# On Spaces with Quasi–Regular–$G_\delta$–Diagonals*

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**Abstract**

This paper studies spaces with quasi–regular–$G_\delta$–diagonal. It is shown that if $X$ is a normal space, then the following are equivalent:

1. $X$ admits a development satisfying the 3–link property.
2. $X$ is a $w\Delta$ with quasi–regular–$G_\delta$–diagonal.
3. $X$ is a $w\Delta$ with regular–$G_\delta$–diagonal.
4. $X$ is $K$–semimetrizable via a semimetric satisfying $(AN)$.
5. There is a semimetric $d$ on $X$ such that:
   a. if $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences both converging to the same point, then $\lim d(x_n, y_n) = 0$, and
   b. if $x$ and $y$ are distinct points of $X$ and $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences converging to $x$ and $y$, respectively, then there are integers $L$ and $M$ such that if $n > L$, then $d(x_n, y_n) > \frac{1}{M}$.

## 1 Introduction

Diagonal properties play a very important role as a factor of metrizability and developability. For example, a countably compact space with a quasi–$G_\delta$–diagonal is compact metrizable (Shiraki [11]), a linearly ordered topological space (= $LOTS$) with $G_\delta$–diagonal is metrizable (Lutzer [5]), a paracompact

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Cech–complete space with a $G_\delta$–diagonal is metrizable (Nagata [9]), a locally compact locally connected space with $G_\delta^*$–diagonal is developable (Gruenhage [3]), a locally compact locally connected space with quasi–$G_\delta^*$–diagonal is quasi–developable (Gartside and Mohamad [2]), a locally compact locally connected space with a quasi–regular–$G_\delta$–diagonal is metrizable (Gartside and Mohamad [2]), a space $X$ is $w\Delta$ and has a quasi–$G_\delta^*$–diagonal if and only if it is developable (Mohamad [8]).

The purpose of this paper is to investigate the effects of a quasi–regular–$G_\delta$–diagonal.

A space $X$ has a **quasi–regular–$\boldsymbol{G_\delta}$–diagonal** [2] if and only if there is a countable sequence $\langle U_n : n \in \ N \rangle$ of open subsets in $X^2$, such that for all $(x, y) \notin \Delta$, there is $n \in \ N$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$.

Recall that a subset $H$ of the space $X$ is a **regular $\boldsymbol{G_\delta}$–set** if there is a sequence $\langle U_n : n \in \ N \rangle$ of open sets in $X$ such that $H = \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U_n}$. We say that $X$ has a **regular–$\boldsymbol{G_\delta}$–diagonal** [3] if $\Delta = \{(x, x) : x \in X\}$ is a regular $G_\delta$–set in $X^2$.

A space $X$ has a **quasi–$\boldsymbol{G_\delta^*}$–diagonal** [7] if there exists a countable family $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of collections of open subsets of $X$ such that for any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that $x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\}$.

A space $X$ is called **quasi–developable** (**developable**) if $X$ has a countable family $\langle \mathcal{G}_n : n \in \ N \rangle$ of collections of open subsets (open covers) of $X$ such that the sets of the form $st(x, \mathcal{G}_n)$ [i.e. the union of all sets in $\mathcal{G}_n$ which contain $x$] form a local base at $x$ for all $x \in X$. The sequence $\langle \mathcal{G}_n : n \in \ N \rangle$ is called a quasi–development (development) for $X$. A space $X$ is called Moore if $X$ is a regular developable. A development $\langle \mathcal{G}_n : n \in \ N \rangle$ for the space $X$ is said to satisfy the **3–link property** if whenever $p$ and $q$ are distinct points of $X$, there is an integer $n$ such that no member of $\mathcal{G}_n$ intersects both $st(p, \mathcal{G}_n)$ and $st(q, \mathcal{G}_n)$.

A space $X$ has a property $\boldsymbol{(AN)}$ [4] if each point of $X$ has a neighborhood of arbitrarily small diameter.

For terminologies which are not defined in this paper, the readers should consult books [3] and [1].

## 2 The main results

**Theorem 2.1** *A space $X$ has quasi–regular–$G_\delta$–diagonal if and only if there is a sequence $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ of collections of open subsets of $X$ such that*

1. *for all $x \in X$ we have $\{x\} = \bigcap_{n \in c(x)} \overline{st(x, \mathcal{G}_n)}$ where $c(x) = \{n \in \mathbb{N} : x \in \mathcal{G}_n^*\}$, where $\mathcal{G}_n^* = \bigcup \{G : G \in \mathcal{G}_n\}$;*
2. *for all $x, y \in X$ with $x \neq y$, there are an integer $n$ and open sets $U$ and $V$ containing $x$ and $y$ respectively such that $x \in \mathcal{G}_n^*$ and no member of $\mathcal{G}_n$ meets both $U$ and $V$.*

*Proof.* Suppose that $X$ has a quasi–regular–$G_\delta$–diagonal. Let $\{U_n\}_{n \in \mathbb{N}}$ be a countable sequence of open subsets in $X^2$ such that for all $(x, y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$. For each $n \in \mathbb{N}$, let $\mathcal{G}_n = \{G : G \text{ is open in } X \text{ and } G \times G \subset U_n\}$. Then,

claim 1 for all $x \in X$ we have $\{x\} = \bigcap_{n \in c(x)} \overline{st(x, \mathcal{G}_n)}$.

Let $x, y \in X$ with $x \neq y$. Then, there is some $n \in \mathbb{N}$ such that $(x, x) \in U_n$, but $(x, y) \notin \overline{U_n}$. This implies that $y \notin \overline{st(x, \mathcal{G}_n)}$ and $st(x, \mathcal{G}_n) \neq \emptyset$.

claim 2 for all $x, y \in X$ with $x \neq y$, there are an integer $n$ and open sets $U$ and $V$ containing $x$ and $y$ respectively such that no member of $\mathcal{G}_n$ meets both $U$ and $V$. Let $x, y \in X$ with $x \neq y$. Then there is an integer $n$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$. Let $U$ and $V$ be open sets in $X$ that contain $x$ and $y$ respectively such that $U \times V$ does not intersect $U_n$. To see that no member of $\mathcal{G}_n$ intersects both $U$ and $V$, suppose instead that $G$ is a member of $\mathcal{G}_n$ such that $r \in G \cap U$ and $s \in G \cap V$. It follows that $(r, s) \in U_n \cap (U \times V)$ which is a contradiction.

Conversely, suppose that $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ is a sequence of collections of open subsets of $X$ as described in the theorem. Define $U_n = \bigcup \{G \times G : G \in \mathcal{G}_n\}$ for each $n \in \mathbb{N}$. Then the sets $U_n$ are open in $X^2$. Further, if $(x, y) \in X^2$ such that $x \neq y$, then there are an integer $n$ and open sets $U$ and $V$ containing $x$ and $y$ respectively such that no member of $\mathcal{G}_n$ meets both $U$ and $V$. It must be the case that $U_n$ does not intersect $U \times V$. Therefore $(x, y) \notin \overline{U_n}$. ■

**Corollary 2.2** *[2] Every space with a quasi–regular–$G_\delta$–diagonal has a quasi–$G_\delta^*$–diagonal.*

**Theorem 2.3** *[8] A space $X$ is $w\Delta$ and has a quasi–$G_\delta^*$–diagonal if and only if it is developable.*

**Theorem 2.4** *A normal perfect space $X$ has a regular–$G_\delta$–diagonal if and only if it has a quasi–regular–$G_\delta$–diagonal.*

*Proof.* Every regular–$G_\delta$–diagonal is a quasi–regular–$G_\delta$–diagonal. Conversely suppose that $X$ has a quasi–regular–$G_\delta$–diagonal. Then there is a countable family $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ of collections of open subsets of $X$ such that, for any distinct points $x, y \in X$, there exists $n \in \mathbb{N}$ such that

$$x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\}$$

and for all $x, y \in X$ with $x \neq y$, there are an integer $n$ and open sets $U$ and $V$ containing $x$ and $y$ respectively such that no member of $\mathcal{G}_n$ meets both $U$ and $V$ and $x \in \mathcal{G}_n^*$.

For each $n \in \mathbb{N}$, the set $\mathcal{G}_n^*$ is open and hence, since $X$ is perfect, $\mathcal{G}_n^*$ is an $F_\sigma$-set. Thus $\mathcal{G}_n^* = \bigcup_j F_{n,k}$, where $F_{n,k}$ is a closed subset of $X$, for each $k \in N$.

By normality, find $U_{n,k}$ open with $F_{n,k} \subseteq U_{n,k} \subseteq \bar{U}_{n,k} \subseteq \mathcal{G}_n^*$.

For each ordered pair $(n, k)$ of natural numbers, let

$$\mathcal{H}_{n,k} = \mathcal{G}_n \cup \{X - \bar{U}_{n,k}\}.$$

Then $\langle \mathcal{H}_{n,k} : n, k \in \mathbb{N} \rangle$ is a countable family of open covers of $X$.

Now, $x \in \mathcal{G}_n^*$, so $x \in F_{n,k}$ for some $k$. It follows, $x \in U_{n,k}$. Put $U' = U_{n,k} \cap U$. We know that, there is no element of $\mathcal{G}_n$ meets both $U$ and $V$. So, there is no element of $\mathcal{G}_n$ meets both $U'$ and $V$. Since $X - \bar{U}_{n,k}$ does not meet $U'$, there is no element of $\mathcal{H}_{n,k}$ meets both $U'$ and $V$. Hence, from Theorem 1 [12], $X$ has a regular–$G_\delta$–diagonal. ■

**Question 2.5** *Does every perfect space with a quasi–regular–$G_\delta$–diagonal have a regular–$G_\delta$–diagonal?*

In the proof of Theorem 2.4, we did not use perfect normality, but the following condition: every open set is a union of countably many regular closed sets (recall that a set $C$ is a regular closed if $C = \overline{V}$ for some open set $V$). Let us call such a space regularly perfect.

**Question 2.6** *Is there an example of regularly perfect space which is not perfectly normal?*

**Question 2.7** *Is there an example of regular and perfect space which is not regularly perfect?*

**Theorem 2.8** *A normal space $X$ is $w\Delta$ with quasi–regular–$G_\delta$–diagonal if and only if it admits a development satisfying the 3–link property.*

*Proof.* The sufficiency of the condition is obvious.

Necessity. Suppose that $X$ is a $w\Delta$ with quasi–regular–$G_\delta$–diagonal. Then, by Corollary 2.2 and Theorem 2.3, $X$ is developable. Then, $X$ is a perfect. By Theorem 2.4, $X$ has a regular–$G_\delta$–diagonal. Applying [12, Theorem 2] will complete the proof. ■

From Theorem 2.8 and [4, Theorem 5.4.] we get the following result:

**Corollary 2.9** *Let $X$ be a normal space. Then the following conditions are equivalent:*

1. *$X$ admits a development satisfying the 3–link property.*
2. *$X$ is a $w\Delta$ with quasi–regular–$G_\delta$–diagonal.*
3. *$X$ is a $w\Delta$ with regular–$G_\delta$–diagonal.*
4. *$X$ is $K$–semimetrizable via a semimetric satisfying $(AN)$.*
5. *There is a semimetric $d$ on $X$ such that:*
   - a. *if $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences both converging to the same point, then lim $d(x_n, y_n) = 0$, and*
   - b. *if $x$ and $y$ are distinct points of $X$ and $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences converging to $x$ and $y$, respectively, then there are integers $L$ and $M$ such that if $n > L$, then $d(x_n, y_n) > \frac{1}{M}$.*

# 3 Pseudocompatness and quasi–regular–$G_\delta$–diagonals

Recall that a space $X$ is pseudocompact if it is Tychonoff and every continuous real–valued function defined on $X$ is bounded.

In [10] Reed observes that a pseudocompact Moore spaces with regular–$G_\delta$–Diagonal is metrizable. Note that pseudocompact spaces with regular–$G_\delta$–Diagonal are Moore spaces by the following result of McArthur in [6]:

**Lemma 3.1** *Let $X$ be a pseudocompact space. Suppose $\langle U_n : n \in \mathbb{N} \rangle$ is a decreasing sequence of open sets such that $\bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U_n} = \{x\}$ for a point $x \in X$. Then the sets $U_n$ form a local neighborhood base at $x$.*

**Lemma 3.2** *Let $X$ be a pseudocompact space with quasi–$G^*_\delta$–diagonal. Then $X$ is quasi–developable.*

*Proof.* Let $\langle \mathcal{V}_n : n \in \mathbb{N} \rangle$ be a quasi–$G^*_\delta$–diagonal sequence for $X$. Without loss of generality we may assume that $\mathcal{V}_1 = \{X\}$. Set $c_\mathcal{V}(x) = \{n : st(x, \mathcal{V}_n) \neq \emptyset\}$. Then $\bigcap_{n \in c_\mathcal{V}(x)} \overline{st(x, \mathcal{V}_n)} = \{x\}$. Let $\mathcal{F}$ denote the non–empty finite subsets of $\mathbb{N}$. For each $F \in \mathcal{F}$ set

$$\mathcal{G}_F = \{\bigcap\nolimits_{i \in F} V_i : V_i \in \mathcal{V}_i\}.$$

We show that $\{\mathcal{G}_F : F \in \mathcal{F}\}$ is a quasi–development of $X$. For each $n \in \mathbb{N}$ put $F_n = c_\mathcal{V}(x) \cap \{1, 2, ..., n\},$. Then $F_n \neq \emptyset$. Note that $st(x, \mathcal{G}_{F_m}) \subseteq st(x, \mathcal{V}_n)$ for each $m \geq n$ and $n \in c(x)$. Note also that

$$\bigcap\nolimits_{n \in \mathbb{N}} \overline{st(x, \mathcal{G}_{F_n})} = \bigcap\nolimits_{n \in \mathbb{N}} st(x, \mathcal{G}_{F_n}) = \{x\}.$$

By Lemma 3.1, $\{st(x, \mathcal{G}_{F_n}) : n \in \mathbb{N}\}$ forms a local neighborhood base at $x$. Hence, $\{st(x, \mathcal{G}_F) : F \in \mathcal{F}\} - \emptyset$ forms a local neighborhood base at $x$. ■

**Question 3.3** *Is every pseudocompact space with a quasi–regular–$G_\delta$–diagonal metrizable?*

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