ACOUSTO-OPTIC SCATTERING
IN A SINGLE-MODE OPTIC FIBER

A.V.Badanin[1] B.S.Pavlov[2,3], A.A.Pokrovski[2], L.V.Prokhorov[4]

[1] *Department of Mathematics,*
Arkhangelsk State Technical University, Nab.Sev.Dviny 17,
Arkhangelsk 163007, Russia
[2] *Laboratory of Complex Systems Theory,*
Institute for Physics, St.-Petersburg State University,
Ulyanovskaya 1, St.Petersburg 198904, Russia
[3] *Department of*
Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand
[4]*Department of Field Theory, Institute for Physics,*
St.-Petersburg State University, Ulyanovskaya 1,
St.Petersburg 198904, Russia

July 1, 1999 – December 31, 1999

ABSTRACT

The analysis of acousto-optic scattering in a single-mode fiber in terms of the effective equation with perturbation caused by variation of the speed of light is done. Using an ansatz based on the Lorentz transform we reduce the corresponding equation to Mathieu equation with a non-canonical small (acousto-optic) parameter. In the lowest order of perturbation theory we calculate the positions and widths of spectral lacunae for the case when the elastic wave is infinite. This result is applied for the estimation of the reflection coefficient R in the lacunae using methods suggested earlier by authors for investigation of periodic nanostructures. We calculate explicitly the reflection coefficient for scattering by

a segment of elastic wave of length L and derive the relation $|R_{max}|^2 = \tanh^2(\pi \frac{L}{c} \Delta\nu)$ for the maximal reflection coefficient (c stands for the phase velocity of light and $\Delta\nu$ denotes the width of the reflection band in Hz).

1 Introduction

In the recent years numerous publications were devoted to acousto-optic interaction in fiber optics [1–9]. Among other results it is worth to mention acousto-optic effects in birefringent fibers [1, 2], amplitude modulation [3], superlattice effects [4] and especially tunable acousto-optic filters on single-mode fiber [5]. Due to universal character of the interaction and its natural built-in-fiber nature this kind of devices is rather promising for industrial production of tunable filters, modulators and frequency shifters[6, 7, 8, 9].

Acousto-optic fiber devices are analogous to Bragg lattices, imprinted in the fiber core by ultraviolet radiation [10, 11]. These lattices are used in particular as an external selective fiber resonators with narrow reflection band for stabilization of semiconductor lasers. In the present paper we propose to use acousto-optic devices as **reflective** filters with tunable central frequency and linewidth; they would be useful as tunable fiber resonators for semiconductor lasers. This is a new branch of acousto-optic fiber science, where in so far all the devices were based on coupling of unidirectional light modes. In frames of empirical equation (2) we estimate the reflection coefficient for scattering by a finite segment of running elastic wave in fiber. With this paper we answer the basic question: What should be the interaction length necessary to provide sufficient back-reflection?

In so far back-reflection was observed for thermal rather than induced phonons[12, 13, 14]; this reflection was even used for the control of fiber diameter [15] and fiber material density [16]. Brillouin scattering in fiber by thermal phonons may have very narrow ($\approx 50kHz$) spectral width [12]. This observation essentially reduces the problem of narrow reflection band to the problem of excitation of longitudinal phonons with suitable wavelength. We can also Theoretically leaky waves with small dissipation [17] would be also interesting for our purposes.

In spite of considerable experimental success theoretical considerations applied in so far to Bragg scattering by elastic waves in fiber waveguides were not satisfactory elaborated [6, 18]. They were mostly restricted by relations between the wavelengths of the elastic and the light waves, whose ratio defines the spectral zones of low transmission when it is integer or semi-integer [19]. In particular, there was no estimates for the strength of back-reflection as the function of the length of interaction.

In this paper we suggest an empirical model equation for longitudinal light propagation in fiber perturbed by running elastic wave. We restrict ourselves by single-mode fiber. We suppose that the main effect of the elastic deformation is the change of the phase velocity of propagating light mode. Thus we do not make any difference between various types of elastic waves that would produce this perturbation. We also neglect the dissipation effects despite their importance in certain experimental settings [5]. We suppose that within a small spectral interval the phase speed of the propagating mode $c = \frac{\omega(k)}{k}$ is constant (here ω is the frequency and k is the longitudinal wave number). Thus in a fiber with no perturbation the electric tension ψ satisfies

$$c^2\psi_{xx} = \psi_{tt},$$

where x stands for the longitudinal variable and t for the time. When the fiber is subject to tension S, the speed of light undergoes a small change $c \to c(1-\varepsilon_0 S/2)$, where ε_0 denotes the effective parameter of acousto-optic coupling (it depends from the fiber geometry and material). Thus for an infinite running elastic wave

$$S(x,t) = S_0 \cos(\Omega t - Kx), \tag{1}$$

where S_0 is the amplitude of the tension wave, Ω is its frequency and K is the wave number. The effective equation for light propagation becomes

$$c^2\psi_{xx}(x,t) = \{1 + \varepsilon\cos(\Omega t - Kx)\psi(x,t)\}_{tt}\,, \tag{2}$$

where small parameter ε incorporates the amplitude of the tension wave and therefore can be varied experimentally. When the perturbation is finite, we have in the right-hand side a truncated perturbation, i.e. ε becomes a function of x and quickly decays outside the interval of interaction $[0, L]$:

$$c^2\psi_{xx} - \psi_{tt} = \varepsilon I_{[0,L]}(x)\left\{\cos(\Omega t - Kx)\psi\right\}, \tag{3}$$

where $I_{[0,L]}(x)$ is a smoothed indicator function of the interval $[0, L]$:

$$I_{[0,L]}(x) = \begin{cases} 1, & \text{if} \quad x \in [0,L], \\ 0, & \text{if} \quad x \notin [-l, L+l], \qquad l << L. \end{cases} \tag{4}$$

Equations (2–4) formally coincide with the collinear case of more general equations with two spatial variables, analyzed by Raman and Nath [20]. Their results give a theoretical background of acoustooptics for more than 50 years [19, 21]. Nevertheless for our purposes they are not sufficient. First, a collinear acoustooptic interaction was always considered with anisotropic materials, which is not the case for fused silica. Second, the approximations involved therein are not satisfactory for us: they explicitly require that the wave vectors of light and elastic waves are NOT collinear [18, 19]. Finally, the resulting expression for reflected intensity [19] shows a periodic dependence on the length of the truncated periodic segment. As it was shown in [22, 23] another formula is more relevant in zones of high reflection, corresponding to spectral lacunae of the infinite system.

Following this motivation we have performed a theoretical attack at equations (2–4). In Appendix 1 we briefly recall our results for static nanostructures [22, 23], modified here for wave equation. In Section 2 we transform the initial problem (2–4) to the form where these results would be useful. Namely, we introduce new variables (ξ, τ) which corresponds to the coordinate system accompanying the running wave (1). This is exactly the Lorents transform with velocity $c_m = \frac{\Omega}{K}$ of the elastic wave in fiber. As a result we reduce the initial system (2–4) of partial differential equations to a spectral problem for an ordinary differential operator.

In Section 3 following the approach [22, 23] we calculate the Lyapunov function (Appendix 2) and estimate the position and size of spectral gaps for infinite periodic structure. Quite naturally these lacunae correspond to the Bragg rule. Next we consider scattering by a segment of running wave of length L. Neglecting the influence of the segment's ends we obtain explicit expression for the reflection coefficient in spectral band. For the first order of interferention we derive the formula that gives the maximal value of the reflection coefficient R for the bandwidth $\Delta\nu(Hz)$ and the length of interaction L:

$$|R_{max}| = \tanh\,(\pi \frac{L}{c} \Delta\nu).$$

2 Transformation of the collinear a/o equation

In this section we use small parameters $\varepsilon << 1$ and $c_m/c << 1$ for transformation of the effective a/o equation (3) to a more convenient form.

Let us make the Lorents transformation into the coordinate system (ξ, τ) that accompanies the running wave, i.e. that moves with velocity c_m:

$$\xi = \frac{x - c_m t}{\sqrt{1 - c_m^2/c^2}}, \qquad \tau = \frac{t - (c_m/c^2)x}{\sqrt{1 - c_m^2/c^2}}; \qquad \psi(x,t) = \tilde{\psi}(\xi, \tau).$$

It modifies the transformation suggested in the previous report [24] and turns out to be more convenient.

After this change of variables the equation (3) acquires the form ($I \equiv 1$)

$$c^2 \frac{\partial^2 \tilde{\psi}}{\partial \xi^2} - \frac{\partial^2 \tilde{\psi}}{\partial \tau^2} = \frac{\varepsilon}{1 - c_m^2/c^2} \left(\frac{\partial}{\partial \tau} - c_m \frac{\partial}{\partial \xi} \right)^2 \left\{ \tilde{\psi} \cos K\xi \right\}.$$

Considering the harmonic solution $\tilde{\psi}(\xi, \tau) = e^{i\omega\tau} \tilde{\psi}(\xi)$ we have

$$c^2 \tilde{\psi}''(\xi) + \omega^2 \tilde{\psi}(\xi) = \frac{\varepsilon}{1 - c_m^2/c^2} \left(i\omega - c_m \frac{d}{d\xi} \right)^2 \left\{ \tilde{\psi} \cos K\xi \right\}.$$

In order to simplify it let us consider in details its right-handside:

$$\frac{\varepsilon}{1 - c_m^2/c^2} \left(c_m^2 \frac{d^2}{d\xi^2} (\tilde{\psi} \cos K\xi) - 2ic_m\omega \frac{d}{d\xi} (\tilde{\psi} \cos K\xi) - \omega^2 \tilde{\psi} \cos K\xi \right).$$

Then let us put the item with $\frac{d^2}{d\xi^2}$ into the left-hand side of (2):

$$\frac{d^2}{d\xi^2} \left[c^2 \tilde{\psi} - \frac{\varepsilon c_m^2}{1 - c_m^2/c^2} \tilde{\psi}(\xi) \cos K\xi \right] = -\frac{\varepsilon}{1 - c_m^2/c^2} \left\{ \omega^2 \tilde{\psi} \cos K\xi + 2ic_m\omega \frac{d}{d\xi} (\tilde{\psi} \cos K\xi) \right\}$$

and introduce the new function

$$\phi(\xi) = [1 - \varepsilon \frac{c_m^2/c^2}{1 - c_m^2/c^2} \cos K\xi] \tilde{\psi}(\xi)$$

which results in the equation

$$c^2 \frac{d^2\phi}{d\xi^2} + \frac{\omega^2 \phi}{1 - \varepsilon \frac{c_m^2/c^2}{1 - c_m^2/c^2} \cos K\xi} =$$

$$-\frac{\varepsilon}{1 - c_m^2/c^2} \left\{ \frac{\omega^2 \phi \cos K\xi}{1 - \varepsilon \frac{c_m^2/c^2}{1 - c_m^2/c^2} \cos K\xi} + 2ic_m\omega \frac{d}{d\xi} \frac{\phi \cos K\xi}{1 - \varepsilon \frac{c_m^2/c^2}{1 - c_m^2/c^2} \cos K\xi} \right\}.$$

Following the standard acoustooptic approximation [19], we neglect the items $O(\varepsilon^2)$ which gives

$$c^2 \frac{d^2\phi}{d\xi^2} + \omega^2 \phi = -\varepsilon \omega^2 \frac{1 + c_m^2/c^2}{1 - c_m^2/c^2} \left\{ \phi \cos K\xi - \frac{2i}{1 + c_m^2/c^2} \frac{c_m}{\omega} \frac{d}{\partial \xi} (\phi \cos K\xi) \right\}.$$

Since we will use the perturbation with respect to ε, we have $\phi \sim e^{\pm i\omega/c\xi} + O(\varepsilon)$, which gives

$$\frac{c_m}{\omega} \frac{d\phi}{\partial \xi} \approx \{ (\pm i\omega/c) \cos K\xi - K \sin K\xi \} \phi \frac{c_m}{\omega}$$

and since $K < \omega/c$, $c_m/c \ll 1$ we can neglect the corresponding item in (2). Thus we have

$$c^2 \frac{d^2\phi}{d\xi^2} + \omega^2\phi = -\varepsilon\omega^2 \frac{1 + c_m^2/c^2}{1 - c_m^2/c^2} \phi \cos K\xi$$

and using the relation $c_m/c \ll 1$ once again we obtain

$$c^2\phi'' + \omega^2\phi = -\varepsilon\omega^2\phi \cos K\xi \tag{5}$$

which is essentially the Mathieu equation [25]. However the small parameter ε participates in the equation in a non-standard way: introducing the variable $y(K\xi/2) = \phi(\xi)$, $s = K\xi/2$ we have the equation (5) in the form

$$y'' + (acos2s + b)y = 0, \qquad a = \varepsilon \left(\frac{\omega/c}{K/2}\right)^2, \quad b = \left(\frac{\omega/c}{K/2}\right)^2.$$

Classical approximation holds when $a \to 0$ [26], whereas in our situation $\frac{\omega/c}{K} = \frac{\Lambda}{\lambda}$ may be rather large (in [5] it is of the order of 10^3).

In order to use the small parameter ε we re-write (5) in the variables

$$\eta = \frac{\omega}{c}\xi, \qquad \chi(\eta) = \phi(\xi)$$

and it acquires the form

$$-\chi'' = \{1 + \varepsilon \cos \frac{\eta}{\kappa}\}\chi,$$

where $\kappa = \frac{\omega/c}{K} = \frac{\Lambda}{\lambda}$ equals the ratio of the elastic wave length to the electromagnetic one. Now the small parameter ε is in a convenient form for the perturbation theory.

We will proceed as following: for infinite ($-\infty < x < \infty$) problem we will calculate the Lyapunov function and find the position and size of spectral lacunae. However scattering by a finite segment of running wave requires certain accuracy of treatment. Indeed, for infinite structure the spectral properties in the accompanying coordinate system (ξ, τ) are essentially the same as in the laboratory system (x, t). When the running wave is present in a finite segment, however, the transition to the accompanying system is more complicated. First, we have evident Doppler shift of the frequency for the falling and reflected waves. Second, the edge of the periodically perturbed segment moves to the point of observation: if $I_{[\xi_1,\xi_2]}(\xi)$ is the indicator function of the segment $[\xi_1, \xi_2]$ in the ξ coordinate, the equation for scattering by the segment is

$$c^2\phi'' + \omega^2\phi = -\varepsilon\omega I_{[c_m t, L+c_m t]}(\xi)\phi \cos K\xi.$$

That is the structure of the inhomogeneity is in rest in the (ξ, τ) coordinate system and only its boundaries move with respect to the observation point. However we can reasonably neglect the effect of this motion since the perturbation is small and the edge effects are not essential. In experimental setting the edges are smoothed (because elastic waves decay gradually) which also reduces their influence.

Thus for scattering by a finite segment we will use the system

$$c^2\phi'' + \omega^2\phi = -\varepsilon\omega^2 I_{[0,L]}(\xi)\phi\cos K\xi$$

with evident continuity conditions on the edges $\xi = 0$ and $\xi = L$ of the truncated periodic perturbation. This form allows us to apply the approach developed in the previous section.

3 Transmission coefficient on the lacunae

3.1 The spectrum of the infinite problem

Having the expression (20) for $F(\omega)$ we can find the positions of lacunae, their size and the approximate quasimomentum in the lacunae. Outside these bands reflection is negligible.

Lemma The lacunae of the infinite problem (5) in the first order of the small parameter ε, satisfying

$$\left(\frac{\tilde{\varepsilon}}{4}\right)^2 \frac{\Lambda}{\lambda} << 1, \tag{6}$$

form two series. The first one corresponds to an elastic wavelength equal to an even number of the light half-wavelengths, $n\lambda \approx \Lambda$:

$$\Delta^{2n} = \{\omega : \omega \in [ncK, ncK + \Delta\omega_{2n}]\}, \qquad n = 1, 2, ..., \tag{7}$$

where the spectral width of the lacuna is

$$\Delta\omega_{2n} = 2ncK\frac{\varepsilon^2/4}{1 - 1/(2n)^2}. \tag{8}$$

In the second series the elastic wavelength equals to an odd number of the light half-wavelengths, $(n + 1/2)\lambda \approx \Lambda$:

$$\Delta^{2n+1} = \{\omega : \omega \in [(n + 1/2)cK, ncK + \Delta\omega_{2n+1}]\}, \qquad n = 1, 2, ..., \tag{9}$$

where the spectral width of the lacuna is

$$\Delta\omega_{2n+1} = 2(n + 1/2)cK\frac{\varepsilon^2/4}{1 - \frac{1}{4(n+1/2)^2}}. \tag{10}$$

Proof easily follows from direct calculations.

The case $n = 0$ corresponding to the first order of interferention $\Lambda \approx \lambda/2$ can be calculated by another method. Now the center of lacuna is exactly at $\omega_0 = cK/2$ and its width $\Delta\omega_0 = \varepsilon\pi/2$ which is greater than all other lacunae by $1/\varepsilon$.

Of course we should keep in mind that the spectrum (7–10) refers to the moving coordinate system (ξ, τ); when considering scattering in the laboratory system (x, t) one should take into account the Doppler shift whose sign depends on the position of the observation point.

Practical notice: For communication purposes it is highly desirable to obtain for the wavelength $\lambda \approx 1\mu m$ the spectral width of the reflection zone $\Delta\omega \sim 10kHz$ [28]. Therefore the relative bandwidth $\frac{\Delta\nu}{\nu} \approx \frac{10kHz}{c/\lambda} \approx 3 \cdot 10^{-9}$ and for $n \geq 1$ $\varepsilon \approx 10^{-5}$. However for the first order of interferention for the same bandwidth $\varepsilon \approx 10^{-9}$ which allows us to operate with more wide bands or lower acoustic power. This estimate justifies the condition (6) $\left(\frac{\varepsilon}{4}\right)^2 << \frac{\lambda}{\Lambda}$ which is necessary for application of the last lemma. It is a kind of feedback argument which was supposed in advance and then proved to be self-consistent.

3.2 Estimates of the reflection coefficient

As the next step towards the estimate (17) we will find the quasimomentum p from the equation

$$\frac{1}{2}F(\omega) = \cos p\Lambda,$$

which allows to estimate on the bands (7–10) the value $\mu_\pm = e^{\pm ip\Lambda}$ controlling transmission and reflection coefficients.

Introducing for each band Δ_n the central frequency ω_n we write down a frequency ω in the lacuna in the form $\omega = \omega_n + \delta\omega$. Then on a band we have (for all orders of interferention)

$$p = \frac{\pi n}{\Lambda} \pm \frac{i}{c}\sqrt{(\Delta\omega)^2/4 - (\delta\omega)^2}, \qquad n = 1, 2, ..$$

and therefore $e^{\pm ip\Lambda} = e^{\pm(\Lambda/c)\sqrt{(\Delta\omega)^2/4-(\delta\omega)^2}}$. Substituting this expression into the formula (17) estimating the transmission coefficient for scattering by N periods one has

$$|T|^2 \leq e^{\pm 2(N\Lambda/c)\sqrt{(\Delta\omega)^2/4-(\delta\omega)^2}}.$$

Thus in the center of a lacuna we have the maximal reflection

$$|T|^2 \leq e^{-\frac{N\Lambda\Delta\omega_n}{c}}.$$

The estimate shows that for the desired width $\Delta\omega = 10kHz$ estimating the phase velocity $c \approx 2 \cdot 10^8 m/sec$ the length $L = N\Lambda$ of the device required for the reflection $|T|^2 < 1/e$ is

$$L = N\Lambda \approx 2 \cdot 10^4 m,$$

which is evidently an exceedingly high value.

For the first lacuna it is easy to find $|T|^2$ and $|R|^2$ explicitly:

$$|T|^2 = \frac{1 - (\frac{\delta\nu}{\Delta\nu/2})^2}{\cosh^2\left(\gamma\sqrt{1 - (\frac{\delta\nu}{\Delta\nu/2})^2}\right) - (\frac{\delta\nu}{\Delta\nu/2})^2},$$

$$|R|^2 = \frac{\sinh^2\left(\gamma\sqrt{1 - (\frac{\delta\nu}{\Delta\nu/2})^2}\right)}{\cosh^2\left(\gamma\sqrt{1 - (\frac{\delta\nu}{\Delta\nu/2})^2}\right) - (\frac{\delta\nu}{\Delta\nu/2})^2}.$$

Here $\Delta\nu$ is the width of the first band in Herts, $\delta\nu$ is the spectral distance from the center of the band, $-\Delta\nu/2 \le \delta\nu \le \Delta\nu/2$. The shape of the lines is illustrated on Fig.1 for various values of the dimensionless parameter $\gamma = \pi\frac{L}{c}\Delta\nu$.

4 Conclusion

We have analyzed the empirical 1-D equation for light propagation in a single-mode fiber with a vibrationally imposed running periodic structure. The equations formally coincide with those for collinear acoustooptic interaction in isotropic media. We suggested an anzats reducing the initial partial differential equation to an ordinary differential equation with periodic coefficients. The anzats is the key instrument for an analysis of the zone structure and spatial decay properties of stationary solutions. In the lowest order of the acoustooptic coupling constant ε we analyzed the fully periodic problem. This gives the basis for calculations of parameters of scattering by a segment containing N periods, using the methods worked out for nanoelectronic periodic structures.

In particular, we calculated the reflection in higher orders of Bragg interferention. The strength of reflection has the same relation to the width of the reflection bandwidth as in the first order of the interferention. The only difference between the first and the higher orders is the dependence from the coupling parameter. Therefore a reflecting Bragg filter may be based on higher orders of Bragg interferention as well as on the first one. The first order has the advantage of low acoustic power required for modulation. Higher orders have the advantage of easier implementation for long segments of fiber up to a few meters.

5 Acknowledgments

The authors thank the Commission of European Communities for financial support in the frame of EC-Russia Exploratory Collaborative Activity under EU ESPRIT Project 28890 NTCONGS.

Appendix 1. Scattering by a segment in terms of infinite periodic problem

In this section we will outline the interconnection between the solutions of the fully periodic problem (2) and scattering by a segment $[0, L]$ of the same periodic structure of finite length. Following the approach of [23] we will make it in a general form, suitable for the equation

$$-\psi''(x) = \frac{\omega^2}{c^2} n^2(x)\psi(x) I_{[0,Na]}(x), \tag{11}$$

where $n^2(\xi)$ is a periodic refraction coefficient, $n(\xi + a) = n(\xi)$, and the segment contains exactly N periods. In the following sections we will use its specific form $n^2(\xi) = 1 + \varepsilon \cos K\xi$.

Suppose that from the right-hand side a wave e^{-ikx} of frequency ω falls onto the scattering region. Then the solution may be represented as

$$\psi = \begin{cases} Te^{-ikx}, & x \le 0, \\ \alpha\chi_+ + \beta\chi_-, & 0 \le x \le Na, \\ e^{-ikx} + Re^{ikx}, & Na \le x, \end{cases} \tag{12}$$

where T and R stand for the transmission and reflection coefficient respectively and $\chi_\pm$ denotes the Weyl solutions for fully periodic problem

$$-c_0^2 n^2(x) \frac{d^2\psi(x)}{dx^2} = \omega^2 \psi(x); \tag{13}$$

here $n(x + a) = n(x)$ holds for all $x \in \mathbf{R}$. $\chi_\pm$ denotes square integrable solutions of (13) on the right $(+)$ or left $(-)$ semiaxis for complex ω. We have according to the Floquet theorem:

$$\chi_\pm(x + a, \omega) = e^{\pm ipa}\chi_\pm(x, \omega), \qquad e^{\pm ipa} = \mu_\pm, \qquad \Im p \ge 0.$$

The Weyl solutions are connected with the standard basis of solutions of Eq. (13) on $[0, a]$:

$$\begin{cases} \vartheta(0, \omega) = 1, \\ \vartheta'(0, \omega) = 0, \end{cases} \qquad \begin{cases} \varphi(0, \omega) = 0, \\ \varphi'(0, \omega) = 1 \end{cases} \tag{14}$$

through the Weyl functions $m_\pm$,

$$\chi_+ = \vartheta + m_+\varphi, \qquad \chi_- = \vartheta + m_-\varphi,$$

where θ, φ are real. Our notations follow the notations in [23]. If the periodic structure (13) admits light propagation with frequency ω then ω belongs to *spectral band*, p is real, $|\mu_\pm| = 1$, $\overline{\mu_+} = \mu_-$, $\overline{\chi_+} = \chi_-$ and therefore $\overline{m_+} = m_-$.

If propagation is impossible then (real) ω is in a *spectral lacuna*, p is purely imaginary, $\mu_+ = \mu_-^{-1}$ is real so that $\overline{\chi_\pm} = \chi_\pm$ and $\overline{m_\pm} = m_\pm$.

Now let us calculate in terms of m and μ the transmission coefficient for the problem (11–12) and show that in a spectral lacuna of Eq. (13) the transmission coefficient is exponentially small with respect to N, that is, $|T| \sim e^{-\gamma N}$ where γ is positive.

The condition of continuity at $x = 0$ gives

$$\begin{cases} T = \alpha + \beta, \\ -ikT = \alpha m_+ + \beta m_-. \end{cases} \tag{15}$$

The condition at $x = aN$ becomes

$$\begin{cases} e^{-ikNa} + Re^{ikNa} = \alpha\mu_+^N + \beta\mu_-^N, \\ -ike^{-ikNa} + ikRe^{ikNa} = \alpha m_+\mu_+^N + \beta m_-\mu_-^N. \end{cases} \tag{16}$$

Eq. (15) gives

$$\frac{\alpha}{\beta} = -\frac{m_- + ik}{m_+ + ik},$$

and Eq. (16) implies

$$\beta = \frac{2ike^{-ikNa}(ik + m_+)}{\mu_-^N(ik - m_-)(ik + m_+) - \mu_+^N(ik - m_+)(ik + m_-)},$$

$$\alpha = -\frac{2ike^{-ikNa}(ik + m_-)}{\mu_-^N(ik - m_-)(ik + m_+) - \mu_+^N(ik - m_+)(ik + m_-)}.$$

Therefore, we have for $T = \alpha + \beta$

$$T = \frac{2ike^{-ikNa}(m_+ - m_-)}{\mu_-^N(ik - m_-)(ik + m_+) - \mu_+^N(ik - m_+)(ik + m_-)}.$$

On the spectral band of Eq. (13) $\overline{\chi_+} = \chi_-$, p is real, $\mu_\pm = e^{\pm ipa}$, $\overline{m_+} = m_-$. Thus

$$T = \frac{4k^2e^{-ikNa}\Im m_+}{\mu_-^N|ik + m_+|^2 - \mu_+^N|ik - m_+|^2},$$

and

$$|T|^2 = \frac{4k^2|\Im m_+|^2}{(k^2+|m_+|^2)\sin^2 Npa + 4k^2|\Im m_+|^2\cos^2 Npa}.$$

In a spectral lacuna $|\mu_+| < 1$, $\mu_+ = \mu_-^{-1}$ is real, $m_\pm$ are real and

$$T = \frac{ie^{-ikNa}}{\mu_+^N \frac{(m_+-ik)(m_-+ik)}{2k(m_+-m_-)} - \mu_-^N \frac{(m_--ik)(m_++ik)}{2k(m_+-m_-)}}.$$

If the perturbation is symmetrical then $\theta(a,\omega) = \varphi(a,\omega)$, therefore $m_+ = -m_-$ and

$$|T|^2 = \frac{4}{(\mu_+^N + \mu_-^N)^2 + \frac{1}{4}(m_- - \frac{1}{m_-})^2(\mu_+^N - \mu_-^N)^2}.$$

For sufficiently large N: $\mu_+^N << \mu_-^N \to \infty$ and

$$|T|^2 \le \frac{1}{\mu_-^{2N}} = e^{-2\gamma N}, \qquad \text{where} \quad |\mu_-| = e^\gamma \ge 1. \tag{17}$$

Thus the main proposition of the Appendix is proved.

Appendix 2. Lyapunov function in ε^2-approximation

We will use for the spectral analysis the standard dispersion equation [27]

$$F(\varepsilon,\kappa) = 2\cos\Lambda p, \tag{18}$$

where F is the Lyapunov function which can be expressed as

$$F = \vartheta(2\pi\kappa) + \varphi'(2\pi\kappa). \tag{19}$$

Let us calculate them up to the terms proportional to ε^2.

Lemma. The relation $\cos K(\Lambda/2 - \xi) = \cos K\xi$ implies that $\vartheta(2\pi\kappa) = \varphi'(2\pi\kappa)$.

For ϑ we have the following integral equation

$$\vartheta(\eta) = \cos\eta - \varepsilon\int_0^\eta \sin(\eta - s)\cos\frac{s}{\kappa}\vartheta(s)\,ds,$$

which we will solve by iterations: the zero iteration is

$$\vartheta_0(\eta) = \cos\eta,$$

and the first and the second iterations are

$$\vartheta_n(\eta) = \cos\eta - \varepsilon\int_0^\eta \sin(\eta - s)\cos\frac{s}{\kappa}\vartheta_{n-1}(s)\,ds,$$

where $n = 1, 2$. Thus

$$\vartheta(\eta) = \cos\eta - \varepsilon F_1(\eta) + \varepsilon^2 F_2(\eta),$$

where

$$F_1(\eta) = \int_0^\eta \sin(\eta - s)\cos\frac{s}{\kappa}\cos s\, ds =$$

$$-\frac{\kappa^2}{4\kappa^2 - 1}\cos\eta + \frac{\kappa^2/2}{2\kappa - 1}\cos\eta(1 - 1/\kappa) - \frac{\kappa^2/2}{2\kappa + 1}\cos\eta(1 + 1/\kappa),$$

$$F_2(\eta) = \int_0^\eta \sin(\eta - s)\cos\frac{s}{\kappa}F_1(s)\, ds =$$

$$-\frac{\kappa^2}{4\kappa^2 - 1}\int_0^\eta \sin(\eta - s)\cos\frac{s}{\kappa}\cos s\, ds + \frac{\kappa^2/2}{2\kappa - 1}\int_0^\eta \sin(\eta - s)\cos\frac{s}{\kappa}\cos s(1 - 1/\kappa)\, ds -$$

$$\frac{\kappa^2/2}{2\kappa + 1}\int_0^\eta \sin(\eta - s)\cos\frac{s}{\kappa}\cos s(1 + 1/\kappa)\, ds.$$

Let us introduce the notation

$$I_\kappa(\eta) = \int_0^\eta \sin(\eta - s)\cos\frac{s}{\kappa}\cos s(1 - 1/\kappa)\, ds.$$

Many simple calculations give for I_κ

$$I_\kappa(\eta) = \frac{\eta\sin\eta}{4} + \frac{\kappa^2/8}{\kappa - 1}\left(-\cos\eta + \cos\eta(1 - 2/\kappa)\right).$$

Therefore for F_2

$$F_2(\eta) = -\frac{\kappa^2}{4\kappa^2 - 1}F_1(\eta) + \frac{\kappa^2/2}{2\kappa - 1}I_\kappa(\eta) - \frac{\kappa^2/2}{2\kappa + 1}I_{-\kappa}(\eta).$$

For the Lyapunov function we will need $\vartheta\Big|_{\eta=2\pi\kappa}$ so we will use the formulae

$$F_1(2\pi\kappa) = 0, \qquad I_\kappa(2\pi\kappa) = \frac{2\pi\kappa}{4}\sin 2\pi\kappa,$$

and

$$F_2(2\pi\kappa) = \frac{\kappa^2/2}{4\kappa^2 - 1}\pi\kappa\sin 2\pi\kappa.$$

Therefore

$$F(2\pi\kappa) = \vartheta(2\pi\kappa) = \cos 2\pi\kappa - \varepsilon F_1\big|_{\eta=2\pi\kappa} + \varepsilon^2 F_2\Big|_{\eta=2\pi\kappa} =$$

$$\cos 2\pi\kappa + \frac{\varepsilon^2\kappa^2/2}{4\kappa^2 - 1}\pi\kappa\sin 2\pi\kappa. \qquad (20)$$

References

[1] R.C.Youngquist, J.L.Brooks, H.J.Shaw. Birefringent-fiber polarization coupler. Optics Lett., V.8, No.12, pp.656–658, 1983.

[2] W.P.Risk, R.C.Youngquist, G.S.Kino et. al.Acousto-optic frequency shifting in birefringent fiber. .Lett., V.9, No.7, 1984.

[3] S.G.Farwell, D.O.Culverhouse, T.A.Birks, P.St.J.Russell. Low-loss all-fibre amplitude modulator at 1,55μm. Electron.Lett., V.32, No.6, p.577, 1996.

[4] W.F.Lieu, P.St.J.Russell, L.Dong. Acousto-optic superlattice modulator using a fiber Bragg grating. Optic Letters, V.22, No.19, pp.1515–1517, 1997.

[5] H.S.Kim, S.H.Yun, I.K.Kuang et.al. All-fiber acousto-optic tunable notch filter with electronically controllable spectral profile. Opt.Lett., V.22, No.19,pp.1476–1478, 1997.

[6] Yu.V.Gulyaev, M.Ya.Mesh, V.V.Proklov. Modulation effects in fiber waveguides and their applications. Moscow, Radio i Svaz, 1991.

[7] A.Korpel. Acousto-optics. Moscow, mir, 1993.

[8] D.O.Culverhouse, S.H.Yun, D.J.Richardson et al. Low-loss all-fiber acousto-optic tunable filter. Opt.Lett., V.22, No.2, pp.96–98, 1997.

[9] H.S.Chung, D.Lee, S.J.Ahn et.al. Demonstration of 37nm gain bandwidth from 40 cascaded EDFA system using gain equalizer based on all-fibre acousto-optic tunable filters. Electron.Lett., V.34, No.21, p.2045, 1998.

[10] A.Iocco, H.G.Limberger, R.P.Salathe. Bragg grating fast tunable filter. Electron.Lett., V.33, No.25, pp.2147–2148, 1997.

[11] M.M.Ohn, A.T.Alavie, R.Maaskant et.al. Dispersion variable fibre Bragg grating using a piezoelectric stack. Electron.Lett., V.32, No.21, pp.2000–2001, 1996.

[12] R.M.Shelby, M.D.Levinson, P.W.Bayer. Guided acoustic-wave Brillouin scattering. Phys.Rev.B, 1985, V.31, pp.5244–5252.

[13] P.J.Thomas et.al. Normal acoustic modes and Brillouin scattering in single-mode acoustic fiber. Phys.Rev.B, 1979, V.19, pp.4986–4998.

[14] N.Shibata, K.Okamoto, Y.Azuma. Longitudinal acoustic modes and Brillouin-gain spectra for GeO_2-doped-core single-mode fibers. J.O.S.A B, V.6, No.6, pp.1167–1174, 1989.

[15] M.Ohashi, N.Shibata, K.Shiraki. Fibre diameter estimation based on guided acoustic wave Brillouin scattering. Electron.Lett., V.28, No.10, pp.900–902, 1992.

[16] R.W.Tkach, A.R.Chaplyvy, R.M.Derosier. Spontaneous Brillouin scattering for single-mode optic fiber characterization. Electron.Lett., V.22, No.19, pp.1011–1013, 1986.

[17] K.Tajima. Exact acoustic leaky wave solutions for single-mode fibers. Electron.Lett., V.27, No.3, pp.251–253, 1991.

[18] L.N.Magdich, V.Ya.Molchanov. Acoustooptical devices and their applications. Moscow, Soviet Radio, !978.

[19] V.I.Balakshii, V.N.Parygin, L.E.Chirkov. Physical Foundations of Acoustooptics. Moscow, Radio & Communication, 1985.

[20] V.C.Raman, N.S.Nath. The diffraction of light by high frequency sound waves. Proc.Ind.Acad.Sci., V.2A, pt.1, 406–412(1935); V.2A, pt.2, 413–420(1935); V.3A, pt.3, 75–84(1936;) V.3A, pt.4,119–125(1936); V.3A, pt.5,

459–469(1936).

[21] M.Born, E.Wolf. Foundations of Optics. Pergamon Press, 1956.

[22] B.Pavlov, N.Smirnov. Resonance scattering by one-dimensional crystal and thin film. Vestnik of the Leningrad University, V.13, No.3, 1977 (in Russian).

[23] B.Pavlov, G.Roach, A.Yafyasov. Landauer formula and forming of spectral bands. University of Auckland, Dept. of Mathematics, Report series 389, September 1997, New Zealand.

[24] B.S.Pavlov, A.A.Pokrovski, L.V.Prokhorov. Light Propagation in Truncated Periodic Waveguide. Progress Report (July 1, 1998 – December 31, 1998) ESPRIT Project 28890 NTCONGS.

[25] E.T.Whittaker, G.N.Watson. A course of Modern Analysis. V.2, Cambridge, 1927.

[26] E.Kamke. Differential Gleichungen. Leipzig, 1959.

[27] B.M.Levitan, I.S.Sargsan. Operators of Stourm-Liouville and Dirac. Moscow, Nauka, 1988 (in Russian).

[28] M.S.Klimov, A.S.Svakhin, V.A.Sychugov. Technology and some applications of corrugated structures in integrated optics. In: *Waveguide corrugated structures in integrated and fiber optics.* (Proc. IOFAN V.34) Moscow, Nauka,1991 (in Russian).