A TWO POINT LINEARIZATION METHOD FOR THE ANALYSIS OF PIPE NETWORKS

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Abstract: This paper proposes a new method for snapshot analysis of water distribution systems based on the commonly used Gradient method. The proposed method uses a secant (intersecting the headloss function in two points) instead of a tangent to approximate the pipe headloss function. A theoretical model is developed for the flow range in which the secant approximates the headloss function without exceeding a given allowable error. This scheme allows a tradeoff to be made between the allowable error and the number of iterations required to achieve convergence. The proposed method is applied to an example network to illustrate its application and benefits. It is argued that the number of iterations required to find a solution can be reduced significantly in both snapshot and extended-period simulations.

INTRODUCTION

Hydraulic models of water distribution networks normally incorporate a large amount of uncertainty in input parameters such as water demand, pipe roughness values and even network

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configuration parameters. The calculation accuracy of a hydraulic solution of the network does not add value beyond the uncertainty incorporated in the solution. For this reason, it might be beneficial to trade off accuracy against speed of convergence. While efficient solution algorithms and fast computers mean that simulation time is not a critical aspect of most hydraulic analyses, there are certain applications (such as optimization studies) where simulation speed is at a premium.

This paper proposes a new method for snapshot analysis of water distribution systems (under assumptions of incompressible and inertia-less flow) that approximates the headloss function with a secant instead of the tangent used in the Newton-Raphson method. The proposed method was implemented by amending the Newton-Raphson based method used in the public domain Epanet software package (Rossman 2000). The method used in Epanet is based on work by Todini and Pilati (1987) and is referred to as the Gradient method in this paper.

GRADIENT METHOD

The Gradient method is well known and is documented in detail in the Epanet Users Manual (Rossman 2000). Only aspects of the Gradient method important for explaining the proposed method are included here. In the Gradient method the network equations are developed by applying the mass balance principle to all nodes in a network. Pipe headlosses are incorporated using a generic equation that includes a term for secondary losses:

$$h_{ij} = H_i - H_j = Sign(Q_{ij})\left(r\left|Q_{ij}\right|^n + mQ_{ij}^2\right) \quad (1)$$

Where h_{ij} is the headloss in link ij, H_i and H_j the hydraulic heads at nodes i and j respectively, Q_{ij} the flow rate in the pipe, r the pipe resistance coefficient, n the frictional headloss exponent, and

m the secondary loss coefficient. To apply the Newton-Raphson method, this function has to be written in the form $f(Q_{ij}) = 0$:

$$f(Q_{ij}) = Sign(Q_{ij})\left(r\left|Q_{ij}\right|^{n} + mQ_{ij}^{2}\right) - \left(H_i - H_j\right) \tag{2}$$

Initial pipe flow rates are assumed for all pipes, typically based on a flow velocity of 0.3048 m/s (1 ft/s), and the flow rates are then corrected in an iterative way using the Newton-Raphson method. At each iteration new nodal heads are found by solving the matrix equation:

$$\mathbf{AH} = \mathbf{F} \tag{3}$$

Where, for a network with N nodes, **A** is an N x N Jacobian matrix, **H** is an N x 1 matrix of nodal heads, and **F** is an N x 1 right hand side matrix. Cholesky's method, a variant of Gaussian elimination tailored for symmetric positive definite matrixes, is used to solve the equations.

The make-up of the Jacobian matrix elements are of importance for this paper. The diagonal and non-zero, off diagonal elements are described by (4) and (5) respectively:

$$A_{ii} = \sum_j p_{ij} \tag{4}$$

$$A_{ij} = -p_{ij} \tag{5}$$

Where p_{ij} is the inverse of the first derivative of the headloss function with respect to flow. For pipes this is given by (from (1)):

$$p_{ij} = \frac{1}{nr\left|Q_{ij}\right|^{n-1} + 2m\left|Q_{ij}\right|} \tag{6}$$

TWO POINT LINEARIZATION METHOD

In the proposed method, the tangent used in the Gradient method is replaced by a secant that intersects the link's headloss curve in two points (see Fig. 1), ($Q_{ij,0}$; $f(Q_{ij,0})$) and ($Q_{ij,s}$; $f(Q_{ij,s})$), such that the maximum error between the secant and the pipe's headloss curve does not exceed E_{max}. Let the equation of the secant be of the form:

$$f_{\sec}(Q_{ij}) = aQ_{ij} + b \tag{7}$$

Where a and b are constants. The error made in this approximation is calculated as the difference between the secant and the headloss curve

$$E = aQ_{ij} + b - \left[Sign(Q_{ij})\left(r\left|Q_{ij}\right|^{n} + mQ_{ij}^{2}\right) - (H_i - H_j)\right] \tag{8}$$

To find the flow rate Q_{Emax} at which the interpolation error is at a maximum, the derivative of E with respect to Q is determined and set to zero. Assuming $n = 2$ for the Darcy-Weisbach equation:

$$Q_{E\max} = \frac{a}{2(r+m)} \tag{9}$$

The maximum interpolation error E_{max} can now be calculated from (8) as:

$$E_{\max} = \frac{a^2}{4(r+m)} + b + (H_i - H_j) \tag{10}$$

For a given E_{max}, the slope of the secant can now be determined by fitting it through the points ($Q_{ij,0}$; $f(Q_{ij,0})$) and (Q_{Emax} ; $f(Q_{Emax}) + E_{max}$). The equation becomes a quadratic in a, and can be solved to determine the slopes of the secant, such that a given maximum interpolation error is not exceeded:

$$a = 2(r+m)\left\{Q_{ij,0} \pm \sqrt{\frac{E_{\max}}{(r+m)}}\right\} \tag{11}$$

The two slopes correspond to the cases where $Q_{ij,0} > Q_{ij,s}$ and $Q_{ij,0} < Q_{ij,s}$. One of the slopes is always larger than the first derivative of the function, while the other is always smaller. The function $f(Q_{ij})$ can be classified into four regions, depending on whether the Q_{ij} and $f(Q_{ij})$ are positive or negative. In each region, the slope a can be chosen such that a corrected flow rate better than that of the Newton-Raphson method is obtained, thus creating the potential for faster convergence. However, to do this it is also necessary to know the correct link flow rate (the objective of the iteration process), making impossible to consistently predict the slope that will improve convergence. Further work in this area may find ways of exploiting the improved convergence slope, but this was not pursued in this study. In this study $Q_{ij,s}$ was chosen such that $Q_{ij,s} < Q_{ij,0}$ when $Q_{ij,0} > 0$, and $Q_{ij,s} > Q_{ij,0}$ when $Q_{ij,0} < 0$.

The error made by the secant when interpolating between the two intersection points does not exceed E_{max}. However, the secant can also be extrapolated for a distance in both directions without exceeding E_{max}. The range where the error does not exceed E_{max} is defined as $\{Q_{min}; Q_{max}\}$ (refer to Fig. 1). To determine Q_{min} and Q_{max}, the error function (8) is set to E_{max} to obtain a quadratic equation in Q_{ij} with solution:

$$Q_{\min,\max} = \frac{a \pm \sqrt{a^2 - 4(r+m)(aQ_{ij,0} - (r+m)Q_{ij,0}^2 - E_{\max})}}{2(r+m)} \tag{12}$$

For $Q_{ij} \geq 0$, the range is given by the equations:

$$Q_{\min} = Q_{ij,0} - (\sqrt{2}+1)\sqrt{\frac{E_{\max}}{(r+m)}} \tag{13}$$

$$Q_{\max} = Q_{ij,0} + (\sqrt{2}-1)\sqrt{\frac{E_{\max}}{(r+m)}} \tag{14}$$

Similarly for $Q_{ij} < 0$, the range is given by:

$$Q_{\min} = Q_{ij,0} - (\sqrt{2}-1)\sqrt{\frac{E_{\max}}{(r+m)}} \tag{15}$$

$$Q_{\max} = Q_{ij,0} + (\sqrt{2}+1)\sqrt{\frac{E_{\max}}{(r+m)}} \tag{16}$$

Hazen-Williams equation

The Hazen Williams equation is not a quadratic equation and thus does not lend itself to symbolic analysis as well as the Darcy Weisbach equation. To determine the flow rate at the maximum interpolation error, the first derivative of the error equation (8) is taken with respect to Q_{ij} and set equal to zero to obtain (ignoring secondary losses):

$$Q_{E\max} = \left(\frac{a}{rn}\right)^{1/n-1} \tag{17}$$

Replacing into the error equation, the maximum error is determined as:

$$E_{\max} = a\left(\frac{a}{rn}\right)^{1/n-1} + b - r\left(\frac{a}{rn}\right)^{n/n-1} + H_i - H_j \tag{18}$$

An expression for b can be determined from the intersection point and replaced in (18) to obtain:

$$E_{\max} = \left(\frac{1}{rn}\right)^{1/n-1}\left(a^{n/n-1}\right) - r\left(\left(\frac{1}{rn}\right)^{n/n-1}\left(a^{n/n-1}\right)\right) + rQ_{ij,0}^{n} - aQ_{ij,0} \tag{19}$$

It is now assumed that the slope of the secant and the allowable error are proportional the headloss, i.e. $a = crQ_{ij,0}^{n}$ and $E_{\max} = \varepsilon\, rQ_{ij,0}^{n}$, where c is a constant and ε is the allowable error fraction. This results in the expression:

$$c^{n/n-1}\left(\left(\frac{1}{n}\right)^{1/n-1} - \left(\frac{1}{n}\right)^{n/n-1}\right) - c + (1-\varepsilon) = 0 \tag{20}$$

c is determined numerically for a given n and ε. For instance, using the Hazen Williams equation (n = 1.852) and an allowable error of 1 % (ε = 0.01), c can be determined as 1.673847.

The range where the secant approximates the headloss function within the allowable error can now be determined. The analysis can be done in first quadrant without loss of generality. The secant is below the headloss curve when extrapolating to determine Q_{min} and Q_{max}, and thus the error will be negative. Considering that E_{max} is always positive, the expression for E_{max} is:

$$-E_{\max} = aQ_{ij} + rQ_{ij,0}^{n} - aQ_{ij,0} - rQ_{ij}^{n} \tag{21}$$

Since this equation cannot be solved explicitly for the Hazen Williams equation, a numerical solution has to be sought. It is assumed that $Q_{min,max} = kQ_{ij,0}$, making it possible to obtain two values of k corresponding to Q_{min} and Q_{max} respectively. Replacing $E_{\max} = \varepsilon\, rQ_{ij}^{n}$ and $a = crQ_{ij}$ the following expression is obtained:

$$k^{n} - kc + c - (1+\varepsilon) = 0 \tag{22}$$

For the example above, the values of k can be determined as 0 .730962 and 1.046552 for Q_{min} and Q_{max} respectively.

Discussion

The proposed method is simple to implement in the Gradient method by using the slope of the secant a instead of the Newton-Raphson tangent to estimate the corrected flow rate, thus (6) becomes:

$$p_{ij} = \frac{1}{a} \tag{23}$$

The Gradient method employs an iterative method that is terminated when acceptable convergence has been achieved. Various convergence tests can be employed, but in each instance the results will include some level of error. Epanet checks whether the sum of absolute flow changes between successive iterations divided by the sum of absolute flows in all links is smaller than some tolerance (Rossman 2000). The proposed method used the same test for convergence, except that the flow change was replaced by the absolute of the difference between the flow rate and the acceptable flow range $\{Q_{min}; Q_{max}\}$ (for flow rates falling within the acceptable range, the distance was set to zero). This test for convergence becomes identical to the Epanet test when $E_{max} = 0$.

It is important to distinguish between the convergence error and the allowable error used in the proposed method. The progression of the convergence error through the iteration process cannot be predicted with certainty, and thus it cannot be used to trade off accuracy and convergence speed. The purpose of the convergence test is to ensure that the convergence error is sufficiently small.

We believe that the proposed method offers two main advantages over the conventional Gradient method. Firstly, by incorporating an allowable error in the solution, it allows the user to trade off accuracy and convergence speed. Secondly, by defining a flow range for each link where the results are within the required accuracy, it is possible to check the accuracy of a solution without performing another full iteration (as is required by the Gradient method). This is advantageous when a snapshot solution is required from a good initial estimate, for instance when only small changes occur in demands or tank levels between two snapshot simulations. In the Gradient method, the accuracy of a solution is calculated by comparing the solution found in successive iterations. Thus, once a change in tank level or demand occurs in a network, it is necessary to do at least one full iteration to update and calculate the accuracy of the solution. In the proposed method it is possible to solve and test the accuracy of a solution for a changed network using the current Jacobian matrix coefficients. A full iteration is only required if the current solution does not comply to the accuracy requirements, and thus the potential exists for a further reduction in the number of full iterations performed when multiple snapshot simulations are required, such as optimization runs or extended period simulations.

EXAMPLE

The proposed method was applied to an example water network model to compare its performance to that of the conventional Gradient method. The Exnet network (Farmani et al 2005) was originally proposed as a realistic benchmark network for multi-objective design optimization. It is available online (UOECWS 2005) and its size and complexity makes it ideal for testing the efficacy of the proposed method. The Exnet distribution network is based on a calibrated network model of a system that serves a population of approximately 400 000 people. It consists of approximately 2 500 pipes varying from 50 to 1073 mm, 1 900 junctions, two control valves and is supplied from two sources.

Both the Darcy-Weisbach and Hazen-Williams hydraulic resistance formulae were applied. Since the results of difference resistance formula were not compared to each other, it was not necessary to have equivalent friction factors in the Darcy-Weisbach and Hazen-Williams networks. The Darcy-Weisbach hydraulic friction formula is used in the original example network and a Hazen-Williams network was created from it by assuming typical Hazen-Williams flow coefficients for all pipes. The proposed method was applied to the example network with allowable errors varying between 1 and 10 %. The Gradient method solution is equivalent to an allowable error of zero. Initial solutions were identical based on pipe flow velocities of 0.3048 m/s and the standard Epanet convergence tolerance of 0.001 was used in all simulations.

The results for the example system are shown in Fig. 2. The numbers of iterations required in the conventional approach are 9 and 8 for the Darcy-Weisbach and Hazen-Williams equations, respectively. As the allowable error increases, the number of iterations decreases for both networks. The Hazen-Williams network shows the fastest reduction and settles at six iterations (a 33 % reduction) at an allowable error of only 2 %. The Darcy-Weisbach network also shows a reduction to six iterations (a 25 % reduction), although this is only achieved at an allowable error of 4 %. The results show that the proposed method can significantly reduce the computational effort by allowing a larger computational error.

CONCLUSIONS

A new method for snapshot simulation of water distribution systems is proposed. The method uses a secant to approximate the pipe headloss function, instead of the tangent used in the Newton-Raphson based Gradient method. The benefits of the proposed method lie in the possibility to trade off accuracy and convergence rate, and in testing whether convergence has been achieved. It is shown that the proposed method can significantly reduce the computational effort by allowing a larger computational error.

APPENDIX I. REFERENCES

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APPENDIX II. NOTATION

The following symbols are used in this paper:

a = coefficient of linear flow rate function;

$\mathbf{A}$ = nodal Jacobean matrix;

A_{ij} = element ij of $\mathbf{A}$;

b = coefficient of linear flow rate function;

c = a constant;

k = a constant ;

D_j = demand at node j;

E_{max} = maximum interpolation error;

$f(Q_{ij})$ = function of pipe headloss minus the head difference over the start and end nodes;

$f_{sec}(Q_{ij})$ = secant intersecting $f(Q_{ij})$ in two points;

$\mathbf{F}$ = right hand side matrix of mass balance equations;

h_{ij} = total headloss in a link;

H_i = head at node i;

$\mathbf{H}$ = nodal head matrix;

m = secondary headloss coefficient;

n = frictional headloss exponent;

N = number of nodes in a network;

p_{ij} = inverse of the first derivative of the headloss function;

Q_{Emax} = flow rate at which the interpolation error is maximum;

Q_{ij} = flow rate in link ij;

$Q_{ij,0}$ = initial flow rate;

$Q_{ij,c}$ = corrected flow rate;

Q_{min} = the minimum flow rate where the secant error ≤ the maximum error;

Q_{max} = the maximum flow rate where the secant error ≤ the maximum error;

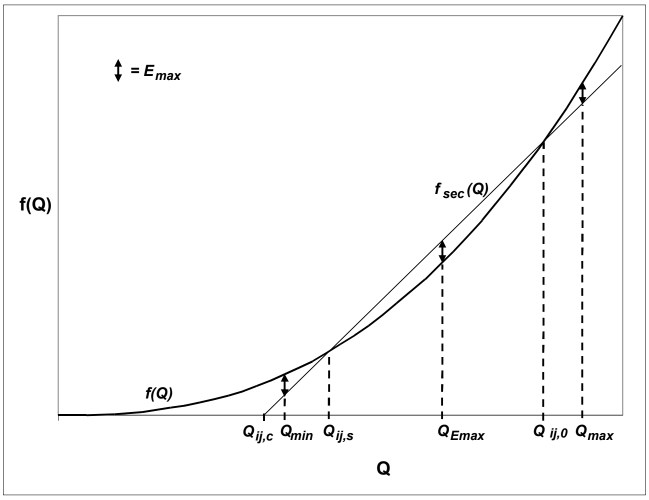
r = pipe resistance coefficient;

ε = allowable error fraction.

FIGURES

Fig. 1 Approximating the first derivative at $Q_{ij,0}$ with a secant

Fig. 2 Number of iterations required to achieve convergence for different allowable errors in the Exnet example network



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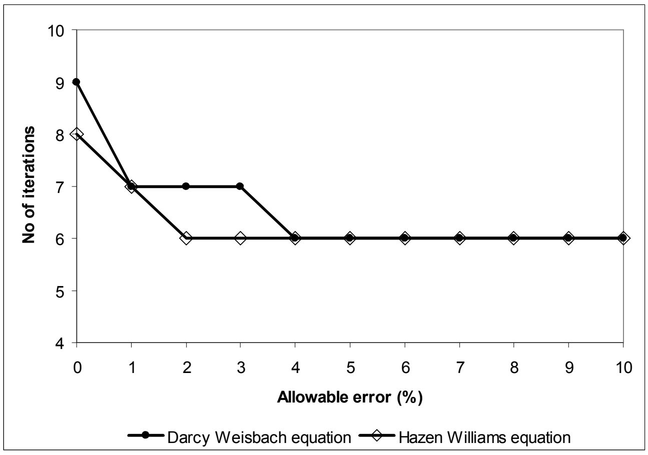

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