) and (6.51) at $r_0 = 0.1$ and $r_0 = 0.5$, respectively. It yields $\min\gamma = 0.10101$ at $r_0 = 0.1$ and $\min\gamma = 0.37822$ at $r_0 = 0.5$, respectively. This verifies that the values of range $\min\gamma$ in Table 6.5 are consistent with the point-wise $\min\gamma$ values we presented.

[10] We first use the default range of `sigma` to determine which frequency range contains the peak singular value. Based on the previous information, we assigned `w = logspace(-1, 2, 2000000)` to `sigma` to ensure the accuracy of $\min\gamma$.

[11] We thank Dr. Suat Gumussoy for providing this information about the `sigma` function in Matlab.

Finally, the estimation problem described in subsection 6.4.4 can be easily applied to the system (6.47) with (6.48) and (6.51) assuming that the value of γ in (6.17) is known. To be specific, consider the system (6.47) with (6.48) and (6.51), let $\gamma = 0.37822$ and $r_0 = 0.5$ with which feasible solutions can be produced by (6.36) with $d = 3$. Now, one can use (6.38)[12] with $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$, $d = 3$ and $\delta_7 = 2$, $\delta_8 = 1$ to find out the minimum r^*, which renders the corresponding system to be stable and satisfying $\gamma = 0.37822$ over $[r^*, r_0]$. Given the results in Table 6.5, it is predictable that the optimal value of r^* here is $r^* = 0.1$.

[12] Note that in this case instead of solving the SoS contraints in (6.18)–(6.20) in (6.38), the equivalent inequalities in (6.35) are directly solved for (6.38) via the property of convex hull.

Chapter 7

Dissipative Stabilization of Linear Systems with Uncertain Bounded Time-Varying Distributed Delays

7.1 Introduction

In the previous chapters, the delay parameters of linear systems are assumed to be constants. However, in certain real-time applications such as the model in [313], delays can be time-varying. A particular class of delays $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$, $0 \le r_1 \le r_2$, where $[r_1, r_2]^{\mathbb{R}}$ is the set containing any function defined between $\mathbb{R}$ onto the bounded interval $[r_1, r_2]$, is of great research interests. Indeed, since $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ can be any function defined between $\mathbb{R}$ onto $[r_1, r_2]$, this type of delays can be applied to model sampled-data [248] or networked control systems (NCSs) [314], or even a delay which is bounded but non-deterministic [315]. This strongly motives one to investigate solutions for the problem of stability analysis and synthesis for linear systems with time-varying delays in the form of $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$.

There has been a significant number of results pertaining to the stability analysis [242, 290, 316–328] and stabilization [329–333] of linear systems in the form of $\dot{\boldsymbol{x}}(t) = A_1\boldsymbol{x}(t) + A_2\boldsymbol{x}(t - r(t))$ with $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$, based on the construction of LKFs. Furthermore, it has been shown in [244, 334] that the fruit of the LKF approach on $\dot{\boldsymbol{x}}(t) = A_1\boldsymbol{x}(t) + A_2\boldsymbol{x}(t - r(t))$ can be successfully applied to handle synthesis problems related to NCSs. It is worthy mention that unlike the case of constant delays, frequency-domain-based approaches [5, 80, 81, 86, 335] may not be easily extended to handle a system with a time-varying delay $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ if the exact expression of $r(t)$ is unknown. This clearly demonstrates the advantage and the adaptability of the LKF approach when it comes to time-varying delays $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ with unknown expressions.

It has been pointed out in [48, 336] that the digital communication channel of NCSs with stochastic packet delays and loss can be modeled by distributed delays. To the best of the author's knowledge however, no existing results can handle the problem of the stability analysis and stabilization of systems with time-varying $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ distributed delays at system state, input, and output, where the distributed delay kernels can be non-constants. In Theorem 2 of [337], a solution of stabilization is proposed for systems in the form of $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \int_{-r(t)}^{0} B(\tau)\boldsymbol{u}(t+\tau)\mathsf{d}\tau$. Nevertheless, all the poles of A in [337] are assumed to be positioned on the imaginary axis, and the delay considered therein is required to be $r(\cdot) \in (0, r_2]^{\mathbb{R}}$. Moreover, the stability of positive linear systems with distributed time-varying delays is investigated in [338, 339]. Although the method in [339] does include criteria which can determine the stability of non-positive linear systems, the delay structure $r(\cdot) \in [0, r_2]^{\mathbb{R}}$ considered in [339] is still rather restrictive. On the other hand, the synthesis (stability analysis) results in [48, 57, 185, 188, 340], which are derived to handle linear distributed delay systems with constant delay values, may not be easily extended to cope with the case of systems with an unknown time-varying delays $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$. This is especially true for the approximation approaches in [48, 185, 188, 340] since the resulting approximation coefficients can become nonlinear with respect to $r(t)$ if distributed delay kernels are approximated over the interval $[-r(t), 0]$. Consequently, it is crucial to develop solutions for the stabilization of linear systems with non-trivial distributed delays where the delay function $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ is unknown but bounded.

In this chapter, new approaches based on the LKF approach for the design of a state feedback controller for a linear system with distributed delays are developed where the time-varying delay $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ is unknown but bounded by given values $0 \leq r_1 \leq r_2$. No discrete time-varying delay $\boldsymbol{x}(t - r(t))$ is considered in this chapter at this stage as its presence can significantly change the manner of constructing LKFs. The distributed delay terms of the system can be found in states, inputs and outputs which ensure the generality of our model. In addition, the distributed delay kernels considered in this chapter follow the same class in Chapter 2. To obtain numerically tractable synthesis (stability) conditions via the construction of an LKF, a novel integral inequality is derived which generalizes Lemma 2 of [341]. Consequently, sufficient conditions for the existence of a state feedback controller, which ensures that the system is stable and dissipative with a supply function, are derived in terms of matrix inequalities summarized in the first theorem of this chapter. For the conditions in our first theorem, there is a bilinear matrix inequality corresponding to the problem of dissipative synthesis, whereas that inequality becomes convex when non-stabilization problems are considered. To tackle the problem of non-convexity, the second theorem of this chapter is derived via the application of Projection Lemma where the dissipative synthesis condition is denoted in terms of LMIs. Furthermore, an iterative algorithm is also derived to solve the bilinear condition in the first theorem and the algorithm can be initiated by feasible solutions of the second theorem. To the best of the author's knowledge, no existing methods can handle the synthesis problem considered in this chapter. Finally, two numerical examples are presented to demonstrate the effectiveness of our proposed methodologies.

The layout of the rest of the chapter is outlined as follows. The model of the closed-loop system is first derived in Section 7.2. Secondly, some important lemmas and definition are presented in Section 7.3 which includes the derivation of a novel integral inequality in Lemma 7.3. Next, the main results on dissipative synthesis for the existence of a state feedback controller are presented in Section 7.4 which are summarized in Theorem 7.1,7.2 and Algorithm 4. Two numerical examples were tested in Section 7.5 prior to the final conclusion.

7.2 Problem formulations

Consider a linear distributed delay system

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A_1\boldsymbol{x}(t) + \int_{-r(t)}^{0} \widetilde{A}_2(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau + B_1\boldsymbol{u}(t) + \int_{-r(t)}^{0} \widetilde{B}_2(\tau)\boldsymbol{u}(t+\tau)\mathsf{d}\tau + D_1\boldsymbol{w}(t), \quad t \geq t_0 \\
\boldsymbol{z}(t) &= C_1\boldsymbol{x}(t) + \int_{-r(t)}^{0} \widetilde{C}_2(\tau)\boldsymbol{x}(t+\tau)\mathsf{d}\tau + B_3\boldsymbol{u}(t) + \int_{-r(t)}^{0} \widetilde{B}_4(\tau)\boldsymbol{u}(t+\tau)\mathsf{d}\tau + D_2\boldsymbol{w}(t) \\
&\forall\theta \in [-r_2, 0], \ \ \boldsymbol{x}(t_0 + \theta) = \boldsymbol{\phi}(\tau), \quad r(\cdot) \in [r_1, r_2]^{\mathbb{R}}
\end{aligned} \tag{7.1}$$

with any time-varying delay satisfying $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ to be stabilized, where $r_2 > r_1 \geq 0$ and $t_0 \in \mathbb{R}$ and $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r_2, 0] ⨾ \mathbb{R}^n)$. Furthermore, $\boldsymbol{x} : [t_0 - r_2, \infty) \to \mathbb{R}^n$ satisfies (7.1), $\boldsymbol{u}(t) \in \mathbb{R}^p$ denotes input signals, $\boldsymbol{w}(\cdot) \in \widehat{\mathbb{L}}^2([t_0, \infty) ⨾ \mathbb{R}^q)$ represents disturbance, $\boldsymbol{z}(t) \in \mathbb{R}^m$ is the regulated output. Note that (7.1) is initiated at $t = t_0$ by the initial condition $\forall\theta \in [-r_2, 0]$, $\boldsymbol{x}(t_0 + \theta) = \boldsymbol{\phi}(\theta)$. The size of the given state space systems parameters in (7.1) is determined by the values of $n \in \mathbb{N}$ and $m; p; q \in \mathbb{N}_0$. The boundaries of the time-varying delay $r(t)$ are determined by the given values $r_2 > r_1 \geq 0$. Finally, the matrices $\widetilde{A}_2(\cdot)$, $\widetilde{C}_2(\cdot)$, $\widetilde{B}_2(\cdot)$ and $\widetilde{B}_4(\cdot)$ satisfy the following assumption:

Assumption 7.1. There exist $\mathsf{Col}_{i=1}^{d} f_i(\tau) = \boldsymbol{f}(\cdot) \in \mathbb{C}^1(\mathbb{R} ; \mathbb{R}^d)$ with $d \in \mathbb{N}$, and $A_2 \in \mathbb{R}^{n\times dn}$, $B_2 \in \mathbb{R}^{n\times dp}$, $C_2 \in \mathbb{R}^{m\times dn}$, $B_4 \in \mathbb{R}^{m\times dp}$ such that for all $\tau \in [-r, 0]$ we have $\widetilde{A}_2(\tau) = A_2(\boldsymbol{f}(\tau)\otimes I_n)$ and $\widetilde{B}_2(\tau) = B_2(\boldsymbol{f}(\tau)\otimes I_p)$ and $\widetilde{C}_2(\tau) = C_2(\boldsymbol{f}(\tau)\otimes I_n)$ and $\widetilde{B}_4(\tau) = B_4(\boldsymbol{f}(\tau)\otimes I_p)$. In addition, $\boldsymbol{f}(\cdot)$ satisfies the following properties:

$$\exists M \in \mathbb{R}^{d\times d} : \frac{\mathsf{d}\boldsymbol{f}(\tau)}{\mathsf{d}\tau} = M\boldsymbol{f}(\tau) \tag{7.2}$$

and

$$\mathsf{F}_1 = \int_{-r_1}^{0} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau \succ 0, \quad \mathsf{F}_2 = \int_{-r_2}^{-r_1} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau \succ 0. \tag{7.3}$$

Remark 7.1. By Theorem 7.2.10 in [258], the matrix inequalities in (7.3) indicate that the functions in $\boldsymbol{f}(\cdot)$ are linearly independent in a Lebesgue sense over $[-r_1, 0]$ and $[-r_2, -r_1]$.

In this chapter, a static feedback controller $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ with $K \in \mathbb{R}^{p\times n}$ is applied to stabilize (7.1), which yields the resulting closed-loop system

$$\begin{aligned}&\dot{\boldsymbol{x}}(t) = \big(\mathbf{A} + \mathbf{B}_1 [(I_{3+3d}\otimes K)\oplus \mathsf{O}_q]\big)\boldsymbol{\chi}(t), \quad \boldsymbol{z}(t) = (\mathbf{C} + \mathbf{B}_2 [(I_{3+3d}\otimes K)\oplus \mathsf{O}_q])\boldsymbol{\chi}(t), \ \ t \geq t_0\\ &\forall\theta \in [-r_2, 0], \ \ \boldsymbol{x}(t_0+\theta) = \boldsymbol{\phi}(\theta)\end{aligned} \tag{7.4}$$

with t_0 and $\boldsymbol{\phi}(\cdot)$ in (7.1), where

$$\mathbf{A} := \begin{bmatrix}\mathsf{O}_n & \mathsf{O}_n & A_1 & A_2\left(\sqrt{\mathsf{F}_1}\otimes I_n\right) & A_2\left(\sqrt{\mathsf{F}_2}\otimes I_n\right) & \mathsf{O}_{n\times\varrho} & D_1\end{bmatrix} \tag{7.5}$$

$$\mathbf{B}_1 := \begin{bmatrix}\mathsf{O}_{n\times p} & \mathsf{O}_{n\times p} & B_1 & B_2\left(\sqrt{\mathsf{F}_1}\otimes I_p\right) & B_2\left(\sqrt{\mathsf{F}_2}\otimes I_p\right) & \mathsf{O}_{n\times dp} & \mathsf{O}_{n\times q}\end{bmatrix} \tag{7.6}$$

$$\mathbf{C} := \begin{bmatrix}\mathsf{O}_{m\times n} & \mathsf{O}_{m\times n} & C_1 & C_2\left(\sqrt{\mathsf{F}_1}\otimes I_n\right) & C_2\left(\sqrt{\mathsf{F}_2}\otimes I_n\right) & \mathsf{O}_{m\times\varrho} & D_2\end{bmatrix} \tag{7.7}$$

$$\mathbf{B}_2 := \begin{bmatrix}\mathsf{O}_{m\times p} & \mathsf{O}_{m\times p} & B_3 & B_4\left(\sqrt{\mathsf{F}_1}\otimes I_p\right) & B_4\left(\sqrt{\mathsf{F}_2}\otimes I_p\right) & \mathsf{O}_{m\times dp} & \mathsf{O}_{m\times q}\end{bmatrix} \tag{7.8}$$

$$\begin{aligned}\boldsymbol{\chi}(t) := \mathsf{Col}\Bigg(&\begin{bmatrix}\boldsymbol{x}(t-r_1)\\ \boldsymbol{x}(t-r_2)\end{bmatrix}, \begin{bmatrix}\boldsymbol{x}(t)\\ \int_{-r_1}^{0}\left(\sqrt{\mathsf{F}_1^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\end{bmatrix}\cdots\\ &\cdots\begin{bmatrix}\int_{-r(t)}^{-r_1}\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\\ \int_{-r_2}^{-r(t)}\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\end{bmatrix}, \boldsymbol{w}(t)\Bigg)\end{aligned} \tag{7.9}$$

with F_1 and F_2 in (7.3). Note that the terms in (7.5)–(7.8) are obtained by the following relations:

$$\begin{aligned}&\int_{-r(t)}^{0} B_2(\boldsymbol{f}(\tau)\otimes I_p)K\boldsymbol{x}(t+\tau)\mathsf{d}\tau = \int_{-r(t)}^{0} B_2(I_d\boldsymbol{f}(\tau)\otimes KI_n)\boldsymbol{x}(t+\tau)\mathsf{d}\tau =\\ &\int_{-r_1}^{0} B_2(I_d\otimes K)\left(\sqrt{\mathsf{F}_1}\sqrt{\mathsf{F}_1^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau + \int_{-r(t)}^{-r_1} B_2(I_d\otimes K)\left(\sqrt{\mathsf{F}_2}\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\\ &= \int_{-r_1}^{0} B_2\left(\sqrt{\mathsf{F}_1}\otimes K\right)\left(\sqrt{\mathsf{F}_1^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau + \int_{-r(t)}^{-r_1} B_2\left(\sqrt{\mathsf{F}_2}\otimes K\right)\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\\ &= \int_{-r_1}^{0} B_2\left(\sqrt{\mathsf{F}_1}\otimes I_p\right)(I_d\otimes K)\left(\sqrt{\mathsf{F}_1^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\\ &\quad + \int_{-r(t)}^{-r_1} B_2\left(\sqrt{\mathsf{F}_2}\otimes I_p\right)(I_d\otimes K)\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\end{aligned} \tag{7.10}$$

and

$$\begin{aligned}&\int_{-r(t)}^{0} B_4(\boldsymbol{f}(\tau)\otimes I_p)K\boldsymbol{x}(t+\tau)\mathsf{d}\tau = \int_{-r(t)}^{0} B_2(I_d\boldsymbol{f}(\tau)\otimes KI_n)\boldsymbol{x}(t+\tau)\mathsf{d}\tau =\\ &\int_{-r_1}^{0} B_4(I_d\otimes K)\left(\sqrt{\mathsf{F}_1}\sqrt{\mathsf{F}_1^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau + \int_{-r(t)}^{-r_1} B_4(I_d\otimes K)\left(\sqrt{\mathsf{F}_2}\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\end{aligned}$$

$$= \int_{-r_1}^{0} B_4 \left(\sqrt{\mathsf{F}_1} \otimes K\right) \left(\sqrt{\mathsf{F}_1^{-1}} \boldsymbol{f}(\tau) \otimes I_n\right) \boldsymbol{x}(t+\tau) \mathsf{d}\tau + \int_{-r(t)}^{-r_1} B_4 \left(\sqrt{\mathsf{F}_2} \otimes K\right) \left(\sqrt{\mathsf{F}_2^{-1}} \boldsymbol{f}(\tau) \otimes I_n\right) \boldsymbol{x}(t+\tau) \mathsf{d}\tau$$

$$= \int_{-r_1}^{0} B_4 \left(\sqrt{\mathsf{F}_1} \otimes I_p\right) (I_d \otimes K) \left(\sqrt{\mathsf{F}_1^{-1}} \boldsymbol{f}(\tau) \otimes I_n\right) \boldsymbol{x}(t+\tau) \mathsf{d}\tau + \int_{-r(t)}^{-r_1} B_4 \left(\sqrt{\mathsf{F}_2} \otimes I_p\right) (I_d \otimes K) \left(\sqrt{\mathsf{F}_2^{-1}} \boldsymbol{f}(\tau) \otimes I_n\right) \boldsymbol{x}(t+\tau) \mathsf{d}\tau \quad (7.11)$$

which themselves are derived via (2.1) with the fact that F_1 and F_2 in (7.3) are invertible[1].

7.3 Important lemmas and definition

In this section, some lemmas and definition are presented which are crucial for the derivation of the synthesis condition in the next section. A novel integral inequality is also derived for the handling of time-varying delay in the context of constructing LKFs.

The following property of the commutation matrix for the Kronecker product will be used throughout this chapter.

Lemma 7.1.

$$\begin{aligned} &\forall X \in \mathbb{R}^{d\times\delta},\ \ \forall Y \in \mathbb{R}^{n\times m}\ \ \mathsf{K}_{(n,d)} (X \otimes Y) \mathsf{K}_{(\delta,m)} = Y \otimes X \\ &\forall m, n \in \mathbb{N},\quad \mathsf{K}_{(n,m)}^{-1} = \mathsf{K}_{(m,n)} = \mathsf{K}_{(n,m)}^{\top} \end{aligned} \quad (7.12)$$

where $\mathsf{K}_{(n,d)}$ is the commutation matrix defined by

$$\forall A \in \mathbb{R}^{n\times d},\ \ \mathsf{K}_{(n,d)} \,\mathbf{vec}(A) = \mathbf{vec}\left(A^{\top}\right)$$

which follows the definition in [342], where $\mathbf{vec}(\cdot)$ stands for the vectorization of a matrix. See Section 4.2 of [343] for the definition and more details of the vectorization of matrices.

Remark 7.2. Note that for the computation matrix for the Kronecker product, we have $\mathsf{K}_{(n,1)} = \mathsf{K}_{(1,n)} = I_n$, $\forall n \in \mathbb{N}$ which supports the identity

$$\mathsf{K}_{(n,d)} (\boldsymbol{f}(\tau) \otimes I_n) = \mathsf{K}_{(n,d)} (\boldsymbol{f}(\tau) \otimes I_n) K_{(1,n)} = I_n \otimes \boldsymbol{f}(\tau) \quad (7.13)$$

with $\boldsymbol{f}(\tau) \in \mathbb{R}^d$. The commutation matrix can be numerically implemented by $\mathsf{K}_{(n,d)} = \mathtt{vecperm(d,n)}$ in Matlab environment where `vecperm` is a function in the Matlab toolbox [344].

Two integral inequalities are presented as follows. The first one can be derived from the proof of Theorem 4.1 in Chapter 4, and the second inequality is specifically derived for the handling of time-varying delays in the next section. Firstly, we define the following weighted Lebesgue function space

$$\mathbb{L}^2_{\varpi}\left(\mathcal{K} \,;\, \mathbb{R}^d\right) := \left\{\boldsymbol{\phi}(\cdot) \in \mathbb{L}_f\left(\mathcal{K} \,;\, \mathbb{R}^d\right) : \|\boldsymbol{\phi}(\cdot)\|_{2,\varpi} < \infty\right\} \quad (7.14)$$

with $d \in \mathbb{N}$ and $\|\boldsymbol{\phi}(\cdot)\|_{2,\varpi} := \int_{\mathcal{K}} \varpi(\tau) \boldsymbol{\phi}^{\top}(\tau) \boldsymbol{\phi}(\tau) \mathsf{d}\tau$ where $\varpi(\cdot) \in \mathbb{L}_f(\mathcal{K} \,;\, \mathbb{R}_{\geq 0})$ and the function $\varpi(\cdot)$ has only countably infinite or finite number of zero values. Furthermore, $\mathcal{K} \subseteq \mathbb{R} \cup \{\pm\infty\}$ and the Lebesgue measure of $\mathcal{K}$ is non-zero.

Lemma 7.2. *Let $\varpi(\cdot)$ in (7.14) be given with $d \in \mathbb{N}$ and let $U \in \mathbb{S}^n_{\succeq 0}$ with $n \in \mathbb{N}$ and $\boldsymbol{f}(\cdot) \in \mathbb{L}^2_{\varpi}\left(\mathcal{K} \,;\, \mathbb{R}^d\right)$ satisfying*

$$\int_{\mathcal{K}} \varpi(\tau) \boldsymbol{f}(\tau) \boldsymbol{f}^{\top}(\tau) \mathsf{d}\tau \succ 0, \quad (7.15)$$

then we have

$$\forall \boldsymbol{x}(\cdot) \in \mathbb{L}^2_{\varpi}(\mathcal{K} \,;\, \mathbb{R}^n),\ \int_{\mathcal{K}} \varpi(\tau) \boldsymbol{x}^{\top}(\tau) U \boldsymbol{x}(\tau) \mathsf{d}\tau \geq \int_{\mathcal{K}} \varpi(\tau) \boldsymbol{x}^{\top}(\tau) F^{\top}(\tau) \mathsf{d}\tau \left(U \otimes \mathsf{F}^{-1}\right) \int_{\mathcal{K}} \varpi(\tau) F(\tau) \boldsymbol{x}(\tau) \mathsf{d}\tau \quad (7.16)$$

where $F(\tau) := I_n \otimes \boldsymbol{f}(\tau)$ and $\mathsf{F} = \int_{\mathcal{K}} \varpi(\tau) \boldsymbol{f}(\tau) \boldsymbol{f}^{\top}(\tau) \mathsf{d}\tau$.

[1] Note that $\sqrt{X^{-1}} = \left(\sqrt{X}\right)^{-1}$ for any $X \succ 0$ based on the application of eigendecomposition of $X \succ 0$

Proof. (7.16) can be easily proved by changing the order of the Kronecker product for relevant expressions in the proof of Theorem 4.1. Note that the definition of F in Lemma 7.2 is different from the definition of F in Theorem 4.1. Nevertheless, (7.16) is essentially equivalent to (4.3). ■

Lemma 7.3. *Let $\varpi(\cdot)$ in (7.14) with $d \in \mathbb{N}$ and $\mathcal{K} = [-r_2, -r_1]$ with $0 \le r_1 < r_2$ be given. Assume $U \in \mathbb{S}^n_{\succeq 0}$ with $n \in \mathbb{N}$ and $\boldsymbol{f}(\tau) := \mathbf{Col}_{i=1}^d f_i(\tau) \in \mathbb{L}^2_\varpi\left([-r_2, -r_1] ; \mathbb{R}^d\right)$ satisfying*

$$\int_{-r_2}^{-r_1} \varpi(\tau)\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau \succ 0, \tag{7.17}$$

then we have

$$\begin{aligned}\int_{-r_2}^{-r_1} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau &\ge [*]\left(\begin{bmatrix} U & Y \\ * & U \end{bmatrix} \otimes \mathsf{F}^{-1}\right)\begin{bmatrix} \int_{-\varrho}^{-r_1} (I_n \otimes \boldsymbol{f}(\tau))\, \boldsymbol{x}(\tau)\varpi(\tau)\mathsf{d}\tau \\ \int_{-r_2}^{-\varrho} (I_n \otimes \boldsymbol{f}(\tau))\, \boldsymbol{x}(\tau)\varpi(\tau)\mathsf{d}\tau \end{bmatrix} \\ &= [*]\left(\begin{bmatrix} \mathsf{K}_{(d,n)} & \mathsf{O}_{dn} \\ * & \mathsf{K}_{(d,n)} \end{bmatrix}\left(\begin{bmatrix} U & Y \\ * & U \end{bmatrix} \otimes \mathsf{F}^{-1}\right)\begin{bmatrix} \mathsf{K}_{(n,d)} & \mathsf{O}_{dn} \\ * & \mathsf{K}_{(n,d)} \end{bmatrix}\right)\begin{bmatrix} \int_{-\varrho}^{-r_1} (\boldsymbol{f}(\tau) \otimes I_n)\, \boldsymbol{x}(\tau)\varpi(\tau)\mathsf{d}\tau \\ \int_{-r_2}^{-\varrho} (\boldsymbol{f}(\tau) \otimes I_n)\, \boldsymbol{x}(\tau)\varpi(\tau)\mathsf{d}\tau \end{bmatrix}\end{aligned} \tag{7.18}$$

for all $\boldsymbol{x}(\cdot) \in \mathbb{L}^2_\varpi(\mathcal{K} ; \mathbb{R}^n)$ and for any $Y \in \mathbb{R}^{n\times n}$ satisfying $\left[\begin{smallmatrix} U & Y \\ * & U \end{smallmatrix}\right] \succeq 0$, where $\varrho \in [r_1, r_2]$ and $\mathsf{F} := \int_{-r_2}^{-r_1} \varpi(\tau)\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$.

Proof. See Appendix E for details. ■

Remark 7.3. Note that the matrix F in (7.18) is independent of ρ which can be a function of other variables. This is extremely important in deriving tractable dissipative conditions in the next section.

A stability criterion based on the LKF approach and the definition of dissipativity are presented as follows.

Lemma 7.4. *Let $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ in (7.4) and $r_2 > r_1 \ge 0$ be given, then the equilibrium point of (7.4) is uniformly globally asymptotically (exponentially)[2] stable if there exist $\epsilon_1; \epsilon_2; \epsilon_3 > 0$ and a differentiable functional $v : \mathbb{C}([-r_2, 0] ; \mathbb{R}^n) \to \mathbb{R}$ with $v(\mathbf{0}_n) = 0$ such that*

$$\epsilon_1 \|\boldsymbol{\phi}(0)\|_2^2 \le v(\boldsymbol{\phi}(\cdot)) \le \epsilon_2 \|\boldsymbol{\phi}(\cdot)\|_\infty^2, \tag{7.19}$$

$$\left.\frac{\mathsf{d}^+}{\mathsf{d}t} v(\mathbf{x}_t(\cdot))\right|_{t=t_0, \mathbf{x}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} \le -\epsilon_3 \|\boldsymbol{\phi}(0)\|_2^2 \tag{7.20}$$

for any $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r_2, 0] ; \mathbb{R}^n)$ in (7.4), where t_0 is given in (7.4) and $\|\boldsymbol{\phi}(\cdot)\|_\infty^2 := \sup_{-r_2 \le \tau \le 0} \|\boldsymbol{\phi}(\tau)\|_2^2$ and $\frac{\mathsf{d}^+}{\mathsf{d}x} f(x) := \operatorname{limsup}_{\eta \downarrow 0} \frac{f(x+\eta) - f(x)}{\eta}$. Furthermore, $\mathbf{x}_t(\cdot)$ in (7.20) is defined by $\forall t \ge t_0$, $\forall \theta \in [-r_2, 0]$, $\mathbf{x}_t(\theta) = \boldsymbol{x}(t+\theta)$ in which $\boldsymbol{x} : [t_0 - r_2, \infty) \to \mathbb{R}^n$ satisfies (7.4) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$.

Proof. Let the functions $u(\cdot), v(\cdot), w(\cdot)$ in Theorem 3 of [64] to be quadratic function with $\epsilon_1; \epsilon_2; \epsilon_3 > 0$, respectively, then Lemma 7.4 can be obtained accordingly since (7.4) is a special case of the general time-varying system considered in Theorem 3 of [64]. ■

The following definition of dissipativity is presented based on the general definition of dissipativity in [261].

Definition 7.1. Given $r_2 > r_1 \ge 0$, the closed-loop system (7.4) with a supply rate function $s(\boldsymbol{z}(t), \boldsymbol{w}(t))$ is said to be dissipative if there exists a differentiable functional $v : \mathbb{C}([-r_2, 0] ; \mathbb{R}^n) \to \mathbb{R}_{\ge 0}$ such that

$$\forall t \ge t_0, \quad \dot{v}(\mathbf{x}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \le 0, \tag{7.21}$$

with t_0 in (7.4), where $\mathbf{x}_t(\cdot)$ is defined by the equality $\forall t \ge t_0$, $\forall \theta \in [-r_2, 0]$, $\mathbf{x}_t(\theta) = \boldsymbol{x}(t+\theta)$. Moreover, $\boldsymbol{x}(t)$ and $\boldsymbol{z}(t)$ here follow the equation in (7.4) with $\boldsymbol{w}(\cdot) \in \widehat{\mathbb{L}}^2([t_0, \infty) ; \mathbb{R}^q)$.

[2] See [10] for the explanation on the equivalence bewteen uniform asymptotic and exponential stability for a linear coupled differential functional system

Note that (7.21) is an equivalent condition of the original definition of dissipativity, given $v : \mathbb{C}([-r_2, 0] ; \mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ is differentiable. Furthermore, the following quadratic supply function

$$s(\boldsymbol{z}(t), \boldsymbol{w}(t)) = \begin{bmatrix} \boldsymbol{z}(t) \\ \boldsymbol{w}(t) \end{bmatrix}^\top \mathbf{J} \begin{bmatrix} \boldsymbol{z}(t) \\ \boldsymbol{w}(t) \end{bmatrix}, \ \mathbf{J} = \begin{bmatrix} \widetilde{J}^\top J_1^{-1} \widetilde{J} & J_2 \\ * & J_3 \end{bmatrix} \in \mathbb{S}^{m+q}, \quad \widetilde{J}^\top J_1^{-1} \widetilde{J} \preceq 0, \ J_1^{-1} \prec 0, \ J_3 \succeq 0 \quad (7.22)$$

is considered in this chapter to characterize dissipativity.

7.4 Dissipative controller synthesis

The results on dissipative controller synthesis are presented in this section summarized in Theorem 7.1, 7.2 and Algorithm 4.

Theorem 7.1. *Let $r_2 > r_1 > 0$ and $\boldsymbol{f}(\cdot)$, M in Assumption 7.1 be given, then the closed-loop system* (7.4) *with the supply rate function in* (7.22) *is dissipative and the origin of* (7.4) *with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is globally uniformly asymptotically stable if there exists $K \in \mathbb{R}^{p \times n}$ and $P_1 \in \mathbb{S}^n$, $P_2 \in \mathbb{R}^{n \times 2\varrho}$, $P_3 \in \mathbb{S}^{2\varrho}$ with $\rho = dn$ and $Q_1; Q_2; R_1; R_2 \in \mathbb{S}^n$ and $Y \in \mathbb{R}^{n \times n}$ such that*

$$\begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} + \left(\mathsf{O}_n \oplus \left[I_d \otimes Q_1 \right] \oplus \left[I_d \otimes Q_2 \right] \right) \succ 0, \quad (7.23)$$

$$Q_1 \succeq 0, \ Q_2 \succeq 0, \ R_1 \succeq 0, \ \begin{bmatrix} R_2 & Y \\ * & R_2 \end{bmatrix} \succeq 0, \quad (7.24)$$

$$\begin{bmatrix} \boldsymbol{\Psi} & \boldsymbol{\Sigma}^\top \widetilde{J}^\top \\ * & J_1 \end{bmatrix} = \mathbf{Sy}\left[\mathbf{P}^\top \boldsymbol{\Pi} \right] + \boldsymbol{\Phi} \prec 0 \quad (7.25)$$

where $\boldsymbol{\Sigma} = \mathbf{C} + \mathbf{B}_2 \left[(I_{3+3d} \otimes K) \oplus \mathsf{O}_q \right]$ with $\mathbf{C}$ and $\mathbf{B}_2$ in (7.7) *and* (7.8), *and*

$$\boldsymbol{\Psi} = \mathbf{Sy} \left(\begin{bmatrix} \mathsf{O}_{2n \times n} & \mathsf{O}_{2n \times 2\varrho} \\ I_n & \mathsf{O}_{n \times 2\varrho} \\ \mathsf{O}_{3\varrho \times n} & \widehat{I}^\top \\ \mathsf{O}_{q \times n} & \mathsf{O}_{q \times 2\varrho} \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} \begin{bmatrix} \mathbf{A} + \mathbf{B}_1 \left[(I_{3+3d} \otimes K) \oplus \mathsf{O}_q \right] \\ \mathbf{F} \end{bmatrix} - \begin{bmatrix} \mathsf{O}_{(3n+3\varrho) \times m} \\ J_2^\top \end{bmatrix} \boldsymbol{\Sigma} \right)$$
$$- \Bigg([Q_1 - Q_2 - r_3 R_2] \oplus Q_2 \oplus (-Q_1 - r_1 R_1) \oplus (I_d \otimes R_1)$$
$$\oplus \left(\begin{bmatrix} \mathsf{K}_{(d,n)} & \mathsf{O}_{dn} \\ * & \mathsf{K}_{(d,n)} \end{bmatrix} \left(\begin{bmatrix} R_2 & Y \\ * & R_2 \end{bmatrix} \otimes I_d \right) \begin{bmatrix} \mathsf{K}_{(n,d)} & \mathsf{O}_{dn} \\ * & \mathsf{K}_{(n,d)} \end{bmatrix} \right) \oplus J_3 \Bigg) \quad (7.26)$$

with $\mathbf{A}$ and $\mathbf{B}_1$ in (7.5) *and* (7.6), *and* $\widehat{I} = \begin{bmatrix} I_\varrho & \mathsf{O}_\varrho & \mathsf{O}_\varrho \\ \mathsf{O}_\varrho & I_\varrho & I_\varrho \end{bmatrix}$ *and* $\mathbf{F} = \begin{bmatrix} \widehat{\mathbf{F}} \otimes I_n & \mathsf{O}_{2\varrho \times q} \end{bmatrix}$ *with*

$$\widehat{\mathbf{F}} = \begin{bmatrix} -\sqrt{\mathsf{F}_1^{-1}} \boldsymbol{f}(-r_1) & \mathbf{0}_d & \sqrt{\mathsf{F}_1^{-1}} \boldsymbol{f}(0) & -\sqrt{\mathsf{F}_1^{-1}} M \sqrt{\mathsf{F}_1} & \mathsf{O}_d & \mathsf{O}_d \\ \sqrt{\mathsf{F}_2^{-1}} \boldsymbol{f}(-r_1) & -\sqrt{\mathsf{F}_2^{-1}} \boldsymbol{f}(-r_2) & \mathbf{0}_d & \mathsf{O}_d & -\sqrt{\mathsf{F}_2^{-1}} M \sqrt{\mathsf{F}_2} & -\sqrt{\mathsf{F}_2^{-1}} M \sqrt{\mathsf{F}_2} \end{bmatrix} \quad (7.27)$$

with F_1, F_2 in (7.3). *Moreover, the rest of the parameters in* (7.25) *is defined as*

$$\mathbf{P} := \begin{bmatrix} \mathsf{O}_{n \times 2n} & P_1 & P_2 \widehat{I} & \mathsf{O}_{n \times q} & \mathsf{O}_{n \times m} \end{bmatrix} \qquad \boldsymbol{\Pi} := \begin{bmatrix} \mathbf{A} + \mathbf{B}_1 \left[(I_{3+3d} \otimes K) \oplus \mathsf{O}_q \right] & \mathsf{O}_{n \times m} \end{bmatrix} \quad (7.28)$$

and

$$\boldsymbol{\Phi} := \mathbf{Sy} \left(\begin{bmatrix} \mathsf{O}_{2n \times 2\varrho} \\ P_2 \\ \widehat{I}^\top P_3 \\ \mathsf{O}_{(q+m) \times 2\varrho} \end{bmatrix} \begin{bmatrix} \mathbf{F} & \mathsf{O}_{2\varrho \times m} \end{bmatrix} + \begin{bmatrix} \mathsf{O}_{(3n+3\varrho) \times m} \\ -J_2^\top \\ \widetilde{J} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma} & \mathsf{O}_m \end{bmatrix} \right) - \Bigg([Q_1 - Q_2 - r_3 R_2] \oplus Q_2$$

$$\oplus(-Q_1 - r_1R_1)\oplus\left[I_d\otimes R_1\right]\oplus\left([*]\left(\begin{bmatrix}R_2 & Y\\ * & R_2\end{bmatrix}\otimes I_d\right)\begin{bmatrix}\mathsf{K}_{(n,d)} & \mathsf{O}_{dn}\\ * & \mathsf{K}_{(n,d)}\end{bmatrix}\right)\oplus J_3\oplus(-J_1)\bigg). \tag{7.29}$$

Proof. The proof of Theorem 7.1 is based on the construction of the LKF:

$$\begin{aligned} v(\mathbf{x}_t(\cdot)) &= \boldsymbol{\eta}^\top(t)\begin{bmatrix}P_1 & P_2\\ * & P_3\end{bmatrix}\boldsymbol{\eta}(t) + \int_{-r_1}^{0}\boldsymbol{x}^\top(t+\tau)\left[Q_1 + r_1(\tau+r_1)R_1\right]\boldsymbol{x}(t+\tau)\mathsf{d}\tau \\ &+ \int_{-r_2}^{-r_1}\boldsymbol{x}^\top(t+\tau)\left[Q_2 + r_3(\tau+r_2)R_2\right]\boldsymbol{x}(t+\tau)\mathsf{d}\tau, \end{aligned} \tag{7.30}$$

where $\mathbf{x}_t(\cdot)$ follows the same definition in (7.21) and $P_1\in\mathbb{S}^n$, $P_2\in\mathbb{R}^{n\times 2\varrho}$, $P_3\in\mathbb{S}^{2\varrho}$ and $Q_1;Q_2;R_1;R_2\in\mathbb{S}^n$ and

$$\boldsymbol{\eta}(t) := \mathbf{Col}\left[\boldsymbol{x}(t),\ \int_{-r_1}^{0}\left(\sqrt{\mathsf{F}_1^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau,\ \int_{-r_2}^{-r_1}\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\right] \tag{7.31}$$

with F_1 and F_2 in (7.3).

Given $t_0\in\mathbb{R}$ in (7.4), differentiate $v(\mathbf{x}_t(\cdot))$ along the trajectory of (7.4) and consider (7.22), it produces

$$\begin{aligned} &\forall t\geq t_0,\ \ \dot{v}(\mathbf{x}_t(\cdot)) - s(\boldsymbol{z}(t),\boldsymbol{w}(t)) \\ &= \boldsymbol{\chi}^\top(t)\,\mathbf{Sy}\left(\begin{bmatrix}\mathsf{O}_{2n\times n} & \mathsf{O}_{2n\times 2\varrho}\\ I_n & \mathsf{O}_{n\times 2\varrho}\\ \mathsf{O}_{3\varrho\times n} & \widehat{I}^\top\\ \mathsf{O}_{q\times n} & \mathsf{O}_{q\times 2\varrho}\end{bmatrix}\begin{bmatrix}P_1 & P_2\\ * & P_3\end{bmatrix}\begin{bmatrix}\mathbf{A}\\ \mathbf{F}\end{bmatrix} - \begin{bmatrix}\mathsf{O}_{(3n+3\varrho)\times m}\\ J_2^\top\end{bmatrix}\boldsymbol{\Sigma}\right)\boldsymbol{\chi}(t) \\ &+ \boldsymbol{x}^\top(t)\left(Q_1 + r_1R_1\right)\boldsymbol{x}(t) - [*]\left(Q_1 - Q_2 - r_3R_2\right)\boldsymbol{x}(t-r_1) - [*]Q_2\boldsymbol{x}(t-r_2) - \boldsymbol{w}^\top(t)J_3\boldsymbol{w}(t) \\ &- \int_{-r_1}^{0}\boldsymbol{x}^\top(t+\tau)R_1\boldsymbol{x}(t+\tau)\mathsf{d}\tau - \int_{-r_2}^{-r_1}\boldsymbol{x}^\top(t+\tau)R_2\boldsymbol{x}(t+\tau)\mathsf{d}\tau - \boldsymbol{\chi}^\top(t)\boldsymbol{\Sigma}^\top\widetilde{J}^\top J_1^{-1}\widetilde{J}\boldsymbol{\Sigma}\boldsymbol{\chi}(t) \end{aligned} \tag{7.32}$$

where $\boldsymbol{\chi}(t)$ is given in (7.9) and $\boldsymbol{\Sigma}$, $\widehat{I}$ and $\mathbf{F}$ are defined in the statements of Theorem 7.1. Note that $\widehat{\mathbf{F}}$ in (7.27) is obtained by the relations

$$\begin{aligned} \int_{-r_1}^{0}\left(\sqrt{\mathsf{F}_1}\boldsymbol{f}(\tau)\otimes I_n\right)\dot{\boldsymbol{x}}(t+\tau)\mathsf{d}\tau &= \left(\sqrt{\mathsf{F}_1}\boldsymbol{f}(0)\otimes I_n\right)\boldsymbol{x}(t) - \left(\sqrt{\mathsf{F}_1}\boldsymbol{f}(-r_1)\otimes I_n\right)\boldsymbol{x}(t-r_1) \\ &- \left(\sqrt{\mathsf{F}_1}M\sqrt{\mathsf{F}_1^{-1}}\otimes I_n\right)\int_{-r_1}^{0}\left(\sqrt{\mathsf{F}_1}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau \end{aligned} \tag{7.33}$$

$$\begin{aligned} \int_{-r_2}^{-r_1}\left(\sqrt{\mathsf{F}_2}\boldsymbol{f}(\tau)\otimes I_n\right)\dot{\boldsymbol{x}}(t+\tau)\mathsf{d}\tau &= \left(\sqrt{\mathsf{F}_2}\boldsymbol{f}(-r_1)\otimes I_n\right)\boldsymbol{x}(t-r_1) - \left(\sqrt{\mathsf{F}_2}\boldsymbol{f}(-r_2)\otimes I_n\right)\boldsymbol{x}(t-r_2) \\ &- \left(\sqrt{\mathsf{F}_2}M\sqrt{\mathsf{F}_2^{-1}}\otimes I_n\right)\int_{-r(t)}^{-r_1}\left(\sqrt{\mathsf{F}_2}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau \\ &- \left(\sqrt{\mathsf{F}_2}M\sqrt{\mathsf{F}_2^{-1}}\otimes I_n\right)\int_{-r_2}^{-r(t)}\left(\sqrt{\mathsf{F}_2}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau \end{aligned} \tag{7.34}$$

which are derived via the application of (2.1) and (2.2). Note that also the parameters $\mathbf{A}$,$\mathbf{B}_1$,$\mathbf{C}$ and $\mathbf{B}_2$ in (7.32) are defined in (7.5)–(7.8).

Now let $R_1\succeq 0$ and $\left[\begin{smallmatrix}R_2 & Y\\ * & R_2\end{smallmatrix}\right]\succeq 0$ with $Y\in\mathbb{R}^{n\times n}$, and apply (7.16) and (7.18) to the integral terms $\int_{-r_1}^{0}\boldsymbol{x}^\top(t+\tau)R_1\boldsymbol{x}(t+\tau)\mathsf{d}\tau$ and $\int_{-r_2}^{-r_1}\boldsymbol{x}^\top(t+\tau)R_2\boldsymbol{x}(t+\tau)\mathsf{d}\tau$ in (7.32), respectively. Then we have

$$\begin{aligned} &\int_{-r_1}^{0}\boldsymbol{x}^\top(t+\tau)R_1\boldsymbol{x}(t+\tau)\mathsf{d}\tau \geq [*]\left(I_d\otimes R_1\right)\left[\int_{-r_1}^{0}\left(\sqrt{\mathsf{F}_1^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\right], \\ &\int_{-r_2}^{-r_1}\boldsymbol{x}^\top(t+\tau)R_2\boldsymbol{x}(t+\tau)\mathsf{d}\tau \geq [*]\left(\begin{bmatrix}R_2 & Y\\ * & R_2\end{bmatrix}\otimes I_d\right)\begin{bmatrix}\int_{-r(t)}^{-r_1}\left(I_n\otimes\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau)\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\\ \int_{-r_2}^{-r(t)}\left(I_n\otimes\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau)\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\end{bmatrix} \\ &= [*]\left([*]\left(\begin{bmatrix}R_2 & Y\\ * & R_2\end{bmatrix}\otimes I_d\right)\begin{bmatrix}\mathsf{K}_{(n,d)} & \mathsf{O}_{dn}\\ * & \mathsf{K}_{(n,d)}\end{bmatrix}\right)\begin{bmatrix}\int_{-r(t)}^{-r_1}\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\\ \int_{-r_2}^{-r(t)}\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\end{bmatrix} \end{aligned} \tag{7.35}$$

Applying (7.35) with (7.24) to (7.32) produces

$$\forall t \geq t_0,\ \ \dot{v}(\mathbf{x}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \leq \boldsymbol{\chi}^\top(t)\left(\boldsymbol{\Psi} - \boldsymbol{\Sigma}^\top \widetilde{J}^\top J_1^{-1}\widetilde{J}\boldsymbol{\Sigma}\right)\boldsymbol{\chi}(t), \tag{7.36}$$

where $\boldsymbol{\Psi}$ is given in (7.25) and $\boldsymbol{\chi}(t)$ is given in (7.9). It is obvious to conclude that given (7.24) with $\boldsymbol{\Psi} - \boldsymbol{\Sigma}^\top \widetilde{J}^\top J_1^{-1}\widetilde{J}\boldsymbol{\Sigma} \prec 0$, we have

$$\exists \epsilon_3 > 0 : \forall t \geq t_0,\ \ \dot{v}(\mathbf{x}_t(\cdot)) - s(\boldsymbol{z}(t), \boldsymbol{w}(t)) \leq -\epsilon_3 \left\|\boldsymbol{x}(t)\right\|_2. \tag{7.37}$$

Moreover, by (7.37) with $t = t_0$ and $\boldsymbol{w}(t) \equiv \mathbf{0}_q$, we have

$$\exists \epsilon_3 > 0,\ \left.\frac{\mathsf{d}^+}{\mathsf{d}t} v(\mathbf{x}_t(\cdot))\right|_{t=t_0, \mathbf{x}_{t_0}(\cdot)=\boldsymbol{\phi}(\cdot)} \leq -\epsilon_3 \left\|\boldsymbol{\phi}(0)\right\|_2 \tag{7.38}$$

for any $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r_2, 0] ⨾ \mathbb{R}^n)$ in (7.4). Note that $\mathbf{x}_t(\cdot)$ in (7.38) is in line with the definition in (7.20). As a result, it is obvious that (7.24) and $\boldsymbol{\Psi} - \boldsymbol{\Sigma}^\top \widetilde{J}^\top J_1^{-1}\widetilde{J}\boldsymbol{\Sigma} \prec 0$ infer that (7.30) satisfies the dissipative condition in (7.21) and there exists $\epsilon_3 > 0$ such that (7.30) satisfies (7.20). Finally, applying the Schur complement to $\boldsymbol{\Psi} - \boldsymbol{\Sigma}^\top \widetilde{J}^\top J_1^{-1}\widetilde{J}\boldsymbol{\Sigma} \prec 0$ with (7.24) and $J_1^{-1} \prec 0$ gives (7.25). Therefore we have proved that the existence of the feasible solutions of (7.24) and (7.25) infer the existence of a functional (7.30) and $\epsilon_3 > 0$ satisfying (7.21) and (7.20).

Now we start to prove that (7.23) and (7.24) infer that there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that (7.30) satisfies (7.19). Let $\left\|\boldsymbol{\phi}(\cdot)\right\|_\infty^2 := \sup_{-r_2 \leq \tau \leq 0} \left\|\boldsymbol{\phi}(\tau)\right\|_2^2$ and consider the structure of (7.30) with $t = t_0$, it follows that $\exists \lambda > 0$:

$$\begin{aligned}
&v(\mathbf{x}_{t_0}(\cdot)) = v(\boldsymbol{\phi}(\cdot)) \leq \boldsymbol{\eta}^\top(t_0)\lambda\boldsymbol{\eta}(t_0) + \int_{-r_2}^{0} \boldsymbol{\phi}^\top(\tau)\lambda\boldsymbol{\phi}(\tau)\mathsf{d}\tau \leq \lambda\left\|\boldsymbol{\phi}(0)\right\|_2^2 + \lambda r_2 \left\|\boldsymbol{\phi}(\cdot)\right\|_\infty^2 \\
&+ \int_{-r_1}^{0} \boldsymbol{\phi}^\top(\tau)\left(\sqrt{\mathsf{F}_1^{-1}}\boldsymbol{f}(\tau) \otimes I_n\right)^\top \mathsf{d}\tau\lambda \int_{-r_1}^{0}\left(\sqrt{\mathsf{F}_1^{-1}}\boldsymbol{f}(\tau) \otimes I_n\right)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \\
&+ \int_{-r_2}^{-r_1} \boldsymbol{\phi}^\top(\tau)\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau) \otimes I_n\right)^\top \mathsf{d}\tau\lambda \int_{-r_2}^{-r_1}\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau) \otimes I_n\right)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \\
&\leq (\lambda + \lambda r_2)\left\|\boldsymbol{\phi}(\cdot)\right\|_\infty^2 + [*](\lambda I_d \otimes I_n)\left(\int_{-r_1}^{0}\left(\sqrt{\mathsf{F}_1^{-1}}\boldsymbol{f}(\tau) \otimes I_n\right)\boldsymbol{\phi}(\tau)\mathsf{d}\tau\right) \\
&+ \left[\int_{-r_2}^{-r_1}\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau) \otimes I_n\right)\boldsymbol{\phi}(\tau)\mathsf{d}\tau\right]^\top (\lambda I_d \otimes I_n)\int_{-r_2}^{-r_1}\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau) \otimes I_n\right)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \\
&\leq (\lambda + \lambda r_2)\left\|\boldsymbol{\phi}(\cdot)\right\|_\infty^2 + \lambda\int_{-r_2}^{0} \boldsymbol{\phi}^\top(\tau)\boldsymbol{\phi}(\tau)\mathsf{d}\tau \leq (\lambda + 2\lambda r_2)\left\|\boldsymbol{\phi}(\cdot)\right\|_\infty^2
\end{aligned} \tag{7.39}$$

for any $\boldsymbol{\phi}(\cdot) \in \mathbb{C}([-r_2, 0] ⨾ \mathbb{R}^n)$ in (7.4), where (7.39) is derived via the property of quadratic forms: $\forall X \in \mathbb{S}^n, \exists \lambda > 0 : \forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbf{x}^\top(\lambda I_n - X)\mathbf{x} > 0$ together with (7.16). Then (7.39) shows that it is possible to find an upper bound for (7.30) which satisfies (7.19) with a $\epsilon_2 > 0$.

Now we want to construct a lower bound for $v(\mathbf{x}_t(\cdot))$ to formulate LMI conditions inferring that (7.30) satisfies (7.19) with certain $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Apply (7.16) to (7.30) at $t = t_0$. Then one can obtain the inequalities

$$\begin{aligned}
&\int_{-r_1}^{0} \boldsymbol{\phi}^\top(\tau)Q_1\boldsymbol{\phi}(\tau)\mathsf{d}\tau \geq [*]\left(I_d \otimes Q_1\right)\int_{-r_1}^{0}\left(\sqrt{\mathsf{F}_1^{-1}}\boldsymbol{f}(\tau) \otimes I_n\right)\boldsymbol{\phi}(\tau)\mathsf{d}\tau, \\
&\int_{-r_2}^{-r_1} \boldsymbol{\phi}^\top(\tau)Q_2\boldsymbol{\phi}(\tau)\mathsf{d}\tau \geq [*]\left(I_d \otimes Q_2\right)\int_{-r_2}^{-r_1}\left(\sqrt{\mathsf{F}_2^{-1}}\boldsymbol{f}(\tau) \otimes I_n\right)\boldsymbol{\phi}(\tau)\mathsf{d}\tau,
\end{aligned} \tag{7.40}$$

provided that (7.24) holds.

By using (7.40) with (7.24) to (7.30) at $t = t_0$, it is clearly to see that (7.23) and (7.24) infer that (7.30) satisfies (7.19) with some $\epsilon_1 > 0$ and $\epsilon_2 > 0$. This demonstrates that the existence of the feasible solutions of (7.23)–(7.25) infers the existence of a functional (7.30) and $\epsilon_1; \epsilon_2 > 0$ satisfying the dissipative condition

(7.21), and the stability criteria in (7.19) and (7.20). As a result, it shows that the existence of the feasible solutions of (7.23)–(7.25) infers that the trivial solution $\boldsymbol{x}(t) \equiv \mathbf{0}_n$ of the closed-loop system (7.4) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is globally uniformly asymptotically stable, and (7.4) with (7.22) is dissipative. This finishes the proof of this theorem. ■

Remark 7.4. By analyzing the derivation of Theorem 7.1, the significance of the application of (7.18) can be easily grasped. Indeed, consider the situation when (7.16) is applied to $\int_{-r(t)}^{-r_1} \boldsymbol{x}^\top(t+\tau)Q_2\boldsymbol{x}(t+\tau)\mathsf{d}\tau$ and $\int_{-r_2}^{-r(t)} \boldsymbol{x}^\top(t+\tau)Q_2\boldsymbol{x}(t+\tau)\mathsf{d}\tau$ which gives the inequalities

$$\begin{aligned} \int_{-r(t)}^{-r_1} \boldsymbol{x}^\top(t+\tau)Q_2\boldsymbol{x}(t+\tau)\mathsf{d}\tau &\geq [*]\left(\mathsf{F}_1^{-1}(r(t))\otimes Q_2\right)\left[\int_{-r(t))}^{-r_1} \left(\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\right] \\ \int_{-r_2}^{-r(t)} \boldsymbol{x}^\top(t+\tau)Q_2\boldsymbol{x}(t+\tau)\mathsf{d}\tau &\geq [*]\left(\mathsf{F}_2^{-1}(r(t))\otimes Q_2\right)\left[\int_{-r(t))}^{-r(t)} \left(\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau\right] \end{aligned} \tag{7.41}$$

where $\mathsf{F}_1(r(t)) = \int_{-r(t)}^{-r_1} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$ and $\mathsf{F}_2(r(t)) = \int_{-r_2}^{-r(t)} \boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$. Now combine the inequalities in (7.41), we have

$$\begin{aligned} \int_{-r_2}^{-r_1} \boldsymbol{x}^\top(t+\tau)Q_2\boldsymbol{x}(t+\tau)\mathsf{d}\tau \geq &\begin{bmatrix} \int_{-r(t)}^{-r_1} \left(\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau \\ \int_{-r_2}^{-r(t)} \left(\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau \end{bmatrix}^\top \times \\ &\begin{bmatrix} \mathsf{F}_1^{-1}(r(t))\otimes Q_2 & \mathsf{O}_n \\ \mathsf{O}_n & \mathsf{F}_2^{-1}(r(t))\otimes Q_2 \end{bmatrix}\begin{bmatrix} \int_{-r(t)}^{-r_1} \left(\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau \\ \int_{-r_2}^{-r(t)} \left(\boldsymbol{f}(\tau)\otimes I_n\right)\boldsymbol{x}(t+\tau)\mathsf{d}\tau \end{bmatrix} \end{aligned} \tag{7.42}$$

which also provides a lower bound for $\int_{-r_2}^{-r_1} \boldsymbol{x}^\top(t+\tau)Q_2\boldsymbol{x}(t+\tau)\mathsf{d}\tau$. Conventionally, the reciprocally convex combination lemma or its derivatives [324, 345–347] can be applied to a matrix in the form of $\left[\begin{smallmatrix} \frac{1}{1-\alpha}X & \mathsf{O}_n \\ \mathsf{O}_n & \frac{1}{\alpha}X \end{smallmatrix}\right]$ to construct a tractable lower bound in the context of semidefinite programmings. However, the structure of $\left[\begin{smallmatrix} \frac{1}{1-\alpha}X & \mathsf{O}_n \\ \mathsf{O}_n & \frac{1}{\alpha}X \end{smallmatrix}\right]$ may not be guaranteed by

$$\begin{bmatrix} \mathsf{F}_1^{-1}(r(t))\otimes Q_2 & \mathsf{O}_n \\ \mathsf{O}_n & \mathsf{F}_2^{-1}(r(t))\otimes Q_2 \end{bmatrix} \tag{7.43}$$

since the terms of $\mathsf{F}_1^{-1}(r(t))$ and $\mathsf{F}_2^{-1}(r(t))$ might be nonlinear in general.[3]. On the other hand, if (7.42) is applied directly to replace the step of (7.35), then the corresponding resulting (7.25) will become infinite dimensional and nonlinear in general with respect to $r(t) \in [r_1, r_2]$. This shows the significance of (7.18) by which one can derive (7.35) where the matrix parameters are of finite dimensional.

Remark 7.5. In Theorem 7.1, it is assumed $r_2 > r_1 > 0$ which implies that there is no obvious redundancy in the matrix parameters of (7.30) and no redundant zero vectors in (7.31) and (7.9). With $r_1 = 0$, the functional in (7.30) can be simplified into having decision variables with a simpler structure, by which one can derive a synthesis condition for the case of $r_2 > 0$, $r_1 = 0$. Therefore, our proposed method in Theorem 7.1 can handle the situation of $r_2 > 0$, $r_1 = 0$ with certain modifications. Note we do not present the synthesis condition corresponding to $r_2 > 0$, $r_1 = 0$ in this chapter given the fact that the condition can be easily derived based on the proof of Theorem 7.1. Finally, we emphasize here that Theorem 7.1 is specifically derived to handle the stabilization problem for (7.1) with a genuine time-vary delay. If $r_2 = r_1$, then (7.1) becomes a system with a constant delay where the corresponding synthesis problem should be handled by the proposed methods in Chapter 2.

[3]If $\boldsymbol{f}(\cdot)$ contains only Legendre polynomials with appropriate structures, then the reciprocally convex combination lemma or its derivatives can be applied to (7.43). Nevertheless, this is a special case of the $\boldsymbol{f}(\cdot)$ we considered in this chapter.

The inequality in (7.25) is nonconvex if a genuine synthesis problem is concerned, thus it cannot be solved directly via standard semidefinite programming solvers. In the following theorem, a synthesis condition denoted via convex matrix inequalities, which requires certain given parameters,[4] is presented whose feasible solutions ensure that the trivial solution $\boldsymbol{x}(t) \equiv \mathbf{0}_n$ of (7.4) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is globally uniformly asymptotically stable and (7.4) with (7.22) is dissipative.

Theorem 7.2. *Given $r_2 > r_1 > 0$ and $\boldsymbol{f}(\cdot)$, M in Assumption 7.1 and $\{\alpha_i\}_{i=1}^{3+3d} \subset \mathbb{R}$, then the closed-loop system (7.4) with the supply rate function in (7.22) is dissipative and the trivial solution $\boldsymbol{x}(t) \equiv \mathbf{0}_n$ of (7.4) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is globally uniformly asymptotically stable if there exists $\acute{P}_1 \in \mathbb{S}^n$, $\acute{P}_2 \in \mathbb{R}^{n\times 2\varrho}$, $\acute{P}_3 \in \mathbb{S}^{2\varrho}$ and $\acute{Q}_1; \acute{Q}_2; \acute{R}_1; \acute{R}_2 \in \mathbb{S}^n$ and $X, \acute{Y} \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{p\times n}$ such that*

$$\begin{bmatrix} \acute{P}_1 & \acute{P}_2 \\ * & \acute{P}_3 \end{bmatrix} + \left(\mathsf{O}_n \oplus \left[I_d \otimes \acute{Q}_1\right] \oplus \left[I_d \otimes \acute{Q}_2\right]\right) \succ 0, \tag{7.44}$$

$$\acute{Q}_1 \succeq 0,\ \acute{Q}_2 \succeq 0,\ \acute{R}_1 \succeq 0,\ \begin{bmatrix} \acute{R}_2 & \acute{Y} \\ * & \acute{R}_2 \end{bmatrix} \succeq 0, \tag{7.45}$$

$$\mathsf{Sy}\left(\begin{bmatrix} I_n \\ \mathsf{Col}_{i=1}^{3+3d}\, \alpha_i I_n \\ \mathsf{O}_{(q+m)\times n} \end{bmatrix} \begin{bmatrix} -X & \acute{\mathbf{\Pi}} \end{bmatrix}\right) + \begin{bmatrix} \mathsf{O}_n & \acute{\mathbf{P}} \\ * & \acute{\mathbf{\Phi}} \end{bmatrix} \prec 0 \tag{7.46}$$

where

$$\acute{\mathbf{P}} = \begin{bmatrix} \mathsf{O}_{n\times 2n} & \acute{P}_1 & \acute{P}_2 \widehat{I} & \mathsf{O}_{n\times q} & \mathsf{O}_{n\times m} \end{bmatrix},\ \widehat{I} = \begin{bmatrix} I_\varrho & \mathsf{O}_\varrho & \mathsf{O}_\varrho \\ \mathsf{O}_\varrho & I_\varrho & I_\varrho \end{bmatrix} \tag{7.47}$$

$$\acute{\mathbf{\Pi}} = \begin{bmatrix} \mathbf{A}\left[(I_{3+3d} \otimes X) \oplus I_q\right] + \mathbf{B}_1\left[(I_{3+3d} \otimes V) \oplus \mathsf{O}_q\right] & \mathsf{O}_{n\times m} \end{bmatrix} \tag{7.48}$$

$$\begin{aligned}\acute{\mathbf{\Phi}} = \mathsf{Sy}&\left(\begin{bmatrix} \mathsf{O}_{2n\times 2\varrho} \\ \acute{P}_2 \\ \widehat{I}^\top \acute{P}_3 \\ \mathsf{O}_{(q+m)\times 2\varrho} \end{bmatrix} \begin{bmatrix} \mathbf{F} & \mathsf{O}_{2\varrho\times m} \end{bmatrix} + \begin{bmatrix} \mathsf{O}_{(3n+3\varrho)\times m} \\ -J_2^\top \\ \widetilde{J} \end{bmatrix} \begin{bmatrix} \acute{\mathbf{\Sigma}} & \mathsf{O}_m \end{bmatrix}\right) - \left(\left[\acute{Q}_1 - \acute{Q}_2 - r_3 \acute{R}_2\right] \oplus \acute{Q}_2\right. \\ &\left.\oplus (-\acute{Q}_1 - r_1 \acute{R}_1) \oplus \left[I_d \otimes \acute{R}_1\right] \oplus \left([*]\left(\begin{bmatrix} \acute{R}_2 & \acute{Y} \\ * & \acute{R}_2 \end{bmatrix} \otimes I_d\right) \begin{bmatrix} \mathsf{K}_{(n,d)} & \mathsf{O}_{dn} \\ * & \mathsf{K}_{(n,d)} \end{bmatrix}\right) \oplus J_3 \oplus (-J_1)\right)\end{aligned} \tag{7.49}$$

with $\acute{\mathbf{\Sigma}} = \mathbf{C}\left[(I_{3+3d} \otimes X) \oplus I_q\right] + \mathbf{B}_2\left[(I_{3+3d} \otimes V) \oplus \mathsf{O}_q\right]$ and $\mathbf{A}$,$\mathbf{B}_1$,$\mathbf{B}_2$,$\mathbf{C}$ are given in (7.5)–(7.8). Finally, the controller gain is calculated via $K = VX^{-1}$.

Proof. First of all, note that the inequality $\mathsf{Sy}\left(\mathbf{P}^\top \mathbf{\Pi}\right) + \mathbf{\Phi} \prec 0$ in (7.25) can be reformulated into

$$\mathsf{Sy}\left(\mathbf{P}^\top \mathbf{\Pi}\right) + \mathbf{\Phi} = \begin{bmatrix} \mathbf{\Pi}^\top & I_{3n+3\varrho+q+m} \end{bmatrix} \begin{bmatrix} \mathsf{O}_n & \mathbf{P} \\ * & \mathbf{\Phi} \end{bmatrix} \begin{bmatrix} \mathbf{\Pi} \\ I_{3n+3\varrho+q+m} \end{bmatrix} \prec 0. \tag{7.50}$$

It is easy to observe that the structure of (7.50) is similar to one of the inequalities in (2.27) as part of the statements of Projection Lemma. Given the fact that two matrix inequalities are presented in (2.27), thus a new matrix inequality must be constructed accordingly. Now consider the following inequality

$$\Upsilon^\top \begin{bmatrix} \mathsf{O}_n & \mathbf{P} \\ * & \mathbf{\Phi} \end{bmatrix} \Upsilon \prec 0 \tag{7.51}$$

with $\Upsilon^\top := \begin{bmatrix} \mathsf{O}_{(q+m)\times(4n+3\varrho)} & I_{q+m} \end{bmatrix}$, which can be further simplified into

$$\Upsilon^\top \begin{bmatrix} \mathsf{O}_n & \mathbf{P} \\ * & \mathbf{\Phi} \end{bmatrix} \Upsilon = \begin{bmatrix} -J_3 - \mathsf{Sy}(D_2^\top J_2) & D_2^\top \widetilde{J} \\ * & J_1 \end{bmatrix} \prec 0. \tag{7.52}$$

[4]It is illustrated later in Remark 7.8 that it possible to only adjust the value of one parameter with others parameter assigned to be zeros.

Note that (7.52) is the very matrix produced by extracting the 2×2 block matrix at the right bottom of the matrices $\mathbf{Sy}\left(\mathbf{P}^\top\mathbf{\Pi}\right)+\mathbf{\Phi}$ or $\mathbf{\Phi}$. As a result, it is clear that (7.52) is automatically satisfied if (7.50) holds thus the constructed inequality (7.52) introduces no additional conservatism to the original inequality $\mathbf{Sy}\left(\mathbf{P}^\top\mathbf{\Pi}\right)+\mathbf{\Phi}\prec 0$. On the other hand, one can conclude that

$$\begin{gathered}\begin{bmatrix}-I_n & \mathbf{\Pi}\end{bmatrix}\begin{bmatrix}\mathbf{\Pi}\\ I_{3n+3\varrho+q+m}\end{bmatrix}=\mathsf{O}_{n\times(3n+3\varrho+q+m)},\quad \begin{bmatrix}-I_n & \mathbf{\Pi}\end{bmatrix}_\perp=\begin{bmatrix}\mathbf{\Pi}\\ I_{3n+3\varrho+q+m}\end{bmatrix}\\ \begin{bmatrix}I_{4n+3\varrho} & \mathsf{O}_{(4n+3\varrho)\times(q+m)}\end{bmatrix}\begin{bmatrix}\mathsf{O}_{(4n+3\varrho)\times(q+m)}\\ I_{q+m}\end{bmatrix}=\begin{bmatrix}I_{4n+3\varrho} & \mathsf{O}_{(4n+3\varrho)\times(q+m)}\end{bmatrix}\Upsilon=\mathsf{O}_{(4n+3\varrho)\times(q+m)}\\ \begin{bmatrix}I_{4n+3\varrho} & \mathsf{O}_{(4n+3\varrho)\times(q+m)}\end{bmatrix}_\perp=\begin{bmatrix}\mathsf{O}_{(4n+3\varrho)\times(q+m)}\\ I_{q+m}\end{bmatrix}=\Upsilon\end{gathered}\tag{7.53}$$

which satisfies $\operatorname{rank}\left(\begin{bmatrix}-I_n & \mathbf{\Pi}\end{bmatrix}\right)=n$ and $\operatorname{rank}\left(\begin{bmatrix}I_{4n+3\varrho} & \mathsf{O}_{(4n+3\varrho)\times(q+m)}\end{bmatrix}\right)=4n+3\varrho$. By using the rank nullity theorem, this implies that the relations in (7.53) can be applied with Lemma 2.4.

Applying Lemma 2.4 to (7.50) and (7.52) considering (7.53) yields the conclusion that (7.50) and (7.52) are true if and only if

$$\exists\mathbf{W}\in\mathbb{R}^{(4n+3\varrho)\times n}:\mathbf{Sy}\left(\begin{bmatrix}I_{4n+3\varrho}\\ \mathsf{O}_{(q+m)\times(4n+3\varrho)}\end{bmatrix}\mathbf{W}\begin{bmatrix}-I_n & \mathbf{\Pi}\end{bmatrix}\right)+\begin{bmatrix}\mathsf{O}_n & \mathbf{P}\\ * & \mathbf{\Phi}\end{bmatrix}\prec 0.\tag{7.54}$$

Now the inequality in (7.54) is nonconvex due to the product between $\mathbf{W}$ and $\mathbf{\Pi}$. To convexify (7.54), consider

$$\mathbf{W}:=\mathbf{Col}\left[W,\ \mathbf{Col}_{i=1}^{3+3d}\,\alpha_i W\right]\tag{7.55}$$

with $W\in\mathbb{R}^{n\times n}_{[n]}$ and $\{\alpha_i\}_{i=1}^{3+3d}\subset\mathbb{R}$. Note that having the structural constraints in (7.55) infers that the corresponding (7.54) is no longer an equivalent but only a sufficient condition implying (7.50). Now consider the proposition in (7.54) with (7.55), we can conclude that (7.50) holds if

$$\mathbf{\Theta}=\mathbf{Sy}\left(\begin{bmatrix}W\\ \mathbf{Col}_{i=1}^{3+3d}\,\alpha_i W\\ \mathsf{O}_{(q+m)\times n}\end{bmatrix}\begin{bmatrix}-I_n & \mathbf{\Pi}\end{bmatrix}\right)+\begin{bmatrix}\mathsf{O}_n & \mathbf{P}\\ * & \mathbf{\Phi}\end{bmatrix}\prec 0\tag{7.56}$$

holds with $W\in\mathbb{R}^{n\times n}_{[n]}$ and $\{\alpha_i\}_{i=1}^{3+3d}\subset\mathbb{R}$. It is important to stress here that a full rank W is implied by (7.56) since the expression $-W-W^\top$ is the only element at the first diagonal block of $\mathbf{\Theta}$.

Now use congruence transformations on (7.23),(7.24) and (7.56) with the fact that a full rank W is implied by (7.56). One can conclude that the inequalities

$$\begin{gathered}X^\top Q_1X\succ 0,\ \ X^\top Q_2X\succ 0,\ \ X^\top R_1X\succ 0,\ \ \begin{bmatrix}X^\top & \mathsf{O}_n\\ * & X^\top\end{bmatrix}\begin{bmatrix}R_2 & Y\\ * & R_2\end{bmatrix}\begin{bmatrix}X & \mathsf{O}_n\\ * & X\end{bmatrix}\succ 0\\ \left[\left(I_{4+3d}\otimes X^\top\right)\oplus I_{q+m}\right]\mathbf{\Theta}\left[(I_{4+3d}\otimes X)\oplus I_{q+m}\right]\prec 0,\ \ \left(I_{1+2d}\otimes X^\top\right)\begin{bmatrix}P_1 & P_2\\ * & P_3\end{bmatrix}(I_{1+2d}\otimes X)\succ 0\end{gathered}\tag{7.57}$$

hold if and only if (7.23),(7.24) and (7.56) hold, where $X^\top:=W^{-1}$. Moreover, by letting $\acute{Y}=X^\top YX$ and

$$\begin{bmatrix}\acute{P}_1 & \acute{P}_2\\ * & \acute{P}_3\end{bmatrix}:=[*]\begin{bmatrix}P_1 & P_2\\ * & P_3\end{bmatrix}(I_{1+2d}\otimes X),\quad \begin{bmatrix}\acute{Q}_1 & \acute{Q}_2 & \acute{R}_1 & \acute{R}_2\end{bmatrix}:=X^\top\begin{bmatrix}Q_1X & Q_2X & R_1X & R_2X\end{bmatrix}\tag{7.58}$$

and considering (2.1) with (7.57), the inequalities in (7.57) can be rewritten into (7.23) and (7.24) and

$$\left[\left(I_{4+3d}\otimes X^\top\right)\oplus I_{q+m}\right]\mathbf{\Theta}\left[(I_{4+3d}\otimes X)\oplus I_{q+m}\right]=\acute{\mathbf{\Theta}}=\mathbf{Sy}\left(\begin{bmatrix}I_n\\ \mathbf{Col}_{i=1}^{3+3d}\,\alpha_i I_n\\ \mathsf{O}_{(q+m)\times n}\end{bmatrix}\begin{bmatrix}-X & \acute{\mathbf{\Pi}}\end{bmatrix}\right)$$

$$+\begin{bmatrix}\mathsf{O}_n & \acute{\mathbf{P}}\\ * & \acute{\mathbf{\Phi}}\end{bmatrix} \prec 0 \quad (7.59)$$

where $\acute{\mathbf{P}} = X\mathbf{P}\left[(I_{3+3d}\otimes X)\oplus I_{q+m}\right] = \begin{bmatrix}\mathsf{O}_{n\times 2n} & \acute{P}_1 & \acute{P}_2\widehat{I} & \mathsf{O}_{n\times q} & \mathsf{O}_{n\times m}\end{bmatrix}$ and

$$\begin{aligned}\acute{\mathbf{\Pi}} = \mathbf{\Pi}\left[(I_{3+3d}\otimes X)\oplus I_{q+m}\right] &= \begin{bmatrix}\mathbf{A}\left[(I_{3+3d}\otimes X)\oplus I_q\right] + \mathbf{B}_1\left[(I_{3+3d}\otimes KX)\oplus \mathsf{O}_q\right] & \mathsf{O}_{n\times m}\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{A}\left[(I_{3+3d}\otimes X)\oplus I_q\right] + \mathbf{B}_1\left[(I_{3+3d}\otimes V)\oplus \mathsf{O}_q\right] & \mathsf{O}_{n\times m}\end{bmatrix} \quad (7.60)\end{aligned}$$

with $V = KX$, and

$$\begin{aligned}&\acute{\mathbf{\Phi}} = \left[\left(I_{3+3d}\otimes X^\top\right)\oplus I_{q+m}\right]\mathbf{\Phi}\left[(I_{3+3d}\otimes X)\oplus I_{q+m}\right] =\\ &\left[\left(I_{3+3d}\otimes X^\top\right)\oplus I_{q+m}\right]\mathsf{Sy}\left(\begin{bmatrix}\mathsf{O}_{2n\times 2\varrho}\\ P_2\\ \widehat{I}^\top P_3\\ \mathsf{O}_{(q+m)\times 2\varrho}\end{bmatrix}\begin{bmatrix}\mathbf{F} & \mathsf{O}_{2\varrho\times m}\end{bmatrix} + \begin{bmatrix}\mathsf{O}_{(3n+3\varrho)\times m}\\ -J_2^\top\\ \widetilde{J}\end{bmatrix}\begin{bmatrix}\mathbf{\Sigma} & \mathsf{O}_m\end{bmatrix}\right)\left[(I_{3+3d}\otimes X)\oplus I_{q+m}\right]\\ &-\left[\left(I_{3+3d}\otimes X^\top\right)\oplus I_{q+m}\right]\Bigg(\left[Q_1 - Q_2 - r_3R_2\right]\oplus Q_2\oplus(-Q_1 - r_1R_1)\oplus(I_d\otimes R_1)\\ &\oplus\left(\begin{bmatrix}\mathsf{K}_{(d,n)} & \mathsf{O}_{dn}\\ * & \mathsf{K}_{(d,n)}\end{bmatrix}\left(\begin{bmatrix}R_2 & Y\\ * & R_2\end{bmatrix}\otimes I_d\right)\begin{bmatrix}\mathsf{K}_{(n,d)} & \mathsf{O}_{dn}\\ * & \mathsf{K}_{(n,d)}\end{bmatrix}\right)\oplus J_3\oplus(-J_1)\Bigg)\left[(I_{3+3d}\otimes X)\oplus I_{q+m}\right]\\ &= \mathsf{Sy}\left(\begin{bmatrix}\mathsf{O}_{2n\times 2\varrho}\\ \acute{P}_2\\ \widehat{I}^\top \acute{P}_3\\ \mathsf{O}_{(q+m)\times 2\varrho}\end{bmatrix}\begin{bmatrix}\mathbf{F} & \mathsf{O}_{2\varrho\times m}\end{bmatrix} + \begin{bmatrix}\mathsf{O}_{(3n+3\varrho)\times m}\\ -J_2^\top\\ \widetilde{J}\end{bmatrix}\begin{bmatrix}\acute{\mathbf{\Sigma}} & \mathsf{O}_m\end{bmatrix}\right)\\ &-\Bigg(\left[\acute{Q}_1 - \acute{Q}_2 - r_3\acute{R}_2\right]\oplus\acute{Q}_2\oplus\left(-\acute{Q}_1 - r_1\acute{R}_1\right)\oplus\left(I_d\otimes\acute{R}_1\right)\\ &\oplus\left(\begin{bmatrix}\mathsf{K}_{(d,n)} & \mathsf{O}_{dn}\\ * & \mathsf{K}_{(d,n)}\end{bmatrix}\left(\begin{bmatrix}\acute{R}_2 & \acute{Y}\\ * & \acute{R}_2\end{bmatrix}\otimes I_d\right)\begin{bmatrix}\mathsf{K}_{(n,d)} & \mathsf{O}_{dn}\\ * & \mathsf{K}_{(n,d)}\end{bmatrix}\right)\oplus J_3\oplus(-J_1)\Bigg) \quad (7.61)\end{aligned}$$

which is identical to (7.49). Note that (7.59) is the same as (7.46), and the form of $\acute{\mathbf{\Phi}}$ is derived via the relations

$$\begin{aligned}&\begin{bmatrix}\mathbf{F} & \mathsf{O}_{2\varrho\times m}\end{bmatrix}\left[(I_{3+3d}\otimes X)\oplus I_{q+m}\right] = \begin{bmatrix}\widehat{\mathbf{F}}\otimes I_n & \mathsf{O}_{2\varrho\times(q+m)}\end{bmatrix}\left[(I_{3+3d}\otimes X)\oplus I_{q+m}\right]\\ &\quad = \begin{bmatrix}I_{2d}\widehat{\mathbf{F}}\otimes XI_n & \mathsf{O}_{2\varrho\times(q+m)}\end{bmatrix} = (I_{2d}\otimes X)\begin{bmatrix}\widehat{\mathbf{F}}\otimes I_n & \mathsf{O}_{2\varrho\times(q+m)}\end{bmatrix} = (I_{2d}\otimes X)\begin{bmatrix}\mathbf{F} & \mathsf{O}_{2\varrho\times m}\end{bmatrix} \quad (7.62)\end{aligned}$$

$$\widehat{I}\,(I_{3d}\otimes X) = \begin{bmatrix}I_\varrho & \mathsf{O}_\varrho & \mathsf{O}_\varrho\\ \mathsf{O}_\varrho & I_\varrho & I_\varrho\end{bmatrix}(I_{3d}\otimes X) = (I_{2d}\otimes X)\,\widehat{I} \quad (7.63)$$

$$\begin{aligned}\begin{bmatrix}\mathsf{K}_{(n,d)} & \mathsf{O}_{dn}\\ * & \mathsf{K}_{(n,d)}\end{bmatrix}\begin{bmatrix}I_d\otimes X & \mathsf{O}_{dn}\\ * & I_d\otimes X\end{bmatrix} &= \begin{bmatrix}X\otimes I_d & \mathsf{O}_{dn}\\ * & X\otimes I_d\end{bmatrix}\begin{bmatrix}\mathsf{K}_{(n,d)} & \mathsf{O}_{dn}\\ * & \mathsf{K}_{(n,d)}\end{bmatrix}\\ &= \left(\begin{bmatrix}X & \mathsf{O}_n\\ * & X\end{bmatrix}\otimes I_d\right)\begin{bmatrix}\mathsf{K}_{(n,d)} & \mathsf{O}_{dn}\\ * & \mathsf{K}_{(n,d)}\end{bmatrix}. \quad (7.64)\end{aligned}$$

which are derived from the properties of matrices with (2.1),(2.2) and (7.12). Furthermore, since $-X - X^\top$ is the only element at the first diagonal block of $\acute{\mathbf{\Theta}}$ in (7.59) or (7.46), thus $X \in \mathbb{R}_{[n]}^{n\times n}$ if (7.59) or (7.46) hold. This is consistent with the fact that a full rank W is implied by the matrix inequality in (7.56).

As a result, we have shown the equivalence between (7.23)–(7.24) and (7.44)–(7.45). Meanwhile, it has been shown via (7.59) that (7.46) is equivalent to (7.56) which infers (7.50). Consequently, (7.23)–(7.25) are satisfied if (7.44)–(7.46) hold with some W and $\{\alpha_i\}_{i=1}^{3+3d} \subset \mathbb{R}$. Since the existence of the feasible solutions of (7.23)–(7.25) infer the existence of $\epsilon_1; \epsilon_2; \epsilon_3 > 0$ and an LKF (7.30) satisfying (7.19),(7.20) and (7.21), thus it demonstrates that feasible solutions of (7.44)–(7.46) infers the existence of $\epsilon_1; \epsilon_2; \epsilon_3 > 0$ and a functional in (7.30) satisfying the corresponding stability and dissipativity criteria. This further shows that the existence of the feasible solutions of (7.44)–(7.46) ensures that the trivial solution $\boldsymbol{x}(t) \equiv \mathbf{0}_n$ of the closed-loop system (7.4) with $\boldsymbol{w}(t) \equiv \mathbf{0}_q$ is globally uniformly asymptotically stable and (7.4) with (7.22) is dissipative. The proof is finished. ■

Remark 7.6. Theorem 7.2 is specifically derived to handle a genuine synthesis problem for (7.4). If an open-loop system is considered with $B_1 = B_2(\tau) = \mathsf{O}_{n\times p}$ and $B_3 = B_4(\tau) = \mathsf{O}_{n\times p}$, then Theorem 7.1 should be applied instead of Theorem 7.2 since the introduction of the slack variables in Theorem 7.2 does not render its conditions more feasible in comparison with the conditions in 7.1.

Remark 7.7. Similar to what has been explained in Remark 7.5, one can derive a synthesis condition for the case of $r_2 > 0$, $r_1 = 0$ with fewer variables based on the proof of Theorem 7.2 with certain modifications.

Remark 7.8. For the structure in (7.46), some values of $\{\alpha_i\}_{i=1}^{3+3d} \subset \mathbb{R}$ may be more significant than others in terms of their influence on the feasibility of (7.46). Specifically, ϵ_3 is the most crucial one since it may determine the feasibility of the diagonal related to A_1 in (7.46). A simple assignment of $\{\alpha_i\}_{i=1}^{3+3d} \subset \mathbb{R}$ can be $\alpha_1 = \alpha_2 = 0$ and $\alpha_i = 0, i = 3\cdots 3+3d$ which allows one to only adjust the value of α_3 to use Theorem 7.2.

7.4.1 An inner convex approximation solution of Theorem 7.1

By fixing the values of $\{\alpha_i\}_{i=1}^{3+3d} \subset \mathbb{R}$, Theorem 7.2 provides a synthesis solution of (7.4) via solving LMIs. Nevertheless, the simplification we have applied in (7.55) can introduce certain conservatism to the feasibility of 7.2. In this subsection, an iterative algorithm is derived based on the results in [257] to further reduce the potential conservatism of Theorem 7.2. The resulting algorithm avoids the introduction of slack variables and its initial data can be supplied by a feasible solution of Theorem 7.2.

First of all, note that the inequality in (7.25) is nonconvex in general whereas (7.23) and (7.24) remain convex even when a genuine synthesis problem is considered. Now it is obvious that (7.25) can be rewritten into

$$\mathfrak{U}(\mathbf{H}, K) := \mathbf{Sy}\left[\mathbf{P}^\top \mathbf{\Pi}\right] + \mathbf{\Phi} = \mathbf{Sy}\left(\mathbf{P}^\top \mathbf{B}\left[(I_{3+3d}\otimes K)\oplus \mathsf{O}_{p+m}\right]\right) + \widehat{\mathbf{\Phi}} \prec 0 \tag{7.65}$$

with $\mathbf{B} = \begin{bmatrix}\mathbf{B}_1 & \mathsf{O}_{n\times m}\end{bmatrix}$ and $\widehat{\mathbf{\Phi}} = \mathbf{Sy}\left(\mathbf{P}^\top\begin{bmatrix}\mathbf{A} & \mathsf{O}_{n\times m}\end{bmatrix}\right) + \mathbf{\Phi}$, where $\mathbf{A}$ and $\mathbf{B}_1$ are given in (7.5)–(7.6) and $\mathbf{H} := \begin{bmatrix}P_1 & P_2\end{bmatrix}$ with P_1 and P_2 in Theorem 7.1. Note that no products between variables are involved in $\widehat{\mathbf{\Phi}}$ in (7.65) thus $\widehat{\mathbf{\Phi}}$ contains no non-convexities. Now considering Example 3 in [257], one can conclude that the function $\Delta\left(\bullet, \widetilde{\mathbf{G}}, \bullet, \widetilde{\mathbf{\Gamma}}\right)$ which is defined as

$$\Delta\left(\mathbf{G}, \widetilde{\mathbf{G}}, \mathbf{\Gamma}, \widetilde{\mathbf{\Gamma}}\right) := \begin{bmatrix}\mathbf{G}^\top - \widetilde{\mathbf{G}}^\top & \mathbf{\Gamma}^\top - \widetilde{\mathbf{\Gamma}}^\top\end{bmatrix}\left[Z\oplus(I_n - Z)\right]^{-1}\begin{bmatrix}\mathbf{G}-\widetilde{\mathbf{G}}\\ \mathbf{\Gamma}-\widetilde{\mathbf{\Gamma}}\end{bmatrix} + \mathbf{Sy}\left(\widetilde{\mathbf{G}}^\top\mathbf{\Gamma} + \mathbf{G}^\top\widetilde{\mathbf{\Gamma}} - \widetilde{\mathbf{G}}^\top\widetilde{\mathbf{\Gamma}}\right) + \mathbf{T} \tag{7.66}$$

with $Z\oplus(I_n - Z)\succ 0$ satisfying

$$\mathbf{T} + \mathbf{Sy}\left(\mathbf{G}^\top\mathbf{\Gamma}\right) \preceq \Delta\left(\mathbf{G}, \widetilde{\mathbf{G}}, \mathbf{\Gamma}, \widetilde{\mathbf{\Gamma}}\right), \quad \mathbf{T} + \mathbf{Sy}\left(\mathbf{G}^\top\mathbf{\Gamma}\right) = \Delta(\mathbf{G}, \mathbf{G}, \mathbf{\Gamma}, \mathbf{\Gamma}) \tag{7.67}$$

$\forall\mathbf{G}; \widetilde{\mathbf{G}} \in \mathbb{R}^{n\times\kappa}$ and $\forall\mathbf{\Gamma}; \widetilde{\mathbf{\Gamma}} \in \mathbb{R}^{n\times\kappa}$, is a psd-convex overestimate of $\acute{\Delta}(\mathbf{G}, \mathbf{\Gamma}) = \mathbf{T} + \mathbf{Sy}\left[\mathbf{G}^\top\mathbf{\Gamma}\right]$ with respect to the parameterization

$$\begin{bmatrix}\mathbf{vec}(\widetilde{\mathbf{G}})\\ \mathbf{vec}(\widetilde{\mathbf{\Gamma}})\end{bmatrix} = \begin{bmatrix}\mathbf{vec}(\mathbf{G})\\ \mathbf{vec}(\mathbf{\Gamma})\end{bmatrix}. \tag{7.68}$$

Let $\kappa = 3n + 3\varrho + q + m$ and $Z \oplus (I_n - Z) \succ 0$ and

$$\begin{aligned} \mathbf{T} = \widehat{\boldsymbol{\Phi}},\ \ \mathbf{G} = \mathbf{P} = \begin{bmatrix} \mathsf{O}_{n\times 2n} & P_1 & P_2\widehat{I} & \mathsf{O}_{n\times q} & \mathsf{O}_{n\times m} \end{bmatrix},\ \ \mathbf{H} = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \\ \widetilde{\mathbf{G}} = \widetilde{\mathbf{P}} = \begin{bmatrix} \mathsf{O}_{n\times 2n} & \widetilde{P}_1 & \widetilde{P}_2\widehat{I} & \mathsf{O}_{n\times q} & \mathsf{O}_{n\times m} \end{bmatrix},\ \ \widetilde{\mathbf{H}} := \begin{bmatrix} \widetilde{P}_1 & \widetilde{P}_2 \end{bmatrix},\ \ \widetilde{P}_1 \in \mathbb{S}^n,\ \widetilde{P}_2 \in \mathbb{R}^{n\times dn} \\ \boldsymbol{\Gamma} = \mathbf{B}\mathbf{K},\ \ \mathbf{K} = \left[(I_{3+3d}\otimes K)\oplus \mathsf{O}_{p+m}\right],\ \ \widetilde{\boldsymbol{\Gamma}} = \mathbf{B}\widetilde{\mathbf{K}},\ \ \widetilde{\mathbf{K}} = \left[\left(I_{3+3d}\otimes \widetilde{K}\right)\oplus \mathsf{O}_{p+m}\right] \end{aligned} \tag{7.69}$$

with $\widehat{\boldsymbol{\Phi}}$, $\mathbf{H}$ and K in line with the definition in (7.65), we have

$$\begin{aligned} \widehat{\boldsymbol{\Phi}} + \mathsf{Sy}\left[\mathbf{P}^\top \mathbf{B}\left[(I_{3+3d}\otimes K)\oplus \mathsf{O}_{p+m}\right]\right] \preceq \mathcal{S}\left(\mathbf{H}, \widetilde{\mathbf{H}}, K, \widetilde{K}\right) := \widehat{\boldsymbol{\Phi}} + \mathsf{Sy}\left(\widetilde{\mathbf{P}}^\top \mathbf{B}\mathbf{K} + \mathbf{P}^\top \mathbf{B}\widetilde{\mathbf{K}} - \widetilde{\mathbf{P}}^\top \mathbf{B}\widetilde{\mathbf{K}}\right) \\ + \begin{bmatrix} \mathbf{P}^\top - \widetilde{\mathbf{P}}^\top & \mathbf{K}^\top\mathbf{B}^\top - \widetilde{\mathbf{K}}^\top\mathbf{B}^\top \end{bmatrix} \left[Z\oplus (I_n - Z)\right]^{-1} \begin{bmatrix} \mathbf{P} - \widetilde{\mathbf{P}} \\ \mathbf{B}\mathbf{K} - \mathbf{B}\widetilde{\mathbf{K}} \end{bmatrix} \end{aligned} \tag{7.70}$$

by (7.67), where $\mathcal{S}(\bullet, \widetilde{\mathbf{H}}, \bullet, \widetilde{K})$ is a psd-convex overestimate of the term in (7.65) with respect to the parameterization

$$\begin{bmatrix} \mathbf{vec}(\widetilde{\mathbf{H}}) \\ \mathbf{vec}(\widetilde{K}) \end{bmatrix} = \begin{bmatrix} \mathbf{vec}(\mathbf{H}) \\ \mathbf{vec}(K) \end{bmatrix}. \tag{7.71}$$

By (7.70), it is obvious that (7.65) is inferred by $\mathcal{S}\left(\mathbf{H}, \widetilde{\mathbf{H}}, K, \widetilde{K}\right) \prec 0$. Moreover, we have $\mathcal{S}\left(\mathbf{H}, \widetilde{\mathbf{H}}, K, \widetilde{K}\right) \prec 0$ holds if and only if

$$\begin{bmatrix} \widehat{\boldsymbol{\Phi}} + \mathsf{Sy}\left(\widetilde{\mathbf{P}}^\top \mathbf{B}\mathbf{K} + \mathbf{P}^\top \mathbf{B}\widetilde{\mathbf{K}} - \widetilde{\mathbf{P}}^\top \mathbf{B}\widetilde{\mathbf{K}}\right) & \mathbf{P}^\top - \widetilde{\mathbf{P}}^\top & \mathbf{K}^\top\mathbf{B}^\top - \widetilde{\mathbf{K}}^\top\mathbf{B}^\top \\ * & -Z & \mathsf{O}_n \\ * & * & Z - I_n \end{bmatrix} \prec 0 \tag{7.72}$$

holds given $Z \oplus (I_n - Z) \succ 0$. Note that (7.65) is inferred by (7.72) which can be handled by standard interior algorithms of semidefinite programmings provided that the values of $\widetilde{\mathbf{H}}$ and $\widetilde{K}$ are given. To initialize the algorithm, one has to determine an initial data for $\widetilde{\mathbf{H}}$ and $\widetilde{K}$ whose values must be included by the corresponding elements in the relative interior of the feasible set of the original conditions in Theorem 7.1. Namely, $\widetilde{P}_1 \leftarrow P_1$, $\widetilde{P}_2 \leftarrow P_2$ and $\widetilde{K} \leftarrow K$ can be used for the initial data of $\widetilde{\mathbf{H}}$ and $\widetilde{K}$ where P_1, P_2 and K are the feasible solutions of (7.23)–(7.25).

By compiling all the aforementioned procedures according to the expositions in [257], Algorithm 4 can be constructed as follows where $\mathbf{x}$ consists of all the variables in P_3, Q_1, Q_2 R_1, R_2 in Theorem 7.1 and Z in (7.72). Furthermore, $\mathbf{H}$, $\widetilde{\mathbf{H}}$, $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ in Algorithm 4 are defined in (7.69) and ρ_1, ρ_2 and ε are given constants to achieve regularizations and determine error tolerance, respectively.

Remark 7.9. When a convex objective function is concerned in Theorem 7.1, for instance $\mathbb{L}^2$ gain $\gamma > 0$ minimization, a termination condition [257] might be added to Algorithm 4 concerning the improvement of objective function between two successive iterations. Nonetheless, such a condition has not been applied with the tests of our numerical examples in this chapter.

Remark 7.10. The most challenging step in using Algorithm 4 is its initialization. Generally speaking, acquiring a feasible solution of Theorem 7.1 may not be an easy task. Nevertheless, as what has been proposed in Theorem 7.2, initial values of $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{K}$ can be supplied by solving (7.44)–(7.46) with given values[5] of $\{\alpha_i\}_{i=1}^{3+3d}$.

Remark 7.11. Similar to what has been stated in Remark 7.5 and 7.7, an iterative algorithm can be constructed for the case of $r_2 > 0$, $r_1 = 0$ with fewer decision variables involved based on the structure of Algorithm 4.

[5]Note that as we have elaborated in Remark 7.8 that one may apply Theorem 7.2 with $\alpha_1 = \alpha_2 = 0$ and $\alpha_i = 0, i = 4\cdots 3+3d$ which allow users to only adjust the value of ϵ_3 to solve the conditions in Theorem 7.2

Algorithm 4: An inner convex approximation solution for Theorem 7.1

begin

solve Theorem 7.2 with given α_i to obtain K and then **solve** Theorem 7.1 with the previous K to obtain $\mathbf{H} = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$.

update $\widetilde{\mathbf{H}} \longleftarrow \mathbf{H}, \quad \widetilde{K} \longleftarrow K$,

solve $\min_{\mathbf{x},\mathbf{H},K} \mathsf{tr}\left[\rho_1[*]\big(\mathbf{H} - \widetilde{\mathbf{H}}\big) + \rho_2[*]\big(K - \widetilde{K}\big)\right]$ subject to (7.23) and (7.72) obtain $\mathbf{H}$ and K

while $\dfrac{\left\| \begin{bmatrix} \mathbf{vec}(\mathbf{H}) \\ \mathbf{vec}\,(K) \end{bmatrix} - \begin{bmatrix} \mathbf{vec}(\widetilde{\mathbf{H}}) \\ \mathbf{vec}(\widetilde{K}) \end{bmatrix} \right\|_\infty}{\left\| \begin{bmatrix} \mathbf{vec}(\widetilde{\mathbf{H}}) \\ \mathbf{vec}(\widetilde{K}) \end{bmatrix} \right\|_\infty + 1} \geq \varepsilon$ **do**

update $\widetilde{\mathbf{H}} \longleftarrow \mathbf{H}, \quad \widetilde{K} \longleftarrow K$;

solve $\min_{\mathbf{x},\mathbf{H},K} \mathsf{tr}\left[\rho_1[*]\big(\mathbf{H} - \widetilde{\mathbf{H}}\big) + \rho_2[*]\big(K - \widetilde{K}\big)\right]$ subject to (7.23) and (7.72) to obtain $\mathbf{H}$ and K;

end

end

7.5 Numerical examples

In this section, we present two numerical examples to demonstrate the effectiveness of our proposed methodologies. The test is conducted in Matlab environment using Yalmip [266] as the optimization interface. Moreover, SDPT3 [270] is applied as the numerical solver for semidefinite programmings.

7.5.1 Stability analysis of a linear system with a time-varying distributed delay

Given $t_0 \in \mathbb{R}$, consider a distributed delay system

$$\dot{x}(t) = 0.395x(t) - 5\int_{-r(t)}^{0} \cos(12\tau)x(t+\tau)\mathsf{d}\tau, \quad t \geq t_0 \quad r(\cdot) \in [r_1, r_2]^{\mathbb{R}}, \tag{7.73}$$

with $r_2 > r_1 > 0$, which corresponds to $A_1 = 0.395$, $\widetilde{A}_2(\tau) = -5\cos(12\tau)$ for the model in (7.1) with $n = 1$ and $p = m = q = 0$. Note that the rest of the state space matrices in (7.1) corresponding to (7.73) are empty matrices. The function $\boldsymbol{f}(\cdot)$ in Assumption 7.1 for (7.73) is chosen to be

$$\boldsymbol{f}(\tau) = \begin{bmatrix} 1 \\ \sin(12\tau) \\ \cos(12\tau) \end{bmatrix} \quad \text{with} \quad M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 12 \\ 0 & -12 & 0 \end{bmatrix}. \tag{7.74}$$

which gives $A_2 = \begin{bmatrix} 0 & 0 & -5 \end{bmatrix}$ for the distributed delay term in (7.73) satisfying $\widetilde{A}_3(\tau) = A_2 F(\tau)$ with $d = 3, n = 1$. Consider the case that $r(t)$ is a constant, then the corresponding stable delay intervals $[0.104, 0.1578]$, $[0.6276, 0.6814]$, $[1.1512, 1.205]$, $[1.6748, 1.7286]$ and $[2.1984, 2.2522]$ can be obtained by the method in [81]. Since a constant $r(t)$ is one option for $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$, thus the stable delay intervals of (7.73) with $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ cannot be larger than the stable delay intervals of the same system with $r(t)$ being a constant. This clearly establishes a 'boundary' in terms of how far a time-varying stability result can be.

To the best of the author's knowledge, no existing methods, neither time nor frequency-domain-based approaches, are capable of analyzing the stability of (7.73) with an uncertain function $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ and given $r_2 > r_2 > 0$. By applying Theorem 7.1 with (7.74) to (7.73) with $A_2 = \begin{bmatrix} 0 & 0 & -5 \end{bmatrix}$, the results of time-varying stability with $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ are presented in Table 7.1 where $r_3 = r_2 - r_1$ and NoDVs stand for the numbers of decision variables. Furthermore, a leftmost stable interval in Table 7.1 is produced by Theorem 7.1 with a given value of r_3 and the smallest value of r_1 to render Theorem 7.1 to be feasible. On the other hand, a rightmost stable interval produced by Theorem 7.1 is the interval whose upper bound is the largest value of r_2 to render Theorem 7.1 to be feasible with a given r_3.

Methodologies	first interval	second interval	third interval	forth interval	fifth interval	NoDVs
[81] $r(t)$ is constant	$[0.104, 0.1578]$	$[0.6276, 0.6814]$	$[1.1512, 1.205]$	$[1.6748, 1.7286]$	$[2.1984, 2.2522]$	-
Theorem 7.1 leftmost $r_3 = 0.0272$	$[0.1051, 0.1323]$	$[0.629, 0.6562]$	$[1.1533, 1.1805]$	$[1.6786, 1.7058]$	$[2.2073, 2.2345]$	33
Theorem 7.1 rightmost $r_3 = 0.0272$	$[0.1295, 0.1567]$	$[0.6528, 0.68]$	$[1.1756, 1.2028]$	$[1.6976, 1.7248]$	$[2.2161, 2.2433]$	33
Theorem 7.1 leftmost $r_3 = 0.0273$	$[0.1052, 0.1325]$	$[0.629, 0.6563]$	$[1.1534, 1.1807]$	$[1.6786, 1.7059]$	$[2.2074, 2.2347]$	33
Theorem 7.1 rightmost $r_3 = 0.0273$	$[0.1293, 0.1566]$	$[0.6527, 0.68]$	$[1.1755, 1.2028]$	$[1.6974, 1.7247]$	$[2.2159, 2.2432]$	33
Theorem 7.1 leftmost $r_3 = 0.0274$	$[0.1052, 0.1326]$	$[0.629, 0.6564]$	$[1.1534, 1.1808]$	$[1.6787, 1.7061]$	$[2.2075, 2.2349]$	33
Theorem 7.1 rightmost $r_3 = 0.0274$	$[0.1292, 0.1566]$	$[0.6525, 0.6799]$	$[1.1754, 1.2028]$	$[1.6973, 1.7247]$	$[2.2157, 2.2431]$	33

Table 7.1: Detectable stable delay intervals $[r_1, r_2]$ for (7.73) for any $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$

The stability results in Table 7.1 are not complete in the sense that one may test more values of r_2 and r_1 to search stable intervals for (7.73) with $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$. Moreover, it is worthy to point out that all the results in Table 7.1 produced by Theorem 7.1 for $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ are consistent with the "boundaries" of the stable delay intervals calculated by [81] with $r(t)$ being a constant. Finally, it is imperative to stress that the stability results produced by Theorem 7.1 hold for any $r(t)$ bounded by $[r_1, r_2]$ with no further limitation on the structure of $r(t)$.

7.5.2 Dissipative stabilization of systems with time-varying distributed delays

Consider a system of the form (7.1) with any $r(\cdot) \in [0.5, 1]^{\mathbb{R}}$ and the state space matrices

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad \widetilde{A}_2(\tau) = \begin{bmatrix} -0.4 - 0.1\mathrm{e}^{\tau} + 0.3\mathrm{e}^{2\tau} & 1 - 0.1\mathrm{e}^{\tau} + 0.01\mathrm{e}^{2\tau} \\ -1 & 0.4 - 0.3\mathrm{e}^{2\tau} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\widetilde{B}_2(\tau) = \begin{bmatrix} 0.1\mathrm{e}^{\tau} - 0.1 \\ 0.12\mathrm{e}^{2\tau} + 0.1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \tag{7.75}$$
$$\widetilde{C}_2(\tau) = \begin{bmatrix} -0.11 + 0.2\mathrm{e}^{\tau} & 0.1 \\ 0.1\mathrm{e}^{\tau} & -0.2\mathrm{e}^{\tau} + 0.14\mathrm{e}^{2\tau} \end{bmatrix}, \quad \widetilde{B}_4(\tau) = \begin{bmatrix} 0 \\ 0.1 + 0.15\mathrm{e}^{\tau} \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.12 \\ 0.1 \end{bmatrix}.$$

Moreover, let

$$J_1 = -\gamma I_m, \quad \widetilde{J} = I_m, \quad J_2 = \mathsf{O}_{m \times q}, \quad J_3 = \gamma I_q \tag{7.76}$$

for the supply rate function in (7.22) to calculate the minimum value of $\mathbb{L}^2$ gain γ.

According to our best knowledge, no existing methods can find a controller for (7.1) with the parameters in (7.75). Note that since $r(t)$ is time-varying and the expression of $r(t)$ is unknown, hence the distributed delay kernels in (7.75) may not be approximated via the approaches in [185, 188].

By observing the functions inside of $\widetilde{A}_3(\cdot), \widetilde{B}_2(\cdot), \widetilde{C}_3(\cdot)$ and $\widetilde{B}_4(\cdot)$, we choose $\boldsymbol{f}(\cdot)$ in Assumption 7.1 to be

$$\boldsymbol{f}(\tau) = \begin{bmatrix} 1 \\ \mathrm{e}^{\tau} \\ \mathrm{e}^{2\tau} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{7.77}$$

with $d = 3$, $n = m = 2$, $q = 1$, which gives

$$\begin{aligned} A_2 &= \begin{bmatrix} -0.4 & 1 & -0.1 & -0.1 & 0.3 & 0.01 \\ -1 & 0.4 & 0 & 0 & 0 & -0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0.1 & 0 & 0.12 \end{bmatrix} \\ C_2 &= \begin{bmatrix} -0.11 & 0.1 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & -0.2 & 0 & 0.14 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0.1 & 0.15 & 0 \end{bmatrix} \end{aligned} \tag{7.78}$$

satisfying $\widetilde{A}_3(\tau) = A_3 (\boldsymbol{f}(\tau) \otimes I_2)$, $\widetilde{B}_2(\tau) = B_2 \boldsymbol{f}(\tau)$, $\widetilde{C}_3(\tau) = C_3 (\boldsymbol{f}(\tau) \otimes I_2)$ and $\widetilde{B}_4(\tau) = B_4 \boldsymbol{f}(\tau)$ as in Assumption 7.1.

Given $\alpha_1 = \alpha_2 = \alpha_i = 0$, $i = 4 \cdots 12$ and $\alpha_3 = 0.5$ for Theorem 7.2, now apply Algorithm 4 to (7.4) with the parameters in (7.75)–(7.78), which yields the results summarized in Table 7.2 with different resulting values of $\min \gamma$, where NoIs stands for the number of iterations in the while loop of Algorithm 4.

Controller gain K	$\begin{bmatrix} -0.3370 \\ -5.0007 \end{bmatrix}^\top$	$\begin{bmatrix} -0.9268 \\ -8.7625 \end{bmatrix}^\top$	$\begin{bmatrix} -1.8644 \\ -14.8565 \end{bmatrix}^\top$	$\begin{bmatrix} -2.7223 \\ -20.5042 \end{bmatrix}^\top$
$\min \gamma$	0.16395	0.16028	0.15827	0.1577
NoIs	10	20	30	40

Table 7.2: $\min \gamma$ produced by different iterations

Since $r(t)$ in this subsection is time-varying and its expression is unknown, hence existing frequency-domain-based approaches may not be applied to analyze the stability of the corresponding closed-loop systems obtained by our methods in Table 7.2. To partially verify our synthesis results, confine $r(t)$ to be an unknown constant. This allows one to apply the method in [81] to calculate the spectral abscissa of the spectrum of the closed-loop systems with a constant delay. Note that since our synthesis results indicate that the closed-loop systems are stable with any $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$, thus the stable delay intervals of the closed-loop systems with $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ cannot be larger than the stable intervals of the same systems with $r(t)$ being a constant. The numerical results produced by [81] show that the stable delay interval $[r_1, r_2] = [0.5.1]$ with $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ of the corresponding closed-loop systems in Table 7.2 is the proper subset of all the stable delay intervals of the closed-loop systems with $r(t)$ being a constant.

Chapter 8

Conclusions and Future Works

In this thesis, we have presented solutions for the stability analysis and stabilization of linear systems with distributed delays possessing non-trivial kernels. Solutions for the stability (dissipativity) analysis and stabilization have been derived vian LKF approach for which novel integral inequalities are proposed. The conclusions of the results presented in this thesis are summarized in the following section.

8.1 Conclusions

- In Chapter 2, a solution has been presented concerning stabilizing a linear distributed delay system with distributed delays in states, input and output. Our proposed synthesis scenario also includes a dissipative constraint, which is characterized by the quadratic supply function in (2.15), to secure the performances of resulting controllers. By constructing the LKF in (2.28) with the newly proposed inequality (2.17), sufficient conditions for the existence of a dissipative stabilizing controller are derived in terms of matrix inequalities in Theorem 2.1. To solve the resulting BMI (2.30) in Theorem 2.1, we developed convex conditions via the application of Projection Lemma in Theorem 2.2 whose feasible solutions infer the existence of the feasible solutions of Theorem 2.1. To reduce the potential conservatism of Theorem 2.2 due to the simplification of $\mathbf{W}$ at (2.54), an iterative algorithm is presented in Algorithm 1 based on the ideas developed in [257]. Due to the generality of our LKF in (2.28) and the novel integral inequality in (2.17), the proposed synthesis solutions can produce nonconservative results with fewer decision variables compared to existing literature. Finally, two numerical examples have been investigated which can clearly demonstrate the advantage of the proposed methodology over existing approaches.

- Dissipative stabilization conditions for uncertain linear distributed delay systems have been developed in Chapter 3 where uncertainties with general form exist among system's state space matrices. An instrumental mathematical device to tackle the general uncertainties in (3.1) is presented in Lemma 3.1 under the framework of matrix inequalities. This allows one to derive dissipative synthesis conditions, which have been summarized in Section 3.3, for the uncertain closed-loop system (3.7). It has also been shown in Section 3.4 that the idea presented in Section 3.3 can be further modified to calculate the gains of a non-fragile dynamical state feedback controller for the uncertain input delay system in (3.43), where the controller itself is robust against uncertainties with general form. It is worthy to mention that the design of a resilient controller is made possible due to the mathematical structure of (3.48) with a dynamic state feedback controller. Namely, no matrix products exist among the uncertainties of the controller gains and input gain matrix, which allows one to handle the uncertainties in (3.48) similar to (3.7).

- In Chapter 4, three general integral inequalities with the relations concerning inequality bound gaps have been proposed in Theorem 4.1, 4.2 and 4.3, respectively. The inequalities (4.3),(4.17) and (4.23) generalize almost all the existing integral inequalities in the literature, and many of which in fact are essentially equivalent in terms of inequality bound gaps in the context of semidefinite programmings. Moreover, the proposed inequalities and their properties are demonstrated to be useful in deriving equivalent stability conditions for a linear CDDS with a distributed delay. Finally, our inequalities have great potential to be used in wider contexts such as the stability analysis of PDE-related systems or sampled-data systems or other types of infinite dimensional systems whenever the contexts are suitable.

- In Chapter 5, a new method for the dissipativity and stability analysis of a linear CDDS with distributed delays in state and output equations have been proposed in Theorem 5.1 in terms of LMIs. The proposed approach can handle distributed delay with $\mathbb{L}^2$ functions kernel and simultaneously includes approximation errors in the resulting conditions (5.35)–(5.37) thanks to the novel integral inequality in (5.25). In comparison to existing approach in [188] which depends on the application of Legendre polynomials approximations, the proposed method allows one to apply a broader class of elementary functions $\acute{\boldsymbol{f}}(\cdot)$ and $\grave{\boldsymbol{f}}(\cdot)$ to approximate the distributed delay kernels of (5.1). Because of the fact that the generality of the LKF in (5.44) is also related to the structure of $\acute{\boldsymbol{f}}(\cdot)$ and $\grave{\boldsymbol{f}}(\cdot)$, thus our proposed methods derived from constructing (5.44) can produce less conservative results compared to a functional parameterized by Legendre polynomials such as the one considered in [188]. The results of numerical examples in Chapter 5 have clearly demonstrated the advantage of the proposed methodologies over existing approaches. A potential future direction is to investigate if the hierarchy conclusion in Chapter 5 can be derived without having an orthogonality constraint.

- In Chapter 6, a new solution for the problem of delay range dissipativity and stability analysis of a CDDS with polynomials-kernels-distributed delays (6.1) has been presented in Theorem 6.1 in terms of the SoS constraints (6.18)–(6.21) based on the construction of an LKF (6.24). The superiority of the proposed methodologies is rooted in the form of the functional (6.24) with delay-dependent matrix parameters which mathematically lead to less conservative conditions in terms of delay-parameter-dependent LMIs in (6.31) and (6.35). The difficulty of numerically solving robust LMIs is circumvented by the application of the matrix relaxation technique in [309] giving the SoS constraints (6.18)–(6.21) which are equivalent to (6.31) and (6.35). Meanwhile, it has been shown in Subsection 6.4.3 that in certain occasions there is no need to solve the SoS constraints (6.18)–(6.21) but to solve (6.31) and (6.35) directly via the property of convex hull. Moreover, a solution to the delay margins estimation problem with prescribed performance objectives is also presented in Subsection 6.4.4 which is followed by the feasibility hierarchy established in Theorem 6.2. Finally, the tests of numerical examples in Section 6.5 have demonstrated that less conservative results with less computational burdens can be produced by our methods compared to existing approaches.

- In Chapter 7, solutions for the design of a dissipative state feedback controller of a linear system with distributed delays (7.1) have been proposed where the delay $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$ is time-varying and bounded. The key step for the derivation of the synthesis condition in Theorem 7.1 is the application of the novel inequality proposed in Lemma 7.3 which leads to simple LMI terms as explained in Remark 7.4. Though (7.25) in Theorem 7.1 is bilinear, it has been shown in Theorem 7.2 that convex conditions (7.44)–(7.46) can be derived by the application of Projection Lemma. On the other hand, Algorithm 4 is further proposed to solve the conditions in Theorem 7.1 iteratively, which can be initiated through the feasible solutions of Theorem 7.2. Due to the generality of $r(\cdot) \in [r_1, r_2]^{\mathbb{R}}$, the proposed methodologies have potential to handle a large class of delay functions, through which many delay related problems might be solved. For future works, it is would be interesting to consider systems with more general distributed-delay kernels.

8.2 Future works

The results presented in this thesis can be further extended to tackle new problems in the context of control and optimization, we provide several directions as follows.

- In Chapter 2 and 3, the distributed delay kernel function $\boldsymbol{f}(\cdot)$ can be extended to include any differentiable functions. This requires the modification of Assumption 2.1 to reflect the existence of functions with new types resulted from the differentiation operation $\frac{\mathrm{d}\boldsymbol{f}(\tau)}{\mathrm{d}\tau}$. The inequality in (2.17) can handle any $\boldsymbol{g}(\cdot) \in \mathbb{L}_2(\mathcal{K} \mathbin{⨾} \mathbb{R}^d)$ thus the extension will not introduce difficulties to the construction of lower bounds for the quadratic integral terms in an LKF. Finally, the problems of filtering and dynamical output feedback control may also be considered.

- The results in section Chapter 3 indicate that it is possible to use the LKF approach to construct a dynamical state feedback controller resembling the structure of a predictor controller for an uncertain linear system with input delays. It is of great research interest to extend this strategy to solve the problem of designing dissipative dynamical state feedback controllers for linear systems with discrete delays at both states and inputs. The available results on predictor controllers in [204, 205, 207–210] have indicated that point-wise input delays in a linear system with state point-wise delays can be totally compensated by predictor controllers possessing general distributed-delay integral terms. It is also interesting to look into the problem of designing a dissipative dynamical state feedback controller for a linear system with input distributed delays.

- It is preferable to extend the inequalities proposed in Chapter 4 with more general mathematical settings. For instance, inequalities in Hilbert space.

- The delays in Chapter 5 are assumed to be given constants, it would be of great research value to consider the situation where the delay values are uncertain just like what has been considered in Chapter 6. In that case, the approximation scheme and Lemma 5.2 in Chapter 5 may need to be modified in order to produce constant approximation coefficients which are independent of the uncertain delay r.

- It is natural to consider the possibility of extending the range dissipativity and stability results in Chapter 6 to handle a dissipative range synthesis problem.

- The results in Chapter 6 indicate that using a functional with delay-dependent matrix parameters is beneficial to derived a range dissipativity and stability condition with less conservatism. For the uncertain time-varying delay problem solved in Chapter 7, it would be interesting to consider how to derive a stability (dissipativity) condition which is related to $r(t)$. However, since we do not want $\dot{r}(t)$ to be introduced, thus a functional with $r(t)$-dependent matrix parameters may not be usable.

Appendix A

Proof of Lemma 3.1

Proof. The proof is inspired by the strategies illustrated in [173]. Consider first the situation that $\mathcal{F} = \mathcal{D}$. We first need to find an equivalent condition for the well-posedness of $(I_m - \Delta F)^{-1}$ for all $\Delta \in \mathcal{D}$ with $\mathcal{D}$ in (3.10). Assume first that $F \neq \mathsf{O}_{p\times m}$, it is obvious that $(I_m - \Delta F)^{-1}$ is well defined for all $\Delta \in \mathcal{D}$ if and only if $\forall \Delta \in \mathcal{D}, \mathrm{rank}(I_m - \Delta F) = m$. Furthermore, we know that $\forall \Delta \in \mathcal{D}$, $\mathrm{rank}(I_m - \Delta F) = m$ if and only if $\forall \Delta \in \mathcal{D}$, $(I_m - \Delta F)^\top (I_m - \Delta F) \succ 0$ according to the property of ranks and Gramian matrix .

Let $\mathbb{R}^m \ni \boldsymbol{\mu} := \Delta F \boldsymbol{\theta}$ and $\mathbb{R}^p \ni \boldsymbol{\omega} := F\boldsymbol{\theta}$ with $\boldsymbol{\theta} \in \mathbb{R}^m$, we can conclude that $\forall \Delta \in \mathcal{D}$, $(I_m - \Delta F)^\top (I_m - \Delta F) \succ 0$ if and only if $\forall \boldsymbol{\theta} \in \mathbb{R}^m \backslash \{\mathbf{0}_m\}$, $\forall \Delta \in \mathcal{D}$, $\boldsymbol{\theta}^\top (I_m - \Delta F)^\top (I_m - \Delta F)\boldsymbol{\theta} > 0$ which is equivalent to

$$\begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\mu} \end{bmatrix}^\top \begin{bmatrix} I_m & -I_m \\ * & I_m \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\mu} \end{bmatrix} > 0, \ \forall \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\mu} \end{bmatrix} \in \mathcal{M} \setminus \{\mathbf{0}_{2m}\}, \tag{A.1}$$

$$\text{with } \mathcal{M} \in \left\{ \begin{bmatrix} \acute{\boldsymbol{\theta}} \\ \acute{\boldsymbol{\mu}} \end{bmatrix} \middle| \acute{\boldsymbol{\mu}} = \acute{\Delta} F \acute{\boldsymbol{\theta}} \ \ \& \ \ \acute{\Delta} \in \mathcal{D} \ \ \& \ \ \acute{\boldsymbol{\theta}} \in \mathbb{R}^m \right\} \tag{A.2}$$

Based on the definition of $\mathcal{D}$ and the property of quadratic forms, it is true that $\forall \boldsymbol{\theta} \in \mathbb{R}^m$ and $\forall \Delta \in \mathcal{D}$ we have

$$\boldsymbol{\theta}^\top F^\top \begin{bmatrix} I \\ \Delta \end{bmatrix}^\top \begin{bmatrix} \Theta_1^{-1} & \Theta_2 \\ * & \Theta_3 \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix} F\boldsymbol{\theta} = \begin{bmatrix} F\boldsymbol{\theta} \\ \Delta F \boldsymbol{\theta} \end{bmatrix}^\top \begin{bmatrix} \Theta_1^{-1} & \Theta_2 \\ * & \Theta_3 \end{bmatrix} \begin{bmatrix} F\boldsymbol{\theta} \\ \Delta F \boldsymbol{\theta} \end{bmatrix} \geq 0. \tag{A.3}$$

Now if $\begin{bmatrix} \boldsymbol{\theta}^\top & \boldsymbol{\mu}^\top \end{bmatrix}^\top \in \mathcal{M}$, then we have

$$\begin{bmatrix} F\boldsymbol{\theta} \\ \boldsymbol{\mu} \end{bmatrix}^\top \begin{bmatrix} \Theta_1^{-1} & \Theta_2 \\ * & \Theta_3 \end{bmatrix} \begin{bmatrix} F\boldsymbol{\theta} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} F\boldsymbol{\theta} \\ \Delta F \boldsymbol{\theta} \end{bmatrix}^\top \begin{bmatrix} \Theta_1^{-1} & \Theta_2 \\ * & \Theta_3 \end{bmatrix} \begin{bmatrix} F\boldsymbol{\theta} \\ \Delta F \boldsymbol{\theta} \end{bmatrix} \geq 0 \tag{A.4}$$

for some $\Delta \in \mathcal{D}$ given the property in (A.3). Therefore,

$$\mathcal{M} \subseteq \mathcal{J} := \left\{ \begin{bmatrix} \acute{\boldsymbol{\theta}} \\ \acute{\boldsymbol{\mu}} \end{bmatrix} \in \mathbb{R}^{2m} \middle| \begin{bmatrix} \acute{\boldsymbol{\theta}} \\ \acute{\boldsymbol{\mu}} \end{bmatrix}^\top \begin{bmatrix} F & \mathsf{O}_{p\times m} \\ \mathsf{O}_m & I_m \end{bmatrix}^\top \begin{bmatrix} \Theta_1^{-1} & \Theta_2 \\ * & \Theta_3 \end{bmatrix} \begin{bmatrix} F & \mathsf{O}_{p\times m} \\ \mathsf{O}_m & I_m \end{bmatrix} \begin{bmatrix} \acute{\boldsymbol{\theta}} \\ \acute{\boldsymbol{\mu}} \end{bmatrix} \geq 0 \right\}. \tag{A.5}$$

By (A.5), one can conclude that if

$$\begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\mu} \end{bmatrix}^\top \begin{bmatrix} I_m & -I_m \\ * & I_m \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\mu} \end{bmatrix} > 0, \ \forall \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\mu} \end{bmatrix} \in \mathcal{J} \setminus \{\mathbf{0}_{2m}\} \tag{A.6}$$

holds then (A.1) holds since $\mathcal{M} \subseteq \mathcal{J}$. Invoking S-procedure to (A.6) concludes that (A.6) is true if and only if [1]

[1] Since (A.7) with $\alpha = 0$ cannot be feasible, thus we only need $\alpha > 0$ in (A.15) for the existence of $\alpha \geq 0$ introduced by the application of S-procedure.

$$\exists\alpha > 0: \begin{bmatrix} I_m & -I_m - \alpha F^\top\Theta_2 \\ * & I_m - \alpha\Theta_3 \end{bmatrix} - \alpha \begin{bmatrix} F^\top \\ \mathsf{O}_{m\times p} \end{bmatrix} \Theta_1^{-1} \begin{bmatrix} F & \mathsf{O}_{p\times m} \end{bmatrix}$$

$$\begin{bmatrix} I_m & -I_m - \alpha F^\top\Theta_2 \\ * & I_m - \alpha\Theta_3 \end{bmatrix} - \begin{bmatrix} \alpha F^\top \\ \mathsf{O}_{m\times p} \end{bmatrix} (\alpha\Theta_1)^{-1} \begin{bmatrix} \alpha F & \mathsf{O}_{p\times m} \end{bmatrix} \succ 0. \quad \text{(A.7)}$$

By applying the Schur complement to (A.7) with the fact that $\Theta_1^{-1} \succ 0$, it gives (3.9). Since $\forall\Delta \in \mathcal{D}$, $(I_m - \Delta F)^\top(I_m - \Delta F) \succ 0$ is equivalent to (A.1) which is inferred by (3.9), thus one can conclude that $(I_m - \Delta F)^{-1}$, $F \neq \mathsf{O}_{p\times m}$ is well defined for all $\Delta \in \mathcal{D}$ if (3.9) holds. Finally, it is obvious that the well-posdness of $(I_m - \Delta F)^{-1}$ is automatically ensured if $F = \mathsf{O}_{p\times m}$ where (3.9) does not need to be considered.

Now we shall proceed to prove the rest of the results in Lemma 3.1. Assume that (3.9) is satisfied for a $\alpha > 0$ which infers that $(I_m - \Delta F)^{-1}$ is well defined. by the definition of negative definite matrices, we know that

$$\forall\Delta \in \mathcal{D}, \;\; \Phi + \mathbf{Sy}\left[G(I_m - \Delta F)^{-1}\Delta H\right] \prec 0 \quad \text{(A.8)}$$

if and only if

$$\forall\Delta \in \mathcal{D}, \;\; \forall\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}, \;\; \mathbf{x}^\top \left(\Phi + \mathbf{Sy}\left[G(I_m - \Delta F)^{-1}\Delta H\right]\right)\mathbf{x} < 0. \quad \text{(A.9)}$$

Now let $\mathbb{R}^m \ni \boldsymbol{\varrho} := (I_m - \Delta F)^{-1}\Delta H\mathbf{x}$ and consider the fact that $(I_m - \Delta F)^{-1}$ is well defined, we have $\boldsymbol{\varrho} = \Delta H\mathbf{x} + \Delta F\boldsymbol{\varrho}$. By using $\boldsymbol{\varrho} = \Delta H\mathbf{x} + \Delta F\boldsymbol{\varrho}$, we can reformulate (A.9) into

$$\begin{bmatrix} \mathbf{x} \\ \boldsymbol{\varrho} \end{bmatrix}^\top \begin{bmatrix} \Phi & G \\ * & \mathsf{O}_m \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\varrho} \end{bmatrix} < 0, \;\; \forall \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\varrho} \end{bmatrix} \in \mathcal{X} \setminus \{\mathbf{0}_{n+m}\} \quad \text{(A.10)}$$

$$\text{with } \;\mathcal{X} \in \left\{ \begin{bmatrix} \acute{\mathbf{x}} \\ \acute{\boldsymbol{\varrho}} \end{bmatrix} \in \mathbb{R}^{n+m} \,\middle|\, \acute{\boldsymbol{\varrho}} := (I_m - \acute{\Delta}F)^{-1}\acute{\Delta}H\acute{\mathbf{x}} \;\;\&\;\; \acute{\Delta} \in \mathcal{D} \right\}.$$

Based on the definition of $\mathcal{D}$ and the property of quadratic forms, it is true that $\forall\mathbf{x} \in \mathbb{R}^n$ and $\forall\Delta \in \mathcal{D}$ we have

$$(H\mathbf{x} + F\boldsymbol{\varrho})^\top \begin{bmatrix} I \\ \Delta \end{bmatrix}^\top \begin{bmatrix} \Theta_1^{-1} & \Theta_2 \\ * & \Theta_3 \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix} (H\mathbf{x} + F\boldsymbol{\varrho}) =$$

$$\begin{bmatrix} H\mathbf{x} + F\boldsymbol{\varrho} \\ \Delta\,(H\mathbf{x} + F\boldsymbol{\varrho}) \end{bmatrix}^\top \begin{bmatrix} \Theta_1^{-1} & \Theta_2 \\ * & \Theta_3 \end{bmatrix} \begin{bmatrix} H\mathbf{x} + F\boldsymbol{\varrho} \\ \Delta\,(H\mathbf{x} + F\boldsymbol{\varrho}) \end{bmatrix} = [*]^\top \begin{bmatrix} \Theta_1^{-1} & \Theta_2 \\ * & \Theta_3 \end{bmatrix} \begin{bmatrix} H\mathbf{x} + F\boldsymbol{\varrho} \\ \boldsymbol{\varrho} \end{bmatrix} \geq 0 \quad \text{(A.11)}$$

where $\boldsymbol{\varrho} = (I_m - \Delta F)^{-1}\Delta H\mathbf{x}$ which is equivalent to $\boldsymbol{\varrho} = \Delta H\mathbf{x} + \Delta F\boldsymbol{\varrho}$. Now if $\begin{bmatrix} \mathbf{x}^\top & \boldsymbol{\varrho}^\top \end{bmatrix}^\top \in \mathcal{X}$, then we have

$$\begin{bmatrix} \mathbf{x} \\ \boldsymbol{\varrho} \end{bmatrix}^\top \begin{bmatrix} H & F \\ \mathsf{O}_{m\times p} & I_m \end{bmatrix}^\top \begin{bmatrix} \Theta_1^{-1} & \Theta_2 \\ * & \Theta_3 \end{bmatrix} \begin{bmatrix} H & F \\ \mathsf{O}_{m\times p} & I_m \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\varrho} \end{bmatrix} \geq 0 \quad \text{(A.12)}$$

given the property in (A.11) and the fact that $\forall \begin{bmatrix} \mathbf{x}^\top & \boldsymbol{\varrho}^\top \end{bmatrix}^\top \in \mathcal{X}$, $\exists\Delta \in \mathcal{D}$ such that $\boldsymbol{\varrho} = \Delta H\mathbf{x} + \Delta F\boldsymbol{\varrho}$. As a result, one can derive the following relation

$$\mathcal{X} \subseteq \mathcal{Y} := \left\{ \begin{bmatrix} \acute{\mathbf{x}} \\ \acute{\boldsymbol{\varrho}} \end{bmatrix} \,\middle|\, \begin{bmatrix} \acute{\mathbf{x}} \\ \acute{\boldsymbol{\varrho}} \end{bmatrix}^\top \begin{bmatrix} H & F \\ \mathsf{O}_{m\times p} & I_m \end{bmatrix}^\top \begin{bmatrix} \Theta_1^{-1} & \Theta_2 \\ * & \Theta_3 \end{bmatrix} \begin{bmatrix} H & F \\ \mathsf{O}_{m\times p} & I_m \end{bmatrix} \begin{bmatrix} \acute{\mathbf{x}} \\ \acute{\boldsymbol{\varrho}} \end{bmatrix} \geq 0 \right\}. \quad \text{(A.13)}$$

Now by (A.13) it is obvious to see that (A.10) holds if

$$\begin{bmatrix} \mathbf{x} \\ \boldsymbol{\varrho} \end{bmatrix}^\top \begin{bmatrix} \Phi & G \\ * & \mathsf{O}_m \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\varrho} \end{bmatrix} < 0, \;\; \forall \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\varrho} \end{bmatrix} \in \mathcal{Y} \setminus \{\mathbf{0}_{n+m}\} \quad \text{(A.14)}$$

holds. Clearly, the structure of (A.14) and $\mathcal{Y}$ enables us to apply S-procedure so that we can conclude that (A.14) is true if and only if[2].

$$\exists\kappa > 0: \begin{bmatrix} \Phi & G + \kappa H^\top \Theta_2 \\ * & \kappa F^\top \Theta_2 + \kappa \Theta_2^\top F + \kappa \Theta_3 \end{bmatrix} + \kappa \begin{bmatrix} H^\top \\ F^\top \end{bmatrix} \Theta_1^{-1} \begin{bmatrix} H & F \end{bmatrix}$$

$$= \begin{bmatrix} \Phi & G + \kappa H^\top \Theta_2 \\ * & \kappa F^\top \Theta_2 + \kappa \Theta_2^\top F + \kappa \Theta_3 \end{bmatrix} + \begin{bmatrix} \kappa H^\top \\ \kappa F^\top \end{bmatrix} (\kappa\Theta_1)^{-1} \begin{bmatrix} \kappa H & \kappa F \end{bmatrix} \prec 0. \quad \text{(A.15)}$$

It should be emphasized here that for the cases of having a single constraint like (A.14), S-procedure can produce an equivalent condition. Applying the Schur complement to (A.15) with $\Theta_1^{-1} \succ 0$ yields (3.11) which is equivalent to (A.14). Since (A.14) infers (A.10) which is equivalent to (3.10), hence we have shown that (3.11) is a sufficient condition for (3.10) with $\mathcal{F} = \mathcal{D}$. Finally, it is obvious that (3.10) and the well-posedness of $(I_m - \Delta F)^{-1}$ with $\mathcal{F} \subseteq \mathcal{D}$ are inferred by (3.9) and (3.11), respectively.

If $\Theta_1^{-1} = \mathsf{O}_p$, (A.15) and (A.7) become (3.13) and (3.12), respectively, where Θ_1 does not need to be considered as the Schur complement does not need to be applied at the steps of (A.15) and (A.7).

Finally, it is important to mention that (3.9) and (3.11) can handle an uncertainty term $G\Delta(I_p - F\Delta)^{-1}H$ via Sylvester's determinant identity[3] $\det(I_p - F\Delta) = \det(I_m - \Delta F)$, and $\Delta(I_p - F\Delta)^{-1} = (I_m - \Delta F)^{-1}\Delta$ which is a special case of (A.3) in [173]. ■

[2]Since (A.15) with $\kappa = 0$ cannot be feasible, thus $\kappa \geq 0$ is applied in (A.15) for the condition $\kappa \geq 0$ introduced by the application of S-procedure

[3]See B.1.16 in [348]

Appendix B

Proof of Theorem 4.1

Proof. The proof is similar to the Lemma 2.3 apart from the fact that we have a weight function $\varpi(\cdot)$ here.

Let $\boldsymbol{\varepsilon}(\tau) := \boldsymbol{x}(\tau) - F^\top(\tau)(\mathsf{F}\otimes I_n)\int_{\mathcal{K}}\varpi(\theta)F(\theta)\boldsymbol{x}(\theta)\mathsf{d}\theta$, where $F(\tau) = \boldsymbol{f}(\tau)\otimes I_n$. Considering the expression of $\boldsymbol{\varepsilon}(\cdot)$ with $\int_{\mathcal{K}}\varpi(\tau)\boldsymbol{\varepsilon}^\top(\tau)U\boldsymbol{\varepsilon}(\tau)\mathsf{d}\tau$, we have

$$\int_{\mathcal{K}}\varpi(\tau)\boldsymbol{\varepsilon}^\top(\tau)U\boldsymbol{\varepsilon}(\tau)\mathsf{d}\tau = \int_{\mathcal{K}}\varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau - 2\int_{\mathcal{K}}\varpi(\tau)\boldsymbol{x}^\top(\tau)UF^\top(\tau)\mathsf{d}\tau(\mathsf{F}\otimes I_n)\boldsymbol{\vartheta} + \boldsymbol{\vartheta}^\top\int_{\mathcal{K}}\varpi(\tau)(\mathsf{F}\otimes I_n)^\top F(\tau)UF^\top(\tau)(\mathsf{F}\otimes I_n)\mathsf{d}\tau\boldsymbol{\vartheta}, \quad \text{(B.1)}$$

where $\boldsymbol{\vartheta} := \int_{\mathcal{K}}\varpi(\theta)F(\theta)\boldsymbol{x}(\theta)\mathsf{d}\theta$. Now apply (2.1) to the term $UF^\top(\tau)$ with $F(\tau) = \boldsymbol{f}(\tau)\otimes I_n$ and $U = U^\top$, we have

$$U(\boldsymbol{f}^\top(\tau)\otimes I_n) = \boldsymbol{f}^\top(\tau)\otimes U = \left(\boldsymbol{f}^\top(\tau)\otimes I_n\right)(I_d\otimes U) = F^\top(\tau)(I_d\otimes U). \quad \text{(B.2)}$$

Apply (B.2) to some of the terms in (B.1). It follows that

$$\int_{\mathcal{K}}\varpi(\tau)\boldsymbol{x}^\top(\tau)UF^\top(\tau)\mathsf{d}\tau(\mathsf{F}\otimes I_n)\boldsymbol{\vartheta} = \int_{\mathcal{K}}\varpi(\tau)\boldsymbol{x}^\top(\tau)F^\top(\tau)\mathsf{d}\tau\,(I_d\otimes U)\,(\mathsf{F}\otimes I_n)\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^\top(I_d\otimes U)(\mathsf{F}\otimes I_n)\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^\top(\mathsf{F}\otimes U)\boldsymbol{\vartheta}. \quad \text{(B.3)}$$

By (B.2) and (2.1) and the fact that $\mathsf{F} = \mathsf{F}^\top$, we have

$$\int_{\mathcal{K}}(\mathsf{F}\otimes I_n)^\top\varpi(\tau)F(\tau)UF^\top(\tau)(\mathsf{F}\otimes I_n)\mathsf{d}\tau = (\mathsf{F}\otimes I_n)\int_{\mathcal{K}}\varpi(\tau)F(\tau)F^\top(\tau)\mathsf{d}\tau(\mathsf{F}\otimes U) \\ (\mathsf{F}\otimes I_n)\left[\int_{\mathcal{K}}\varpi(\tau)\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau\otimes I_n\right](\mathsf{F}\otimes U) = \mathsf{F}\otimes U \quad \text{(B.4)}$$

where $F(\tau) = \boldsymbol{f}(\tau)\otimes I_n$ and $\mathsf{F}^{-1} = \int_{\mathcal{K}}\varpi(\tau)\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau$. Substituting (B.4) into (B.1) and also considering the relation in (B.3) yields

$$\int_{\mathcal{K}}\varpi(\tau)\boldsymbol{\varepsilon}^\top(\tau)U\boldsymbol{\varepsilon}(\tau)\mathsf{d}\tau = \int_{\mathcal{K}}\varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau - \int_{\mathcal{K}}\varpi(\tau)\boldsymbol{x}^\top(\tau)F^\top(\tau)\mathsf{d}\tau\,(\mathsf{F}\otimes U)\int_{\mathcal{K}}\varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau. \quad \text{(B.5)}$$

Given $U\succ 0$, (B.5) gives (4.3). This finishes the proof. ■

Appendix C

Proof of Theorem 4.4

Proof. Let $U \succ 0$ and $\varpi(\cdot)$ and $\boldsymbol{f}(\cdot)$ satisfying (4.2) be given which gives $\mathsf{F}^{-1} = \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\mathsf{d}\tau \in \mathbb{S}^d_{\succ 0}$. Using the Schur complement to (4.22) with $U \succ 0$ concludes that $Y \succeq X^\top U^{-1}X$ for any Y, $X = \mathbf{Row}_{i=1}^d X_i \in \mathbb{R}^{n\times\rho dn}$ satisfying (4.22). Now consider W in Theorem 4.3 with $Y \succeq X^\top U^{-1}X$ and (4.21), we have

$$\begin{aligned}\mathsf{W} \succeq \int_{\mathcal{K}} \varpi(\tau)(\boldsymbol{f}^\top(\tau)\otimes I_{nd})X^\top U^{-1}X(\boldsymbol{f}(\tau)\otimes I_{nd})\mathsf{d}\tau = \widehat{X}^\top \int_{\mathcal{K}} \varpi(\tau)(\boldsymbol{f}(\tau)\otimes I_n)U^{-1}\left(\boldsymbol{f}^\top(\tau)\otimes I_n\right)\mathsf{d}\tau\widehat{X}\\ = \widehat{X}^\top\left(\int_{\mathcal{K}} \varpi(\tau)\left(I_d\otimes U^{-1}\right)(\boldsymbol{f}(\tau)\boldsymbol{f}^\top(\tau)\otimes I_n)\mathsf{d}\tau\right)\widehat{X} = \widehat{X}^\top\left(\mathsf{F}^{-1}\otimes U^{-1}\right)\widehat{X}\end{aligned} \tag{C.1}$$

with $\widehat{X} = \mathbf{Col}_{i=1}^d X_i \in \mathbb{R}^{dn\times\rho n}$. By the structures in (C.1), one can also conclude that W in Theorem 4.3 satisfies

$$\mathsf{W} = \widehat{X}^\top\left(\mathsf{F}^{-1}\otimes U^{-1}\right)\widehat{X} \tag{C.2}$$

with

$$\widehat{X} = (\mathsf{F}\otimes U)\,\Upsilon, \quad Y = X^\top U^{-1}X \tag{C.3}$$

for a given $U \succ 0$, where the values of X for Y in (C.3) can be determined by the structural relation $X = \mathbf{Row}_{i=1}^d X_i \in \mathbb{R}^{n\times\rho dn}$ with $\widehat{X} = (\mathsf{F}\otimes U)\,\Upsilon = \mathbf{Col}_{i=1}^d X_i \in \mathbb{R}^{dn\times\rho n}$.

By (C.1)–(C.3) with (4.14), we have

$$\mathbf{Sy}\left(\Upsilon^\top\widehat{X}\right) - \mathsf{W} \preceq \mathbf{Sy}\left(\Upsilon^\top\widehat{X}\right) - \widehat{X}^\top\left(\mathsf{F}^{-1}\otimes U^{-1}\right)\widehat{X} \preceq \Upsilon^\top(\mathsf{F}\otimes U)\,\Upsilon \tag{C.4}$$

holds for any Y and $X = \mathbf{Row}_{i=1}^d X_i \in \mathbb{R}^{n\times\rho dn}$ satisfying (4.22), and

$$\mathbf{Sy}\left(\Upsilon^\top\widehat{X}\right) - \mathsf{W} = \mathbf{Sy}\left(\Upsilon^\top\widehat{X}\right) - \widehat{X}^\top\left(\mathsf{F}^{-1}\otimes U^{-1}\right)\widehat{X} = \Upsilon^\top(\mathsf{F}\otimes U)\,\Upsilon \tag{C.5}$$

if Y and X satisfy the equalities in (C.3). Given $\Upsilon\boldsymbol{z} = \int_{\mathcal{K}} \varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau$ in Theorem 4.3 in light of the results in (C.4) and (C.5), we have

$$\boldsymbol{z}^\top\left[\mathbf{Sy}\left(\Upsilon^\top\widehat{X}\right) - \mathsf{W}\right]\boldsymbol{z} \le \boldsymbol{z}^\top\Upsilon^\top(\mathsf{F}\otimes U)\,\Upsilon\boldsymbol{z} = \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)F^\top(\tau)\mathsf{d}\tau\,(\mathsf{F}\otimes U)\int_{\mathcal{K}} \varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau \tag{C.6}$$

holds for any Y and X satisfying (4.22) with $U \succ 0$, and

$$\boldsymbol{z}^\top\left[\mathbf{Sy}\left(\Upsilon^\top\widehat{X}\right) - \mathsf{W}\right]\boldsymbol{z} = \boldsymbol{z}^\top\Upsilon^\top(\mathsf{F}\otimes U)\,\Upsilon\boldsymbol{z} = \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)F^\top(\tau)\mathsf{d}\tau\,(\mathsf{F}\otimes U)\int_{\mathcal{K}} \varpi(\tau)F(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau \tag{C.7}$$

holds with (C.3).

As a result, the above arguments show that under the same $\varpi(\cdot)$, U and $\boldsymbol{f}(\cdot)$, one can always find X and Y for (4.22) to render (4.23) to become identical to (4.3) in which case it corresponds to the smallest achievable inequality bound gap of (4.23). Since the smallest achievable inequality bound gap of (4.17) is also identical to (4.3), this finishes the proof of this theorem. ■

Appendix D

Proof of Lemma 5.2

Proof. The proof of Lemma 5.2 is inspired by the proof of Lemma 2 in [188] and the proof of Lemma 5 in [57]. Firstly, one can conclude that $\mathcal{E}_d$ in (5.26) is invertible for any $\mathsf{f}(\cdot) \in \mathbb{L}^2_{\varpi}(\mathcal{K} ; \mathbb{R}^d)$, $\mathsf{g}(\cdot) \in \mathbb{L}^2_{\varpi}(\mathcal{K} ; \mathbb{R}^\delta)$ satisfying (5.24) since

$$\mathcal{E}_d = \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{e}(\tau)\boldsymbol{e}^\top(\tau)\mathsf{d}\tau = \begin{bmatrix} I_\delta & -\mathsf{A} \end{bmatrix} \int_{\mathcal{K}} \varpi(\tau) \begin{bmatrix} \mathsf{g}(\tau) \\ \mathsf{f}(\tau) \end{bmatrix} \begin{bmatrix} \mathsf{g}^\top(\tau) & \mathsf{f}^\top(\tau) \end{bmatrix} \mathsf{d}\tau \begin{bmatrix} I_\delta & -\mathsf{A} \end{bmatrix}^\top \succ 0, \tag{D.1}$$

where the positive definite matrix inequality can be derived based on (5.24) and the property of congruence transformations with the fact that rank $\begin{bmatrix} I_\delta & -\mathsf{A} \end{bmatrix} = \delta$. Consequently, $\mathcal{E}_d^{-1}$ is well defined.

Let $\boldsymbol{v}(\tau) := \boldsymbol{x}(\tau) - \mathsf{F}^\top(\tau)(\mathcal{F}_d \otimes I_n) \int_{\mathcal{K}} \varpi(\theta)\mathsf{F}(\theta)\boldsymbol{x}(\theta)\mathsf{d}\theta - \mathsf{E}^\top(\tau)\left(\mathcal{E}_d^{-1} \otimes I_n\right) \int_{\mathcal{K}} \varpi(\theta)\mathsf{E}(\theta)\boldsymbol{x}(\theta)\mathsf{d}\theta$, where $\mathsf{F}(\cdot)$, $\mathsf{E}(\cdot)$ have been given in Lemma 5.2. By $\mathsf{A} = \int_{\mathcal{K}} \varpi(\tau)\mathsf{g}(\tau)\mathsf{f}^\top(\tau)\mathsf{d}\tau\mathcal{F}_d$ and $\boldsymbol{e}(\tau) = \mathsf{g}(\tau) - \mathsf{A}\mathsf{f}(\tau) \in \mathbb{R}^\delta$, we have

$$\begin{aligned}\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{e}(\tau)\mathsf{f}^\top(\tau)\mathsf{d}\tau &= \int_{\mathcal{K}} \varpi(\tau)\left[\mathsf{g}(\tau) - \mathsf{A}\mathsf{f}(\tau)\right]\mathsf{f}^\top(\tau)\mathsf{d}\tau = \int_{\mathcal{K}} \varpi(\tau)\mathsf{g}(\tau)\mathsf{f}^\top(\tau)\mathsf{d}\tau - \mathsf{A}\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}(\tau)\mathsf{f}^\top(\tau)\mathsf{d}\tau \\ &= \int_{\mathcal{K}} \varpi(\tau)\mathsf{g}(\tau)\mathsf{f}^\top(\tau)\mathsf{d}\tau - \left(\int_{\mathcal{K}} \varpi(\tau)\mathsf{g}(\tau)\mathsf{f}^\top(\tau)\mathsf{d}\tau\right)\mathcal{F}_d\mathcal{F}_d^{-1} = \mathsf{O}_{\delta\times d}.\end{aligned} \tag{D.2}$$

Now substituting the expression of $\boldsymbol{v}(\cdot)$ into $\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{v}^\top(\tau)U\boldsymbol{v}(\tau)\mathsf{d}\tau$ and considering (D.2) yields

$$\begin{aligned}&\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{v}^\top(\tau)U\boldsymbol{v}(\tau)\mathsf{d}\tau = \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau - 2\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\mathsf{F}^\top(\tau)\mathsf{d}\tau(\mathcal{F}_d \otimes I_n)\boldsymbol{\zeta} \\ &+ \boldsymbol{\zeta}^\top\int_{\mathcal{K}} \varpi(\tau)(\mathcal{F}_d \otimes I_n)^\top\mathsf{F}(\tau)U\mathsf{F}^\top(\tau)(\mathcal{F}_d \otimes I_n)\mathsf{d}\tau\boldsymbol{\zeta} - 2\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\mathsf{E}^\top(\tau)\mathsf{d}\tau(\mathcal{E}_d^{-1} \otimes I_n)\boldsymbol{\omega} \\ &\qquad + \boldsymbol{\omega}^\top\int_{\mathcal{K}} \varpi(\tau)(\mathcal{E}_d^{-1} \otimes I_n)^\top\mathsf{E}(\tau)U\mathsf{E}^\top(\tau)(\mathcal{E}_d^{-1} \otimes I_n)\mathsf{d}\tau\boldsymbol{\omega}\end{aligned} \tag{D.3}$$

where $\boldsymbol{\zeta} := \int_{\mathcal{K}} \varpi(\theta)\mathsf{F}(\theta)\boldsymbol{x}(\theta)\mathsf{d}\theta$ and $\boldsymbol{\omega} := \int_{\mathcal{K}} \varpi(\theta)\mathsf{E}(\theta)\boldsymbol{x}(\theta)\mathsf{d}\theta$. Apply (2.1) to the term $U\mathsf{F}^\top(\tau)$ and $U\mathsf{E}^\top(\tau)$ and consider $\mathsf{F}(\tau) = \mathsf{f}(\tau) \otimes I_n$ and $\mathsf{E}(\tau) = \boldsymbol{e}(\tau) \otimes I_n$, then we have

$$U\mathsf{F}^\top(\tau) = \mathsf{F}^\top(\tau)(I_d \otimes U), \quad U\mathsf{E}^\top(\tau) = \mathsf{E}^\top(\tau)(I_\delta \otimes U) \tag{D.4}$$

given $(X \otimes Y)^\top = X^\top \otimes Y^\top$. One the other hand, it is true that

$$\begin{aligned}\int_{\mathcal{K}} \varpi(\tau)\mathsf{F}(\tau)\mathsf{F}^\top(\tau)\mathsf{d}\tau &= \left(\int_{\mathcal{K}} \varpi(\tau)\mathsf{f}(\tau)\mathsf{f}^\top(\tau)\mathsf{d}\tau\right) \otimes I_n = \mathcal{F}_d^{-1} \otimes I_n \\ \int_{\mathcal{K}} \varpi(\tau)\mathsf{E}(\tau)\mathsf{E}^\top(\tau)\mathsf{d}\tau &= \left(\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{e}(\tau)\boldsymbol{e}^\top(\tau)\mathsf{d}\tau\right) \otimes I_n = \mathcal{E}_d \otimes I_n\end{aligned} \tag{D.5}$$

since $\mathsf{F}(\tau) = \mathsf{f}(\tau) \otimes I_n$ and $\mathsf{E}(\tau) = \boldsymbol{e}(\tau) \otimes I_n$. By using (D.4) and (D.5) with (2.1) to some of the terms in (D.3), it follows that

$$\begin{aligned}&\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\mathsf{F}^\top(\tau)\mathsf{d}\tau(\mathcal{F}_d \otimes I_n)\boldsymbol{\zeta} = \boldsymbol{\zeta}^\top(\mathcal{F}_d \otimes U)\boldsymbol{\zeta} \\ &\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\mathsf{E}^\top(\tau)\mathsf{d}\tau(\mathcal{E}_d^{-1} \otimes I_n)\boldsymbol{\omega} = \boldsymbol{\omega}^\top(\mathcal{E}_d^{-1} \otimes U)\boldsymbol{\omega}.\end{aligned} \tag{D.6}$$

and

$$
\begin{aligned}
&\int_{\mathcal{K}} (\mathcal{F}_d \otimes I_n)^\top \varpi(\tau)\mathsf{F}(\tau) U \mathsf{F}^\top(\tau)(\mathcal{F}_d \otimes I_n)\mathsf{d}\tau = (\mathcal{F}_d \otimes I_n)\int_{\mathcal{K}} \varpi(\tau)\mathsf{F}(\tau)\mathsf{F}^\top(\tau)\mathsf{d}\tau(\mathcal{F}_d \otimes U) = \mathcal{F}_d \otimes U \\
&\int_{\mathcal{K}} (\mathcal{E}_d^{-1} \otimes I_n)^\top \varpi(\tau)\mathsf{E}(\tau) U \mathsf{E}^\top(\tau)(\mathcal{E}_d^{-1} \otimes I_n)\mathsf{d}\tau = (\mathcal{E}_d^{-1} \otimes I_n)\int_{\mathcal{K}} \varpi(\tau)\mathsf{E}(\tau)\mathsf{E}^\top(\tau)\mathsf{d}\tau \left(\mathcal{E}_d^{-1} \otimes U\right) \\
&\qquad = \mathcal{E}_d^{-1} \otimes U.
\end{aligned}
\tag{D.7}
$$

Substituting (D.7) into (D.3) and also considering the relations in (D.6) yields

$$
\begin{aligned}
\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{v}^\top(\tau) U \boldsymbol{v}(\tau)\mathsf{d}\tau = &\int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau) U \boldsymbol{x}(\tau)\mathsf{d}\tau - \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)\mathsf{F}^\top(\tau)\mathsf{d}\tau\,(\mathcal{F}_d \otimes U)\int_{\mathcal{K}} \varpi(\tau)\mathsf{F}(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau \\
&- \int_{\mathcal{K}} \varpi(\tau)\boldsymbol{x}^\top(\tau)\mathsf{E}^\top(\tau)\mathsf{d}\tau \left(\mathcal{E}_d^{-1} \otimes U\right)\int_{\mathcal{K}} \varpi(\tau)\mathsf{E}(\tau)\boldsymbol{x}(\tau)\mathsf{d}\tau.
\end{aligned}
\tag{D.8}
$$

Given $U \succeq 0$, (D.8) gives (5.25). This finishes the proof. ■

Appendix E

Proof of Lemma 7.3

Proof. The proof is based on the insights illustrated in [341]. Consider the equality

$$\int_{-r_2}^{-r_1} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau = \int_{-\varrho}^{-r_1} \varpi(\tau)\begin{bmatrix}\boldsymbol{x}(\tau)\\ \mathbf{0}_n\end{bmatrix}^\top \begin{bmatrix}U & Y\\ * & U\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(\tau)\\ \mathbf{0}_n\end{bmatrix}\mathsf{d}\tau$$
$$+\int_{-r_2}^{-\varrho} \varpi(\tau)\begin{bmatrix}\mathbf{0}_n\\ \boldsymbol{x}(\tau)\end{bmatrix}^\top \begin{bmatrix}U & Y\\ * & U\end{bmatrix}\begin{bmatrix}\mathbf{0}_n\\ \boldsymbol{x}(\tau)\end{bmatrix}\mathsf{d}\tau = \int_{-r_2}^{-r_1} \boldsymbol{y}^\top(\tau)\begin{bmatrix}U & Y\\ * & U\end{bmatrix}\boldsymbol{y}(\tau)\mathsf{d}\tau \quad \text{(E.1)}$$

which holds for any $Y \in \mathbb{R}^{n\times n}$ with

$$\mathbb{R}^{2n} \ni \boldsymbol{y}(\tau) := \begin{cases}\begin{bmatrix}\boldsymbol{x}(\tau)\\ \mathbf{0}_n\end{bmatrix}, \forall\tau \in [-\varrho, -r_1]\\ \begin{bmatrix}\mathbf{0}_n\\ \boldsymbol{x}(\tau)\end{bmatrix}, \forall\tau \in [-r_2, -\varrho].\end{cases} \quad \text{(E.2)}$$

Assume that $Y \in \mathbb{R}^{n\times n}$ satisfies $\left[\begin{smallmatrix}U & Y\\ * & U\end{smallmatrix}\right] \succeq 0$, then one can apply (7.16) to the rightmost integral in (E.1) with $\mathcal{K} = [-r_2, -r_1]$ and $\boldsymbol{f}(\cdot) \in \mathbb{L}^2_\varpi\big(\mathcal{K} \,;\, \mathbb{R}^d\big)$, which yields

$$\int_{-r_2}^{-r_1} \varpi(\tau)\boldsymbol{x}^\top(\tau)U\boldsymbol{x}(\tau)\mathsf{d}\tau = \int_{-r_2}^{-r_1} \varpi(\tau)\boldsymbol{y}^\top(\tau)\begin{bmatrix}U & Y\\ * & U\end{bmatrix}\boldsymbol{y}(\tau)\mathsf{d}\tau$$
$$\geq \left(\int_{-r_2}^{-r_1} \varpi(\tau)\boldsymbol{y}^\top(\tau)\Big[I_{2n}\otimes \boldsymbol{f}(\tau)\Big]^\top\mathsf{d}\tau\right)\left(\begin{bmatrix}U & Y\\ * & U\end{bmatrix}\otimes \mathsf{F}^{-1}\right)\left(\int_{-r_2}^{-r_1} \varpi(\tau)\Big[I_{2n}\otimes \boldsymbol{f}(\tau)\Big]\boldsymbol{y}(\tau)\mathsf{d}\tau\right). \quad \text{(E.3)}$$

Furthermore, it follows that

$$\int_{-r_2}^{-r_1} \Big[I_{2n}\otimes \boldsymbol{f}(\tau)\Big]\boldsymbol{y}(\tau)\varpi(\tau)\mathsf{d}\tau = \int_{-\varrho}^{-r_1} \begin{bmatrix}I_n\otimes \boldsymbol{f}(\tau) & \mathsf{O}_{dn}\\ \mathsf{O}_{dn} & I_n\otimes \boldsymbol{f}(\tau)\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(\tau)\\ \mathbf{0}_n\end{bmatrix}\varpi(\tau)\mathsf{d}\tau$$
$$+\int_{-r_2}^{-\varrho} \begin{bmatrix}I_n\otimes \boldsymbol{f}(\tau) & \mathsf{O}_{dn}\\ \mathsf{O}_{dn} & I_n\otimes \boldsymbol{f}(\tau)\end{bmatrix}\begin{bmatrix}\mathbf{0}_n\\ \boldsymbol{x}(\tau)\end{bmatrix}\varpi(\tau)\mathsf{d}\tau = \begin{bmatrix}\int_{-\varrho}^{-r_1}[I_n\otimes \boldsymbol{f}(\tau)]\,\boldsymbol{x}(\tau)\varpi(\tau)\mathsf{d}\tau\\ \int_{-r_2}^{-\varrho}[I_n\otimes \boldsymbol{f}(\tau)]\,\boldsymbol{x}(\tau)\varpi(\tau)\mathsf{d}\tau\end{bmatrix}. \quad \text{(E.4)}$$

Substituting (E.4) into (E.3) and using the properties of the commutation matrix in (7.12) yields (7.18). This finishes the proof. ∎

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