A Result on $\aleph_1$-Compact Spaces [*]

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Abstract

In this note, we prove that every countably compact space with quasi–S_1–diagonal is compact. However, it is shown that it need not be metrizable.

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1 Introduction

In this work we are interested in properties $\mathcal{P}$ which satisfy the implication **countably compact + $\mathcal{P}$ $\rightarrow$ compact**, that is, properties $\mathcal{P}$ such that every countably compact space with property $\mathcal{P}$ is compact. For very useful survey of this problem, see [9].

H. Shiraki [8] has proven that a countably compact space with a quasi–G_δ–diagonal is compact metrizable. Here, we prove that the implication above is true for $\mathcal{P}$ = quasi–S_1–diagonal. By an example we show that the implication above does not get metrizability when $\mathcal{P}$ = quasi–S_1–diagonal.

A space X has a **quasi–G_δ–diagonal** (resp. **quasi–S_1–diagonal**) if there exists a countable family $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of collections of open subsets (resp. of collections of subsets

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for which $st(x, \mathcal{G}_n)$ is open for each $x \in X, n \in \mathbb{N}$) such that for any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that $x \in st(x, \mathcal{G}_n) \subset X - \{y\}$. A space X is **closed-complete** if every ultrafilter of closed sets in X with the countable intersection property has nonempty intersection. A space X is **$\aleph_1$-compact** if every uncountable subset of X has a limit point in X. A space X is **real compact** if it is a closed subspace of a product of copies of the real line.

2 Main Results

Theorem 2.1 *An $\aleph_1$-compact space with quasi-S_1–diagonal is closed-complete.*

Proof. Let X be $\aleph_1$-compact having a quasi-S_1-sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$, and let $\mathcal{F}$ be an ultrafilter of closed sets in X with the countable intersection property. For each $n \in \mathbb{N}$, let $A_n = \{x \in X : x \in \bigcup \mathcal{G}_n$ and $X - st(x, \mathcal{G}_n) \in \mathcal{F}\}$. We begin by showing that $X \neq \bigcup_{n\in\mathbb{N}} A_n$. Suppose instead that $X = \bigcup_{n\in\mathbb{N}} A_n$. Then there exists $m \in \mathbb{N}$ such that A_m meets $\bigcap \mathcal{K}$ for every countable subset $\mathcal{K}$ of $\mathcal{F}$, and since $A_m \subset \bigcup \mathcal{G}_m$, there exists $F \in \mathcal{F}$ such that $F \subset \bigcup \mathcal{G}_m$. By Zorn's Lemma, there is a subset D of $A_m \cap F$ such that

(1) if $x, y \in D$ with $x \neq y$, then $x \notin st(y, \mathcal{G}_m)$

(2) $A_m \cap F \subset \bigcup_{x\in D} st(x, \mathcal{G}_m)$.

If D is uncountable, then D has a limit point $p \in X$. Then $p \in F \subset \bigcup \mathcal{G}_m$, so $p \in G$ for some $G \in \mathcal{G}_m$. Then $st(p, \mathcal{G}_m)$ is an open neighbourhood of p and contains infinitely many points of D. Let $x \in D$ one of them, $G' \in \mathcal{G}_m$, $p, x \in G'$. Now $st(x, \mathcal{G}_m)$ is an open neighbourhood of p, hence it contains a point $y \in D$ distinct from x. Let $x, y \in G'' \in \mathcal{G}_m$; this is contrary to (1). Hence D is countable, so $A_m \cap F \cap (\bigcap_{x\in D}(X - st(x, \mathcal{G}_m)) \neq \emptyset$, which contradicts (2). It follows that $X \neq \bigcup_{n\in\mathbb{N}} A_n$, so there exists $x \in X$ such that $x \notin A_n$ for every $n \in \mathbb{N}$. Let $M = \{n \in \mathbb{N} : x \in \bigcup \mathcal{G}_n\}$. Then for every $n \in M$ we have $X - st(x, \mathcal{G}_n) \notin \mathcal{F}$, so there exists $F_n \in \mathcal{F}$ with $F_n \subset st(x, \mathcal{G}_n)$. Therefore

$$\emptyset \neq \bigcap_{n\in M} F_n \subset \bigcap_{n\in M} st(x, \mathcal{G}_n) = \{x\},$$

and hence $\{x\} = \bigcap_{n\in M} F_n \in \mathcal{F}$. But then $x \in \bigcap \mathcal{F}$, and we conclude that X is closed-complete. ■

Corollary 2.2 *A countably compact space with a quasi-S_1–diagonal is compact.*

Proof. Every countably compact space is $\aleph_1$-compact, and every countably compact closed-complete space is compact (see [1, Theorem 3.6] or [4, Theorem 2.2]). ■

Almost real-compact spaces are discussed in [2], [4] and [7].

Corollary 2.3 *An $\aleph_1$-compact, countably paracompact (resp. Tychonoff countably compact) space with a quasi-S_1–diagonal is almost real-compact (resp. real-compact).*

Proof. Every closed-complete, countably paracompact (resp. Tychonoff countably compact) is almost realcompact [4, Theorem 2.1] (resp. real-compact [2, Theorem 1.10]). ■

Corollary 2.4 *An $\aleph_1$-compact, normal countably paracompact space with a quasi-S_1–diagonal is real-compact.*

Proof. Every normal countably paracompact space is countably compact [6, Corollary 2]. ■

Example 2.5 *[5] The Alexandroff double $A(I)$ of the closed unit interval $I = [0,1]$ is a non–separable compact space with quasi–S_1-diagonal. (Refer to [3] for more about the Alexandroff double and the notations.) $A(I)$ has quasi–S_1-diagonal.*

Proof. For each $x \in I$ and $A \subset I$, let x^1 and A^1 be the duplicates of x and A respectively. Let $B_n(x) = (x - \frac{1}{n}, x + \frac{1}{n}) \cap I$ and $S_n(x) = \{\{x,y\} : y \in B_n(x)\} \cup \{\{x, y^1\} : y \in B_n(x) - \{x\}\} \cup \{(B_n(x))^1\}$ for each $x \in I$ and each $n \in \mathbb{N}$. Let $\mathcal{G}_0 = \{\{x^1\} : x \in I\}$ and $\mathcal{G}_n = \bigcup\{S_n(x) : x \in I\}$ for each $n \in \mathbb{N}$. Then, the sequence $\langle \mathcal{G}_n : n \in \mathbb{N} \cup \{0\}\rangle$ is a quasi–S_1–sequence. The following properties of $\langle \mathcal{G}_n : n \in \mathbb{N} \cup \{0\}\rangle$ justify our claim.

1. $\bigcup\{\bigcup \mathcal{G}_n : n \in \mathbb{N} \cup \{0\}\} = A(I)$;
2. for each $x \in I$, $st(x, \mathcal{G}_0) = \emptyset$ and $st(x^1, \mathcal{G}_0) = \{x^1\}$;

3. for each $x \in I$ and $n \in \mathbb{N}$, $st(x, \mathcal{G}_n) = \{B_n(x) \cup (B_n(x))^1\} - \{x^1\}$ and $st(x^1, \mathcal{G}_n) = \{B_n(x) \cup (B_n(x))^1\} - \{x\}$ while $x^1 \notin \overline{st(x, \mathcal{G}_n)}$;

4. For each $x \in I$, we have $\bigcap\{\overline{st(x, \mathcal{G}_n)} : n \in \mathbb{N}\} = \{x\}$ and $\bigcap\{\overline{st(x^1, \mathcal{G}_n)} : n \in \mathbb{N} \cup \{0\}\} = \{x^1\}$.

Thus , a compact space with quasi–S_1-diagonal need not be metrizable. ■

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