Developable Spaces and Problems of Fletcher and Lindgren*

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Abstract

In this paper, we answer two questions of P. Fletcher and W. Lindgren [1]. We prove that a space $X$ is $w\Delta$ and has a quasi-$G^*_\delta$-diagonal if and only if it is developable, a space $X$ is $\beta$-space with a quasi-$G^*_\delta$-diagonal if and only if it is semi-stratifiable, a space $X$ is $\beta$, quasi-$\gamma$-space and has a quasi-$G^*_\delta$-diagonal if and only if $X$ is developable and a space $X$ is metrizable if and only if it is paracompact $\beta$-space with a quasi-$G^*_\delta$-diagonal.

1 Introduction

In this brief note we present some conditions which imply developability, and consequently we give full positive answers for two questions of Fletcher and Lindgren [1]: is every quasi-developable space $c$-semi-stratifiable and is every quasi-developable $\beta$-space developable?

In [12], the author makes it possible to factorize quasi-developability into two parts: a space $X$ is quasi-developable if and only if it is a quasi-$w\Delta$-space with a quasi-$G^*_\delta$-diagonal. This result plays an important role in getting the results in this paper.

A COC-map (= countable open covering map) for a topological space $X$ is a function from $N \times X$ into the topology of $X$ such that for every $x \in X$ and $n \in \mathbb{N}$, $x \in g(n, x)$ and $g(n + 1, x) \subseteq g(n, x)$. It is well known that many important classes of generalized metrizable spaces can be characterized in

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terms of a COC-map. In particular, \( X \) is developable \([4](w\Delta\text{-space})\) if and only if \( X \) has a COC-map \( g \) such that if \( \{p, x_n\} \subseteq g(n, y_n) \) for all \( n \), then \( \langle x_n \rangle \) converges to \( p \) (then \( \langle x_n \rangle \) has a cluster point).

A space \( X \) is called quasi-\( \gamma \) \([9]\) if and only if \( X \) has a COC-map \( g \) such that if \( x_n \in g(n, y_n) \) for each \( n \in \mathbb{N} \), and the sequence \( \langle y_n \rangle \) converges in \( X \), then the sequence \( \langle x_n \rangle \) has a cluster point; a space \( X \) is called semi-stratifiable \([6]\) \((\beta\text{-space} \[5]\)) if and only if \( X \) has a COC-map \( g \) such that if for each \( x \in g(n, x_n) \) for each \( n \in \mathbb{N} \) then \( x \) is a cluster point of \( \langle x_n \rangle \) (\( \langle x_n \rangle \) has a cluster point).

Let \( \mathcal{G} = \{\mathcal{G}_n\}_{n \in \mathbb{N}} \) be a sequence of open families of \( X \). Define \( c(x) = \{n : x \in \mathcal{G}_n^* = \bigcup\{G : G \in \mathcal{G}_n\}\} \). A space \( X \) has a quasi-\( G_\delta^s \)-diagonal \([12]\) if there is such a sequence \( \mathcal{G} \) such that for any distinct \( x, y \in X \), there exists \( n \in \mathbb{N} \) such that \( x \in \text{st}(x, \mathcal{G}_n) \subset X - \{y\} \); a space \( X \) is called a quasi-\( w\Delta\)-space \([12]\) if \( X \) has such a sequence \( \mathcal{G} \) such that

1. for all \( x \), \( c(x) \) is infinite,
2. if \( \langle x_n \rangle \) is a sequence with \( x_n \in \text{st}(x, \mathcal{G}_n) \) for all \( n \in c(x) \) then \( \langle x_n \rangle \) has a cluster point.

A space \( X \) is called \( c \)-semi-stratifiable \([9]\) if there is a sequence \( \langle g(n, x) \rangle \) of open neighborhoods of \( x \) such that for each compact set \( K \subset X \), if \( g(n, K) = \bigcup\{g(n, x) : x \in K\} \) then \( \cap\{g(n, K) : n \geq 1\} = K \). The COC-map \( g : \mathbb{N} \times X \to \tau \) is called a \( c \)-semi-stratification of \( X \). All spaces will be regular, unless we state otherwise.

# 2 Main Results

**Theorem 2.1** A space \( X \) is \( \beta \)-space with a quasi-\( G_\delta^s \)-diagonal if and only if it is semi-stratifiable.

**Proof.** The necessity of the condition is obvious because every regular semi-stratifiable space is \( \beta \) and has a \( G_\delta^s \)-diagonal. To prove the sufficiency of the condition, let \( \{\mathcal{V}_n : n \in \mathbb{N}\} \) be a quasi-\( G_\delta^s \)-diagonal sequence of \( X \) and let \( g : \mathbb{N} \times X \to \tau \) be a \( \beta \)-map of \( X \). Set \( c_{\mathcal{V}}(x) = \{n : \text{st}(x, \mathcal{V}_n) \neq \emptyset\} \). Then \( \cap_{n \in c_{\mathcal{V}}(x)} \text{st}(x, \mathcal{V}_n) = \{x\} \). Let \( \mathcal{F} \) denote the non-empty finite subsets of \( \mathbb{N} \), and for \( F \in \mathcal{F} \) put

\[
\mathcal{G}_F = \bigcap_{i \in F} \mathcal{V}_i \in \mathcal{V}_i.
\]

For \( n \in \mathbb{N} \) and \( F \in \mathcal{F} \), set \( F_n(x) = c_{\mathcal{V}}(x) \cap \{1, 2, \ldots, n\} \). Then \( \text{st}(x, \mathcal{G}_{F_n}) \subseteq \text{st}(x, \mathcal{V}_m) \) for each \( n \in \mathbb{N} \), each \( F \in \mathcal{F} \) and each \( m \in F \). Put \( d(x) = \{F_n(x) : \)
\[ n \in \mathbb{N} \}. \text{ Note that } d(x) \subseteq c_G(x). \text{ Since } c_V(x) \text{ is infinite, } d(x) \text{ is infinite.}

Because \( F_m \subseteq F_n \) for \( m \geq n \), \( st(x, G_{F_m}) \subseteq st(x, G_{F_n}) \) for \( m \geq n \).

Define a map \( h : \mathbb{N} \times X \rightarrow \tau \) by

\[
h(n, x) = \begin{cases} 
g(n, x) \cap st(x, G_{F_n}) & \text{if } x \in G_{F_n}^*, 
g(n, x) & \text{if } x \notin G_{F_n}^*. 
\end{cases}
\]

We prove that \( h(n, x) \) is a semistratifiable-map. Let \( x \in h(n, x_n) \). It is clear that \( h \) is a \( \beta \)-map, so, \( (x_n) \) has a cluster point, say \( p \). Suppose that \( x \neq p \). Choose \( k \) large enough that \( x \in st(x, V_k) \) but \( p \notin st(x, V_k) \).

For each \( n \geq k \), we have \( k \in F_n \) so

\[
x_n \in h(n, x) \subseteq st(x, G_{F_n}) \subseteq st(x, V_k).
\]

Thus the open neighborhood \( X - st(x, V_k) \) of \( p \) contains at most \( k - 1 \) members of the sequence \( \langle x_n : n \in \mathbb{N} \rangle \), which contradicts the fact that \( p \) is a cluster point of \( (x_n) \).  

\textbf{Corollary 2.2} Every quasi-developable \( \beta \)-space is developable.

\textbf{Proof.} This follows from [12, Theorem 3.1 (a space is a quasi-developable if and only if it is a quasi-\( w \Delta \)-space with quasi-\( G_\delta^* \)-diagonal)], Theorem 2.1 and since every semistratifiable space is perfect the proof done.  

\textbf{Corollary 2.3} A space \( X \) is \( w \Delta \) and has a quasi-\( G_\delta^* \)-diagonal if and only if it is developable.

\textbf{Proof.} This follows from [12, Theorem 3.1], Corollary 2.2 and since every \( w \Delta \)-space is \( \beta \)-space.  

From [12, Corollary 3.4 (Every paracompact \( w \Delta \)-space with quasi-\( G_\delta \)-diagonal is metrizable)], and Theorem 2.1 we have the following metrization result:

\textbf{Corollary 2.4} A space \( X \) is metrizable if and only if it is paracompact \( \beta \)-space with a quasi-\( G_\delta \)-diagonal.

\textbf{Lemma 2.5} Let \( X \) be a space with a quasi-\( G_\delta^* \)-diagonal sequence. Then \( X \) has a quasi-\( G_\delta^* \)-diagonal sequence \( \langle G_n : n \in \mathbb{N} \rangle \) such that for each \( x \in X \) there is an infinite subset \( d(x) \subseteq c_G(x) \) such that if \( x_n \in st(x, G_n) \) for each \( n \in d(x) \) then \( (x_n) \) either clusters at \( x \) or it does not cluster at all.
Proof. Let \( \langle \mathcal{H}_n : n \in \mathbb{N} \rangle \) be a quasi-\( \mathcal{G}_d^* \)-diagonal sequence of \( X \). Without loss of generality we may assume that \( c_{\mathcal{H}}(x) \) is infinite for each \( x \in X \) and \( \mathcal{H}_1 = \{X\} \). Let \( \mathcal{F} \) denote the non-empty finite subsets of \( \mathbb{N} \). For each \( F \in \mathcal{F} \) set

\[
\mathcal{G}_F = \{ \bigcap_{i \in F} H_i : H_i \in \mathcal{H}_i \}.
\]

For \( n \in \mathbb{N} \) and \( x \in X \), set \( F_n(x) = c_{\mathcal{H}}(x) \cap \{1, 2, \ldots, n\} \). Put \( d(x) = \{F_n(x) : n \in \mathbb{N}\} \). Note that \( d(x) \subseteq c_\mathcal{G}(x) \). Since \( c_{\mathcal{H}}(x) \) is infinite, \( d(x) \) is infinite. Because \( F_n(x) \subseteq F_m(x) \) for \( m \geq n \), \( st(x, \mathcal{G}_{F_n(x)}) \subseteq st(x, \mathcal{G}_{F_m(x)}) \) for \( m \geq n \).

For each \( n \in \mathbb{N} \) suppose that \( x_n \in st(x, \mathcal{G}_{F_n(x)}) \). Then for \( m \geq n \) we have

\[
x_m \in st(x, \mathcal{G}_{F_m(x)}) \subseteq st(x, \mathcal{G}_{F_n(x)}),
\]

so

\[
\{x_m / m \geq n\} \subseteq st(x, \mathcal{G}_{F_n(x)}).
\]

Since \( \cap_{n \in \mathbb{N}} st(x, \mathcal{G}_{F_n(x)}) = \{x\} \) it follows that either \( \langle x_n \rangle \) clusters at \( x \) or does not cluster at all. \( \blacksquare \)

Remark 2.6 Let \( X \) be a space and \( \langle \mathcal{G}_n : n \in \mathbb{N} \rangle \) a countable family of collections of open subsets of a space \( X \), such that for all \( x \), \( c(x) = \{n \in \mathbb{N} : x \in \mathcal{G}_n^*\} \) is infinite. Consider the following condition on \( \langle \mathcal{G}_n : n \in \mathbb{N} \rangle \): if \( \langle x_n : n \in \mathbb{N} \rangle \) is a sequence with \( x_n \in st(x, \mathcal{G}_n) \) for all \( n \in c(x) \) then \( x \) is a cluster point of \( \langle x_n : n \in \mathbb{N} \rangle \).

For all spaces, this condition is equivalent to the following condition: for each point \( x \in X \) the set \( st(x, \mathcal{G}_n) \) is nonempty for infinitely many \( n \) and the nonempty sets of the form \( st(x, \mathcal{G}_n) \) form a local base at \( x \) for all \( x \in X \). Thus the condition (2) is a characterization of a quasi-developable space.

Theorem 2.7 Every quasi-developable space is a \( c \)-semi-stratifiable space.

Proof. Let \( \langle \mathcal{G}_n : n \in \mathbb{N} \rangle \) be a quasi-development sequence in a space \( X \). Define

\[
g(n, x) = \begin{cases} 
st(x, \mathcal{G}_n) & \text{if } x \in \mathcal{G}_n^*, \\
X & \text{if } x \notin \mathcal{G}_n^*. 
\end{cases}
\]

Let \( h(n, x) = \cap_{i=1}^n g(i, x) \). We prove that \( h(n, x) \) is a \( c \)-semi-stratifiable-map. We claim that \( C = \cap_{n \in \mathbb{N}} h(n, C) \) for any compact \( C \subset X \), where \( h(n, C) = \bigcup_{c \in C} h(n, c) \). As \( \mathcal{G}_1 = \{X\} \) it follows readily that \( C \subset \cap_{n \in \mathbb{N}} h(n, C) \) so it is appropriate concentrate on the reverse inclusion. To prove that, let \( y \in \bigcap h(n, C) \), so \( y \in h(n, c_n) \) for some \( c_n \in C \). Then \( y \in st(c_n, \mathcal{G}_n) \) for infinitely many \( n \in \mathbb{N} \). It follows that \( c_n \in st(y, \mathcal{G}_n) \) for infinitely many \( n \in \mathbb{N} \). From Remark 2.6, \( \langle c_n \rangle \) clusters at \( y \). Hence, \( y \in C \). \( \blacksquare \)
Theorem 2.8 A space $X$ is developable if and only if it is $\beta$, quasi-$\gamma$ and has a quasi-\(G^*_\delta\)-diagonal.

Proof. The necessity of the conditions is obvious. To prove the sufficiency of the conditions, let $f$ be a $\beta$-map and $g$ a quasi-$\gamma$-map of $X$. Define $h(n,x) = f(n,x) \cap g(n,x)$. It is clear that $h$ is a $\beta$ and quasi-$\gamma$-map of $X$. We prove that $h$ is a $w\Delta$-map of $X$. Let $\{x,x_n\} \subset h(n,y_n)$. By the $\beta$-condition, $\langle y_n \rangle$ converges and so by the quasi-$\gamma$-condition $\langle x_n \rangle$ has a cluster point. Thus $h$ is $w\Delta$-map of $X$. From Corollary 2.3, $X$ is a developable space.

Now, it is natural to ask:

Question 2.9 Is every quasi-$w\Delta$-space with $G^*_\delta$-diagonal developable?

We answer this question in negative manner.

Example 2.10 There is a $p$-adic analytic manifold which is separable, submetrizable, quasi-developable, but not perfect ([11, Example 7.4.7]). This example also can serve as a quasi-semi-stratifiable space (see [7] for the definition) which has a $G^*_\delta$-diagonal but which is not semi-stratifiable.

Example 2.11 There is a quasi-developable manifold which has a $G_\delta$-diagonal but not a $G^*_\delta$-diagonal (see [3, Example 2.2]). This example also can serve as a quasi-$w\Delta$ manifold which is not $w\Delta$. (It is not even a $\beta$-manifold).
Figure 1: Relationships between some generalized metric spaces and quasi-$G^*_\delta$-diagonal.

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