Weak bases and Metrizability*

A.M. Mohamad

October 19, 1999

Abstract

In this paper we investigate weak bases. We give a characterization of weakly developable spaces and metrization theorems. The metrization results are: a space $X$ is metrizable if and only if $X$ has a $CWBC$-map $g$ satisfying the following conditions:

1. $g$ is a pseudo–strongly–quasi–$N$–map;
2. for any $A \subseteq X, \overline{A} \subseteq \bigcup \{g(n, x) : x \in A\};$

a space $X$ is metrizable if and only if $X$ has a $CWBC$-map $g$ satisfying the following conditions:

1. if $x \in g(n, y_n), y_n \in g(n, x_n), x_n \in g(n, y_n)$ and $y_n \in g(n, x)$ for all $n \in \mathbb{N}$, then $x_n$ converges to $x$;
2. for any $A \subseteq X, \overline{A} \subseteq \bigcup \{g(n, x) : x \in A\}.$

1 Definitions

A collection $\mathcal{W}$ of subsets of a space $X$ is said to be a weak base for $X$ provided that to each $x \in X$, there exists $\mathcal{W}_x \subset \mathcal{W}$ such that:

(i) Each member of $\mathcal{W}_x$ contains $x$.

(ii) For any two members $W_1$ and $W_2$ of $\mathcal{W}_x$, there is a $W_3 \in \mathcal{W}_x$ such that $W_3 \subseteq W_1 \cap W_2$.

(iii) A subset $U$ of $X$ is open if and only if for every point $x \in U$ there exists a $W \in \mathcal{W}_x$ such that $W \subseteq U$.

---

*The author acknowledges the support of the Marsden Fund Award UOA 611, from the Royal Society of New Zealand. AMS (1991) Subject Classification: 54E30, 54E35. Keywords and phrases: weakly developable; metrizable; weakly first countable; quasi-$G_\delta$-diagonal.
If to each \( x \in X \) we assign a collection \( \mathcal{W}_x \) of supersets of \( \{x\} \) such that \( \mathcal{W} = \bigcup \{\mathcal{W}_x : x \in X\} \) is a weak base by virtue of the collections \( \mathcal{W}_x \), then we say that the collection \( \mathcal{W}_x \) is a \textbf{local weak base} at \( x \) for each \( x \in X \).

**Graded weak bases:**

Let \((X, \tau)\) be a space, let \( g : \mathbb{N} \times X \to \mathcal{P}(X) \) be a function and \( \mathcal{G} = \{ g(n, x) : n \in \mathbb{N}, x \in X \} \). We call \( \mathcal{G} \) a \textbf{graded weak base} for \( X \) and \( g \) is called a \textbf{CWBC–map} (= countable weak base covering map) for \( X \) and \( X \) with a graded weak base is called \textbf{weakly first countable} if the following conditions are satisfied:

(a) \( x \in \bigcap_{n \in \mathbb{N}} g(n, x) \) for all \( x \in X \).

(b) \( g(n + 1, x) \subseteq g(n, x) \) for all \( n \in \mathbb{N} \) and \( x \in X \).

(c) A subset \( U \) of \( X \) is open if and only if for every \( x \in U \) there is an \( n \in \mathbb{N} \) such that \( g(n, x) \) is contained in \( U \).

The map \( g \) is called a \textbf{COC–map} (= countable open covering map) for \( X \) if conditions (a), (b) and for each \( n \in \mathbb{N}, g(n, x) \) is open are satisfied. A space \( X \) is called first countable (resp. \( q \)) if and only if \( X \) has a COC–map \( g \) such that if \( x_n \in g(n, x) \) for every \( n \in \mathbb{N} \), then \( x \) is a cluster point of the sequence \( \{x_n\} \) (then the sequence \( \{x_n\} \) has a cluster point).

**Generalizations of developable spaces:**

Martin in [4] introduced weakly developable spaces. Let \( \mathcal{G} = \{ \mathcal{G}_n : n \in \mathbb{N} \} \) be a countable family of collections of subsets of a space \( X \). Consider the following conditions on \( \mathcal{G} \):

(a) For each \( n \in \mathbb{N}, \mathcal{G}_n \) is a collection of open sets in \( X \).

(b) Each \( \mathcal{G}_n \) is a covering of \( X \);

(c) For each \( x \in X \), \( \{ st(x, \mathcal{G}_n) : n \in \mathbb{N}, x \in \mathcal{G}_n^* \} \) is a local base at \( x \);

(d) For each \( x \in X \) and \( n \in \mathbb{N}, st(x, \mathcal{G}_n) \) is an open subset of \( X \).

(e) For each \( x \in X \), \( \{ st(x, \mathcal{G}_n) : n \in \mathbb{N}, x \in \mathcal{G}_n^* \} \) is a local weak base at \( x \);

(f) For any distinct \( x, y \in X \), there exists \( n \in \mathbb{N} \) such that

\[
x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\};
\]

**Definition 1.1** A space \( X \) is \textbf{developable} if there exists a family \( \mathcal{G} \) satisfying (a), (b) and (c);

A space \( X \) is \textbf{o–semi–developable} if there exists a family \( \mathcal{G} \) satisfying (b), (c) and (d);
A space $X$ is **semi-developable** if there exists a family $\mathcal{G}$ satisfying (b) and (c);

A space $X$ is **weakly-developable space** if there exists a family $\mathcal{G}$ satisfying (b) and (e);

A space $X$ has a **quasi-$G_\delta^*$-diagonal** if there exists a family $\mathcal{G}$ satisfying (a) and (f);

A regular developable space is called a Moore space. Throughout this paper, every space is $T_1$.

## 2 Generalizations of first countable spaces

Recall, a space $X$ is **sequential** [1] if every sequentially open set is open, where a set $U$ is said to be sequentially open if every sequence converging to a point in $U$ is eventually in $U$. A space is **Frechet** [1] if every accumulation point of a set is the limit of a sequence in the set. Every first countable space is Frechet and every weakly first countable space is sequential.

**Lemma 2.1** A $T_2$ space $X$ is first countable if and only if $X$ is Frechet and weakly first countable.

**Proof.**

It is clear that every first countable space is Frechet and weakly first countable. Now, let $g : \mathbb{N} \times X \to \mathcal{P}(X)$ be a CWBC-map. We claim that for each $x \in X, g(n, x)$ is a neighborhood of $x$ for each $n \in \mathbb{N}$. Suppose it is not, so there is a point $x \in X$ and $m \in \mathbb{N}$ for which $g(m, x)$ is not a neighborhood of $x$. Then by the Frechet assumption, there is a sequence $\langle x_k \rangle$ which converges to $x$ such that $x_k \in X - g(m, x)$ . Let $U = X - \{ x, x_1, x_2, x_3, \ldots \}$, which is open (we are assuming our space is Hausdorff), so that for each $p \in U$ there exist a $g(n, p)$ which is contained in $U$. Put $V = U \cup \{ x \}$, for each $p \in V$ there is a $g(n, p)$, such that $g(n, p) \subset V$, since if $p = x$ then $g(n, p) = g(m, x) \subset V$. By the assumption of weak first countability of $g$, $V$ is open in $X$. But $x$ is in $V$, so that we have the contradiction that $\langle x_k \rangle$ does not converge to $x$. Thus $x$ is in the interior of $g(m, x)$. Now, put $h(n, x) = \text{Int} \ g(n, x)$ for each $n \in \mathbb{N}$ and $x \in X$, then $h : \mathbb{N} \times X \to \tau$ satisfies the first countability condition. \qed

From the Lemma 2.1 and the results in [8], we can summarize the relationships between the classes above in the following diagram:
Lemma 2.2 A q space with quasi-\( \mathbb{G}_k^* \)-diagonal is first countable.

Proof. Let \( f \) be a q-map and \( \{ G_n : n \in \mathbb{N} \} \) a quasi-\( \mathbb{G}_k^* \)-sequence on \( X \). Define \( g \) by

\[
g(n, x) = \begin{cases} 
st(x, G_n) & \text{if } x \in G_n^*, \\
X & \text{if } x \notin G_n^*. 
\end{cases}
\]

For each \( x \in X \) and \( n \in \mathbb{N} \), let \( h(n, x) = f(n, x) \cap \bigcap_{i=1}^{n} g(i, x) \). Then \( h \) is a first countable map. Let \( x_n \in h(n, x) \). Then \( \{ x_n \} \) has a cluster point, say \( y \) (because \( g \) is q-map). For all \( n \in \mathbb{N}, y \) is a cluster point of \( \{ x_m : m \geq n \} \), so \( y \in h(n, x) \) as \( x_m \in h(n, x) \) for all \( m \). Thus \( y \in \bigcap_{n \in \mathbb{N}} h(n, x) \subset \bigcap_{n \in \mathbb{N}} \text{st}(x, G_n) = \{ x \} \), so \( y = x \) and \( x \) is a cluster point of \( \{ x_n \} \).

3 Weakly developable spaces

Our next theorem is a characterization of Martin’s weak developably concept.

**Theorem 3.1** A space \( X \) is weakly developable if and only if there is a CWBC-map \( g : \mathbb{N} \times X \rightarrow X \), such that if \( \{ p, x_n \} \subseteq g(n, y_n) \) for all \( n \), then the sequence \( \{ x_n \} \) has a cluster point.

**Proof.** We will prove firstly the sufficiency of the condition. Suppose that there is a CWBC-map \( g : \mathbb{N} \times X \rightarrow X \), such that if \( \{ p, x_n \} \subseteq g(n, y_n) \) for all \( n \), then the sequence \( \{ x_n \} \) has a cluster point. For each \( i \in \mathbb{N}, \) let \( G_i = \{ g(j, x) : x \in X, j \geq i \} \). The sequence \( \{ G_n : n \in \mathbb{N} \} \) of covers constitutes a weak–development. Suppose, conversely, that \( \{ G_n : n \in \mathbb{N} \} \) is a weak–development for \( X \). Define the map \( g \) as follows: for each point \( x \) of \( X \) let
$g(1, x)$ be some member of $G_1$ which contains $x$ and, if $n > 1$, let $g(n, x)$ be a member of $G_n$ such that $x \in g(n, x) \subset g(n-1, x)$. Clearly $g$ is a CWBC-map which satisfies the condition of the theorem. \hfill \blacksquare

**Theorem 3.2** A regular space is Moore if and only if it is weakly developable and Frechet.

*Proof.* Let $g : \mathbb{N} \times X \to \mathcal{P}(X)$ be a weakly developable–map. We can use the same proof as for Lemma 2.1 to prove that for each $x \in X, g(n, x)$ is a neighborhood of $x$ for each $n \in \mathbb{N}$. Thus $x$ is in the interior of $g(n, x)$. Now, put $h(n, x) = \text{Int } g(n, x)$ for each $n \in \mathbb{N}$ and $x \in X$. Then $h : \mathbb{N} \times X \to \tau$ is a developable map. \hfill \blacksquare

We can summarize the relationships between some classes of generalizations of developable spaces in the following diagram:

```
Developable
  ↓
\ o-semidevelopable
  ↓
\ semidevelopable
  ↓
\ weakly developable
  ↓
\ weakly first countable
```

Figure 2: Relationships between some classes of generalizations of developable spaces and weakly developable spaces.

**Definition 3.3** A space $X$ is called a pseudo–strongly–quasi–$\mathbb{N}$–space if there is a CWBC–map $g : \mathbb{N} \times X \to \mathcal{P}(X)$ such that if for each $n \in \mathbb{N}, y_n \in g(n, x_n)$ and the sequence $\langle y_n \rangle$ converges to $p$ in $X$, then $p$ is a cluster point of the sequence $\langle x_n \rangle$. The CWBC–map $g$ is called a pseudo–strongly–quasi–$\mathbb{N}$–map for $X$.

The proof of our next result relies on a metrization theorem of Martin [4].
Theorem 3.4 (Martin) A necessary and sufficient condition that a topological space $X$ be metrizable is that $X$ has a weak development $\{\mathcal{G}_n\}_{n \in \mathbb{N}}$ such that $\{st^2(x, \mathcal{G}_n) : n \in \mathbb{N}, x \in X\}$ is a weak base of $X$.

Theorem 3.5 A space $X$ is metrizable if and only if $X$ has a CWBC-map $g$ satisfying the following conditions:

1. $g$ is a pseudo-strongly-quasi-$N$-map;

2. for any $A \subseteq X$, $\overline{A} \subseteq \bigcup \{g(n,x) : x \in A\}$.

Proof. The only if part is obvious. We now prove the if part. Assume that $X$ has a CWBC-map $g$ satisfying the conditions (1) and (2). Let $h(n, x) = X - \{y \in X : x \notin g(n,y)\}$ and $k(n, x) = g(n, x) \cap h(n, x)$ for each $(n, x) \in \mathbb{N} \times X$. Let $\mathcal{G}_n = \{k(n, x) : (n, x) \in \mathbb{N} \times X\}$. Then $st(x, \mathcal{G}_n) = \bigcup \{k(n, y) : x \in k(n, x)\}$ and $st^2(x, \mathcal{G}_n) = \bigcup \{k(n, y) : k(n, y) \cap st(x, \mathcal{G}_n) \neq \emptyset, (n, x) \in \mathbb{N} \times X\}$.

By condition (2) on $g$, $x \in h(n, x)$. To see this, let $A = \{y : x \notin g(n,y)\}$, and suppose by contradiction that $x \in \overline{A}$. By (2), there exists $y \in A$ such that $x \in g(n,y)$. But $y \in A$ means that $x \notin g(n,y)$, a contradiction. Therefore, $h(n, x)$ is a neighborhood of $x$ and so is $k(n, x)$. Therefore, in virtue of the Martin metrization theorem 3.4, we only need prove that $\{st^2(x, \mathcal{G}_n) : n \in \mathbb{N}, x \in X\}$ is a weak base of $X$. If $\{st^2(x, \mathcal{G}_n) : n \in \mathbb{N}\}$ is not a local weak base for some $x \in X$, then there exists an open neighbourhood $U$ of $x$ such that $st^2(x, \mathcal{G}_n) - U \neq \emptyset$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ take $y_n \in st^2(x, \mathcal{G}_n) - U$. That means we can find $z_n, w_n \in X$ such that $y_n \in k(n, z_n), k(n, z_n) \cap k(n, w_n) \neq \emptyset$ and $x \in k(n, w_n)$. Take $v_n \in k(n, z_n) \cap k(n, w_n)$. By $x \in k(n, w_n) \subseteq g(n, w_n)$ and condition (1), we conclude that $\langle w_n \rangle$ converges to $x$, and by $v_n \in k(n, w_n) \subseteq h(n, w_n)$ and the definition of $h$, we get $w_n \in g(n, v_n)$. Using condition (1) again, we have that $\langle v_n \rangle$ converges to $x$. Similarly, from $v_n \in k(n, z_n) \subseteq g(n, z_n)$, we have that $\langle z_n \rangle$ converges to $x$, and by $y_n \in k(n, z_n) \subseteq h(n, z_n)$, we get that $\langle y_n \rangle$ converges to $x$. But $y_n \notin U$ for each $n \in \mathbb{N}$, which is a contradiction.  

In the next theorem, we use a technique similar to that used in [5, Theorem 2.1].

Theorem 3.6 A space $X$ is metrizable if and only if $X$ has a CWBC-map $g$ satisfying the following conditions:

1. if $x \in g(n, y_n)$, $y_n \in g(n, x_n)$, $x_n \in g(n, y_n)$ and $y_n \in g(n, x)$ for all $n \in \mathbb{N}$, then $x_n$ converges to $x$;
(2) for any $A \subseteq X, \overline{A} \subseteq \bigcup \{g(n, x) : x \in A\}$.

Proof. The only if part is obvious. We now prove the if part. Assume that $X$ has a CWBC-map $g$ satisfying the conditions (1) and (2). By Theorem 3.5, we need only to prove that $X$ has a pseudo-N-map (because every pseudo-N-map is a pseudo-strongly-quasi-N-map) which satisfies condition (2). For each $p \in X$ and each $n \in \mathbb{N}$ let $h(n, p) = X - \{y : p \notin g(n, y)\}$ and $k(n, x) = g(n, x) \cap h(n, x)$. By condition (2) on $g, x \in h(n, x)$. To see this, let $A = \{y : p \notin g(n, y)\}$, and suppose by contradiction that $p \notin A$. By (2), there exists $y \in A$ such that $p \notin g(n, y)$. But $y \in A$ means that $p \notin g(n, y)$, a contradiction. Therefore, $h(n, x)$ is a neighborhood of $x$ and so is $k(n, x)$.

So, $k$ is a CWBC-map. Now, let $y_n \in k(n, x) \cap k(n, x_n)$ for all $n \in \mathbb{N}$. We have, $y_n \in k(n, x), y_n \in g(n, x)$ and $y_n \in h(n, x)$. From the definition of $h$, $x \in g(n, y_n)$. It follows that $y_n \in g(n, x)$ and $x \in g(n, y_n)$ (3).

We have, $y_n \in k(n, x_n), y_n \in g(n, x_n)$ and $y_n \in h(n, x_n)$. From the definition of $h, x_n \in g(n, y_n)$. It follows that $y_n \in g(n, x_n)$ and $x_n \in g(n, y_n)$ (4).

If we now combine (3) and (4), we see that $x \in g(n, y_n)$, $y_n \in g(n, x_n)$, $x_n \in g(n, y_n)$ and $y_n \in g(n, x)$ for all $n \in \mathbb{N}$. Hence by condition (1), $\langle x_n \rangle$ converges to $x$.

\textbf{Acknowledgement:} The author is grateful to Prof. David Gauld for his kind help and suggestions on this paper.

\section*{References}


The Department of Mathematics
The University of Auckland
Private Bag 92019
Auckland
New Zealand.
mohamad@math.auckland.ac.nz