Bounds on the orders of groups of automorphisms of a pseudo-real surface of given genus

E. Bujalance[*], F.J. Cirre[†] and M.D.E. Conder[‡]

Abstract

A compact Riemann surface is called *pseudo-real* if it admits anti-conformal (orientation-reversing) automorphisms, but no anti-conformal automorphism of order 2, or equivalently, if the surface is reflexible but not definable over the reals. In this paper, we consider upper bounds on the order of a group G of automorphisms of a pseudo-real surface S of given genus $g > 1$, in the case where G is cyclic, abelian, or an arbitrary group of automorphisms of S. This is motivated by a number of long-standing theorems about the orders of groups of automorphisms of general compact Riemann surfaces, including theorems of Hurwitz (1893) and Wiman (1896).

We determine, for each integer $g \geq 2$, the orders of the largest cyclic and the largest abelian group of automorphisms of a pseudo-real surface of genus g such that the group contains orientation-reversing elements, and consider the problem of finding similar bounds when the groups contain no orientation-reversing elements.

In the case of arbitrary groups, we show that if $M(g)$ is the order of the largest group of automorphisms of a pseudo-real surface of genus g, then $M(g) \geq 2g$ for every even $g \geq 2$, while $M(g) \geq 4(g-1)$ for every odd $g \geq 3$, and we prove that the latter bound is sharp for a very large and possibly infinite set of odd values of $g \geq 3$. Unfortunately we are not yet able to determine the level of sharpness of the former bound (for even g), so we leave that as an open question. We also give the precise values of $M(g)$ for all g between 2 and 128, together with the signatures for the actions of the corresponding groups of largest order.

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1 Introduction

A compact Riemann surface is called *pseudo-real* if it admits anti-conformal (orientation-reversing) automorphisms, but no anti-conformal automorphism of order 2. Another term used for such surfaces is *asymmetric.* Their importance stems from the fact that in the

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moduli space of compact Riemann surfaces of given genus, pseudo-real surfaces represent the points that have real moduli but are not definable over the reals.

Here we note that there are no pseudo-real surfaces of genus 0 or 1, since in those cases every reflexible surface admits an anti-conformal automorphism of order 2. On the other hand, it was shown in [9] that there exists at least one pseudo-real surface of genus g for every integer $g > 1$. Indeed the pseudo-real surfaces of genus 2 to 4 were classified in [9, 10], and this was extended for genus 5 to 10 in [2], except that five of the entries in the tables in [2] are invalid and should be deleted, as will be explained later.

In this paper, we consider upper bounds on the order of a group of automorphisms of a pseudo-real surface of given genus $g > 1$. This is motivated by a number of theorems about the orders of groups of automorphisms of other kinds of surfaces. The most famous of these are the theorems of Hurwitz (1983) and Wiman (1896) that give an upper bound of $84(g-1)$ on the number of orientation-preserving automorphisms of a compact Riemann surface of genus $g > 1$, and an upper bound of $4g+2$ on the order of any such automorphism, respectively. Other such theorems deal with special cases where the group is abelian, or the surface is non-orientable, or the automorphism reverses orientation; see [6, 14, 15, 16, 17].

It was shown in [9] that the total number of automorphisms of a pseudo-real surface of given genus $g > 1$ is bounded above by $12(g-1)$. This upper bound is attained for infinitely many g, but certainly not for all $g > 1$, and so it makes sense to look for more refined bounds on the group order, in general and special cases. One particular question of interest is to find a sharp lower bound on the order of the largest group of automorphisms of a pseudo-real surface of given genus, akin to the Accola-Maclachlan bound for general compact Riemann surfaces (see [1, 18]). We can find a lower bound for all even g and a lower bound for all odd g, and prove that the latter is sharp for a very large and possibly infinite set of odd values of g, but the corresponding task is much more challenging for even values of g. Nevertheless we can give sharp bounds for every genus $g > 1$ when we restrict to the cases where the group is cyclic or abelian. To explain this in more detail, we define certain parameters.

For every integer $g > 1$, let $M(g)$ be the order of the largest group of automorphisms of a pseudo-real surface of genus g. Note that this group contains elements that reverse the orientation of the surface, but the same is not necessarily true for smaller groups of automorphisms. Accordingly, we also denote by $M_{\mathrm{ab}}(g)$ and $M_{\mathrm{cyc}}(g)$ respectively the orders of the largest abelian and largest cyclic group of automorphisms of a pseudo-real surface of genus g, such that the group contains orientation-reversing elements, and by $M^+_{\mathrm{ab}}(g)$ and $M^+_{\mathrm{cyc}}(g)$ respectively the orders of the largest abelian and largest cyclic group of orientation-preserving automorphisms of a pseudo-real surface of genus g.

In Sections 3 and 4 we show that $M_{\mathrm{ab}}(g) = M_{\mathrm{cyc}}(g) = 2g$ for every even $g \geq 2$. For all but three values of g for which this bound is attained, the surface is hyperelliptic, by a theorem from [7]. We also show that $M_{\mathrm{cyc}}(g) = 2g-2$ for every odd $g \geq 3$, and that in all cases where this bound is attained, the surface is elliptic-hyperelliptic. On the other hand, for odd $g > 1$ we find that $M_{\mathrm{ab}}(g) = 2g+6$ when $g \equiv 1 \bmod 4$, while $M_{\mathrm{ab}}(g) = 2g+2$ when $g \equiv 3 \bmod 4$. In all these cases we give specific details about the surfaces and groups.

Similarly, but somewhat differently, in Sections 3 and 4 we show that $M^+_{\mathrm{cyc}}(g) = g-1$

for infinitely many $g \equiv 3 \bmod 4$, while $M^+_{\rm cyc}(g) \geq g$ for all even $g \geq 2$ and $M^+_{\rm cyc}(g) \geq g+1$ for all $g \equiv 1 \bmod 4$, and hence that $g-1$ is a sharp lower bound for $M^+_{\rm cyc}(g)$ for infinitely many $g \geq 3$. On the other hand, $M^+_{\rm ab}(g) \geq g+1$ for all odd $g \geq 3$, while $M^+_{\rm ab}(g) = g$ for a very large and possibly infinite set of even values of $g \geq 2$.

In Section 5 we turn to the general case (involving arbitrary groups of automorphisms), and show that $M(g) \geq 2g$ for every even $g \geq 2$, while $M(g) \geq 4(g-1)$ for every odd $g \geq 3$, and prove that the latter bound is sharp for a very large and possibly infinite set of odd values of $g \geq 3$. Unfortunately we are not yet able to either improve the former bound (for even g) or prove it is sharp, so we leave that as an open question.

Finally, in an Appendix we give a table of the values of $M(g)$ for all g between 2 and 128, including the signatures for the actions of the corresponding groups of largest order. This considerably extends the determination of $M(g)$ for $2 \leq g \leq 10$ resulting from the work by Artebani, Quispe and Reyes in [2].

Before all of that, we give some further background in Section 2, on Riemann surfaces, NEC groups and their signatures, and some more information about group actions on pseudo-real surfaces. In particular, in subsection 2.2 we mention a related piece of work by the third author and his student Stephen Lo, on finding the smallest genus of a pseudo-real surface admitting a given group of automorphisms. The latter piece of work has some overlap with this one, but we repeat some of the key common features for completeness.

2 Further background

In this section we begin with some background about Riemann surfaces and their groups, a lot of which can be found in [11], and then we provide some further information about group actions on pseudo-real surfaces.

2.1 Riemann surfaces, NEC groups and their signatures

First, let S be any compact Riemann surface of genus $g > 1$. Then S can be represented as the orbit space U/Φ of the upper half-plane U under the action of some surface Fuchsian group Φ (that is, a torsion-free discrete cocompact subgroup of $\mathrm{PSL}(2,\mathbb{R})$). A finite group G acts as a group of automorphisms of the surface S if and only if G is isomorphic to the quotient Γ/Φ for some non-Euclidean crystallographic group Γ (a discrete cocompact subgroup of $\mathrm{PGL}(2,\mathbb{R})$) containing Φ as a normal subgroup of index $|G|$. The canonical epimorphism $\theta\colon \Gamma \to G$ ($\cong \Gamma/\Phi$) is said to be *smooth* (since Φ is torsion-free). In particular, the full automorphism group $\mathrm{Aut}(S)$ of S is isomorphic to the quotient Γ/Φ where Γ is the normaliser in $\mathrm{Aut}(U) = \mathrm{PGL}(2,\mathbb{R})$ of the surface group Φ, and its orientation-preserving subgroup $\mathrm{Aut}^+(S)$ (of conformal automorphisms) is isomorphic to Γ^+/Φ where Γ^+ is the normaliser of Φ in $\mathrm{Aut}^+(U) = \mathrm{PSL}(2,\mathbb{R})$.

The structure of the NEC group Γ is determined by its signature

$$s(\Gamma) = (\gamma; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}),$$

where (a) γ is the topological genus of the orbit space U/Γ, (b) k is the number of its boundary components, (c) the sign is $+$ or $-$ according to whether or not U/Γ is orientable, (d) the integers $m_i \geq 2$ are the orders of the r branch points in the interior of U/Γ, and (e) the $n_{j\ell}$ are the orders of the s_j branch points on the jth boundary component of U/Γ, for $1 \leq j \leq k$.

Associated with every signature $s(\Gamma)$ is a canonical presentation for the group Γ, and a formula for the hyperbolic area of a fundamental domain for Γ. If the sign is $+$ then Γ is generated by elliptic elements $x_1, \dots, x_r$, reflections $c_{10}, \dots, c_{1s_1}, \dots, c_{k0}, \dots, c_{ks_k}$, boundary transformations $e_1, \dots, e_k$, and hyperbolic elements $a_1, b_1, \dots, a_\gamma, b_\gamma$, which satisfy the defining relations

$$\begin{aligned}
&x_i^{m_i} = 1 \quad (\text{for } 1 \leq i \leq r),\\
&c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1, \;\; c_{is_i} = e_i^{-1}c_{i0}e_i \quad (\text{for } 1 \leq i \leq k, 0 \leq j \leq s_i),\\
&x_1 \dots x_r \, e_1 \dots e_k \, a_1 b_1 a_1^{-1} b_1^{-1} \dots a_\gamma b_\gamma a_\gamma^{-1} b_\gamma^{-1} = 1.
\end{aligned}$$

On the other hand, if the sign is $-$, then $\gamma > 0$ and the generators a_i and b_i are replaced by glide reflections $d_1, \dots, d_\gamma$, and the final relation replaced by $x_1 \dots x_r \, e_1 \dots e_k \, d_1^2 \dots d_\gamma^2 = 1$.

In the first case, the orientation-preserving subgroup Γ^+ is the subgroup generated by the elements x_i, e_i, a_i and b_i and their conjugates under the elements c_{ij}, plus the pairwise products of the c_{ij}. This is the subgroup consisting of all elements expressible as words in the generators of Γ such that the total number of occurrences of reflections c_{ij} is even, and hence has index 2 in Γ if $k > 0$, or index 1 otherwise. In the second case, where the sign is $-$, Γ^+ is the index 2 subgroup consisting of all elements expressible as words in the generators of Γ such that the total number of occurrences of reflections c_{ij} and glide reflections d_i is even.

The hyperbolic area of a fundamental region for Γ is given by

$$\mu(\Gamma) = 2\pi \left(\varepsilon\gamma - 2 + k + \textstyle\sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right)\right),$$

where $\varepsilon = 2$ if the sign is $+$, and $\varepsilon = 1$ if the sign is $-$.

Furthermore, if Δ is any subgroup of finite index in Γ, then Δ is an NEC group, and the hyperbolic areas of fundamental regions for Δ and Γ satisfy the Riemann-Hurwitz formula $\mu(\Delta) = |\Gamma : \Delta| \, \mu(\Gamma)$. In particular, if Φ is the surface Fuchsian group for which $S \cong U/\Phi$, then Φ has signature $s(\Phi) = (g; +; [-]; \{-\})$ and so $\mu(\Phi) = 2\pi(2g-2)$ while $|\Gamma : \Phi| = |\Gamma/\Phi| = |G|$, and it follows that

$$2g-2 \;=\; |G| \left(\varepsilon\gamma - 2 + k + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right)\right).$$

Letting R be twice the reciprocal of the bracketed expression on the right-hand side of this equation, we have $|G| = R(g-1)$, where $0 < R \leq 168$, with the maximum value of R attained when $s(\Gamma) = (0; +; [-]; \{(2,3,7)\})$.

Conversely, if G is any finite group that can be generated by elements that satisfy one of the two canonical presentations given above for the NEC group Γ, with the elements x_i

and c_{ij} and the products $c_{ij-1}c_{ij}$ having the orders m_i, 2 and n_{ij} as appropriate, then G is isomorphic to the quotient Γ/Φ, where Φ is a surface Fuchsian group, and G acts faithfully on the compact Riemann surface $S \cong U/\Phi$ of genus g given by $|G| = R(g-1)$. Also in this case we say that G acts with the given signature on S. Any such action can be described by means of a *generating vector*, which consists of the images under the corresponding smooth epimorphism $\theta : \Gamma \to G$ of the canonical generators for Γ.

Later in this paper we consider upper bounds on the order of a single automorphism of a surface of given genus. For a compact Riemann surface of genus $g > 1$, this upper bound is $4g + 2$ (first proved by Wiman [20]), while for a compact non-orientable Klein surface of algebraic genus p, it is $2p + 2$ when p is even, and $2p$ when p is odd (proved independently by Wendy Hall [16] and the first author [6]).

2.2 Group actions on pseudo-real surfaces

Now suppose the surface S is pseudo-real. Then S admits orientation-reversing automorphisms but no reflections, and so $k = 0$, and the sign is $-$, and the signature of Γ is $(\gamma; -; [m_1, \dots, m_r]; \{-\})$ for some $\gamma > 0$ and integers $m_i \geq 2$ for $1 \leq i \leq r$.

Now let G be any group that acts on S with some automorphisms that reverse orientation. Following [2] and [3] we call such an action *essential*. Then the group G is generated by elements $d_1, \dots, d_\gamma$ and $x_1, \dots, x_r$ satisfying the relations

$$x_1^{m_1} = x_2^{m_2} = \cdots = x_r^{m_r} = x_1 \dots x_r \, d_1^2 \dots d_\gamma^2 = 1,$$

or equivalently, by elements $d_1, \dots, d_\gamma$ and $x_1, \dots, x_{r-1}$ such that

$$x_1^{m_1} = x_2^{m_2} = \cdots = x_{r-1}^{m_{r-1}} = (\, d_1^2 \dots d_\gamma^2 \, x_1 \dots x_{r-1})^{m_r} = 1,$$

and its orientation-preserving subgroup G^+ is generated by the elements x_i and their conjugates under the elements d_j, plus products of any even number of the d_j.

Moreover, $|G| = R(g-1)$ where $2/R = \gamma - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)$, and it follows easily that $|G| \leq 12(g-1)$, with the maximum value of 12 for R attained if and only if the signature is $(1; -; [2, 3]; \{-\})$, as proved in [9, Theorem 5.1]. Also if $|G| < 12(g-1)$, then the next highest possibility is $8(g-1)$, which occurs for signature $(1; -; [2, 4]; \{-\})$.

Incidentally, because S admits no orientation-reversing automorphisms of order 2, it also admits no orientation-reversing automorphisms of order $2n$ with n odd, and so the order of every element of $G \setminus G^+$ is divisible by 4. In particular, every involution in $\mathrm{Aut}(S)$ preserves orientation, and $|\mathrm{Aut}(S)|$ is divisible by 4.

Furthermore, it was proved in [3, Theorem 3.3] that a finite group G is the full automorphism group of some pseudo-real surface S if and only if G is a non-split extension of a subgroup H of even order by the cyclic group of order 2.

On the other hand, it can happen that a group $G \cong \Gamma/\Phi$ acts faithfully and essentially on a compact Riemann surface $S \cong U/\Phi$, but is not the full automorphism group of S. When that happens, there exists an NEC group Λ containing Γ and normalising Φ, so Λ/Φ

is a group of automorphisms of S larger than G. If Λ and Γ have the same Teichmüller dimension then the signatures of Γ and Λ must appear in the lists of finite index inclusions of NEC groups, given by the first author [5] for the case where Γ is normal in Λ, and by Estévez and Izquierdo [13] for the case where Γ is not normal in Λ. These inclusions are analysed in further detail in [8].

If a pair (σ, σ') of signatures occurs in any of these lists, then for any NEC group Γ with signature $s(\Gamma) = \sigma$ there always exists another NEC group Λ with signature $s(\Lambda) = \sigma'$ containing Γ with finite index. In this case, if the surface subgroup Φ is normal in Λ then the action of $G = \Gamma/\Phi$ on $S = U/\Phi$ is extendable to the action of the larger group Λ/Φ, but if Φ is not normal in Λ then no such extension (via Λ) is possible.

On the other hand, if a signature σ does not occur as the first entry of a pair in any of the above lists, then there exists a *maximal* NEC group Γ with signature $s(\Gamma) = \sigma$, so Γ is not contained in any other NEC group with finite index. In this case, the action of $G = \Gamma/\Phi$ cannot be extended, and $G = \mathrm{Aut}(S)$.

If the signature of Γ is of the form $(\gamma; -; [m_1, \ldots, m_r]; \{-\})$, then an inspection of those lists shows that we have only the following possibilities to consider for the signatures of Γ and Λ, all coming from [5], with Γ normal in Λ:

(a) $s(\Gamma) = (3; -; [-]; \{-\})$ and $s(\Lambda) = (0; +; [2, 2, 2]; \{(-)\})$,

(b) $s(\Gamma) = (2; -; [t], \{-\})$ and $s(\Lambda) = (0; +; [2, 2]; \{(t)\})$ where $t \geq 2$,

(c) $s(\Gamma) = (1; -; [t, t]; \{-\})$ and $s(\Lambda) = (0; +; [2, t]; \{(-)\})$ where $t \geq 3$,

(d) $s(\Gamma) = (1; -; [t, u]; \{-\})$ and $s(\Lambda) = (0; +; [2]; \{(t, u)\})$ where $\max(t, u) \geq 3$,

(e) $s(\Gamma) = (1; -; [t, t]; \{-\})$ and $s(\Lambda) = (0; +; [-]; \{(2, 2, 2, t)\})$ where $t \geq 3$.

In cases (a) to (d) of this list, the index of Γ in Λ is 2, while in case (e) it is 4, but in that case the extension of Γ to Λ is a combination of an index 2 intermediate extension of type (c) or (d) with a further index 2 extension from [5], but each of those intermediate extensions introduces reflections (of order 2), and so we will ignore this possibility. Hence we consider extensions of types (a), (b), (c) and (d) only.

For type (a), the NEC group Γ is generated by elements d_1, d_2 and d_3 that satisfy the relation $d_1^2 d_2^2 d_3^2 = 1$, and extending from Γ to the NEC group Λ involves adjoining a new generator c_0 that satisfies the relations $(d_1c_0)^2 = (c_0d_2)^2 = (d_2d_3c_0d_2)^2 = [d_1d_2^2d_3c_0d_2, c_0] = 1$, with the elements $x_1 = d_1c_0$, $x_2 = c_0d_2$, $x_3 = d_2d_3c_0d_2$ and c_0 being standard generators for Λ (see [8]). Conjugation by $c_0 \in \Lambda \setminus \Gamma$ induces an automorphism of Γ that takes (d_1, d_2, d_3) to $(d_1^{-1}, d_2^{-1}, d_2^2d_3^{-1}d_2^{-2})$. Hence it is possible to extend the action of the group G to some larger group of automorphisms acting with signature $(0; +; [2, 2, 2]; \{(-)\})$ on the same surface if and only if the group G has an automorphism that inverts the images of each of d_1 and d_2 and takes d_3 to $d_2^2d_3^{-1}d_2^{-2}$ $(= d_1^{-2}d_3^{-1}d_1^2)$.

For type (b), the group Γ is generated by elements d_1, d_2 and x_1 that satisfy the relations $d_1^2d_2^2\,x_1 = x_1^t = 1$, and extending from Γ to Λ involves adjoining a new generator c_0 that satisfies the relations $c_0^2 = (d_1c_0)^2 = (c_0d_2)^2 = [d_1d_2, c_0] = 1$, with $x_1' = d_1c_0$, $x_2' = c_0d_2$ and c_0 being standard generators for Λ (see [8]). Conjugation by $c_0 \in \Lambda \setminus \Gamma$ induces an automorphism of Γ that takes (d_1, d_2) to (d_1^{-1}, d_2^{-1}), and also takes $x_1 = (d_1^2 d_2^2)^{-1}$ to $(d_1^{-2} d_2^{-2})^{-1} = d_2^2 d_1^2 = d_1^{-2} x_1^{-1} d_1^2$. Hence it is possible to extend the action of the group G to some larger group of automorphisms acting with signature $(0; +; [2, 2]; \{(t)\})$ on the same surface if and only if the group G has an automorphism that inverts the images of each of d_1 and d_2.

Note that in particular, this refutes Lemma 4.3 of [3], the proof of which was wrong.

For type (c), the group Γ is generated by elements d_1, x_1 and x_2 that satisfy the relations $d_1^2 x_1 x_2 = x_1^t = x_2^t = 1$, and extending from Γ to Λ involves adjoining a new generator c_0 that satisfies $c_0^2 = (d_1 c_0)^2 = [d_1 c_0 x_1, c_0] = 1$ with $x_1' = d_1 c_0$, $x_2' = x_1$ and c_0 being standard generators for Λ. Then since $(d_1 c_0)^2 = 1$ and $[x_1' x_2', c_0] = 1$, conjugation by the involution c_0 induces an automorphism of Γ that takes (d_1, x_1, x_2) to $(d_1^{-1}, x_2^{-1}, x_1^{-1})$. Hence the action of the group G on the surface can be extended to one of a larger group with signature $(0; +; [2, t]; \{(-)\})$ if and only if G has an automorphism that conjugates the image of d_1 to its inverse, and interchanges the image of each of x_1 and x_2 with the inverse of the image of the other.

Similarly, for type (d), the group Γ is generated by elements d_1, x_1 and x_2 that satisfy the relations $d_1^2 x_1 x_2 = x_1^t = x_2^u = 1$, and extending from Γ to Λ involves adjoining a new generator c_0 that satisfies $c_0^2 = (d_1 c_0)^2 = (c_0 x_1)^2 = 1$, with $x_1' = d_1 c_0$, c_0 and $c_1 = c_0 x_1$ being standard generators for Λ. Then since $(d_1 c_0)^2 = (c_0 x_1)^2 = 1$, conjugation by the involution c_0 induces an automorphism of Γ that takes (d_1, x_1) to (d_1^{-1}, x_1^{-1}), and also takes $x_2 = x_1^{-1} d_1^{-2}$ to $x_1 d_1^2 = x_1 x_2^{-1} x_1^{-1}$. Hence the action of the group G on the surface extends to one of a larger group with signature $(0; +; [2]; \{(t, u)\})$ if and only if G has an automorphism that inverts the images of each of d_1 and x_1.

We collect the above observations into the following proposition, for future reference.

Proposition 2.1 *If a finite group G acts on a Riemann surface with non-maximal signature* (a), (b), (c) *or* (d) *and the corresponding presentation of G admits the respective automorphism as described above, then the surface is not pseudo-real.*

This has a number of consequences, including the following, which are also given in [12].

Proposition 2.2 *Let Γ be a maximal NEC group with signature $(1; -; [i, j, k]; \{-\})$ for some i, j, k. If the finite group $G = \Gamma/\Phi$ has a faithful action on the Riemann surface $S = U/\Phi$, and G has no elements of order 2 lying outside G^+, then S is pseudo-real and $G = \mathrm{Aut}(S)$.*

Proof: First observe that there exist maximal NEC groups Γ with signature $\sigma = (1; -; [i, j, k]; \{-\})$, because there is no pair (σ, σ') in the list of finite index inclusions of NEC groups in [5]. It follows that the action of G on S is not extendable to the action of some larger group, and so $G = \mathrm{Aut}(S)$. Moreover, it follows that S has no orientation-reversing automorphisms of order 2, and therefore S is pseudo-real. ∎

Proposition 2.3 *No finite abelian group G has a faithful essential action on a pseudo-real surface with signature* $(1; -; [j, k]; \{-\})$ *for* $2 \le j \le k$*, or* $(2; -; [k]; \{-\})$ *for* $k \ge 2$*, or* $(3; -; [-]; \{-\})$.

Proof: In the first case, suppose to the contrary that the abelian group G has a faithful essential action on a pseudo-real surface S with signature $(1; -; [j, k]; \{-\})$. Then G can be generated by elements d and x such that x has order j and d^2x has order k. But since G is abelian, G has an automorphism that inverts each of d and x. It follows that the action of G on S can be extended to one of a larger group on S with signature $(0; +; [2]; \{(j, k)\})$. Hence in particular, the surface S admits reflections, contradiction.

The same argument holds for the second signature $(2; -; [k]; \{-\})$, for in this case, G must be generated by elements d and e such that d^2e^2 has order k, and then since the abelian group G has an automorphism that inverts each of d and e, the action is extendable to one of a larger group with signature $(0; +; [2, 2]; \{(k)\})$ on S, and again S admits reflections, contradiction. The third case can be eliminated similarly. ∎

Corollary 2.4 *If the finite abelian group G has a faithful essential action on a pseudo-real surface of genus g where $|G| > 2(g-1)$, then G acts with signature* $(1; -; [2, j, k]; \{-\})$ *where either* $j = 2 \le k$ *or* $\{j, k\} = \{3, 3\}$, $\{3, 4\}$ *or* $\{3, 5\}$.

Proof: It is easy to verify that if $|G| > 2(g-1)$ (so that $2/R < 1$), then the signature must be either $(1; -; [j, k]; \{-\})$ where $2 \le j \le k$, or $(2; -; [k]; \{-\})$ for some $k \ge 2$, or $(1; -; [2, j, k]; \{-\})$ where $2 \le j \le k$. Proposition 2.3 eliminates the first two of these cases, while in the third case $1 > 2/R = 3/2 - 1/j - 1/k$, so $1/j + 1/k > 1/2$ and so $j = 2 \le k$ or $\{j, k\} = \{3, 3\}$, $\{3, 4\}$ or $\{3, 5\}$. ∎

Here we note that Proposition 2.3 above shows that the single entries in the tables at the end of [2] with $(g, \mathrm{Aut}^{\pm}(S)) = (5, C_{12})$, $(9, C_{20})$ and $(9, C_{24})$ and also the two entries with $(g, \mathrm{Aut}^{\pm}(S)) = (9, C_{12} \times C_2)$ are all invalid, and should be deleted.

The origin of this problem with the tables in [2] is the mistaken Lemma 4.3 of [3]. Also the latter was applied in an flawed attempt in [3] to find the smallest genus of a pseudo-real surface with a given cyclic group (of order divisible by 4) as its full automorphism group, and as a result, Theorem 6.1 of [3] is invalid as well. These mistakes were corrected in [12], where the notion of the *pseudo-real genus* of a group was investigated, as follows.

For any finite group G, let $\psi(G)$ be the smallest genus of those pseudo-real Riemann surfaces on which G acts faithfully as a group of automorphisms, possibly (but not necessarily) preserving orientation, and let $\psi^*(G)$ be the smallest genus of those pseudo-real Riemann surfaces on which G acts faithfully and essentially as a group of automorphisms, when this exists. (Recall that an essential action is one that includes orientation-reversing elements.) The main theorem of [12] shows that for every integer $g \ge 2$, there exists a finite group G for which $\psi(G) = \psi^*(G) = g$, and hence that the range of each of the functions ψ and ψ^* is the set of all integers $g \ge 2$.

We will contribute to the theory of these two parameters in what follows.

3 The cyclic case

We begin this section with the following result by Etayo [14], which gives necessary and sufficient conditions for the existence of an action of a finite cyclic group with signature $(\gamma; -; [m_1, \dots, m_r]; \{-\})$ on a Riemann surface.

Proposition 3.1 [14, Theorem 4] *A cyclic group of order* $4n$ *acts with signature* $(\gamma; -; [m_1, \dots, m_r]; \{-\})$ *on a compact Riemann surface if and only if*

(a) m_i *divides* $2n$ *for* $1 \le i \le r$;

(b) *if* $\gamma = 1$ *then* $\mathrm{lcm}(m_1, \dots, m_r) = 2n$ *and* $r \ge 2$;

(c) $\displaystyle\sum_{i=1}^{r} \frac{2n}{m_i} \equiv \gamma \bmod 2.$

Using this, we can prove the following:

Theorem 3.2 *For every even integer* $g \ge 2$*, the largest order of an orientation-reversing automorphism of a pseudo-real surface of genus* g *is* $2g$.

Proof: By Proposition 3.1 we know that the cyclic group C_{2g} of order $2g$ acts as Γ/Φ with signature $(1; -; [2, 2, g]; \{-\})$ on a Riemann surface $S = U/\Phi$, with genus g (as given by the Riemann-Hurwitz formula). Moreover, we may choose the NEC group Γ to be maximal, and note that the unique involution in C_{2g} preserves orientation, because g is even. Hence by Proposition 2.2 we find that S is pseudo-real, with $C_{2g} = \mathrm{Aut}(S)$.

Next, we show that $2g$ is the largest possible order. Suppose there exists an orientation-reversing automorphism of a pseudo-real surface of the given genus g, having order $4n \ge 2g$. Then by Corollary 2.4, we find that the cyclic group G generated by that automorphism acts with signature $(1; -; [m_1, m_2, m_3]; \{-\})$, where $\{m_1, m_2, m_3\} = \{2, 2, m\}$ for some $m \ge 2$, or $\{2, 3, 3\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$. In the first case, condition (c) of Proposition 3.1 tells us that $n + n + 2n/m \equiv 1 \bmod 2$, and hence that m is even, and then condition (b) of Proposition 3.1 implies that $2n = \mathrm{lcm}(2, 2, m) = m$, and then the Riemann-Hurwitz formula gives $g = m = 2n$, and $|G| = 4n = 2g$. Similarly, in the last three cases $\{2, 3, 3\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$ we have $2n = \mathrm{lcm}(m_1, m_2, m_3) = 6$, 12 and 30, and $g = 6$, 12 and 30, respectively, and again $|G| = 4n = 2g$. ∎

Remark 3.3 Compact (but not necessarily pseudo-real) Riemann surfaces of even genus g that admit an orientation-reversing automorphism u of order $2g$ were studied previously in detail in [7]. There it was shown that if $g \ne 6$, 12 or 30, then every such group action (with signature $(1; -; [2, 2, g]; \{-\})$ as shown above) is unique, given by the generating vector $(\theta(d_1), \theta(x_1), \theta(x_2), \theta(x_3)) = (u, u^g, u^g, u^{-2})$, and the surfaces are hyperelliptic.

For odd values of g, the largest orders of cyclic groups acting faithfully on a compact Riemann surface cannot be attained by group actions with signatures of the form $(1; -; [m_1, \dots, m_r]; \{-\})$. This is an easy consequence of Proposition 3.1, as follows.

Proposition 3.4 *A cyclic group of order* $4n$ *cannot act faithfully with signature* $(1; -; [m_1, \dots, m_r]; \{-\})$ *on a compact Riemann surface of odd genus.*

Proof: Let G be a cyclic group of order $4n$ acting with the given signature. Then by Proposition 3.1 we have $|G| = 4n = 2\,\mathrm{lcm}(m_1, \dots, m_r)$, and $\sum_{i=1}^{r} \frac{2n}{m_i}$ is odd. But also the Riemann-Hurwitz formula gives $g - 1 = 2n\,(r - 1 - \sum_{i=1}^{r} \frac{1}{m_i}) = 2n(r-1) - \sum_{i=1}^{r} \frac{2n}{m_i}$, and then since the right-hand side is odd, it follows that the genus g must be even. ∎

Theorem 3.5 *For every odd integer* $g \geq 3$*, the largest order of an orientation-reversing automorphism of a pseudo-real surface of genus* g *is* $2g - 2$*.*

Proof: By Proposition 3.1 we know that the cyclic group C_{2g-2} of order $2g-2$ acts as Γ/Φ with signature $(2; -; [2, 2]; \{-\})$ on a Riemann surface $S = U/\Phi$, with genus g (as given by the Riemann-Hurwitz formula). Again we may choose the NEC group Γ to be maximal, and note that the unique involution in C_{2g-2} preserves orientation, because (this time) g is odd. Hence by Proposition 2.2 we find that S is pseudo-real, with $C_{2g-2} = \mathrm{Aut}(S)$.

Also $2g - 2$ is the largest possible order, because if there exists a cyclic group G with order $4n \geq 2g - 2$ acting on a pseudo-real surface of the given genus g, then by Corollary 2.4 we find that G acts with signature $(1; -; [m_1, m_2, m_3]; \{-\})$ for some $\{m_1, m_2, m_3\}$, but that contradicts Proposition 3.4. ∎

Next, we show that for odd genus $g \geq 3$, the largest cyclic group actions are not unique, and that the corresponding surfaces are elliptic-hyperelliptic (in contrast to the situation for even genus, described in Remark 3.3).

Proposition 3.6 *A pseudo-real surface of odd genus* g *admitting an orientation-reversing automorphism* v *of the largest possible order* $2g - 2$ *is elliptic-hyperelliptic. In this case the cyclic group* $\langle v \rangle$ *acts with signature* $(2; -; [2, 2]; \{-\})$*, and the action is given by the generating vector* $(\theta(d_1), \theta(d_2), \theta(x_1), \theta(x_2)) = (v, v^{-1}, v^{g-1}, v^{g-1})$ *or* $(v, v^{g-2}, v^{g-1}, v^{g-1})$*.*

Proof: Suppose that C_{2g-2} acts with signature $(\gamma; -; [m_1, \dots, m_r]; \{-\})$ on a pseudo-real surface of odd genus g. Then by the Riemann-Hurwitz formula, $\gamma - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) = 1$, which forces $\gamma \leq 3$. If $\gamma = 3$ then $r = 0$, but then the surface is not pseudo-real by Proposition 2.3, so $\gamma \neq 3$. Similarly, $\gamma \neq 1$ by Proposition 3.4. Hence $\gamma = 2$, and it follows easily that $r = 2$ with $m_1 = m_2 = 2$, and the signature is $(2; -; [2, 2]; \{-\})$.

Next, let $\{d_1, d_2, x_1, x_2\}$ be a canonical set of generators for a maximal NEC group Γ with signature $(2; -; [2, 2]; \{-\})$, and let $\theta : \Gamma \to C_{2g-2}$ be a smooth epimorphism whose kernel uniformises a pseudo-real surface. Since C_{2g-2} contains a unique element of order 2, we find that $\theta(x_1) = \theta(x_2)$, and then from the relation $d_1^2 d_2^2 x_1 x_2 = 1$ we obtain $(\theta(d_1)\theta(d_2))^2 = 1$. Hence either $\theta(d_1)\theta(d_2) = 1$, or $\theta(d_1)\theta(d_2) = \theta(x_1)$. But also each of $\theta(d_1)$ and $\theta(d_2)$ reverses orientation and hence has order divisible by 4, so each generates a cyclic subgroup containing the unique involution $\theta(x_1)$, and it follows that in both cases, the element $v = \theta(d_1)$ generates the whole cyclic group. Accordingly, we find there are just two possibilities θ_1 and θ_2 for the epimorphism θ, given by

$$\theta_1(d_1) = v, \quad \theta_1(d_2) = v^{-1}, \quad \theta_1(x_1) = \theta_1(x_2) = v^{g-1},$$
$$\theta_2(d_1) = v, \quad \theta_2(d_2) = v^{g-2}, \quad \theta_2(x_1) = \theta_2(x_2) = v^{g-1}.$$

These give the two generating vectors $(v, v^{-1}, v^{g-1}, v^{g-1})$ and $(v, v^{g-2}, v^{g-1}, v^{g-1})$.

Finally, consider the pre-image $\Psi_i = \theta_i^{-1}(\langle v^{g-1}\rangle)$ in Γ of the subgroup generated by the (orientation-preserving) involution v^{g-1}, for $i \in \{1, 2\}$. By Theorems 2.2.4 and 2.1.3 in [11] and the Riemann-Hurwitz formula, it is easy to see that Ψ_i has signature $(1; +; [2, \overset{2g-2}{\ldots}, 2]; \{-\})$ for both values of i. This implies that the surface $S_i = U/\ker\theta_i$ can be represented as the double covering $S_i \to S_i/\langle v^{g-1}\rangle$ of an orientable surface of genus 1, ramified over $2g-2$ points. In other words, S_i is an elliptic-hyperelliptic surface, for both values of i. ∎

Theorems 3.2 and 3.5 show respectively that $M_{\text{cyc}}(g) = 2g$ for every even $g \geq 2$, and $M_{\text{cyc}}(g) = 2g-2$ for every odd $g \geq 3$. They also give a solution to the minimum genus problem for orientation-reversing cyclic group actions on pseudo-real surfaces, and correct the mistake made about this in [3, Theorem 6.1].

Corollary 3.7 *The minimum genus of a pseudo-real surface admitting an orientation-reversing automorphism of order* $4n$ *is* $2n$.

Proof: By Theorem 3.2, we know there exists a pseudo-real surface of genus $2n$ admitting an orientation-reversing automorphism of order $4n$. Now let S be any pseudo-real surface admitting such an automorphism, and let g be the genus of S. If g is even then $4n \leq 2g$ by Theorem 3.2, while if g is odd then $4n \leq 2g-2$ by Theorem 3.5. In both cases we have $2g \geq 4n$, and so $g \geq 2n$, as required. ∎

In terms of the 'pseudo-real genus' notation introduced in [12] and described briefly at the end of Section 2, the above corollary is equivalent to stating $\psi^*(C_{4n}) = 2n$. This fact was also proved in [12, Proposition 3.1]. Moreover, it was shown in [12, Proposition 3.3] that the assumption that the automorphism reverses orientation can be dropped, to give $\psi(C_{4n}) = 2n$ as well.

Next, we investigate in more detail the case where a cyclic group of automorphisms of a pseudo-real surface preserves orientation. Of course here if G is the full automorphism group of the surface, then the given cyclic group H is a subgroup of G^+ and hence a proper subgroup of G, and G^+ contains all the involutions of G.

Proposition 3.8 *Let* g *be any integer greater than* 1. *Then the following hold*:

(a) *If* $g \equiv 0$ *mod* 2, *then* $H = C_g$ *is the orientation-preserving subgroup of* $G = C_{2g}$ *in its action on a pseudo-real surface of genus* g *with signature* $(1; -; [2, 2, g]; \{-\})$.

(b) *If $g \equiv 1$ mod 4, then $H = C_{g+1} \cong C_{(g+1)/2} \times C_2$ is the orientation-preserving subgroup of the direct product $G = C_{g+1} \times C_2$ in its action on a pseudo-real surface of genus g with signature* $(1; -; [2, 2, (g+1)/2]; \{-\})$.

(c) *If $g \equiv 3$ mod 4 and $g \geq 7$, then $H = C_{(g-1)/2} \times C_2 \cong C_{g-1}$ is the orientation-preserving subgroup of the semi-direct product $G = C_{(g-1)/2} \rtimes_{-1} C_4$ in its action on a pseudo-real surface of genus g with signature* $(2; -; [2, 2]; \{-\})$.

Proof: First, case (a) follows from the action of $G = C_{2g}$ on a pseudo-real surface of genus g as given by Theorem 3.2, with $H = G^+ \cong C_g$.

For case (b), let v and u be generators for $G = C_{g+1} \times C_2$ of orders $g+1$ and 2, respectively, and let Γ be a maximal NEC group with signature $(1; -; [2, 2, (g+1)/2]; \{-\})$, and canonical generators d_1, x_1, x_2, x_3 satisfying $d_1^2 x_1 x_2 x_3 = x_1^2 = x_2^2 = x_3^{(g+1)/2} = 1$. Then there exists an epimorphism $\theta : \Gamma \to G$ taking (d_1, x_1, x_2, x_3) to (v^{-1}, u, u, v^2), and by Proposition 2.2, this gives a pseudo-real surface S with $G = \mathrm{Aut}(S)$, and with G^+ being the index 2 subgroup generated by the θ-images of d_1^2, x_1, x_2 and x_3, namely $H = \langle v^2, u \rangle \cong C_{(g+1)/2} \times C_2 \cong C_{g+1}$.

Finally, for case (c), let w and v be generators for $G = C_{(g-1)/2} \rtimes_{-1} C_4$ of orders $(g-1)/2$ and 4 respectively such that $w^v = w^{-1}$, and let Γ be a maximal NEC group with signature $(2; -; [2, 2]; \{-\})$, and generators d_1, d_2, x_1, x_2 satisfying $d_1^2 d_2^2 x_1 x_2 = x_1^2 = x_2^2 = 1$. Then there exists an epimorphism $\theta : \Gamma \to G$ taking (d_1, d_2, x_1, x_2) to (v, wv, v^2, v^2), and since w has odd order $(g-1)/2$, Proposition 2.2 gives us a pseudo-real surface S with $G = \mathrm{Aut}(S)$, and with G^+ being the index 2 subgroup generated by the θ-images of d_1^2, d_2^2, $d_1 d_2$, x_1 and x_2, namely $H = \langle w, v^2 \rangle \cong C_{(g-1)/2} \times C_2 \cong C_{g-1}$. ∎

This gives $M^+_{\mathrm{cyc}}(g) \geq g$ for $g \equiv 0$ mod 2 and $M^+_{\mathrm{cyc}}(g) \geq g+1$ for $g \equiv 1$ mod 4, while $M^+_{\mathrm{cyc}}(g) \geq g-1$ for $g \equiv 3$ mod 4 with $g \geq 7$. (It is easy to see that also $M^+_{\mathrm{cyc}}(3) = 2 = 3-1$.) In particular, $M^+_{\mathrm{cyc}}(g) \geq g-1$ for all $g \geq 2$. We now show that this bound is sharp for infinitely many g.

Theorem 3.9 *For every integer $g > 59$ of the form $g = 2p+1$ where p is prime, the maximum order of a cyclic group of orientation-preserving automorphisms of a pseudo-real surface of genus g is $g-1$.*

Proof: First, $g = 2p+1 \equiv 3$ mod 4, and so by Proposition 3.8 we know that C_{g-1} is the orientation-preserving subgroup of the automorphism group of a pseudo-real surface of genus g.

Now suppose that H is a larger cyclic group of orientation-preserving automorphisms of some pseudo-real surface S of genus g, and let $G = \mathrm{Aut}(S)$. Then $|G|$ is divisible by 4, and $|H| > g-1$, so $|G| > 2(g-1)$, and it follows that G acts with one of the following signatures:

(1) Signature $(2; -; [m]; \{-\})$ for some $m \geq 2$;
(2) Signature $(1; -; [j, k]; \{-\})$ for some (j, k) with $k \geq j \geq 2$;

(3) Signature $(1; -; [2, j, k]; \{-\})$ for some (j, k) with $k \geq j \geq 2$.

We will consider these three cases in turn, noting that $p > 29$.

Case (1): Signature $(2; -; [m]; \{-\})$

Here the Riemann-Hurwitz formula gives $|G| = 2m(g-1)/(m-1) = 4mp/(m-1)$, and $|G^+| = 2mp/(m-1)$. Then since G^+ must have an element of order m, it follows that $m-1$ divides $2p$. In particular, $m-1$ lies between 1 and $2p$, so $|G| = 4mp/(m-1)$ lies between $2(2p+1) = 2g$ and $8p = 4(g-1)$, and therefore $|G^+|$ lies between g and $2(g-1)$. But also $|H| > g$ and $|H|$ divides $|G|/2$, and it follows that $|G| = 2|H|$, and $H = G^+$.

In fact $m = 2, 3, p+1$ or $2p+1$, and accordingly, $|G| = 8p, 6p, 4p+4$ or $4p+2$, but the second and fourth of these are impossible, since $|G|$ is divisible by 4. Hence either $m = 2$ or $m = p+1$, and $|G| = 8p = 4(g-1)$ or $|G| = 4p+4 = 2(g+1)$, respectively.

Next, the group G is generated by the images d, e and x of the generators d_1, d_2 and x_1 of any NEC group with signature $(2; -; [m]; \{-\})$ such that $1 = d^2e^2x = x^m$. The index 2 orientation-preserving subgroup $H = G^+$ of G is generated by $(a, b, c) = (d^2, e^2, de^{-1})$, with $x = e^{-2}d^{-2} = b^{-1}a^{-1}$, and $x^d = d^{-1}e^{-2}d^{-1} = d^{-2}de^{-1}e^{-2}ed^{-1} = a^{-1}cb^{-1}c^{-1} = x$, and $x^e = e^{-3}d^{-2}e = e^{-4}ed^{-1}d^{-2}de^{-1}e^2 = b^{-2}c^{-1}a^{-1}cb = x$. Conjugation of the generators a, b and c of G^+ by d has the following effect, since H is abelian: $a^d = d^2 = a$; $b^d = d^{-1}e^2d = d^{-2}de^{-1}e^2ed^{-1}d^2 = a^{-1}cbc^{-1}a = b$; and $c^d = e^{-1}d = e^{-2}ed^{-1}d^2 = b^{-1}c^{-1}a = ab^{-1}c^{-1}$. Also conjugation by e has the same effect as d, since $de^{-1} = c$ lies in H.

Next, let v be any generator for the group H, which has order n, say. Then $d^2 = a = v^r$, and $e^2 = b = v^s$, and $de^{-1} = c = v^t$ for some r, s and t. In particular, $c^d = ab^{-1}c^{-1} = v^{r-s-t}$, and then because c^d has the same order as $c = v^t$, we find that $v^{r-s-t} = c^d = v^{tk} = c^k$ for some k, with $r-s-t \equiv tk \mod n$. Moreover, $c = c^a = c^{d^2} = c^{k^2}$ and so $k^2 \equiv 1 \mod o(c)$, where $o(c)$ is the order of c. Also d centralises v^r $(= a)$ and v^s $(= b)$, and conjugates v^t $(= c)$ to v^{r-s-t}.

This information is enough to completely define G, as a non-split extension by C_2 of the cyclic group H of order n generated by v.

It also follows that there exists an automorphism of G that inverts each of d, e, a and b, and takes c to c^{-k}. To see this, note that the relations $a = d^2$, $b = e^2$, $a^d = a$ and $b^d = b$ are all preserved when each of a, b, d and e is replaced by its inverse. Also the same holds for the relations $c = de^{-1}$ and $c^d = ab^{-1}c^{-1}$ when c is taken to c^{-k}, because $d^{-1}e = d^{-2}de^{-1}e^2 = a^{-1}cb = v^{t+s-r} = v^{-tk} = c^{-k}$, and $a^{-1}bc^k = v^{-r+s+tk} = v^{-t} = c^{-1}$, while on the other hand $(c^{-k})^{d^{-1}} = (c^{-k})^d = c^{-k^2} = c^{-1}$ as well.

It follows that the group G admits an automorphism that inverts each of d and e, and hence by Proposition 2.1, the action of G on the surface S extends to an action of a larger group with signature $(0; +; [2, 2]; \{(m)\})$ on S, so S is not pseudo-real. This contradiction shows that case (1) is impossible.

Case (2): Signature $(1; -; [j, k]; \{-\})$

We know that the group G is generated by elements d, x and y, as the images of the generators d_1, x_1 and x_2 of any NEC group with signature $(1; -; [j, k]; \{-\})$ such that $1 = d^2xy = x^j = y^k = 1$, and that its index 2 subgroup G^+ is generated by x, y and $z = x^d$, with $d^2 = (xy)^{-1}$ and $y^d = d^{-1}yd = d^{-1}(d^2x)^{-1}d = d^{-1}x^{-1}dd^{-2} = z^{-1}xy$. Conjugation of the generators x, y and z of G^+ by d has the following effect, since H is abelian: $x^d = z$, $y^d = z^{-1}xy$ (as above), and $z^d = x^{d^2} = x$ (since d^2 lies in H).

Now suppose that $H = G^+$. Then G^+ itself is cyclic, generated by an element v, and $x = v^r$ and $y = v^s$, and $x^d = z = v^t$ for some r, s and t, with $d^2 = (xy)^{-1} = v^{-(r+s)}$. Also conjugation by d interchanges v^r and v^t, and takes v^s to v^{r+s-t}.

This information and the order of v are enough to completely define G, as a non-split extension by C_2 of the cyclic group H generated by v. Moreover, the set all of the relations that are satisfied by d, x, y and v is preserved under a transformation that inverts each of d and x (and takes $y = x^{-1}d^{-2} = v^s$ to $xd^2 = v^{-s}$, and $z = x^d = v^t$ to $dx^{-1}d^{-1} = d^2(d^{-1}xd)^{-1}d^{-2} = v^{-t}$). Hence the group G admits an automorphism that inverts each of d and x, and so by Proposition 2.1 the surface S is not pseudo-real, contradiction.

Hence H is a proper subgroup of G^+. In particular, G^+ is not cyclic. Furthermore, this implies that $|G| = 2|G^+| \geq 4|H| > 4(g-1)$, and it follows from the Riemann-Hurwitz formula that either $j = 2$ and $k > 2$, or $j = 3$ and $k = 3, 4$ or 5.

If $(j, k) = (3, 3)$ then $|G| = 6(g-1) = 12p$. As $p > 29$, we find that G has a cyclic normal Sylow p-subgroup P of order p, with quotient G/P of order 12. Hence the associated NEC group Γ with signature $(1; -; [3, 3]; \{-\})$ has a normal subgroup N of index 12 contained in Γ^+. But an easy computation using Magma shows there is no such subgroup N in Γ, contradiction. Similarly if $(j, k) = (3, 4)$ then $|G| = (24/5))(g-1) = 48p/5$, which gives $p = 5$, while if $(j, k) = (3, 5)$ then $|G| = (30/7)(g-1) = 60p/7$, which gives $p = 7$, and both of these are impossible (because we have assumed that $p > 29$).

Thus $j = 2$ and $k > 2$.

Now the Riemann-Hurwitz formula gives $|G| = (4k/(k-2))(g-1) = 8kp/(k-2)$. Also $|G|$ must be divisible by $2k$ since the element y of order k lies in G^+, but also $|G|$ cannot be $2k$ (for otherwise G^+ is cyclic, generated by y), and therefore $k-2$ strictly divides $4p$, so $k - 2 = 1, 2, 4, p$ or $2p$. This gives us five sub-cases to consider, which we will denote by (2a) to (2e).

In sub-case (2a), we have $k = 3$. Then G acts with signature $(1; -; [2, 3]; \{-\})$, and $|G| = 8kp/(k-2) = 24p$, so $|G^+| = 12p$, and then since $|G^+| > |H| > g - 1 = 2p$ (and $p > 29$), we find that $|H| = 3p, 4p$ or $6p$. Next, as $p > 23$, we know that G has a cyclic normal Sylow p-subgroup P of order p, with quotient G/P of order 24. Accordingly, the NEC group Γ with signature $(1; -; [2, 3]; \{-\})$ has a normal subgroup N of index 24. A Magma computation shows that there is just one possibility for N, and in this case Γ^+/N is isomorphic to A_4. On the other hand, H is a cyclic subgroup of G^+ with $|H| > 2p$, and since the orders of the elements of A_4 are $1, 2$ and 3, we deduce that $|H| = 3p$. In particular, some element of order 3 in G centralises P, and since P is normal in G and the single conjugacy class of cyclic subgroups of order 3 in A_4 generates A_4, it follows that P is central in G^+. Now by Schur's theorem on centre-by-finite groups (see [19, 10.1.4]), the exponent of the derived subgroup of G^+ divides the index of the centre of G^+ in G^+, so divides $|G^+ : P| = 12$. Another computation using Magma shows that the derived subgroup of Γ^+ has index 6 in Γ^+, so the index of the derived subgroup of G^+ in G^+ divides 6, and therefore this derived subgroup contains P. In particular, the exponent of P divides 12, and so $p = 3$, contradiction.

In sub-case (2b), where $k = 4$, we know that G acts with signature $(1; -; [2, 4]; \{-\})$, and $|G| = 16p$, with $|G^+| = 8p$ and $|H| = 4p$ (again since $|G^+| > |H| > 2p$). Hence in particular, H has index 2 in G^+, and index 4 in G. Accordingly, the NEC group Γ with signature $(1; -; [2, 4]; \{-\})$ has a subgroup of index 4 lying inside Γ^+, with abelianisation large enough for it to have the cyclic group H of order $4p$ as a quotient.

A computation with Magma shows that there are seven conjugacy classes of subgroups of index 2 in Γ^+, but for six of them the subgroups have abelianisation of order 32 or 64, and so H must be a quotient of a subgroup in the seventh one. That class consists of a single subgroup of index 4 in Γ, generated by x_1x_2, x_2x_1 and $x_1d_1x_1d_1^{-1}$, and hence H is generated by $a = xy$, $b = yx$ and $c = xdxd^{-1}$ (where d, x and y are the images of d_1, x_1 and x_2). Conjugation by x and d have the following effects: $a^x = b$, and $b^x = a$, and $c^x = c^{-1}$, while $a^d = a$ (since $a = d^{-2}$), and $b^d = d^{-1}yxd = ac^{-1}bca^{-1} = b$ (by an easy exercise), and $c^d = d^{-1}xdx = ac^{-1}b^{-1}$ (by an easier exercise).

Next, let v be a generator for H, and suppose $a = v^r$, and $b = v^s$, and $c = v^t$. Then $d^2 = (xy)^{-1} = a^{-1} = v^{-r}$, and conjugation by x interchanges v^r with v^s and inverts v^t, while conjugation by d fixes v^r and v^s, and takes v^t to v^{r-s-t}. Since x has order 2 and $[d, x] = d^{-1}xdx$ is conjugate to c, which lies in H and therefore centralises H, this information is enough to completely define G, as a non-split extension by $C_2 \times C_2$ of the cyclic group of order $4p$ generated by v. Also the set all of the relations that are satisfied by d, x, y and v is preserved under a transformation that inverts d and x (and takes $y = (d^2x)^{-1}$ to $(d^{-2}x)^{-1} = (xyx)^{-1}$), and $v^r = a = d^{-2}$ to $d^2 = a^{-1} = v^{-r}$, and $v^s = b = a^x$ to $(a^{-1})^x = v^{-s}$, and $v^t = c = xdxd^{-1}$ to $xd^{-1}xd = (d^{-1}xdx)^{-1} = (ac^{-1}b^{-1})^{-1} = bca^{-1} = v^{-(r-s-t)}$. Hence G admits an automorphism that inverts each of d and x, and so by Proposition 2.1, the surface S is not pseudo-real, contradiction.

In sub-case (2c), we have $k = 6$, and G acts with signature $(1; -; [2, 6]; \{-\})$. This case may be eliminated by a similar argument. We have $|G| = 12p$, so $|G^+| = 6p$ and therefore $|H| = 3p$, with $|G\!:\!H| = 4$. A Magma computation shows that there is just one possibility for the pre-image of H in the associated NEC group Γ, and again this subgroup is generated by x_1x_2, x_2x_1 and $x_1d_1x_1d_1^{-1}$. We can now proceed using the same argument as above, to show that G admits an automorphism that inverts each of the images d and x of the generators d_1 and x_1 of Γ, and hence S is not pseudo-real, contradiction.

In sub-case (2d), where $k = p+2$, we have $|G| = 8k = 8p+16$ so $|G^+| = 4k = 4p+8$, and then since $2 \le |G^+ : H| < (4p+8)/(2p) < 5p/(2p) < 3$, we have $|H| = |G^+|/2 = 2p+4 = 2k$. Next, if d, x and y are the usual images of the generators of the associated NEC group Γ, then the elements y, y^x and y^d of G^+ all have the same order and generate the maximal cyclic subgroup of odd order $p+2 = k$ in H, with index 2. In particular, this subgroup is characteristic in H and hence normal in G^+, and it follows that the subgroup generated by y, y^x, y^d and x has order $2k$. But also this subgroup contains $(xy)^{-1} = d^2$ and $y^dy^{-1}x = y^dd^2 = d^{-1}yd^2d = d^{-1}xd = x^d$, and so contains all of the generators d^2, x,

y, x^d and y^d of G^+, so $|G^+| = 2k$, contradiction.

Finally, in sub-case (2e), we have $k = 2p+2$, with $|G| = 4k$, so $|G^+| = 2k = 4p+4$ and $|H| = 2p+2 = k$. We can replace H if necessary and suppose that H is the cyclic subgroup generated by y. Then also H contains y^x (since H is normal in G^+), and so y^x commutes with y, and hence $1 = [y^x, y] = xy^{-1}xy^{-1}xyxy$, which gives $d^{-4} = (xy)^2 = (yx)^2 \in H$.

Let us consider the elements $a = y$, $b = y^x$ and $c = y^d$ of G^+. Using the facts that $(xy)^2 = (yx)^2$ and $d^2xy = 1$ (and hence that xy commutes with d, and also $dxyd = 1$), we find that conjugation of a, b and c by each of d and x has the following effects:

$$\begin{aligned}
a^d &= y^d = c,\\
b^d &= y^{xd} = d^{-1}xyxd = d^{-1}(xy)^2y^{-1}d\\
&= (xy)^2d^{-1}y^{-1}d = (yx)^2(y^d)^{-1} = y(y^x)(y^d)^{-1} = abc^{-1},\\
c^d &= y^{d^2} = y^{y^{-1}x} = y^x = b,\\
a^x &= y^x = b,\\
b^x &= y^{x^2} = y = a,\\
c^x &= y^{dx} = (dx)^{-1}ydx = (yd)ydx = y(dx)xydx\\
&= y(yd)^{-1}xydx = yd^{-1}y^{-1}dxyx = y(y^d)^{-1}(y^x) = ac^{-1}b.
\end{aligned}$$

In particular, since conjugation by $d^2 = y^{-1}x$ has the same effect as x, this implies that $ac^{-1}b = c^x = c^{d^2} = b^d = abc^{-1}$, and therefore $c^{-1}b = bc^{-1}$, so $bc = cb$. But also $ba = (xy)^2 = (yx)^2 = ab$, and conjugating $bc = cb$ by d^{-1} gives $ca = ac$. Hence the subgroup generated by a, b and c is abelian.

We can now show that H contains $c = y^d$. For suppose the contrary. Then the cyclic subgroups of order k generated by $a = y$ and $c = y^d$ are distinct, and both have index 2 in G^+, and so they generate G^+. Hence G^+ is abelian. In particular, since $x \in G^+$ we see that x centralises a, so $b = a^x = a$, and then also $c = c^x = ac^{-1}b = a^2c^{-1}$, so $c^2 = a^2$. Next, consider x once more. We have $x = a^j$ or a^jc for some j, and because x has order 2, either $x = a^{k/2}$ ($= c^{k/2}$ since $k/2 = p+1$ is even), or $x = a^{-1}c$, or $a^{(k/2)-1}c$. But if $x = a^{k/2}$ then $d^{-2} = xy = a^{1+k/2}$, which has order k and so generates H, and it follows that both x and y are powers of d, so G is cyclic, generated by d, but of order $2k$, not $4k$, a contradiction. Similarly, if $x = a^{(k/2)-1}c$, then $d^{-2} = xy = a^{k/2}c = c^{k/2+1}$, so d^{-2} has order k and hence generates H; but then conjugating by d shows also that $d^{-2} = (d^{-2})^d$ is a generator for the cyclic subgroup generated by $c^d = b = a$, and again this implies that G is cyclic of order $2k$, a contradiction. On the other hand, if $x = a^{-1}c$ then $d^{-2} = xy = c$, so $a = b = c^d = c$, but then G^+ is generated by a single element of order k, another contradiction.

Thus H contains y^d, and it follows that H contains all three of a, b and c, but not x or x^d (as in sub-case (2d)). Indeed since H is generated by y, we see that $b = y^s$ for some integer $s < k$, with $s^2 \equiv 1 \bmod k$ (because $x^2 = 1$), and similarly $c = y^t$ for some $t < k$, with $t^4 \equiv 1 \bmod k$ (because $d^4 = (xy)^{-2} \in H$ and so centralises y). Indeed $d^{-4} = (xy)^2 = ba = y^{s+1}$.

Next, conjugation by d takes $y = a$ to $c = y^t$, and $y^t = c$ to $b = y^s$, and $y^s = b$ to $abc^{-1} = y^{1+s-t}$. Hence $y^s = (y^t)^d = (y^d)^t = y^{t^2}$, so $s \equiv t^2 \bmod k$. Similarly, conjugation by x interchanges y with y^s, and takes y^t to y^{1+s-t}, as does conjugation by $d^2 = y^{-1}x$. In particular, $y^{ts} = (y^t)^x = y^{1+s-t}$, so $st \equiv 1 + s - t \bmod k$, which implies that $b^d = c^x = y^{1+s-t} = y^{st}$. Also $d^4 = a^{-1}b^{-1} = y^{-(1+s)} = y^{-(1+t^2)}$, and $[d, x] = d^{-1}xdx = d^{-2}(dx)^2 = d^{-2}(yd)^{-2} = d^{-4}dy^{-1}d^{-1}y^{-1} = ba(d^{-3}a^{-1}d)a^{-1} = ba(abc^{-1})^{-1}a^{-1} = ca^{-1} = y^{t-1}$.

Again this information is enough to completely define G, this time as an extension by C_4 of the cyclic group of order k generated by y, and then the set all of the relations that are satisfied by d, x and y is preserved under a transformation that inverts each of d and x, and takes $a = y = x^{-1}d^{-2}$ to $xd^2 = x(xy)^{-1} = xy^{-1}x = b^{-1} = y^{-s}$, and $b = y^s$ to $y^{-1} = a^{-1}$, and $c = y^d = y^t$ to y^{-st}. It follows that the group G admits an automorphism that inverts each of d and x, and so by Proposition 2.1, the surface S is not pseudo-real, another contradiction. Thus case (2) is impossible as well.

Case (3): Signature $(1; -; [2, j, k]; \{-\})$

Here G is generated by elements d, x, y and z such that $1 = d^2xyz = x^2 = y^j = z^k = 1$, and G^+ is generated by x, y, z, x^d, y^d and z^d (with $d^2 = (xyz)^{-1}$). These elements have orders $2, j, k, 2, j$ and k. Also $|G| = 4jk(g-1)/(3jk - 2j - 2k)$ by the Riemann-Hurwitz formula, and then because $|H| > g - 1 = 2p$, we need $|G| > 2(g-1) = 4p$. Together these things imply that either $j = 2$, or $(j, k) = (3, 3)$, $(3, 4)$ or $(3, 5)$, as earlier.

If $j = 3$ then $|G| = (12k/(7k-6))(g-1) = (12/5)(g-1)$, $(24/11)(g-1)$ or $(60/29)(g-1)$, and therefore $|G| = 24p/5$, $48p/11$ or $120p/29$. But these are all impossible, since the prime p is not 5, 11 or 29.

Thus $j = 2$, and $|G| = 8k(g-1)/(4k-4) = 2k(g-1)/(k-1) = 4kp/(k-1)$. Moreover, since $|G|$ must be divisible by 4, and $k - 1$ is coprime to k, we find that $k - 1$ must divide p and hence $k - 1 = 1$ or p, which gives $k = 2$ or $p + 1$, and therefore $|G| = 8p$ or $4(p + 1)$.

In the first case, $|G| = 8p = 4(g-1)$, and $|H|$ divides $|G^+| = 2(g-1)$, but $|H| > g - 1$, so $|H| = 2(g-1)$. Hence $H = G^+$, so G^+ is cyclic, of order $2(g-1) = 4p$. On the other hand, G^+ is generated by the elements x, y, z, x^d, y^d and z^d, which all have order 2, so this is impossible. In the second case, $|G| = 4p+4 = 2(g+1)$, and so $|H|$ divides $g+1$, and then since $|H| > g - 1$ we find that $|H| = g + 1 = 2p + 2 = 2k$. Thus $|H| = |G|/2$, so $H = G^+$, and again G^+ is cyclic, of order $2k$. On the other hand, G^+ is generated by the elements x, y, z, x^d, y^d and z^d, which all have order 2 or k, and since they generate a cyclic group of order $2k$, we deduce that k is odd. But here $k = p + 1$, which is even, contradiction.

This completes the proof. ∎

We can now summarise the situation for cyclic group actions in the following.

Corollary 3.10

(a) $M_{\text{cyc}}(g) \geq 2g - 2$ *for all* $g \geq 2$*, and this bound is sharp for every odd* $g \geq 3$*;*

(b) $M^+_{\text{cyc}}(g) \geq g - 1$ *for all* $g \geq 2$*, and this bound is sharp for infinitely many* $g \geq 3$*.*

4 The abelian case

In this section, we first consider the parameter $M_{ab}(g)$, in three different cases (namely g even, $g \equiv 1 \bmod 4$ and $g \equiv 3 \bmod 4$), and find its precise value for each $g \geq 2$; see

Corollary 4.7. Then we consider $M^+_{ab}(g)$ in the same three cases, and show that the smallest of these gives a sharp bound for a very large and possibly infinite set of genera g.

Theorem 4.1 *For every even integer $g \geq 2$ with $g \neq 16$, the largest order of an abelian group of automorphisms acting essentially on a pseudo-real surface of genus g is $2g$. When $g = 16$, the largest such order is* 36.

Proof: First, we know that $M_{ab}(g) \geq M_{cyc}(g) \geq 2g$, by Theorem 3.2, and so all we have to do is show that $M_{ab}(g)$ cannot be greater than $2g$, except in the case $g = 16$.

So let G be an abelian group of order $4n \geq 2g$ having a faithful essential action on a pseudo-real surface of even genus $g \geq 2$. Then by Corollary 2.4, we know that G acts with signature $(1; -; [2, j, k]; \{-\})$, where either $j = 2 \leq k$, or $\{j, k\} = \{3, 3\}$, $\{3, 4\}$ or $\{3, 5\}$. In these cases, if d, x, y and z are the images in G of the canonical generators of the associated NEC group Γ, with $d^2xyz = x^2 = y^j = z^k = 1$, then the orientation-preserving subgroup G^+ is generated by x, y and z (with $d^2 = (xyz)^{-1}$, and $(x^d, y^d, z^d) = (x, y, z)$ since G is abelian), and so $\mathrm{lcm}(2, j, k)$ divides $2n = |G^+|$, which in turn divides $2jk$.

If $j = 2$, then by the Riemann-Hurwitz formula $g = 1 + 2n(1 - 1/k) = 1 + 2n - 2n/k$, and so $2n/k$ has to be odd, making k even (and giving $\mathrm{lcm}(2, 2, k) = k$, which divides $2n$). Also $2n = |G^+|$ divides the product $2jk = 4k$, and it follows that $k = n/2$, n or $2n$, and the unique value of k making $2n/k$ odd is $2n$. Accordingly $g = 1 + 2n - 1 = 2n$, so $|G| = 4n = 2g$, which gives no improvement on what comes from Theorem 3.2.

If $\{j, k\} = \{3, 3\}$, then $6 = \mathrm{lcm}(2, j, k)$ divides $2n$, which divides $2jk = 18$, and therefore $2n = 6$, 12 or 18, and $|G| = 12$, 24 or 36, respectively. By the Riemann-Hurwitz formula, $g = 1 + 5|G|/12 = 6$, 11 or 16. In the first case, $|G| = 2g$, giving no improvement, while in the second case, the genus g is odd. In the third case, however, $|G| > 2g$, and indeed this case can be realised, by taking $G = \langle\, u, v \mid u^3 = v^{12} = [u, v] = 1 \,\rangle \cong C_3 \times C_{12}$, and $(d, x, y, z) = (v, v^6, uv^8, u^{-1}v^8)$, for example. Hence $|G|$ can be 36 when $g = 16$.

If $\{j, k\} = \{3, 4\}$, then $12 = \mathrm{lcm}(2, j, k)$ divides $2n$, which divides $2jk = 24$, so $2n = 12$ or 24, and $|G| = 24$ or 48, respectively. Also $g = 1 + 11|G|/24 = 12$ or 23, and the former case gives no improvement, while the latter gives odd genus.

Finally, if $\{j, k\} = \{3, 5\}$, then $\mathrm{lcm}(2, j, k) = 2jk = 30$, so $2n = 30$ and $|G| = 60$, and then $g = 30$ by the Riemann-Hurwitz formula, again giving no improvement. ∎

In fact the largest order described in Theorem 4.1 completely determines the group, and in most cases the group action, as we see below.

Proposition 4.2 *The largest order of an abelian group having a faithful essential action on a pseudo-real surface of even genus g is attained only by the cyclic group C_{2g} when $g \neq 16$, and by $C_3 \times C_{12}$ when $g = 16$. Moreover, when $g \neq 6$, 12 or 30, the group action is also uniquely determined (up to group automorphism): if Γ is the associated NEC group, then the action of G is given by the epimorphism $\theta : \Gamma \to C_{2g}$ described in Remark 3.3 when $g \neq 16$, or the epimorphism $\theta : \Gamma \to C_3 \times C_{12}$ given in the proof of Theorem 4.1 when $g = 16$. For the exceptional genera 6, 12 and 30, the actions are not unique.*

Proof: We use the same notation as in the proof of Theorem 4.1.

If $j = 2$, then $k = 2n = |G^+|$, and so G^+ is cyclic, generated by z, and hence each of x and y is the unique involution in G^+, namely $z^{k/2} = z^n$. It follows that $d^2 = (xyz)^{-1} = z^{-1}$, and therefore G is cyclic of order $4n = 2g$, generated by d. In this case, the action of $G = C_{2g}$ with signature $(1; -; [2, 2, g]; \{-\})$ is unique up to an automorphism of G, as explained in Remark 3.3.

If $\{j, k\} = \{3, 3\}$, then $g = 6$ and $|G| = 12$, or $g = 16$ and $|G| = 36$. In the former case, G^+ is generated by elements x, y and z of orders 2, 3 and 3 and hence is cyclic of order 6, and $d^2 = (xyz)^{-1}$ has order 2 or 6, so G is cyclic of order 12. Similarly, in the latter case, G^+ is isomorphic to $C_3 \times C_6$ (since it has order 18), and $d^2 = (xyz)^{-1}$ has order 6, so d has order 12, and $G \cong C_3 \times C_{12}$. In this case we also note that $x = d^6$ (the unique involution in G), and y can be chosen as any element of order 3 lying outside $\langle d \rangle$, and this makes the action of G unique up to an automorphism of G, as explained in the proof of Theorem 4.1.

If $\{j, k\} = \{3, 4\}$, then $g = 12$ and $|G| = 24$, and G^+ is generated by elements x, y and z of orders 2, 3 and 4, so must be cyclic of order 12, then $d^2 = (xyz)^{-1}$ has order 12, so $G \cong C_{24}$. Similarly, if $\{j, k\} = \{3, 5\}$, then $g = 30$ and $|G| = 60$, and G^+ is cyclic of order $2 \cdot 3 \cdot 5 = 30$, so $G \cong C_{60}$.

In particular, these cases with $j = 3$ show that for genera 6, 12 and 30, there is more than one action of C_{2g} (up to equivalence). In addition to the action with signature $(1; -; [2, 2, g]; \{-\})$ described in Remark 3.3, the extra actions are the following, where u is a generator for C_{2g}. When $g = 6$ there are two more actions, with signature $(1; -; [2, 3, 3]; \{-\})$, and described by the generating vectors (u, u^6, u^{-4}, u^{-4}) and (u^3, u^6, u^4, u^{-4}); when $g = 12$ there is one more action, with signature $(1; -; [2, 3, 4]; \{-\})$ and described by $(u, u^{12}, u^{-8}, u^{-4})$; and when $g = 30$ there is one more action, with signature $(1; -; [2, 3, 5]; \{-\})$, and described by $(u, u^{30}, u^{-20}, u^{-12})$. ∎

Next, we consider essential actions of abelian groups on pseudo-real surfaces of odd genus. We split this into two cases, starting with the case $g \equiv 1 \bmod 4$.

Theorem 4.3 *For every integer $g \equiv 1$ mod 4 with $g \geq 5$, the largest order of an abelian group of automorphisms acting essentially on a pseudo-real surface of genus g is $2(g + 3)$.*

Proof: First, we show that $M_{ab}(g) \geq 2(g + 3)$ for every such g.

Let Γ be a maximal NEC group with signature $(1; -; [2, 2, (g+3)/4]; \{-\})$, and canonical generators d_1, x_1, x_2 and x_3 satisfying $d_1^2 x_1 x_2 x_3 = x_1^2 = x_2^2 = x_3^{(g+3)/4} = 1$.

If $(g + 3)/4$ is even, then there exists a smooth epimorphism θ from Γ to the direct product $G = C_2 \times C_2 \times C_{(g+3)/2}$ generated by commuting elements a, b and c of orders 2, 2 and $(g + 3)/2$, respectively, taking (d_1, x_1, x_2, x_3) to (c, a, b, abc^{-2}). On the other hand, if $(g + 3)/4$ is odd, then there exists a smooth epimorphism θ from Γ to the direct product $G = C_2 \times C_{g+3}$ generated by commuting elements u and v of orders 2 and $g+3$, respectively, taking (d_1, x_1, x_2, x_3) to $(v, u, uv^{(g+3)/2}, v^{(g-1)/2})$.

In both of these two cases, G is an abelian group of order $2(g + 3)$. Moreover, all of the involutions of the chosen group G lie inside the subgroup generated by the images of x_1, x_2 and x_3, and so by Proposition 2.2 we find that G is the automorphism group of a pseudo-real surface S. Also by the Riemann-Hurwitz formula, the genus of S is $1 + |G|(g-1)/(2(g+3)) = g$, and so this gives $M_{ab}(g) \geq |G| = 2(g+3)$.

Next, we show that this bound of $2(g+3)$ is sharp.

Let G be an abelian group of automorphisms of a pseudo-real surface of genus g, of order $4n \geq 2(g+3)$, and acting with signature $(\gamma; -; [m_1, \ldots, m_r]; \{-\})$. Then by Corollary 2.4, we know that $\gamma = 1$ and $r = 3$, with $\{m_1, m_2, m_3\} = \{2, j, k\}$ for some j, k, where either $j = 2 \leq k$, or $\{j, k\} = \{3, 3\}$, $\{3, 4\}$ or $\{3, 5\}$. Also, just as in the proof of Theorem 4.1, we know that $2n = |G^+|$ is divisible by $\mathrm{lcm}(2, j, k)$ and divides $2jk$.

If $j = 2$ and k is even then $2n$ is divisible by $\mathrm{lcm}(2, j, k) = k$ and divides $4k$, so $2n = k$, $2k$ or $4k$, but then substituting into the Riemann-Hurwitz formula gives $4n = 2g$, $2g + 2$ or $2g + 6$ respectively, which gives no improvement. Similarly if $j = 2$ and k is odd then $2n$ is divisible by $\mathrm{lcm}(2, j, k) = 2k$ and divides $4k$, so $2n = 2k$ or $4k$, and then $4n = 2g + 2$ or $2g + 6$, which again gives no improvement. On the other hand, if $\{j, k\} = \{3, 3\}$, $\{3, 4\}$ or $\{3, 5\}$, then the same arguments give $2n \in \{6, 18\}$, $\{12, 24\}$ or $\{30\}$, and $g \in \{6, 16\}$, $\{12, 23\}$ or $\{30\}$, respectively, but in none of these cases is $g \equiv 1 \bmod 4$, so these three cases can be eliminated. Accordingly, we cannot do better than $|G| = 2(g + 3)$. ∎

Once again, the largest order completely determines the group, and this time also the group action.

Proposition 4.4 *The largest order of an abelian group G with a faithful essential action on a pseudo-real surface of genus $g \equiv 1 \bmod 4$ is attained only for $G = C_2 \times C_2 \times C_{(g+3)/2}$ when $(g+3)/4$ is even, and for $C_2 \times C_{g+3}$ when $(g+3)/4$ is odd, and in both cases, the group acts with signature $(1; -; [2, 2, (g+3)/4]; \{-\})$. Moreover, the group action is also uniquely determined (up to group automorphism): if Γ is the associated NEC group, then the action of G is given by the epimorphism $\theta\colon \Gamma \to G$ as described in the proof of Theorem 4.3.*

Proof: First, the calculations in the proof of Theorem 4.3 show that the only cases for which $|G| = 2(g+3)$ are those with $|G|/2 = 4k$, giving signature $(1; -; [2, 2, (g+3)/4]; \{-\})$. In these cases, the orientation-preserving subgroup G^+ (of order $g + 3$) is generated by the images of the elements x_1, x_2 and x_3 of the associated NEC group Γ, which have orders 2, 2 and $(g + 3)/4$ respectively. It follows that $G^+ \cong C_2 \times C_2 \times C_{(g+3)/4}$, which is isomorphic to $C_2 \times C_{(g+3)/2}$ when $(g + 3)/4$ is odd. Also the relation $d_1^2 x_1 x_2 x_3 = 1$ implies that the image in G of d_1 has order $2 \cdot \mathrm{lcm}(2, 2, (g+3)/4)$, which is $(g+3)/2$ when $(g+3)/4$ is even, and $g+3$ when $(g+3)/4$ is odd. It follows that G is isomorphic to $C_2 \times C_2 \times C_{(g+3)/2}$ when $(g + 3)/4$ is even, and to $C_2 \times C_{g+3}$ when $(g + 3)/4$ is odd.

Next, suppose the action of G is given by the smooth epimorphism $\varphi\colon \Gamma \to G$.

If $(g+3)/4$ is even, with $G \cong C_2 \times C_2 \times C_{(g+3)/2}$, then the images $\varphi(d_1)$, $\varphi(x_1)$ and $\varphi(x_2)$ must be elements of G of orders $(g+3)/2$, 2 and 2 respectively, in order to generate G. But also $|G| = 2(g+3)$ is the product of those orders, and it follows that $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ where $(a, b, c) = (\varphi(x_1), \varphi(x_2), \varphi(d_1))$, and then also $\varphi(x_3) = \varphi(d_1^2 x_1 x_2)^{-1} = (c^2 ab)^{-1} = abc^{-2}$. This makes φ equivalent to the epimorphism $\theta\colon \Gamma \to G$ given in the proof of Theorem 4.3.

On the other hand, if $(g+3)/4$ is odd, with $G \cong C_2 \times C_{g+3}$, then $\varphi(d_1)$ must have order $g+3$, while each of $\varphi(x_1)$ and $\varphi(x_2)$ is one of the three involutions in G, and $\varphi(x_3) = \varphi(d_1^2 x_1 x_2)^{-1}$ has to be an element of order $(g+3)/4$. In particular, $\varphi(x_3)$ lies in $\langle \varphi(d_1) \rangle \cong C_{g+3}$, since its order $(g+3)/4$ is odd, and it follows that both $\varphi(x_1)$ and $\varphi(x_2)$ lie outside $\langle \varphi(d_1) \rangle$. Letting $u = \varphi(x_1)$ and $v = \varphi(d_1)$, we find that $\varphi(x_2) \in \{u, uv^{(g+3)/2}\}$. But $\varphi(x_2) \neq u$, for otherwise $\varphi(x_3) = v^{-2}$ which has order $(g+3)/2$, and therefore $\varphi(x_2) = uv^{(g+3)/2}$, which gives $\varphi(x_3) = (v^2 uuv^{(g+3)/2})^{-1} = (v^{(g+7)/2})^{-1} = v^{(g-1)/2}$. Again this makes φ equivalent to the epimorphism $\theta \colon \Gamma \to G$ chosen in the proof of Theorem 4.3. ∎

We now turn to the case $g \equiv 3 \bmod 4$.

Theorem 4.5 *For every positive integer $g \equiv 3 \bmod 4$, the largest order of an abelian group of automorphisms acting essentially on a pseudo-real surface of genus g is $2(g+1)$.*

Proof: First, we show that $M_{ab}(g) \geq 2(g+1)$ for every such g.

Let Γ be a maximal NEC group with signature $(1; -; [2, 2, (g+1)/2]; \{-\})$ and canonical generators d_1, x_1, x_2 and x_3 satisfying $d_1^2 x_1 x_2 x_3 = x_1^2 = x_2^2 = x_3^{(g+1)/2} = 1$, and let $G = C_2 \times C_{g+1}$, generated by commuting elements u and v of orders 2 and $g+1$ respectively. Then there exists a smooth epimorphism $\theta \colon \Gamma \to G$ taking (d_1, x_1, x_2, x_3) to (v, u, u, v^{-2}). Moreover, since $(g+1)/2$ is even, all the involutions of G lie inside the subgroup generated by the images of x_1, x_2 and x_3, and so by Proposition 2.2 we find that G is the automorphism group of a pseudo-real surface S. Also by the Riemann-Hurwitz formula, the genus of S is $1 + |G|(g-1)/(2(g+1)) = g$, and so this gives $M_{ab}(g) \geq |G| = 2(g+1)$.

Next, we show that this bound of $2(g+1)$ is sharp.

Let G be an abelian group of automorphisms of a pseudo-real surface of genus g, of order $4n \geq 2(g+1)$, and acting with signature $(\gamma; -; [m_1, \ldots, m_r]; \{-\})$. Then by Corollary 2.4, we know that $\gamma = 1$ and $r = 3$, with $\{m_1, m_2, m_3\} = \{2, j, k\}$ for some j, k, where either $j = 2 \leq k$, or $\{j, k\} = \{3, 3\}$, $\{3, 4\}$ or $\{3, 5\}$. Also, just as in the proof of Theorem 4.1, we know that $2n = |G^+|$ is divisible by $\mathrm{lcm}(2, j, k)$ and divides $2jk$.

If $j = 2$ and k is even then $2n$ is divisible by $\mathrm{lcm}(2, j, k) = k$ and divides $4k$, so $2n = k$, $2k$ or $4k$, and then substituting into the Riemann-Hurwitz formula gives $g = 2n$, or $g = 2n - 1$, or $g = 2n - 3 = 4k - 3$ respectively. The only possibility with $g \equiv 3 \bmod 4$ is the second, which gives $|G| = 4n = 2(g+1)$ and hence no improvement. Similarly if $j = 2$ and k is odd then $2n$ is divisible by $\mathrm{lcm}(2, j, k) = 2k$ and divides $4k$, and so $2n = 2k$ or $4k$, but then $g = 2n - 1 = 2k - 1 \equiv 1 \bmod 4$ or $g = 2n - 3 = 4k - 3$, and both of these are impossible. On the other hand, if $\{j, k\} = \{3, 3\}$, $\{3, 4\}$ or $\{3, 5\}$, then once again we find that $2n \in \{6, 18\}$, $\{12, 24\}$ or $\{30\}$, and $g \in \{6, 16\}$, $\{12, 23\}$ or $\{30\}$, respectively. Here there is one case with $g \equiv 3 \bmod 4$, namely the one with $g = 23$ and $|G| = 4n = 48$ (occurring when $\{j, k\} = \{3, 4\}$), and again $|G| = 2(g+1)$, so there is no improvement.

Accordingly, we cannot do better than $|G| = 2(g+1)$. ∎

Here the abelian group G realising the largest action is unique, and the signature with which it acts is unique for $g \neq 23$, but in contrast to previous cases, there are several inequivalent actions (namely three when $g \equiv 3 \bmod 8$ and four when $g \equiv 7 \bmod 8$), as we show below.

Proposition 4.6 *The largest order of an abelian group G with a faithful essential action on a pseudo-real surface of genus $g \equiv 3 \bmod 4$ is attained only for $G = C_2 \times C_{g+1}$. When $g \neq 23$, the largest such group G acts with signature $(1; -; [2, 2, (g+1)/2]; \{-\})$, and if u and v are generators for $G = C_2 \times C_{g+1}$ of orders 2 and $g+1$ respectively, then its action is given by $(\theta(d_1), \theta(x_1), \theta(x_2), \theta(x_3)) = (v, u, u, v^{-2})$, $(v, u, v^{(g+1)/2}, uv^{(g-3)/2})$ or $(v, v^{(g+1)/2}, u, uv^{(g-3)/2})$, or $(v, u, uv^{(g+1)/2}, v^{(g-3)/2})$ when $g \equiv 7 \bmod 8$. In the first of these three or four cases, the surface is hyperelliptic. When $g = 23$, the signature can be $(1; -; [2, 2, 12]; \{-\})$ or $(1; -; [2, 3, 4]; \{-\})$, and in addition to the same four possibilities given in the previous sentence for the former signature, when the signature is $(1; -; [2, 3, 4]; \{-\})$ the action is given by $(\theta(d_1), \theta(x_1), \theta(x_2), \theta(x_3)) = (v, u, v^{16}, uv^6)$.*

Proof: The calculations in the proof of Theorem 4.5 show that if $g \neq 23$ then the only signature with which the bound is attained is $(1; -; [2, 2, (g+1)/2]; \{-\})$.

So now let G be an abelian group of order $2(g+1)$ acting with this signature on a pseudo-real surface of genus g, and let d, x, y and z be the images of the canonical generators d_1, x_1, x_2 and x_3 of the associated NEC group Γ. Then the subgroup G^+ has order $g+1$ and is generated by x, y and z, of orders 2, 2 and $(g+1)/2$ respectively, and it follows that $G^+ \cong C_2 \times C_{(g+1)/2}$, and also that at least one of x and y lies outside $\langle z \rangle$. On the other hand, G^+ is also generated by d^2, x and y (with $z = (d^2xy)^{-1}$), and so d^2 must have order $(g+1)/2$ as well, and therefore d has order $g+1$ (for otherwise $d \in \langle d^2, x, y \rangle = G^+$), and at least one of x and y lies outside $\langle d \rangle$. Accordingly, $G \cong C_2 \times C_{g+1}$, as required.

Next, let u be any involution of G lying outside $\langle d \rangle$, so that $G = \langle u \rangle \times \langle d \rangle$. Then each of x and y is one of the three involutions u, $d^{(g+1)/2}$, $ud^{(g+1)/2}$, and at least one of them is not $d^{(g+1)/2}$. By replacing u by $ud^{(g+1)/2}$ if necessary (via an automorphism of G), we may suppose at least one of x and y is u. Now if $x = y$ then $x = y = u$, and $z = (d^2xy)^{-1} = d^{-2}$. Similarly, if $(x, y) = (u, d^{(g+1)/2})$ or $(d^{(g+1)/2}, u)$, then $z = d^{(g-3)/2}u$, which has order $(g+1)/2$ for all $g \equiv 3 \bmod 4$, while if $(x, y) = (u, ud^{(g+1)/2})$, then $z = d^{(g-3)/2}$, which has order $(g+1)/2$ only when $g \equiv 7 \bmod 8$ (but order $(g+1)/4$ when $g \equiv 3 \bmod 8$). These give the three or four possibilities for the generating vector $(\theta(d_1), \theta(x_1), \theta(x_2), \theta(x_3))$ listed in the statement of the theorem, and all others are equivalent to one of them.

In the first case, where $(\theta(d_1), \theta(x_1), \theta(x_2), \theta(x_3)) = (d, u, u, v^{-2})$, the pre-image $\theta^{-1}(\langle u \rangle)$ of the orientation-preserving cyclic subgroup $\langle u \rangle$ has signature $(0; +; [2, \overset{2g+2}{\ldots}, 2]; \{-\})$, by [11, Theorems 2.2.4 and 2.1.3] and the Riemann-Hurwitz formula. This implies that the surface $S = U/\ker\theta$ can be represented as the double covering $S \to S/\langle u \rangle$ of the Riemann sphere, ramified over $2g+2$ points; in other words, the surface S is hyperelliptic.

Finally, we deal with the exceptional case where $g = 23$. For this, we know from the proof of Theorem 4.5 that an abelian group G of order $2(g+1) = 48$ can act with signature $(1; -; [2, 2, (g+1)/2]; \{-\})$, or with signature $(1; -; [2, 3, 4]; \{-\})$. In the former case, $G \cong C_2 \times C_{g+1} = C_2 \times C_{24}$, and up to equivalence there are four possibilities for the generating vector $(\theta(d_1), \theta(x_1), \theta(x_2), \theta(x_3))$, as given earlier. So now consider the latter case.

Here the subgroup G^+ has order 24 and is generated by $(x, y, z) = (\theta(x_1), \theta(x_2), \theta(x_3))$, of orders 2, 3 and 4 respectively, and so $G^+ \cong C_2 \times C_3 \times C_4 \cong C_2 \times C_{12}$. Also $d = \theta(d_1)$ has order 24, because $d^2 = (xyz)^{-1}$ has order 12, and therefore $G \cong C_2 \times C_{24} = C_2 \times C_{g+1}$, as before. Next, the element y has order 3 and so $y = d^{\pm 8}$, and it follows that both x and z lie outside $\langle d \rangle$, so $G = \langle x \rangle \times \langle d \rangle$, and then $z = (d^2xy)^{-1} = xd^{-2\pm 8} = xd^6$ or xd^{14}, but the latter is impossible since it has order 12, not 4, so $y = d^{-8} = d^{16}$ and $z = xd^6$. This gives an action that is unique up to equivalence, with generating vector $(\theta(d_1), \theta(x_1), \theta(x_2), \theta(x_3)) = (d, x, d^{16}, xd^6)$, as required. ∎

We can now summarise the situation for essential actions of abelian groups in the following.

Corollary 4.7

(a) $M_{\mathrm{ab}}(g) = 2g$ *for every even* $g \neq 16$, *while* $M_{\mathrm{ab}}(16) = 36$.

(b) $M_{\mathrm{ab}}(g) = 2g + 6$ *for every* $g \equiv 1 \bmod 4$.

(c) $M_{\mathrm{ab}}(g) = 2g + 2$ *for every* $g \equiv 3 \bmod 4$.

Now we turn to the case where the abelian group preserves orientation. Just like in the previous section, we let G be the full automorphism group of the surface, and consider the largest order of an abelian subgroup H of G^+. Once again, H is a proper subgroup of G, and G^+ contains all the involutions of G.

By Theorems 4.1, 4.3 and 4.5 (or Corollary 4.7), there exist examples of such an orientation-preserving abelian group H of order g for all positive even $g \neq 16$, order 18 for $g = 16$, order $g + 3$ for all $g \equiv 1 \bmod 4$ (and $g \geq 5$), and order $g + 1$ for all $g \equiv 3 \bmod 4$. Thus $M^+_{ab}(g) \geq g$ for all $g \geq 2$. We complete this section by considering the sharpness of the latter bound.

Theorem 4.8 *For every element g in a very large and possibly infinite set of positive even integers, the maximum order of an abelian group of orientation-preserving automorphisms of a pseudo-real surface of genus g is g.*

Proof: First, let $\mathcal{A}$ be the set of all positive even integers g of the form $p + 1$, where p is an odd prime with the property that $p + 2$ has no prime divisor congruent to 1 mod 4, so that $\mathrm{Aut}(C_{p+2})$ has no elements of order 4. This is a very large and possibly infinite set. Its ten smallest members are 6, 8, 18, 20, 30, 32, 42, 48, 62 and 68, and it contains 19969 members less than 10^6, and 10345920 members less than 10^9. Proving it is infinite, however, would be about as difficult as proving the Twin Primes Conjecture.

Now let g be any element of $\mathcal{A}$, with $p = g - 1 > 29$. Then since g is even, we know that C_g is the orientation-preserving subgroup of C_{2g} in its action on a pseudo-real surface of genus g with signature $(1; -; [2, 2, g]; \{-\})$.

Next, suppose that H is a larger abelian group of orientation-preserving automorphisms of some pseudo-real surface S of genus g, and let G be the full automorphism group of S.

Then $|H| > g$, so $|G| > 2g$, and so G acts with one of the three signatures denoted by (1) to (3) in the proof of Theorem 3.9. We will consider these three cases again, in turn.

Case (1): Signature $(2; -; [m]; \{-\})$

Here the Riemann-Hurwitz formula gives $|G| = 2m(g-1)/(m-1) = 2mp/(m-1)$, so $|G^+| = mp/(m-1)$, and then since G^+ must have an element of order m, it follows that $m-1$ divides p, so $m-1 = 1$ or p, giving $m = 2$ or $p+1$. In the latter case, however, $|G| = 2m$ and so $|H| \le m = p+1 = g$, contradiction. Thus $m = 2$, which gives $|G| = 4p$ and $|G^+| = 2p$, and hence also $|H| = 2p$, and $G^+ = H$. Moreover, since H is abelian of twice odd prime order, it is cyclic. Accordingly, the same argument as in case (1) of the proof of Theorem 3.9 applies, to show that the group G admits an automorphism that inverts each of d and e, and hence that the surface S is not pseudo-real.

This contradiction shows that case (1) is impossible.

Case (2): Signature $(1; -; [j, k]; \{-\})$

Here G is generated by elements d, x and y such that $d^2xy = x^j = y^k = 1$.

First suppose that $H = G^+$, so that G^+ itself is abelian. Then some automorphism α of G^+ inverts every element of G^+. Also d^2 lies in G^+ and so centralises every element of G^+, and hence conjugation by d^{-1} has the same effect on G^+ (and on inverses of elements of G^+) as conjugation by d. As also $(d^{-1})^2x^\alpha y^\alpha = (d^{-1})^2x^{-1}y^{-1} = d^{-2}x^{-1}y^{-1} = xyx^{-1}y^{-1} = 1$, we find that α extends to an automorphism α^* of G that takes d to d^{-1}. But now since the automorphism α^* inverts each of d and x, it follows from Proposition 2.1 that S is not pseudo-real, contradiction. Hence H is a proper subgroup of G^+, and G^+ is not abelian.

Thus $|G| = 2|G^+| \ge 4|H| > 4g > 4(g-1)$, and so either $j = 2$ and $k > 2$, or $j = 3$ and $k = 3$, 4 or 5. If $(j,k) = (3,3)$ then $|G| = 6(g-1) = 6p$, which is not divisible by 4, while if $(j,k) = (3,4)$ or $(3,5)$, then $|G| = (24/5)(g-1) = 48p/5$ or $|G| = (30/7)(g-1) = 60p/7$, both of which are impossible since $p > 7$. Hence $j = 2$.

Now the Riemann-Hurwitz formula gives $|G| = (4k/(k-2))(g-1) = 4kp/(k-2)$, and $|G|$ must be divisible by $2k$ since the element y of order k lies in G^+, but also $|G|$ cannot be $2k$ (for otherwise G^+ is cyclic, generated by y), and therefore $k-2$ strictly divides $2p$, so $k-2 = 1$, 2 or p. This gives us three sub-cases to consider, which we will denote by (2a) to (2c).

In sub-case (2a), we have $k-2 = 1$, so $k = 3$, and G acts with signature $(1; -; [2,3]; \{-\})$. Here $|G| = 4kp/(k-2) = 12p$, so $|G^+| = 6p$, and then since $p > 12$, we know that G has a cyclic normal Sylow p-subgroup P of order p, with quotient G/P of order 12. Accordingly, the pre-image of P in the associated NEC group Γ is a normal subgroup N of index 12, contained also as a normal subgroup of index 6 in Γ^+. A Magma computation shows that there is just one such normal subgroup N contained in Γ^+, and for this one, the quotient Γ/N is isomorphic to C_{12}, and so $\Gamma^+/N \cong C_6$.

Also $|G^+| > |H| > g-1 = p > 7$, and therefore $|H| = 2p$ or $3p$, and so H contains P, and it follows that the pre-image of H in Γ is a subgroup of index 6 or 4 in Γ, and index 3 or 2 in Γ^+, and contains N. A further Magma computation, however, shows there is just one such subgroup of index 6 and just one such subgroup of index 4 in Γ, the abelianisations of which have orders 32 and 27. Hence Γ^+ has no abelian quotient of order $2p$ or $3p$, contradiction, and therefore sub-case (2a) is impossible.

In sub-case (2b), we have $k = 4$, and G acts with signature $(1; -; [2, 4]; \{-\})$. Here $|G| = 8p$ and $p > 9$, so G has a cyclic normal Sylow p-subgroup P of order p, with quotient G/P of order 8. Accordingly, the pre-image of P in the associated NEC group Γ is a normal subgroup N of index 8, contained also as a normal subgroup of index 4 in Γ^+. A Magma computation shows that there are five such normal subgroups in Γ. Next, also $4p = |G^+| > |H| > g - 1 = p > 4$, and therefore $|H| = 2p$, and so H contains P, and hence the pre-image of H in Γ is a subgroup of index 4 in Γ, and index 2 in Γ^+, and containing N. A further Magma computation shows there are five conjugacy classes of subgroups of index 4 in Γ contained in Γ^+, but in all but one case, the abelianisation of the subgroup has order 32 or 64, and so the subgroup cannot have a cyclic quotient of order $2p$. The exception is the (normal) subgroup of index 4 generated by x_1x_2, x_2x_1 and $x_1d_1x_1d_1^{-1}$, which implies that H is generated by $(a, b, c) = (xy, yx, xdxd^{-1})$, where d, x and y are the images of d_1, x_1 and y_1. This subgroup contains only three of the above possibilities for N, the images of which in G are generated by $\{xdxd^{-1}, xyd^{-2}, y^2, d^{-1}yxd^{-1}, d^2yx\}$, $\{xdy^{-1}d^{-1}, xyd^{-2}, y^2, d^{-1}yxd^{-1}, d^2yx\}$ and $\{xy, yx, dyxd^{-1}, xdyxd^{-1}x, xdxdxd^{-1}xd^{-1}\}$. But each of these images in G contains y^2 $(= (yx)(xy))$, and it follows that $y^2 \in P$ and therefore y has order 2 or $2p$ (and not 4), contradiction. Hence sub-case (2b) is impossible.

Finally, in sub-case (2c), we have $k = p+2$, and G has order $4p+8 = 4(p+2)$ and acts with signature $(1; -; [2, p+2]; \{-\})$, with $|G^+| = 2p+4 = 2(p+2)$ and $|H| = p+2 = k$. Without loss of generality we might as well suppose that H is the cyclic subgroup generated by y. A Magma computation in the NEC group Γ associated with the action of G on S shows that there is just one subgroup of index 4 in Γ contained in Γ^+, and it follows that H is normal in G and can be generated by $(a, b, c) = (y, y^x, y^d)$. Moreover, just as in case (2) of the proof of Theorem 3.9, conjugation by each of d and x has the following effects on these elements: $a^d = c$, $b^d = abc^{-1}$ and $c^d = b$, and $a^x = b$, $b^x = a$ and $c^x = ac^{-1}b$. In particular, conjugation by d^2 interchanges a with b. Next, by choice of p we know that $|H| = k = p+2$ has no prime divisor congruent to 1 mod 4, and hence H has no automorphism of order 4, so d^2 must centralise H. This implies that $y^x = b = a^{d^2} = a = y$, and so x centralises y, but then xy is an element of order $2k = 2(p+2)$, contradicting the assumption that H is the largest abelian subgroup of G^+. Hence also sub-case (2c) is impossible, and this rules out signature $(1; -; [j, k]; \{-\})$ completely.

Case (3): Signature $(1; -; [2, j, k]; \{-\})$

In this case, the Riemann-Hurwitz formula gives $|G| = 4jk(g-1)/(3jk-2j-2k)$, and then because $|H| > g$, we need $|G| > 2g > 2(g-1)$, so that $2jk > 3jk-2j-2k$, and therefore $(j-2)(k-2) = jk-2j-2k+4 < 4$. It follows that either $j = 2$, or $j = 3$ and $k = 3$, 4 or 5 (just as before). If $j = 3$, however, then $|G| = 12p/5$, $24p/11$ or $60p/29$, and all of these are impossible because the prime p is not 5, 11 or 29. Hence $j = 2$.

We now have $|G| = 8k(g-1)/(4k-4) = 2kp/(k-1)$, and so $|H|$ divides $kp/(k-1)$. In particular, $k-1$ divides kp, but is coprime to k and hence divides p. Thus $k-1 = 1$ or p, and so $k = 2$ or $p+1$. In the latter case, however, $|G| = 2k = 2(p+1) = 2g$ and therefore $|H|$ divides $|G^+| = g$, which is too small. Hence $j = k = 2$, and $|G| = 2kp/(k-1) = 4p$, so $|H|$ divides $|G^+| = 2p$. But $|H| > g > p$, and it follows that $|H| = 2p = |G^+|$.

Also the group G acts with signature $(1; -; [2, 2, 2]; \{-\})$ on S, and hence we know that G is generated by elements d, x, y and z such that $1 = d^2xyz = x^2 = y^2 = z^2 = 1$, and that G^+ is generated by x, y, z, x^d, y^d and z^d (with $d^2 = (xyz)^{-1}$). These six elements all have order 2, however, and so they cannot generate an abelian group of order $2p$, a contradiction.

This completes the proof. ∎

We can now summarise the situation for abelian group actions in the following.

Corollary 4.9

(a) *$M_{\text{ab}}(g) \geq 2g$ for all $g \geq 2$, and this bound is sharp for every even $g \neq 16$;*

(b) *$M^+_{\text{ab}}(g) \geq g$ for all $g \geq 2$, and this bound is sharp for a large number of even $g \geq 2$.*

5 The general case

In this final section, we consider lower bounds on the order $M(g)$ of the largest arbitrary group of automorphisms of a pseudo-real surface of given genus $g \geq 2$, akin to the Accola-Maclachlan bound for general compact Riemann surfaces, [1, 18]. Here we need not do anything in the orientation-preserving case, because the maximum order of an orientation-preserving automorphism group of a pseudo-real surface of genus $g \geq 2$ is always $M(g)/2$.

We begin with the following easily-proved fact.

Proposition 5.1 *If G is a split metacyclic finite group $C_r \rtimes_s C_2$, generated by two elements a and b of orders 2 and r respectively such that $a^{-1}ba = b^s$, then G has an automorphism that inverts each of the two generators a and b.*

Proof: The elements a^{-1} and b^{-1} satisfy the same defining relations for G as the elements a and b, because a^{-1} and b^{-1} have the same orders 2 and r, and $ab^{-1}a^{-1} = a^{-1}b^{-1}a = (b^s)^{-1} = (b^{-1})^s$. Hence there exists an automorphism of G taking (a, b) to (a^{-1}, b^{-1}). ∎

Next, we give lower bounds on $M(g)$ for all $g \geq 2$, according to the parity of g.

Theorem 5.2

(a) *$M(g) \geq 2g$ for every even integer $g \geq 2$;*

(b) *$M(g) \geq 4(g-1)$ for every odd integer $g \geq 3$.*

Proof: Part (a) follows immediately from Theorem 3.2. For part (b), let Γ be a maximal NEC group with signature $(1; -; [2, 2, 2]; \{-\})$, and for any positive integer n, let G be the semi-direct product $C_{4n} \rtimes_{2n-1} C_2$ of order $8n$, generated by two elements u and v of orders 2 and $4n$ such that $u^{-1}vu = v^{2n-1}$. Then we may construct an epimorphism from Γ to G, taking (d_1, x_1, x_2, x_3) to (uv, u, u, v^{2n}). To see this, note the images of d_1 and x_1 generate G, that each of u and v^{2n} has order 2, and that the relation $d_1^2 x_1 x_2 x_3 = 1$ is preserved since $(uv)^2 = (v^u)v = v^{2n}$. Also the subgroup generated by the images of d_1^2, x_1, x_2, x_3 and their conjugates by d_1 is the index 2 subgroup H generated by u and v^2, noting that $u^v = v^{-1}uv = uv^{2n+1}v = uv^{2n+2}$ and the element v^{2n} is central in G. Hence G acts with signature $(1; -; [2, 2, 2]; \{-\})$ on a Riemann surface S, with genus $g = 2n + 1$ given by the Riemann-Hurwitz formula. Next, we note that uv has order 4, since $(uv)^2 = (v^u)v = v^{2n}$. So now if w is any element of G lying outside $H = \langle u, v^2 \rangle$, necessarily of the form v^{2j+1} or $v^{2j}uv$ for some j, then the order of w is divisible by 4. Hence all involutions of G lie in H, and it follows from Proposition 2.2 that S is pseudo-real and $G = \mathrm{Aut}(S)$. In particular, $|G| = 8n = 4(g - 1)$, and this completes the proof. ∎

Next, we show that the bound in part (b) of Theorem 5.2, is sharp for a very large and possibly infinite set of odd genera $g \geq 3$.

Specifically, we take $\mathcal{G}$ as the set of all integers g of the form $2p + 1$ where p is a prime such that $p \equiv 3 \bmod 8$ and $p \equiv 2$ or $5 \bmod 9$ (so that $p \equiv 11$ or $59 \bmod 72$), and also $p \not\equiv 5 \bmod 7$, and $p+1$ is not divisible by 11, 23, 47 or any prime q such that $q \equiv 1 \bmod 3$ or $q \equiv 1 \bmod 4$. The smallest 10 integers in this set $\mathcal{G}$ are 2567, 3143, 4007, 6023, 14087, 15815, 17255, 19847, 20135 and 30215, and then there are a further 188 such integers less than 10^6, plus a further 91895 of them less than 10^9. But again, however, proving that this set is infinite would be about as difficult as proving the Twin Primes Conjecture.

Theorem 5.3 *$M(g) = 4(g - 1)$ for all $g \in \mathcal{G}$.*

Proof: Assume the contrary. Then some finite group G of order greater than $4(g-1)$ has a faithful action on a pseudo-real surface of genus g, where $g \in \mathcal{G}$. Also an easy argument using the Riemann-Hurwitz formula shows that G must act with signature $(1; -; [2, k]; \{-\})$ for some $k \geq 3$, or $(1; -; [3, k]; \{-\})$ for some $k < 6$.

For the former signature $(1; -; [2, k]; \{-\})$, we have $|G| = 4k(g-1)/(k-2) = 8kp/(k-2)$, and as G must contain an element of order k that sits inside the orientation-preserving subgroup G^+ of index 2 in G, we find that $|G|$ is divisible by $2k$. Hence $k - 2$ divides $4p$, so $k - 2 = 1$, 2, 4, p, $2p$ or $4p$. Accordingly $k = 3$, 4, 6, $p + 2$, $2p + 2$ or $4p + 2$, and then $|G| = 24p$, $16p$, $12p$, $8(p + 2)$, $8(p + 1)$ or $8p + 4 = 4(2p + 1)$. This gives us six cases to consider, which we will call (a) to (f). Similarly, for the latter signature $(1; -; [3, k]; \{-\})$, we have $|G| = 6k(g - 1)/(2k - 3) = 12kp/(2k - 3)$, and again $|G|$ is divisible by $2k$, so $2k - 3$ divides $6p$. But $2 \leq k < 6$, and it follows that $k = 2$ or 3 (since $p \notin \{5, 7\}$), and then $|G| = 24p$ or $12p$, respectively. The case where $k = 2$ (and $|G| = 24p$) gives signature $(1; -; [3, 2]; \{-\})$, which is equivalent to signature $(1; -; [2, 3]; \{-\})$, and so we are left with the possibility that $k = 3$ (and $|G| = 12p$), which we will call case (g).

We will eliminate each of the seven cases (a) to (g) in turn, using properties of the prime p (for which $g = 2p + 1$) as needed. In all seven cases, we let d, x and y be the images in G of the canonical generators d_1, x_1 and x_2 of the associated NEC group Γ with signature $(1; -; [2, k]; \{-\})$ or $(1; -; [3, k]; \{-\})$ under some smooth epimorphism $\theta\colon \Gamma \to G$. Accordingly, G^+ is generated by d^2, x and x^d, with $y = (d^2x)^{-1}$. Sometimes it is helpful to use the facts that $[d^2, x^d] = d^{-3}xd^2xd = [d^2, x]^d$ and $[x^d, x] = d^{-1}xdxd^{-1}xdx = [d, x]^2$.

In cases (a) to (c), because p is large we find by Sylow theory that G has a cyclic normal subgroup N of order p, and then G/N has a faithful action with the same signature $(1; -; [2, k]; \{-\})$ on a surface of smaller genus (which might not be pseudo-real). In those cases, standard group theory shows that $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$, which is cyclic of order $p - 1$ since $N \cong C_p$. In particular, $|G/C_G(N)|$ divides $p - 1$, which is congruent to 10 mod 12 and hence is not divisible by 3 or 4. On the other hand, $|G/C_G(N)|$ divides $|G : N| = 24$, 16 or 12, and it follows that $|G/C_G(N)| = 1$ or 2. Thus $C_G(N) = G$ or is a subgroup of index 2 in G. The same also holds in case (g), since in this case $|G/C_G(N)|$ divides both $p - 1$ and $|G : N| = 12$. On the other hand, in cases (d) to (f) we use properties of p to show that G has a cyclic normal subgroup N of some order other than p, and proceed similarly.

Case (a): Signature $(1; -; [2, 3]; \{-\})$

Here $|G| = 24p$, and G has a cyclic normal subgroup N of order p, which must be the image under the epimorphism from Γ to G of a normal subgroup of index 24 in Γ. A Magma computation, however, shows that this NEC group Γ has only one normal subgroup of index 24, and in the quotient (isomorphic to G/N), the images of the canonical generators d_1, x_1 and x_2 have orders 6, 2 and 3, and hence $d = \theta(d_1)$ has order 6 or $6p$, neither of which is divisible by 4, contradiction.

Case (b): Signature $(1; -; [2, 4]; \{-\})$

Here $|G| = 16p$, and G has a cyclic normal subgroup N of order p, which must be the image under the epimorphism from Γ to G of a normal subgroup of index 16 in Γ. Also the images of x_1 and x_2 in G/N must have orders 2 and 4 (since $|N| = p$), and because $p \equiv 11 \bmod 12$ we know that $p - 1$ is even but not divisible by 4, and therefore $\gcd(|G/N|, |\mathrm{Aut}(N)|) = \gcd(16, p - 1) = 2$.

Now let $H = C_G(N)$. Then G/H is isomorphic to a subgroup of order at most 2 in $\mathrm{Aut}(N)$, and so H has index 1 or 2 in G. An easy exercise (considering homomorphisms onto C_2) shows that either $H = G$, or $H = G^+ = \langle d^2, x, x^d\rangle$, or $H = \langle d, d^x\rangle$ or $\langle dx, xd\rangle$.

Next, a Magma computation shows that the NEC group Γ has seven normal subgroups of index 16, but only two of them have the property that the images of x_1 and x_2 in the quotient have the required orders 2 and 4. The corresponding two quotients of order 16 are isomorphic to $C_8 \times C_2$ and the semi-direct product $C_8 \rtimes_5 C_2$, and the images of d_1 and d_1x_1 have order 8 in both cases, while the images of $[d_1, x_1]$ and $[d_1^2, x_1]$ are trivial in the first case (obviously) and have orders 2 and 1 in the second (because if u and v are the images in $C_8 \rtimes_5 C_2$ of d_1 and x_1 respectively, then $u^2 = v^8 = 1$ and $u^{-1}vu = v^5$, so $(vu)^2 = vv^u = v^6 = v^{-2}$, and therefore $[v, u] = v^{-1}v^u = v^4$, and $[v^2, u] = v^{-2}(v^2)^u = v^8 = 1$). Hence either G/N is isomorphic to $C_8 \times C_2$, with the images of d, dx, $[d, x]$ and $[d^2, x]$ having orders 8, 8, 1 and 1, or G/N is isomorphic to $C_8 \rtimes_5 C_2$, with the images of d, dx, $[d,x]$ and $[d^2,x]$ having orders 8, 8, 2 and 1.

Now suppose that $H = \langle d, d^x \rangle$. Then d has order divisible by 8, and centralises the cyclic normal subgroup N of order p, so H itself is cyclic (of order $8p$). Moreover, since H is generated by d and its conjugate d^x, it follows that H is generated by d. Hence $G = \langle d, x \rangle \cong \langle d \rangle \rtimes \langle x \rangle \cong C_{8p} \rtimes C_2$. By Proposition 5.1, however, this gives an automorphism of G inverting each of d and x, and so by Proposition 2.1, the surface S is not pseudo-real.

Similarly, if $H = \langle dx, xd \rangle$, then since the order of dx is divisible by 8, and $x^{-1}(xd)x = dx$, we find that H is cyclic and generated by dx, so $G = \langle d, x \rangle \cong \langle dx \rangle \rtimes \langle x \rangle \cong C_{8p} \rtimes C_2$. This time Proposition 5.1 gives an automorphism of G that inverts each of dx and x, and then the composite of this with conjugation by x takes x to x^{-1} and $d = (dx)x$ to $x(dx)^{-1} = d^{-1}$, which gives the same contradiction as above.

Thus $H = G$ or $H = \langle d^2, x, x^d \rangle = G^+$, and in either case, H contains d^2, x and x^d.

Next, we observe that N is a central subgroup of index $|H : N| = 8$ or 16 in H, and so $|H : Z(H)|$ divides 16. Hence by Schur's centre-by-finite theorem (see [19, 10.1.4]), the exponent of H' divides 16. It follows that the orders of $[d^2,x]$ and $[x^d,x] = [d,x]^2$ divide 16, and so the order of $[d,x]$ divides 32.

If G/N is isomorphic to $C_8 \times C_2$, we have $[x,d] \in N$ and so $[x,d]$ has order 1 or p, and it follows that $[x,d] = 1$. But in that case $G = \langle d, x \rangle$ is abelian and so $1 = y^4 = (d^2x)^4 = d^8$, which makes the order of G divide 16, contradiction.

On the other hand, if G/N is isomorphic to $C_8 \rtimes_5 C_2$, then both $[d,x]^2$ and $[d^2,x]$ have order 1 or p, and hence both $[d,x]^2$ and $[d^2,x]$ are trivial. Consequently, also the elements $[d^2,x^d] = [d^2,x]^d$ and $[x^d,x] = [d,x]^2$ are trivial, and so the three generators d^2, x and x^d of G^+ commute with each other, and therefore G^+ is abelian. But now since G^+ has order $8p$ and is generated by d^2 and the two involutions x and x^d, the order of d^2 has to be divisible by p, and moreover, since we already know that the order of d is divisible by 8, it follows that d has order $8p$. But again this implies that $G = \langle d, x \rangle \cong \langle d \rangle \rtimes \langle x \rangle \cong C_{8p} \rtimes C_2$, and so from Proposition 5.1 we obtain another contradiction.

Case (c): Signature $(1; -; [2,6]; \{-\})$

This is similar to case (a). Here $|G| = 12p$, and G has a cyclic normal subgroup N of order p, which must be the image under the epimorphism from Γ to G of a normal subgroup of index 12 in Γ. Moreover, the images of x_1 and x_2 in G/N must have orders 2 and 6, since $|N| = p$ is coprime to 2 and 6. A computation with MAGMA, however, shows that this NEC group Γ has only one normal subgroup of index 12 with the property that the images of x_1 and x_2 in the quotient have orders 2 and 6, and for that one, the image of d_1 has order 6, which is not divisible by 4, contradiction.

Case (d): Signature $(1; -; [2, p+2]; \{-\})$

Here $|G| = 8(p+2)$, and so y generates a cyclic subgroup N of odd order $p+2$ and index 8. As $g \in \mathcal{G}$ we know that $p \not\equiv 1 \bmod 3$ and $p \not\equiv 5 \bmod 7$, and therefore $|N| = p+2$ is not divisible by 3 or 7. In particular, no non-trivial divisor of $|G : N| = 8$ can be congruent to 1 modulo a prime divisor of $|N|$, and it follows that every Sylow subgroup of N is normal in G. Thus N itself is normal in G, and we can proceed as in the above cases.

Under the epimorphism $\theta\colon \Gamma \to G$, the subgroup N of G must be the image of a normal subgroup of index 8 in Γ, with the images of x_1 and x_2 in G/N having orders 2 and 1 (since $|G/N| = 8$ but $|N| = o(x_2) = p+2$ is odd). If we drop the relation $x_2^{p+2} = 1$ from the presentation for Γ, however, we have the group $\langle\, d_1, x_1, x_2 \mid d_1^2 x_1 x_2 = x_1^2 = 1 \,\rangle$, and a computation with Magma shows that this group has exactly five normal subgroups of index 8, and for all of them, the order of the image of x_2 is either 2 or 4, contradiction.

Case (e): Signature $(1; -; [2, 2p+2]; \{-\})$

In some ways this is the most challenging of the seven cases, but in fact it is substantially similar to case (b). Here $|G| = 8(p+1)$, and y generates a cyclic subgroup of order $2(p+1)$, which is congruent to 24 mod 48, so the image of x_2^{24} generates a cyclic subgroup N of odd order $(p+1)/12$, and index 96.

Our choice of the set $\mathcal{G}$ ensures that no divisor of 96 is congruent to 1 modulo a prime divisor of $(p+1)/12$, and it follows that every Sylow subgroup of N is normal in G, and hence N itself is normal in G. Furthermore, $\mathrm{Aut}(N)$ is a direct product of cyclic groups of the form $C_{q^{r-1}(q-1)}$ where q is a prime and q^r is a maximal prime-power divisor of $|N| = (p+1)/12$, and then since $q \equiv 2$ mod 3 and $q \equiv 3$ mod 4 (by the definition of $\mathcal{G}$), we find that $\mathrm{Aut}(N)$ has no elements of order 3 or 4, so $|G/C_G(N)| = 1$ or 2 as before.

But also N is contained in the cyclic subgroup of G generated by y, and so $H = C_G(N)$ contains $y = (d^2x)^{-1}$. It follows that if $|G : H| = 2$, then $H \neq \langle d, d^x \rangle$, for otherwise H contains $d^{-2}y^{-1} = x$ and then $H = \langle d, d^x, x \rangle = G$, and similarly $H \neq \langle dx, xd \rangle$, for otherwise H contains $(dx)y = d^{-1}$ and again $H = \langle d, d^x, x \rangle = G$. Hence either $H = G$ or $H = G^+ = \langle d^2, x, x^d \rangle$, and in both cases, H contains d^2, x and x^d.

Next, if we drop the relation $x_2^{2p+2} = 1$ from the presentation for Γ, we have the group $\langle\, d_1, x_1, x_2 \mid d_1^2 x_1 x_2 = x_1^2 = 1 \,\rangle$ as in case (d), and another computation with Magma shows that this group has exactly 77 normal subgroups of index 96, but only two of them have the property that the images of x_1 and x_2 in the quotient have orders 2 and 24.

These two quotients of order 96 are isomorphic to $C_{48} \times C_2$ and the semi-direct product $C_{48} \rtimes_{25} C_2$, with the images of d_1 and d_1x_1 having order 48 in both cases, and also (just as in case (b)) with the images of $[d_1, x_1]$ and $[d_1^2, x_1]$ being trivial in the first case and having orders 2 and 1 in the second. Hence either G/N is isomorphic to $C_{48} \times C_2$, with the images of d, dx, $[d, x]$ and $[d^2, x]$ having orders 48, 48, 1 and 1, or G/N is isomorphic to $C_{48} \rtimes_{25} C_2$, with the images of d, dx, $[d, x]$ and $[d^2, x]$ having orders 48, 48, 2 and 1.

On the other hand, N is a central subgroup of index $|H : N| = 48$ or 96 in H, and so $|H : Z(H)|$ divides 96, and hence by Schur's centre-by-finite theorem, the exponent of H' divides 96. It follows that the orders of $[d^2, x]$ and $[x^d, x] = [d, x]^2$ divide 96, and so the order of $[d, x]$ divides 192.

If G/N is isomorphic to $C_{48} \times C_2$, then $[x, d] \in N$ and so the order of $[d, x]$ divides $(p+1)/12$, which is coprime to 2 and 3, and so $[d, x]$ is trivial. But then $G = \langle d, x \rangle$ is abelian, and so again admits an automorphism that inverts each of d and x, contradiction. On the other hand, if G/N is isomorphic to $C_{48} \rtimes_{25} C_2$, then the orders of $[d, x]^2$ and $[d^2, x]$ divide $(p+1)/12$, and so both $[d, x]^2$ and $[d^2, x]$ are trivial, and hence also the elements $[d^2, x^d] = [d^2, x]^d$ and $[x^d, x] = [d, x]^2$ are trivial. Again this shows that the three generators d^2, x and x^d of G^+ commute with each other, so G^+ is abelian. Moreover, since G^+ has order $4(p+1)$ and is generated by d^2 and the two involutions x and x^d, the order of d^2 has to be divisible by $(p+1)/12$, and since we already know that the order of d is divisible by 48, we find that d has order $48(p+1)/12$. Once again this implies that $G = \langle d, x\rangle \cong \langle d\rangle \rtimes \langle x\rangle \cong C_{4(p+1)} \rtimes C_2$, and Proposition 5.1 gives another contradiction.

(Note: It is tempting to try to drop the part of the definition of $\mathcal{G}$ that requires no prime divisor of $p+1$ to be congruent to 1 mod 3 or to 1 mod 4. Without it, however, in case (e) we get groups of order $8(p+1) = 4(g+1)$ that satisfy all the usual conditions, but with $H = C_G(N)$ having index 4 in G, and some of these are not extendable to larger groups on the same surface. Accordingly, for those examples the lower bound on the genus g is not sharp, and so this part of the definition of $\mathcal{G}$ cannot be ignored.)

Case (f): Signature $(1; -; [2, 4p+2]; \{-\})$

Here $|G| = 8p+4 = 4(2p+1)$, and y^2 generates a cyclic subgroup N of odd order $2p+1$ and index 4. Then since $2p+1$ is not divisible by 3, every Sylow subgroup of N is normal in G (since it is already central in N), and hence N itself is normal in G, and we can proceed as before. Under the epimorphism $\theta : \Gamma \to G$, the subgroup N must be the image of a normal subgroup of index 4 in Γ, with the the images of x_1 and x_2 in G/N having orders 2 and 2. If we drop the relation $x_2^{4p+2} = 1$ from the presentation for Γ, however, we have the group $\langle\, d_1, x_1, x_2 \mid d_1^2 x_1 x_2 = x_1^2 = 1 \,\rangle$ once more, and a computation with Magma shows that this group has exactly three normal subgroups of index 4, but for two of them, the orders of the images of x_1 and x_2 are 1 and 2, and 2 and 1, while for the third, the order of the image of d_1 is 2 and so the order of the image of d_1 in G cannot be divisible by 4, another contradiction.

Case (g): Signature $(1; -; [3, 3]; \{-\})$

This case is a little different from the others, but easier. Here $|G| = 6(g-1) = 12p$, and again G has a cyclic normal subgroup N of order p, which must be the image under the epimorphism from Γ to G of a normal subgroup of index 12 in Γ, and be contained in the index 2 subgroup G^+. A Magma computation, however, shows that this NEC group Γ has only two normal subgroups of index 12, but neither of them is contained in the index 2 subgroup Γ^+, and hence for both of them, the image of the subgroup Γ^+ has index 1 (not 2) in the quotient G, yet another contradiction.

As we have eliminated all seven cases, this completes the proof. ∎

Remark 5.4 In contrast, for even genus g we are not able to say much more about the bound $M(g) \geq 2g$. It is sharp for $g = 2$ and $g = 8$, but not for other small even values of g, as can be seen in the Appendix (described in the next section). On the other hand, for every prime $p \equiv 1 \bmod 4$ we can construct an essential action with signature $(1; -; [2, 2, 2]; \{-\})$ of a semi-direct product $C_p \rtimes C_4$ on a pseudo-real surface of even genus $g = p+1$, and this gives $M(g) \geq 4(g-1)$ for infinitely many even values of g. These actions, however, do not cover an infinite sequence of consecutive even values for g, and so do not give an improved bound that works for all but finitely many even g. We leave this matter as an open question.

6 Some computations

Using the capabilities of MAGMA [4] to find all quotients of a given finitely-presented group up to a given order and check for automorphisms of a finite group G, it is possible to determine actions of all groups of small order on pseudo-real surfaces of small genus $g > 1$, and hence to determine $M(g)$ for small values of g.

When testing whether a particular smooth quotient G of Γ is the (full) automorphism group of a pseudo-real surface, we must check two things: one is that the image in G of the subgroup Γ^+ is a subgroup of index 2 in G (which we may denote by G^+), and the other is that the action of G on the surface is not extendable to that of a larger group. These can be checked relatively easily, using the information provided for cases (a) to (d) in Section 2.2.

In a table in the Appendix we give a list of the values of $M(g)$ for all g between 2 and 128, including also the signatures for the actions of the corresponding groups of largest order. Note that this considerably extends the determination of $M(g)$ for $2 \leq g \leq 10$ that easily follows from the work by Michela Artebani, Saúl Quispe and Cristian Reyes in [2].

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Emilio Bujalance:
Departamento de Matemáticas Fundamentales, Facultad de Ciencias, UNED,
c/ Senda del Rey s/n, 28040 Madrid, Spain
Email: eb@mat.uned.es

Javier Cirre:
Departamento de Matemáticas Fundamentales, Facultad de Ciencias, UNED,
c/ Senda del Rey s/n, 28040 Madrid, Spain
Email: jcirre@mat.uned.es

Marston Conder:
Department of Mathematics, University of Auckland,
Private Bag 92019 Auckland, New Zealand
Email: m.conder@auckland.ac.nz

Appendix

The table below gives the largest order $M(g)$ of group actions on pseudo-real Riemann surfaces of given genus g, for $2 \leq g \leq 128$:

g	$M(g)$	Signature(s) of group actions of largest order
2	4	$(1; -; [2,2,2]; \{-\})$
3	8	$(1; -; [2,2,2]; \{-\})$
4	20	$(1; -; [2,5]; \{-\})$
5	16	$(1; -; [2,2,2]; \{-\})$
6	20	$(1; -; [2,2,2]; \{-\})$
7	24	$(1; -; [2,2,2]; \{-\})$
8	16	$(1; -; [2,2,8]; \{-\})$
9	40	$(1; -; [2,10]; \{-\})$
10	72	$(1; -; [2,4]; \{-\})$
11	40	$(1; -; [2,2,2]; \{-\})$
12	52	$(1; -; [2,13]; \{-\})$
13	48	$(1; -; [2,2,2]; \{-\})$
14	156	$(1; -; [2,3]; \{-\})$
15	56	$(1; -; [2,2,2]; \{-\})$
16	100	$(1; -; [2,5]; \{-\})$
17	128	$(1; -; [2,4]; \{-\})$
18	136	$(1; -; [2,4]; \{-\})$
19	144	$(1; -; [2,4]; \{-\})$
20	60	$(1; -; [5,6]; \{-\})$
21	80	$(1; -; [2,2,2]; \{-\})$, $(1; -; [4,4]; \{-\})$, $(2; -; [2]; \{-\})$
22	84	$(1; -; [3,6]; \{-\})$
23	88	$(1; -; [2,2,2]; \{-\})$
24	100	$(1; -; [2,25]; \{-\})$
25	144	$(1; -; [2,6]; \{-\})$
26	300	$(1; -; [2,3]; \{-\})$
27	104	$(1; -; [2,2,2]; \{-\})$
28	116	$(1; -; [2,29]; \{-\})$
29	168	$(1; -; [2,6]; \{-\})$
30	116	$(1; -; [2,2,2]; \{-\})$
31	120	$(1; -; [2,2,2]; \{-\})$
32	80	$(1; -; [8,10]; \{-\})$
33	256	$(1; -; [2,4]; \{-\})$
34	140	$(1; -; [2,35]; \{-\})$
35	272	$(1; -; [2,4]; \{-\})$
36	148	$(1; -; [2,37]; \{-\})$
37	288	$(1; -; [2,4]; \{-\})$
38	444	$(1; -; [2,3]; \{-\})$
39	160	$(1; -; [2,40]; \{-\})$
40	180	$(1; -; [2,15]; \{-\})$

g	$M(g)$	Signature(s) of group actions of largest order
41	224	$(1;-;[2,7];\{-\})$
42	328	$(1;-;[2,4];\{-\})$
43	168	$(1;-;[2,2,2];\{-\})$
44	180	$(1;-;[2,45];\{-\})$
45	176	$(1;-;[2,2,2];\{-\})$
46	216	$(1;-;[2,12]),(1;-;[3,4];\{-\})$
47	184	$(1;-;[2,2,2];\{-\})$
48	136	$(1;-;[4,17];\{-\})$
49	384	$(1;-;[2,4];\{-\})$
50	588	$(1;-;[2,3];\{-\})$
51	400	$(1;-;[2,4];\{-\})$
52	272	$(1;-;[2,8];\{-\})$
53	624	$(1;-;[2,3];\{-\})$
54	220	$(1;-;[2,55];\{-\})$
55	432	$(1;-;[2,4];\{-\})$
56	200	$(1;-;[4,5];\{-\})$
57	336	$(1;-;[2,6];\{-\})$
58	228	$(1;-;[3,6];\{-\})$
59	240	$(1;-;[2,60];\{-\})$
60	244	$(1;-;[2,61];\{-\})$
61	240	$(1;-;[2,2,2];\{-\})$, $(1;-;[4,4];\{-\})$, $(2;-;[2];\{-\})$
62	732	$(1;-;[2,3];\{-\})$
63	248	$(1;-;[2,2,2];\{-\})$
64	260	$(1;-;[2,65];\{-\})$
65	768	$(1;-;[2,3];\{-\})$
66	312	$(1;-;[3,4];\{-\})$
67	272	$(1;-;[2,68];\{-\})$
68	180	$(1;-;[5,18];\{-\})$
69	544	$(1;-;[2,4];\{-\})$
70	216	$(1;-;[3,36];\{-\})$
71	336	$(1;-;[2,12];\{-\})$
72	292	$(1;-;[2,73];\{-\})$
73	576	$(1;-;[2,4];\{-\})$
74	876	$(1;-;[2,3];\{-\})$
75	296	$(1;-;[2,2,2];\{-\})$
76	500	$(1;-;[2,5];\{-\})$
77	456	$(1;-;[2,6];\{-\})$
78	220	$(1;-;[5,10];\{-\})$
79	320	$(1;-;[2,80];\{-\})$
80	204	$(1;-;[6,17];\{-\})$
81	640	$(1;-;[2,4];\{-\})$
82	648	$(1;-;[2,4];\{-\})$
83	656	$(1;-;[2,4];\{-\})$
84	340	$(1;-;[2,85];\{-\})$

g	$M(g)$	Signature(s) of group actions of largest order
85	504	$(1; -; [2, 6]; \{-\})$
86	408	$(1; -; [2, 12]; \{-\})$
87	344	$(1; -; [2, 2, 2]; \{-\})$
88	356	$(1; -; [2, 89]; \{-\})$
89	440	$(1; -; [2, 10]; \{-\})$
90	712	$(1; -; [2, 4]; \{-\})$
91	720	$(1; -; [2, 4]; \{-\})$
92	468	$(1; -; [2, 9]; \{-\})$
93	368	$(1; -; [2, 2, 2]; \{-\})$
94	380	$(1; -; [2, 95]; \{-\})$
95	376	$(1; -; [2, 2, 2]; \{-\})$
96	388	$(1; -; [2, 97]; \{-\})$
97	768	$(1; -; [2, 4]; \{-\})$
98	1164	$(1; -; [2, 3]; \{-\})$
99	400	$(1; -; [2, 100]; \{-\})$
100	404	$(1; -; [2, 101]; \{-\})$
101	1200	$(1; -; [2, 3]; \{-\})$
102	404	$(1; -; [2, 2, 2]; \{-\})$
103	816	$(1; -; [2, 4]; \{-\})$
104	420	$(1; -; [2, 105]; \{-\})$
105	1248	$(1; -; [2, 3]; \{-\})$
106	336	$(1; -; [3, 24]; \{-\})$
107	424	$(1; -; [2, 2, 2]; \{-\})$
108	436	$(1; -; [2, 109]; \{-\})$
109	864	$(1; -; [2, 4]; \{-\})$
110	1308	$(1; -; [2, 3]; \{-\})$
111	440	$(1; -; [2, 2, 2]; \{-\})$
112	468	$(1; -; [2, 39]; \{-\})$
113	896	$(1; -; [2, 4]; \{-\})$
114	904	$(1; -; [2, 4]; \{-\})$
115	480	$(1; -; [2, 40]; \{-\})$
116	500	$(1; -; [2, 25]; \{-\})$
117	480	$(1; -; [2, 60]; \{-\})$
118	1404	$(1; -; [2, 3]; \{-\})$
119	480	$(1; -; [2, 120]; \{-\})$
120	408	$(1; -; [4, 6]; \{-\})$
121	720	$(1; -; [3, 3]; \{-\})$, $(1; -; [2, 6]; \{-\})$
122	968	$(1; -; [2, 4]; \{-\})$
123	488	$(1; -; [2, 2, 2]; \{-\})$
124	820	$(1; -; [2, 5]; \{-\})$
125	744	$(1; -; [2, 6]; \{-\})$
126	1000	$(1; -; [2, 4]; \{-\})$
127	1008	$(1; -; [2, 4]; \{-\})$
128	312	$(1; -; [6, 52]; \{-\})$