SOME RESULTS ON QUASI–σ AND θ SPACES

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Abstract. In this paper we show that a quasi–G^*_δ–diagonal plays a central role in metrizability.

We prove that: if X is a first–countable GO–space, then X is metrizable if and only if X is quasi–σ–space; a $w\theta$–space is metrizable if and only if it is a quasi–Nagata space with a quasi–$G^*_\delta(2)$–diagonal; a linearly ordered space X with a quasi–$G^*_\delta(2)$–diagonal is a Θ–space; a space X is developable if and only if it is a $w\theta$, β–space with a quasi–$G^*_\delta(2)$–diagonal.

1. Introduction

Let (X,τ) be a space, let $g : \mathbb{N} \times X \to \tau$ be a function and $\mathcal{G} = \{g(n,x) : n \in \mathbb{N}, x \in X\}$. We call $\mathcal{G}$ a **graded system of open covers** for X and g is called COC–map (= countable open covering map) for a topological space X if the following conditions are satisfied:

1. $x \in \bigcap_{n\in\mathbb{N}} g(n,x)$ for all $x \in X$.
2. $g(n+1,x) \subset g(n,x)$ for all $n \in \mathbb{N}$ and $x \in X$.

Consider the following conditions on g.

(A) If $x \in g(n, x_n)$ for every $n \in \mathbb{N}$, then x is a cluster point of the sequence $\langle x_n \rangle$.

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- (B) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n, y_n)$, then x is a cluster point of the sequence $\langle x_n \rangle$.
- (C) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n, y_n)$ and $y_n \in g(n, x)$, then x is a cluster point of the sequence $\langle x_n \rangle$.
- (D) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n, y_n)$ and the sequence $\langle y_n \rangle$ has a cluster point, then x is a cluster point of the sequence $\langle x_n \rangle$.
- (E) If for each $n \in \mathbb{N}$, $x_n \in g(n, y_n)$ and the sequence $\langle y_n \rangle$ converges in X, then the sequence $\langle x_n \rangle$ has a cluster point.

Let (S) be any of the conditions (A), (B), (C), (D) or (E) and (S^{-1}) be the statement obtained by formally interchanging all memberships (e.g., (B^{-1}) is the condition: If for each $n \in \mathbb{N}$, $y_n \in g(n, x) \cap g(n, x_n)$, then x is a cluster point of the sequence $\langle x_n \rangle$). If the graded system $\mathcal{G} = \{g(n, x) : n \in \mathbb{N}, x \in X\}$ of open covers satisfies condition (S)(resp. (S^{-1})) for $S = A$, B, C, D or E, we say that g is an S–map (resp. S^{-1}–map). If there is an S–map (resp. S^{-1}-map) for X then we say that (X, τ) is an S-space (resp. S^{-1}-space). Corresponding to each of the conditions S (except E) above is the weaker condition, denoted by wS, in which 'then x is a cluster point of the sequence $\langle x_n \rangle$' is replaced by 'then the sequence $\langle x_n \rangle$ has a cluster point'. If g satisfies wS, we say that g is a wS–map. If there is a wS–map for X then we say that (X, τ) is a wS-space. wS^{-1}–maps and wS^{-1}-spaces are defined analogously. The following are known, $A =$**semi-stratifiable** space, $B =$**developable** space, $C =$$\boldsymbol{\theta}$–space, $D =$$\boldsymbol{\Theta}$–space, $A^{-1} =$**first-countable** space, $B^{-1} =$**Nagata** space, $E^{-1} =$**quasi–Nagata** space, $wA =$$\boldsymbol{\beta}$–space, $wB =$$\boldsymbol{w\Delta}$–space, $wC =$$\boldsymbol{w\theta}$–space, $wD =$$\boldsymbol{w\Theta}$–space, $wA^{-1} =$$\boldsymbol{q}$–space.

A countable family $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ of collections of open subsets of a space X is called a **quasi–G_δ-diagonal** (**quasi-G^*_δ-diagonal** [11]) [**quasi-G^*_δ(2)-diagonal**], if for each $x \in X$ we have $\bigcap_{n \in c(x)} st(x, \mathcal{G}_n) = \{x\}$ ($\bigcap_{n \in c(x)} \overline{st(x, \mathcal{G}_n)} = \{x\}$) [$\bigcap_{n \in c(x)} \overline{st^2(x, \mathcal{G}_n)} = \{x\}$] where $c(x) =$

$\{n : x \in \text{some } G \in \mathcal{G}_n\}$ [by $st(x, \mathcal{G}_n)$ is meant the union of all sets in $\mathcal{G}_n$ which contain x].

A space X has a **quasi–S_2–diagonal** if there exists a countable family $\mathcal{G} = \langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ of collections of subsets and for each $x \in X, st(x, \mathcal{G}_n)$ is open for all $n \in \mathbb{N}$ such that for any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that $x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\}$.

A space X is called **quasi–developable** if X has a countable family $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ of collections of open subsets of X such that the nonempty sets of the form $st(x, \mathcal{G}_n)$ form a local base at x for all $x \in X$.

Let X be a space and $\mathcal{G} = \langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ a countable family of collections of open subsets of X. Then $\mathcal{G}$ is called a quasi–$w\Delta$–sequence for X if and only if :

(1) for all x, $c(x)$ is infinite.
(2) if $\langle x_n : n \in \mathbb{N} \rangle$ is a sequence with $x_n \in st(x, \mathcal{G}_n)$ for all $n \in c(x)$ then $\langle x_n : n \in \mathbb{N} \rangle$ has a cluster point.

If the space X has a such countable family then it said to be a quasi–$w\Delta$–space [11].

If X is a space and W is a relation on X such that for each $x \in X$, $W(x)$ is a neighborhood of x, then W is called a **neighbornet** [3].

Recall that a LOTS (=linearly ordered topological space) is a topological space whose topology agrees with the topology induced by some linear ordering $<$. A GO–space (=generalized ordered space) is a subspace of a LOTS. By $\mathcal{G}_n^*$ is meant the union of all sets in $\mathcal{G}_n$.

2. Quasi–σ–spaces

Definition 1. A space X is called σ (resp. quasi–σ, resp. weak–σ) if and only if there exists a σ–disjoint network $\mathcal{M} = \bigcup_{k=1}^{\infty} \mathcal{M}_k$ such that, for each $k \in \mathbb{N}, \mathcal{M}_k$ is discrete with respect to X (resp. $\mathcal{M}_k$ is discrete with respect to $\overline{\mathcal{M}_k^*}$, resp. $\mathcal{M}_k$ is discrete with respect to $\mathcal{M}_k^*$).

In COC–map terms, we can define a σ–space as follows: A space X is a σ–space if and only if there is COC–map g such that, if for each $n \in \mathbb{N}, x \in g(n, y_n)$ and $y_n \in g(n, x_n)$, then x is a cluster point of the sequence $\langle x_n \rangle$.

Lemma 2.1. *Let $\mathcal{G} = \{G_\lambda : \lambda \in \Lambda\}$ be a collection of subsets of X. If $\mathcal{G}$ is discrete with respect to $\overline{\mathcal{G}^*}$, then $\bigcup_{\lambda \in \Lambda} \overline{G_\lambda} = \overline{\bigcup_{\lambda \in \Lambda} G_\lambda}$.*

Proof. Let $x \in \overline{\bigcup_{\lambda \in \Lambda} G_\lambda}$. There exists a neighborhood $U(x)$ of x such that $U(x)$ meets exactly one member $G_0 \in \mathcal{G}$. We show that $x \in \overline{G_0}$. If x does not belong to $\overline{G_0}$, then there exists a neighborhood $V(x)$ such that $V(x) \cap G_0 = \emptyset$. Since $U(x) \cap V(x)$ is a neighborhood of x, we have $(U(x) \cap V(x)) \cap (\bigcup_{\lambda \in \Lambda} G_\lambda) \neq \emptyset$. This implies $(\bigcup_{\lambda \in \Lambda}(U(x) \cap V(x) \cap G_\lambda) \neq \emptyset$. It is a contradiction. Hence $\overline{\bigcup_{\lambda \in \Lambda} G_\lambda} \subset \bigcup_{\lambda \in \Lambda} \overline{G_\lambda}$. The converse is obvious. ∎

Theorem 2.2. *Every regular quasi–σ–space is perfect.*

Proof. Let U be a non–empty open set of X. By the assumption, for any $x \in U$, there exists a neighborhood V_x of x such that $\overline{V_x} \subset U$. Let $\mathcal{M} = \bigcup_{k=1}^{\infty} \mathcal{M}_k$ be a σ–disjoint network such that $\mathcal{M}_k$ is discrete with respect to $\overline{\mathcal{M}_k^*}$ for each $k \in \mathbb{N}$. There exist $k \in \mathbb{N}$ and $M \in \mathcal{M}_k$ such that $x \in M \subset V_x$. Thus $\overline{M} \subset \overline{V_x} \subset U$. Set $F_k = \bigcup\{\overline{M} : M \in \mathcal{M}_k, \overline{M} \subset U\}$. Then F_k is a closed set by Lemma 2.1, and F_k is contained in U. Therefore $U = \bigcup_{k=1}^{\infty} F_k$. Hence X is perfect. ∎

Theorem 2.3. *If X is a first–countable GO–space, then X is metrizable if and only if X is quasi–σ–space.*

Proof. Apply [1, Theorem 7] (If X is a first–countable GO–space, then X is quasi–developable if and only if X is weak–σ–space) and

Theorem 2.2. Since a space X is developable if and only if it is quasi–developable and perfect; every quasi–σ–space is weak–σ–space; and every developable GO–space is metrizable, the proof is complete. ■

From [1, Corollary 7.1](A GO–space is quasi–developable if and only if it is a weak–σ–space with a quasi–G_δ–diagonal) and Theorem 2.2 we have the following corollary:

Corollary 2.4. *A GO–space is metrizable if and only if it is a quasi–σ–space with a quasi–G_δ–diagonal.*

3. θ–SPACES

Theorem 3.1. *A linearly ordered space X with a quasi–G_δ–diagonal is a θ–space.*

Proof. Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a sequence of families satisfying the condition for a quasi–G_δ–diagonal. Without loss of generality we assume that $\mathcal{G}_n \subset \mathcal{G}_{n+1}$ and $st(x, \mathcal{G}_m)$ refines $st(x, \mathcal{G}_n)$ if $m > n$ and $m, n \in c(x) = \{k : x \in \mathcal{G}_k^*\}$ and that $st(x, \mathcal{G}_n)$ is an open interval that contains x.

Define $h : \mathbb{N} \times X \to \tau$ as follows:

$$h(n, x) = \begin{cases} st(x, \mathcal{G}_n) & \text{if } x \in \mathcal{G}_n^*. \\ X & \text{if } x \notin \mathcal{G}_n^* \ . \end{cases}$$

We show that h is a θ–map. Suppose that for each $n \in \mathbb{N}, y_n \in h(n, x), \{x, z_n\} \subset h(n, y_n)$ and x is not a cluster point of $\{z_n\}$. Then there is an open interval (a, b) about x and an $M \in \mathbb{N}$ such that if $n > M$, then $z_n \notin (a, b)$. Moreover, without loss of generality there is a subsequence $\{z_{j_n}\}$ of $\{z_n\}$ such that $z_{j_n} \leq a$ for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}, a \in [z_{j_n}, x] \subset h(j_n, y_{j_n}) \subset st(x, \mathcal{G}_{j_n}) \subset st(x, \mathcal{G}_n)$.

It follows that $a \in \bigcap_{n\in c(x)} st(x, \mathcal{G}_n) = \{x\}$, a contradiction. ■

Lemma 3.2. *A q–space with quasi–S_2–diagonal is first countable.*

Proof. Let f be a q–map and $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ a quasi–S_2–sequence on X. Define g by

$$g(n,x) = \begin{cases} st(x,\mathcal{G}_n) & \text{if } x \in \mathcal{G}_n^*. \\ X & \text{if } x \notin \mathcal{G}_n^* . \end{cases}$$

For each $x \in X$ and $n \in \mathbb{N}$, let $h(n,x) = f(n,x) \cap g(n,x)$. Then h is a first countable map. Let $x_n \in h(n,x)$. Then $\langle x_n\rangle$ has a cluster point, say y (because f is q–map). For all $n \in \mathbb{N}, y$ is a cluster point of $\langle x_m : m \geq n\rangle$, so $y \in \overline{h(n,x)}$ as $x_m \in h(n,x)$ for all m. Thus $y \in \bigcap_{n\in\mathbb{N}}\overline{h(n,x)} \subset \bigcap_{n\in\mathbb{N}}\overline{st(x,\mathcal{G}_n)} = \{x\}$, so $y = x$ and x is a cluster point of $\langle x_n\rangle$. ■

Theorem 3.3. *Every $w\theta$–space with a quasi–G_δ^*–diagonal is a θ–space.*

Proof. Every $w\theta$–space X is q–space, so by Lemma 3.2, X is first countable. By [2, Proposition 2.2] [a space X is a θ–space if and only if it is a first countable $w\theta$–space that satisfies condition I (I = there is a sequence $\langle U_n\rangle$ of neighbornets such that if K is a compact set and $x \notin K$, then there is an $n \in \mathbb{N}$ such that $U_n^*(x) \cap K = \emptyset$, where $U_n^*(x) = \bigcup\{U_n(y) : y \in U_n(x) \text{ and } x \in U_n(y)\}$)], we need to prove that X satisfies condition I.

Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a quasi–G_δ^*–diagonal sequence. Without loss of generality we may assume that $st(x,\mathcal{G}_m)$ refines $st(x,\mathcal{G}_n)$ if $m > n$ and $m,n \in c(x) = \{k : x \in \mathcal{G}_k^*\}$.

Define $h : \mathbb{N} \times X \to \tau$ as follows:

$$h(n,x) = \begin{cases} st(x,\mathcal{G}_n) & \text{if } x \in \mathcal{G}_n^*. \\ X & \text{if } x \notin \mathcal{G}_n^* . \end{cases}$$

For each $(n,x) \in \mathbb{N} \times X$ define $V_n(x) = \bigcap_{i=1}^{n} h(i,x)$. Let $z \in X$. Then $V_1(z) = h(1,z) = V_1^*(z)$ (from the definition of $V_n(x)$ and because $z \in st(x,\mathcal{G}_n)$ if and only if $x \in st(z,\mathcal{G}_n)$). Suppose that for $1 \leq j \leq n, V_j(z) = V_j^*(z)$. Then $V_{j+1}(z) = h(j+1,z) \cap V_j(z) =$

$h(j+1,z)\cap V_j^*(z)=V_{j+1}^*(z)$. Therefore, by induction for each $z\in X$ and $n\in\mathbb{N}$, $V_n(z)=V_n^*(z)$. We note that for each $n\in\mathbb{N}, V_{n+1}\subset V_n$. Suppose that K is a compact set, $p\in X-K$ and for each $n\in\mathbb{N}, V_n^*(p)\cap K\neq\emptyset$. Then for each $n\in\mathbb{N}$, there is a point $x_n\in V_n(p)\cap K$. There is a point $q\in K$ that is a cluster point of $\langle x_n\rangle$. For each $n\in\mathbb{N}$ such that $p\in\mathcal{G}_n^*, q\in\overline{\{x_i : i\geq n\}}\subset\overline{V_n(p)}\subset\overline{st(p,\mathcal{G}_n)}$.

Since $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a quasi–G_δ^*–diagonal sequence, $p=q$, a contradiction. Therefore $\langle V_n\rangle$ satisfies condition I. ■

Theorem 3.4. *A $w\Theta$–space is metrizable if and only if it is a quasi–Nagata space with a quasi–$G_\delta^*(2)$–diagonal.*

Proof. Necessity. Every metrizable space is Nagata [6] and has a quasi–$G_\delta^*(2)$–diagonal, and every Nagata space is a quasi–Nagata.

Sufficiency. Let X be a quasi–Nagata, $w\Theta$–space. Then X has a quasi–Nagata–map h and a $w\Theta$–map k. Define g such that $g(n,x)=h(n,x)\cap k(n,x)$ for each $(n,x)\in\mathbb{N}\times X$. Then g is a COC–map. We now prove that g is a $w\Delta$–map. For any two sequences $\langle x_n\rangle$ and $\langle y_n\rangle$ let $\{p,x_n\}\subset g(n,y_n)$ for each $n\in\mathbb{N}$. Consider $\langle p\rangle$ as a constant sequence. We have that $p\in g(n,y_n)$ and $\langle p\rangle$ is converges to p. Since g is a quasi–Nagata–map, $\langle y_n\rangle$ has a cluster point. Now, we have that $\{p,x_n\}\in g(n,y_n)$ and the sequence $\langle y_n\rangle$ has a cluster point. Since g is a $w\Theta$–map, the sequence $\langle x_n\rangle$ has a cluster point, and hence X is $w\Delta$–space. By [10, Corollary 2.3] (a space X is $w\Delta$ and has a quasi–G_δ^*–diagonal if and only if it is developable), X is developable. From [11, Theorem 4.10] (a space X with quasi–$G_\delta^*(2)$–diagonal is Nagata if and only if it is a q–space and a quasi–Nagata–space), X is a Nagata–space. Thus X is metrizable by the well–known fact that every developable, Nagata space is metrizable. ■

Theorem 3.5. *A space X with a quasi–$G_\delta^*(2)$–diagonal is Θ if and only if it is $w\theta$.*

Proof. Every Θ–space is $w\theta$. Now suppose that X is a $w\theta$–space with a quasi–G_δ^*–diagonal. From Theorem 3.3, X is θ–space.

We know that every θ–space is a q–space. Since the space X has a quasi–G^*_δ–diagonal, by Lemma 3.2, X is a first countable space. Let f be a θ–map. Define $g : \mathbb{N} \times X \to \tau$ as follows:

$$g(n,x) = \begin{cases} st(x,\mathcal{G}_n) & \text{if } x \in \mathcal{G}^*_n. \\ X & \text{if } x \notin \mathcal{G}^*_n \,. \end{cases}$$

Let $h(n,x) = \bigcap_{i=1}^{n} g(i,x)$. Set $k(n,x) = f(n,x) \cap h(n,x)$. We show that k is a Θ–map for X. Let $\{x_n, p\} \in k(n, y_n)$ and suppose q is a cluster point of the sequence $\langle y_n \rangle$. Since X is first countable, there is a convergent subsequence $\langle y_{j_n} \rangle$ of $\langle y_n \rangle$ such that for each $n \in \mathbb{N}, y_{j_n} \in k(n,q)$. Then $\{x_{j_n}, p\} \in k(j_n, y_{j_n}) \subset k(n, y_{j_n}) \subset f(n, y_{j_n})$. If $p = q$, it follows from the definition of θ–map that p is a cluster point of $\langle x_n \rangle$. Suppose that $p \neq q$.

Fix $n \in c(x)$. Then there are infinitely many integers $m \geq n$ such that $x_m \in k(n,q)$. Let $m \geq n$ with $x_m \in k(n,q)$. Then $x_m \in g(n,q) = st(q,\mathcal{G}_n)$. Thus $\overline{\{y_m : m \geq n\}} \subseteq st^2(q,\mathcal{G}_n)$ for all $n \in c(x)$. So, $p \in \overline{\{y_m : m \geq n\}} \subseteq \overline{st^2(q,\mathcal{G}_n)}$ for all $n \in c(x)$. It follows that $p \in \bigcap_{n \in c_{\mathcal{G}}(x)} st^2(q,\mathcal{G}_n) = \{q\}$. Thus $p = q$, and k is a Θ–map. ■

From Theorem 3.4 and Theorem 3.5 we get the following result:

Theorem 3.6. *A $w\theta$–space is metrizable if and only if it is a quasi–Nagata space with a quasi–$G^*_\delta(2)$–diagonal.*

From Theorem 3.1 and Theorem 3.5 we get the following result:

Theorem 3.7. *A linearly ordered space X with a quasi–$G^*_\delta(2)$–diagonal is a Θ–space.*

Theorem 3.8. *A space X is developable if and only if it is a $w\theta$, β–space with a quasi–$G^*_\delta(2)$–diagonal.*

Proof. From Theorem 3.5, X is Θ–space. It is easily verified that a space is developable if and only if it is a β, Θ–space. ■

The relationships between the classes of spaces considered in this section can be summarized in the following diagram:

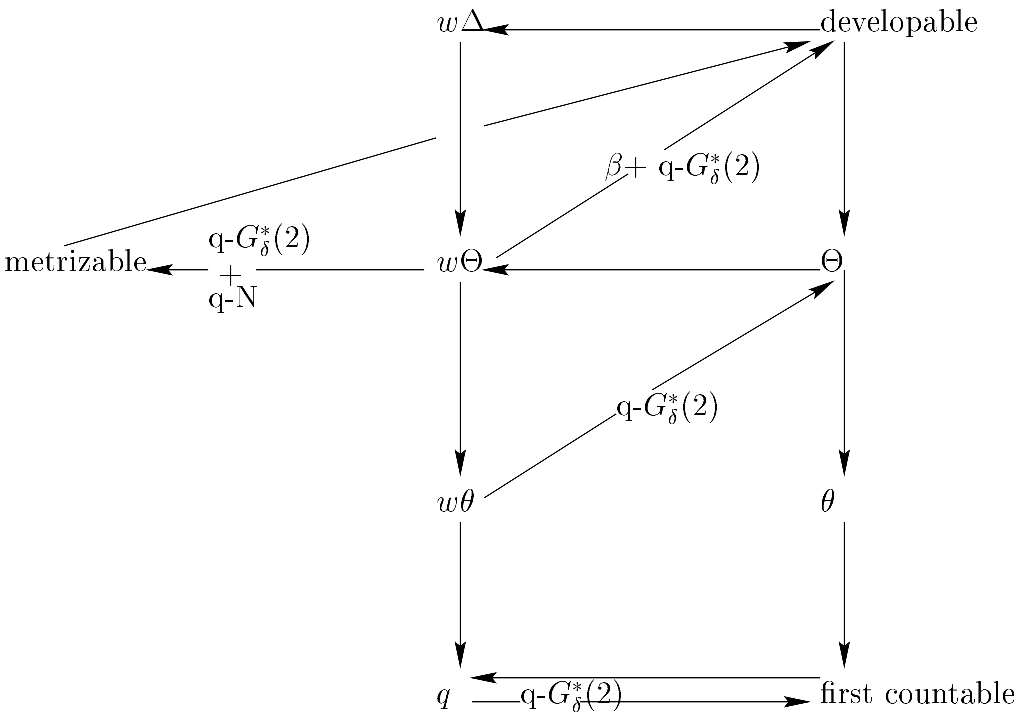

FIGURE 1. Relationships between the classes of spaces considered in this section.

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