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On regular dessins d'enfants with $4g$ automorphisms and a curve of Wiman

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Abstract. In this article we show that with a few exceptions, every regular dessin d'enfant with genus g having exactly $4g$ automorphisms is embedded in Wiman's curve of type II.

1. Introduction

In 1896, Wiman [**19**] gave two (smooth, irreducible) complex algebraic curves for each genus $g \geq 2$: one with equation $y^2 = x^{2g+1} - 1$ admitting an automorphism of order $4g + 2$, and another with equation $y^2 = x(x^{2g} - 1)$ admitting an automorphism of order $4g$. These curves are known as Wiman's curves of type I and II respectively. In 1997 Kulkarni [**16**] showed that, with one exception for genus $g = 3$, Wiman's curve of type II is the only Riemann surface of given genus $g \geq 2$ admitting an automorphism of order $4g$, the exception being Picard's curve ($y^3 = x^4 - 1$); see also [**15**]. Wiman's curves of type II have exactly $8g$ automorphisms, except in the case $g = 2$, when the curve has 48 automorphisms (and is the curve of genus 2 having the maximum number of automorphisms). Recently Bujalance, Costa and Izquierdo [**6**] showed that for $g \geq 31$ the curves admitting exactly $4g$ automorphisms form an open curve $\mathcal{F}$ in moduli space. (In fact, this is the complex projective line (or Riemann sphere) with three punctures.)

The methods used to prove the results above were combinatorial. By the works of Riemann, Poincaré, Klein and others, every complex (real) algebraic curve can be uniformised by a class of Fuchsian (NEC) groups. This provides a well defined hyperbolic structure on the surface underlying the algebraic curve.

On the other hand, in 1980 Belyi [**1**] made an influential discovery now known as Belyi's Theorem: a complex curve X is defined on a number field if and only if X is a covering of the projective line ramified at most over three points, say 0, 1 and ∞. The covering map is called the Belyi map. In combinatorial language,

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a complex curve X is defined over a number field if and only if its uniformising (surface) Fuchsian group Γ is a subgroup of a triangle group $\Delta(l,m,n)$. The Belyi map induces a cell-decomposition of X: the *dessin d'enfant* $\mathcal{H}$, also called a map (if $m = 2$) or a hypermap [**11**]. The pre-images of 0 give the *hypervertices,* the pre-images of 1 the *hyperedges*, and the pre-images of ∞ the *hyperfaces* of $\mathcal{H}$. The genus of $\mathcal{H}$ is the genus of X, and the uniformising group H is called the *hypermap group*. The dessin d'enfant $\mathcal{H}$ is *regular* if the subgroup Γ is normal in $\Delta(l,m,n)$.

Given a regular dessin d'enfant $\mathcal{H}$ on a curve X, then by uniformisation one has $\mathrm{Aut}(\mathcal{H}) \le \mathrm{Aut}(X)$. This lets us show here that dessins d'enfants have the same property as curves, namely as in the following, which is a generalisation to each genus $g \ge 2$ of an earlier result of Girondo [**12**].

THEOREM 1.1. *For all integer values of $g \ge 2$ other than 3, 6, 12 and 30, there are exactly two regular dessins d'enfant of genus g with orientation-preserving automorphism group of order $4g$. In the exceptional cases $g = 3$, 6, 12 and 30, there are one, three, two and two additional dessins respectively. Moreover, for every $g \ge 2$ the regular map $\mathcal{W}_g$ with orientation-preserving automorphism group of order $8g$ corresponding to Wiman's curve of type II with equation $y^2 = x(x^{2g}-1)$ can be obtained as a medial subdivision of each of the two non-sporadic dessins with $4g$ orientation-preserving automorphisms.*

To prove this theorem, we follow closely the methods used in [**6**].

2. Background

2.1. Fuchsian Groups and Riemann Surfaces. Here we follow [**17**].

A *Fuchsian group* Γ is a discrete group of conformal isometries of the hyperbolic plane $\mathbb{D}$. We shall consider here only Fuchsian groups with compact orbit space $\mathbb{D}/\Gamma$ (which is then a closed surface). If Δ is any such group, then its algebraic structure is determined by its signature

$$(h; m_1, \ldots, m_r). \tag{2.1}$$

The number h is the topological type of $\mathbb{D}/\Gamma$, called the genus of Γ, and the integers $m_i \ge 2$ (for $1 \le i \le r$) are the branch indices over points of $\mathbb{D}/\Gamma$ in the natural projection $\pi : \mathbb{D} \to \mathbb{D}/\Gamma$. A Fuchsian group with signature $(g; -)$ is called a *surface Fuchsian group.*

Associated with each Fuchsian group Γ with signature $(h; m_1, \ldots, m_r)$, there exists a *canonical presentation* for Γ, with generators

$x_1, \ldots, x_r$ (elliptic elements) and
$a_1, b_1, \ldots, a_g, b_g$ (hyperbolic elements),

subject to the defining relations

$x_i^{m_i} = 1$ (for $1 \le i \le r$), and
$x_1 \ldots x_r\, e_1 \ldots e_k\, a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1} = 1.$

In the rest of this paper, we will denote by $\Delta(l,m,n)$ a Fuchsian group with signature $(0; l,m,n)$, otherwise known as the ordinary (l,m,n) triangle group. This has the somewhat simpler presentation $\langle\, x, y \mid x^l = y^m = (xy)^n = 1 \,\rangle$.

The hyperbolic area of an arbitrary fundamental region of a Fuchsian group Γ with signature (2.1) is given by

$$\mu(\Delta) = 2\pi \left(2h - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right). \tag{2.2}$$

Furthermore, any discrete group Δ of conformal isometries of $\mathbb{D}$ containing Γ as a subgroup of finite index is also a Fuchsian group, and the hyperbolic area of a fundamental region for Λ is given by the Riemann-Hurwitz formula:

$$|\Delta : \Gamma| = \mu(\Gamma)/\mu(\Delta). \tag{2.3}$$

In particular, if Γ is a surface Fuchsian group of genus g then $\mu(\Gamma) = 2\pi(2g-2)$ and hence the Riemann-Hurwitz formula becomes

$$2g - 2 = |\Delta : \Gamma| \left(2h - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right). \tag{2.4}$$

A Riemann surface is a surface endowed with a complex analytical structure. There is a well-known functorial equivalence between Riemann surfaces and complex algebraic smooth curves.

Let X be a compact Riemann surface of genus $g > 1$. Then there exists a surface Fuchsian group Γ such that $X = \mathbb{D}/\Gamma$, and if G is any group of automorphisms of X, then there exists a Fuchsian group Δ containing Γ and a *surface epimorphism* $\theta : \Delta \to G$ such that $\ker\theta = \Gamma$. This epimorphism θ is the monodromy of the regular (orbifold-)covering $\mathbb{D}/\Gamma \to \mathbb{D}/\Delta$. In particular, the full automorphism group $\mathrm{Aut}(X)$ is isomorphic to Δ/Γ for some Fuchsian group Δ containing Γ.

In general, given Fuchsian groups Λ and Δ with $\Lambda \leq \Delta$, Singerman's Theorem (in [**17**]) tells us that the structure of Λ (and hence also of $\mathbb{D}/\Gamma$) is determined by the structure of Δ and the monodromy $\theta : \Delta \to \Sigma_{|\Delta:\Lambda|}$, where $\Sigma_{|\Delta:\Lambda|}$ denotes the symmetric group on the cosets of Λ in Δ. In fact θ is a transitive representation, and Λ is the pre-image under θ of the stabiliser $\mathrm{Stab}(1)$ of the trivial coset.

2.2. Dessins d'enfants, maps and hypermaps. Here we follow the seminal papers on maps and hypermaps on Riemann surfaces by Jones and Singerman [**14**], and Corn and Singerman [**11**]; see also [**13**].

Belyi's Theorem (from Belyi's influential paper [**1**] in 1980) states that a plane complex curve X is defined over a number field if and only if there is a finite N-sheeted covering $\beta : X \to \widehat{\mathbb{C}}$ of the projective line ramified on at most three points $\{0, 1, \infty\}$. The meromorphic function β is called the *Belyi function*.

Translating this into the world of Fuchsian groups and hyperbolic 2-orbifolds, we have an orbifold-covering $\beta : \mathbb{D}/\Gamma \to \mathbb{D}/\Delta(l, m, n)$, where $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. The meromorphic function β induces a cell-decomposition $\mathcal{H}$ of the Riemann surface X called a *dessin d'enfant*. In general this is a *hypermap*, with the pre-images of 0 providing the *hypervertices*, the pre-images of 1 the *hyperedges*, and the pre-images of ∞ the *hyperfaces*. It can also be viewed as a bipartite graph, with 'black' vertices representing the hypervertices, and 'white' vertices representing the hyperedges, and edges between them representing the pre-images of the line segment $[0, 1]$. Also if $l = 2$ then this hypermap is a *map*. From now on, we will use the terms dessin d'enfant and hypermap interchangeably. The order of the parameters l, m, n is not important for dessins d'enfants, but we will usually suppose that $l \leq m \leq n$.

The dessin $\mathcal{H}$ is said to have *type* (l, m, n), and if $\Gamma = H = \theta_\beta^{-1}(\mathrm{Stab}(1))$ then $\mathcal{H}$ has monodromy $\theta_\beta : \Delta(l, m, n) \to G$, with image G being a subgroup of $\Sigma_{|\Delta:H|}$ called the *monodromy group* of $\mathcal{H}$ and denoted by $\mathrm{Mon}(\mathcal{H})$. In particular, G has a presentation of the form $\langle\, a, s \mid a^l = s^m = (as)^n = \cdots = 1\,\rangle$. We will be interested only when $H = \Gamma$ is a surface Fuchsian group, and in such cases $H = \Gamma$ is called the *hypermap group*, and is the uniformising group (and fundamental group) of the Riemann surface X. Note that $\mathrm{Mon}(\mathcal{H})$ is the monodromy group of the covering $\mathbb{D}/\Gamma \to \mathbb{D}/\Delta(l, m, n)$. Cycles of the permutation a are the cycles around hypervertices, while those of s are the cycles around (hyper)edges, and those of as are the cycles around hyperfaces, consistent with the orientation of X.

Two dessins d'enfants of type (l, m, n) are isomorphic if their hypermap groups are conjugate in $\Delta(l, m, n)$, in which case they define the same complex structure of $X = \mathbb{H}/H = \mathbb{H}/\Gamma$. Also note that for any dessin d'enfant $\mathcal{H}$ on a Riemann surface $X = \mathbb{H}/\Gamma$, one has $\mathrm{Aut}(\mathcal{H}) \le \mathrm{Aut}(X)$. In particular $\mathrm{Aut}(\mathcal{H}) \le \mathrm{Mon}(\mathcal{H}) = G$.

A dessin d'enfant $\mathcal{H}$ with hypermap group H is called *regular* if $\mathrm{Aut}(\mathcal{H})$ acts transitively on the cosets of H, so that $\mathrm{Aut}(\mathcal{H}) = \mathrm{Mon}(\mathcal{H}) = G$. In that case, $H = \Gamma$ is a normal subgroup of $\Delta(l, m, n)$, and G is isomorphic to $\Delta(l, m, n)/\Gamma$ and hence to a subgroup of $\mathrm{Aut}(X)$.

A regular dessin d'enfant $\mathcal{H}$ with monodromy group $G = \langle a, s\rangle \cong \Delta(l, m, n)/\Gamma$ is said to be *reflexible* if it is isomorphic to its mirror image, in which case the group G has an automorphism taking $a \mapsto a^{-1}$ and $s \mapsto s^{-1}$; and otherwise $\mathcal{H}$ is said to be *chiral*. Equivalently, a dessin d'enfant is reflexible if and only if it is embedded in a *symmetric* Riemann surface (which means that the surface admits an anti-conformal automorphism of order 2, called an anti-conformal involution or a *symmetry* of the surface); see [**3**]. Symmetric Riemann surfaces are also called *real* Riemann surfaces, because they correspond to real algebraic curves.

Finally, we explain how to construct a *medial* (or *medial subdivision*) $\mathrm{Med}(\mathcal{H})$ of a regular dessin or hypermap $\mathcal{H}$ of type (m, m, n), as in [**12**]. Every black or white vertex of the bipartite graph associated with $\mathcal{H}$ becomes a white vertex of $\mathrm{Med}(\mathcal{H})$, and the black vertices of $\mathrm{Med}(\mathcal{H})$ are taken as the midpoints of edges of $\mathcal{H}$. In this way, every black vertex of $\mathrm{Med}(\mathcal{H})$ is joined to just two white vertices (incident in $\mathcal{H}$ with the edge it came from), while each white vertex is joined with m black vertices, coming from its incident edges in $\mathcal{H}$. The medial $\mathrm{Med}(\mathcal{H})$ is then a regular hypermap (indeed a regular map) of type $(2, m, 2n)$, and $\mathrm{Aut}(\mathcal{H})$ is isomorphic to a subgroup of index 2 in $\mathrm{Aut}(\mathrm{Med}(\mathcal{H}))$; see [**12**].

3. Regular dessins d'enfants with $4g$ automorphisms

In this paper we are interested in finding all regular hypermaps with automorphism group of order $4g$, where g is the genus. We will identify a hypermap $\mathcal{H}$ with its monodromy (or algebraic hypermap) $\theta : \Delta(l, m, n) \to G$; see [**14, 11, 8**].

PROPOSITION 3.1. Every regular hypermap of genus $g \ge 2$ with automorphism group of order $4g$ is isomorphic to one of those described in the list below:

(1) $\theta : \Delta(2, 4g, 4g) \to C_{4g}$ for any $g \geq 2$;
(2) $\theta : \Delta(4, 4, 2g) \to C_{2g} \diamond C_4$ (central product), for any $g \geq 2$;
(3) $\theta : \Delta(3, 4, 12) \to C_{12}$, for $g = 3$;
(4) $\theta : \Delta(3, 8, 8) \to C_3 \rtimes C_8$, for $g = 6$;
(5) $\theta : \Delta(4, 6, 6) \to \mathrm{SL}(2, 3)$, for $g = 6$;
(6) $\theta : \Delta(4, 6, 6) \to D_4 \times C_3$, for $g = 6$;
(7) $\theta : \Delta(4, 6, 8) \to \langle 2, 3, 4\rangle$ (the binary octahedral group), for $g = 12$;
(8) $\theta : \Delta(4, 6, 8) \to (C_3 \rtimes C_8) \rtimes C_2$, for $g = 12$;
(9) $\theta : \Delta(4, 6, 10) \to \mathrm{SL}(2, 5)$, when $g = 30$;
(10) $\theta : \Delta(4, 6, 10) \to C_{15} \rtimes D_4$, for $g = 30$.

Moreover, every one of the above hypermaps is reflexible.

Note that in items (5) and (9), the group G happens to be the binary tetrahedral group and the binary icosahedral group respectively, just as G is the binary octahedral group in item (7).

Proof. By an easy calculation using the Riemann-Hurwitz formula 2.4, all possible triples (l, m, n) and corresponding genera g with $|G| = 4g$ are given in Table 1 below.

(l, m, n)	genus g	(l, m, n)	genus g	(l, m, n)	genus g
$(2, 4g, 4g)$	any $g \geq 2$	$(3, 6, 2g)$	any $g \geq 2$	$(4, 4, 2g)$	any $g \geq 2$
$(6, 6, 5)$	15	$(6, 6, 4)$	6	$(6, 6, 3)$	3
$(6, 11, 4)$	66	$(6, 10, 4)$	30	$(6, 9, 4)$	18
$(6, 8, 4)$	12	$(6, 4, 4)$	3	$(5, 19, 4)$	190
$(5, 18, 4)$	90	$(5, 16, 4)$	40	$(5, 15, 4)$	30
$(5, 12, 4)$	15	$(5, 10, 4)$	10	$(5, 5, 8)$	20
$(5, 5, 9)$	45	$(5, 5, 5)$	5	$(4, 7, 9)$	126
$(4, 7, 8)$	28	$(4, 7, 7)$	14	$(3, 4, 12)$	3
$(3, 11, 11)$	33	$(3, 11, 12)$	66	$(3, 11, 13)$	429
$(3, 10, 14)$	105	$(3, 10, 12)$	30	$(3, 10, 10)$	15
$(3, 9, 17)$	153	$(3, 9, 16)$	72	$(3, 9, 15)$	45
$(3, 9, 12)$	18	$(3, 9, 9)$	9	$(3, 8, 23)$	276
$(3, 8, 22)$	132	$(3, 8, 21)$	84	$(3, 8, 20)$	60
$(3, 8, 18)$	36	$(3, 8, 16)$	24	$(3, 8, 12)$	12
$(3, 8, 8)$	6	$(3, 7, 41)$	841	$(3, 7, 40)$	420
$(3, 7, 39)$	273	$(3, 7, 36)$	126	$(3, 7, 35)$	105
$(3, 7, 28)$	42	$(3, 7, 21)$	21		

Table 1. Triples (l, m, n) giving $|G| = 4g$

For example, if $2 = l \leq m \leq n$ then $2g - 2 = 4g(0 - 2 + 3 - (\frac{1}{2} + \frac{1}{m} + \frac{1}{n}))$, from which it follows that $4g(\frac{1}{m} + \frac{1}{n}) = 2$, and then since each of m and n must divide $|G| = 4g$ we find the only solution is $(m, n) = (4g, 4g)$. Similarly, if $3 = l \leq m \leq n$ then 3 divides $|G| = 4g$ and $2g - 2 = 4g(0 - 2 + 3 - (\frac{1}{3} + \frac{1}{m} + \frac{1}{n})) = \frac{8g}{3} - 4g(\frac{1}{m} + \frac{1}{n})$, from which it follows that $4g(\frac{1}{m}+\frac{1}{n}) = 2+\frac{2g}{3}$, and hence either $\{m,n\} = \{6,2g\}$, or (m,n) is one of a number of small sporadic possibilities as given in the table.

Next, for each candidate for the parameters l, m, n and g, we need to check if there exists an epimorphism $\theta : \Delta(l,m,n) \to G$ to some group G of order $4g$.

This is easy in the first case, where $(l,m,n) = (2,4g,4g)$, because the image as of the element xy of $\Delta(2,4g,4g) = \langle\, x,y \mid x^2 = y^{4g} = (xy)^{4g} = 1\,\rangle$ has order $4g$ and so G is cyclic, and then the image a of x must be $(as)^{2g}$, and this determines the epimorphism θ uniquely. In the second case, where $(l,m,n) = (3,6,2g)$, the image as of xy has order $2g$ and so generates a subgroup of index 2 in G, but then that subgroup must contain a (since it has odd order 3) and hence also $s = a^{-1}as$, which is impossible since θ is surjective. In the third case, where $(l,m,n) = (4,4,2g)$, the element as generates a cyclic subgroup of index 2 containing both a^2 and s^2, and it follows that $a^2 = (as)^g = s^2$, making G a central product of C_{2g} and C_4.

These three cases were also studied in [**6**], and they give items (1) and (2) in the statement of the Proposition.

Type $(3,4,12)$ was dealt with in [**19, 16**] when considering Picard's curve of genus 3, and gives item (3). The other sporadic cases can be handled using the `LowIndexNormalSubgroups` facility in the Magma computation system [**2**] to determine whether or not the relevant triangle group $\Delta(l,m,n)$ has a smooth quotient of the expected order. This gives the remaining items (4) to (10). For the cases with genus $g \le 101$, the required computations were already done some years ago by the second author in the search for regular maps and hypermaps; see [**8, 9, 10**].

Presentations for the group G in terms of the generating pair $(a,s) = (x^\theta, y^\theta)$ in the ten items in the resulting list (for this Proposition) are as follows:

(1) $G = \langle\, a,s \mid a^2 = 1,\ a = (as)^{2g}\,\rangle \cong C_{4g}$ for every $g \ge 2$;
(2) $G = \langle\, a,s \mid a^4 = 1,\ a^2 = s^2 = (as)^g\,\rangle \cong C_{2g} \diamond C_4$ for every $g \ge 2$;
(3) $G = \langle\, a,s \mid a^3 = s^4 = [a,s] = 1\,\rangle \cong C_{12}$;
(4) $G = \langle\, a,s \mid a^3 = s^8 = 1,\ s^{-1}as = a^{-1}\,\rangle \cong C_3 \rtimes C_8$;
(5) $G = \langle\, a,s \mid a^4 = 1,\ a^2 = s^3 = (as)^3\,\rangle \cong \mathrm{SL}(2,3)$;
(6) $G = \langle\, a,s \mid a^4 = s^6 = 1,\ s^2 = (as)^2\,\rangle \cong D_4 \times C_3$;
(7) $G = \langle\, a,s \mid a^4 = 1,\ a^2 = s^3 = (as)^4\,\rangle \cong \langle 2,3,4\rangle$;
(8) $G = \langle\, a,s \mid a^4 = s^6 = a^{-1}s^2as^2 = as^{-1}a^{-1}sasas^{-1} = 1\,\rangle \cong (C_3 \rtimes C_8) \rtimes C_2$;
(9) $G = \langle\, a,s \mid a^4 = 1,\ a^2 = s^3 = (as)^5\,\rangle \cong \mathrm{SL}(2,5)$;
(10) $G = \langle\, a,s \mid a^4 = s^6 = (as)^{10} = [a^2,s] = a^{-1}s^2as^2 = 1\,\rangle \cong C_{15} \rtimes D_4$.

Note that in many cases at least one of the relations $a^l = s^m = (as)^n = 1$ is missing but still holds in the group G, and is redundant. Similarly, in item (8) the relation $[a^2,s] = 1$ holds in G but is redundant.

It is now an easy exercise to verify that in each of the above cases, the group G admits an automorphism taking $a \mapsto a^{-1}$ and $s \mapsto s^{-1}$, and hence the associated hypermap $\mathcal{H}$ is reflexible, as required. (In some cases this also follows from the content of [**9, 10, 6, 7**].) □

Next, we consider in more detail the two infinite families of regular dessins given in items (1) and (2) of Proposition 3.1.

Item (1) is a family of 'cyclic' regular maps $\mathcal{M}_g$ of type $(4g, 4g)$, each with monodromy $\theta_1 : \Delta(2, 4g, 4g) \to C_{4g}$, for all $g \geq 2$. The monodromy group $\mathrm{Mon}(\mathcal{M}_g)$ can be taken as the permutation group of degree $4g$ generated by the permutations

$$a = (1, 2g+1)(2, 2g+2)(3, 2g+3)\dots(2g, 4g) \quad \text{and} \quad as = (1, 2, 3, \dots, 4g).$$

Item (2) is a family of regular hypermaps $\mathcal{H}_g$ of type $(4, 4, 2g)$, each with monodromy $\theta_2 : \Delta(4, 4, 2g) \to C_{2g} \diamond C_4$, and with $\mathrm{Mon}(\mathcal{H}_g)$ generated by

$$a = (1, 2g+1, g+1, 3g+1)(2, 4g, g+2, 3g)(3, 4g-1, g+3, 3g-1)\dots$$
$$(g-1, 3g+3, 2g-1, 2g+3)(g, 3g+2, 2g, 2g+2).$$
$$as = (1, 2, 3, \dots, 2g-1, 2g)(2g+1, 2g+2, \dots 4g-1, 4g) \quad \text{and}$$

The associated signatures $(0; 2, 4g, 4g)$ and $(0; 4, 4, 2g)$ are both in Singerman's list [**18**] of non-maximal signatures for Fuchsian groups, and each forms an index 2 'normal' pair with the signature $(0; 2, 4, 4g)$, for every $g \geq 2$. For genus $g \geq 3$ the signature $(0; 2, 4, 4g)$ is maximal, while for $g = 2$, Singerman's list of non-maximal signatures includes the pair $((0; 2, 4, 8), (2, 3, 8))$ as well.

The signature $(0; 2, 4, 4g)$ is closed related to Wiman's curve of type II with equation $y^2 = x(x^{2g} - 1)$ mentioned in Section 1. In [**16**] it was shown that this curve $\mathcal{W}_g$ is determined by a regular map of type $(4, 4g)$ with automorphism group G of order $8g$. In this case G is isomorphic to the semi-direct product $C_{4g} \rtimes_{2g-1} C_2$, with presentation $\langle\, a, s \mid a^2 = s^4 = (as)^{4g} = 1,\ a(as)a = (as)^{2g-1} \,\rangle$, realisable by the permutations

$$a = (1, 4g+1)(3, 8g-1)(5, 8g-3)(7, 8g-5)\dots(4g-3, 4g+5)(4g-1, 4g+3)$$
$$(2, 6g)(4, 6g-2)(6, 6g-4)(8, 6g-6)\dots(2g-2, 4g+4)(2g, 4g+2)$$
$$(2g+2, 8g)(2g+4, 8g-2)(2g+6, 8g-4)\dots(4g-2, 6g+4)(4g, 6g+2)$$

and

$$as = (1, 2, 3, \dots, 4g-2, 4g-1, 4g)(4g+1, 4g+2, 4g+3, \dots, 8g-2, 8g-1, 8g).$$

Wiman's curve of type II for genus $g = 2$ is also known as Bolza's curve, and is determined by the regular map $\mathcal{W}_2$ of type $(3, 8)$ with automorphism group $GL(2, 3)$ of order 48, having presentation $\langle\, a, s \mid a^2 = s^3 = (as)^8 = (asasas^{-1})^2 = 1 \,\rangle$.

In [**12**] Girondo showed that the dessin associated with Wiman's curve of genus 2 can be constructed as a medial of each of the dessins $\mathcal{M}_2$ and $\mathcal{H}_2$ (defined above). We can now complete the proof of Theorem 1.1, which generalises Girondo's discovery to every genus $g \geq 2$.

Proof. By Proposition 3.1 and the comments following it, we need only show that the epimorphisms $\theta_1 : \Delta(2, 4g, 4g) \to C_{4g}$ and $\theta_2 : \Delta(4, 4, 2g) \to C_{2g} \diamond C_4$ given earlier both extend to an epimorphism $\theta : \Delta(2, 4, 4g) \to C_{2g} \rtimes_{2g-1} C_2$.

Such an extension of θ_1 was proved by Kulkarni in [**16**], and also by Bujalance and Conder in the final section of [**4**], and both extensions were proved to exist by Bujalance, Costa and Izquierdo in [**6**]. Here we give a direct verification, by showing that the epimorphism $\theta : \Delta(2, 4, 4g) \to C_{4g} \rtimes_{2g-1} C_2$ detemined by the presentation $\langle\, a, s \mid a^2 = s^4 = (as)^{4g} = 1,\ a(as)a = (as)^{2g-1} \,\rangle$ for the group $G = C_{2g} \rtimes_{2g-1} C_2$ restricts to each of the unique epimorphisms θ_1 and θ_2, using material from [**5**].

Before doing that, we note that s^2 is an involution in the index 2 subgroup generated by as, and so $s^2 = (as)^{2g} = (as)^{-2g}$.

Now let x, y and z be the standard generators for $\Delta = \Delta(2, 4, 4g)$, satisfying $x^2 = y^4 = z^{4g} = xyz = 1$. Then we can proceed as follows:

Case (1). By case N8 in [**5**, Section 3], there is a unique Fuchsian subgroup of index 2 in Δ with signature $(0; 2, 4g, 4g)$, namely the subgroup Λ_1 generated by y^2 and z^{-1}. The images of these elements in $G = C_{2g} \rtimes_{2g-1} C_2$ are s^2 and as, which generate a cyclic group of order $4g$, and hence we have a restriction to the given epimorphism $\theta_1 : \Delta(2, 4g, 4g) \to C_{4g}$. In particular, also the map $\mathcal{W}_g$ of type $(2, 4, 4g)$ corresponding to Wiman's curve of type II and genus g is obtained from the map $\mathcal{M}_g$ of type $(2, 2g, 2g)$ by a $(1, \infty)$-subdivision.

Case (2). By a different application of case N8 in [**5**, Section 3], there is a unique Fuchsian subgroup of index 2 in Δ with signature $(0; 4, 4, 2g)$, namely the subgroup Λ_2 generated by y and z^2. The images of these elements in $G = C_{2g} \rtimes_{2g-1} C_2$ are s and $(as)^{-2}$, which generate a central product of C_4 and C_{2g} of order $4g$ (with the involution $s^2 = ((as)^{-2})^g$ generating the centre). Hence we also have a restriction to the given epimorphism $\theta_2 : \Delta(4, 4, 2g) \to C_{2g} \diamond C_4$. In particular, also the map $\mathcal{W}_g$ of type $(2, 4, 4g)$ corresponding to Wiman's curve is obtained from the hypermap $\mathcal{H}_g$ of type $(4, 4, 2g)$ by a $(0, 1)$-subdivision.

This completes the proof of Theorem 1.1. □

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