The Characterization of Finite Simple Groups with no Elements of Order Six by Their Element Orders

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ABSTRACT. We show that all but $A_6$ non-abelian finite simple groups with no elements of order 6 are characterized by their element orders.

INTRODUCTION

Let $G$ be a finite group and $\pi_e(G)$ the set of orders of all elements in $G$. Then $\pi_e(G)$ is a subset of positive integers $\mathbb{Z}^+$. Let $\Omega$ be a subset of $\mathbb{Z}^+$ and $h(\Omega)$ the number of non-isomorphic finite groups $K$ such that $\pi_e(K) = \Omega$. Then $h(\pi_e(G)) \geq 1$ for any finite group $G$. Following [16] we say that a finite group $G$ is non-distinguishable if $h(\pi_e(G)) = \infty$, and distinguishable if $h(\pi_e(G)) < \infty$. Moreover, $G$ is $k$-distinguishable if $h(\pi_e(G)) = k$; and characterizable if $h(\pi_e(G)) = 1$. From [4], [14, Theorem 1.1] and [5], we have the following propositions.

**Proposition 1.** Suppose a finite group $G$ has an elementary abelian normal subgroup. Then $G$ is non-distinguishable. In particular, a solvable finite group is non-distinguishable.

**Proposition 2.** The following non-abelian finite simple groups are characterizable.

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A CARACTERIZATION OF SOME FINITE SIMPLE GROUPS

(a) Alternating groups $A_n$ for $n \in \{5, 7, 8, 9, 11, 12, 13\}$.
(b) All sporadic simple groups except $J_2$.
(c) Simple groups $L_2(q)$ with $q \neq 9$, Suzuki-Ree groups, $L_3(4)$, $L_3(7)$, $L_3(8)$, $L_4(3)$, $S_4(7)$, $U_3(4)$, $U_4(3)$, $U_6(2)$, $G_2(3)$, $O_{8^-}(2)$ and $O_{10^-}(2)$.

In this paper, we proved that, except for $A_6$, all non-abelian finite simple groups with no elements of order 6 are characterizable, that is we proved the following Theorem.

**Theorem.** Let $G$ be a finite group and $H$ a finite non-abelian simple group with no elements of order 6 such that $H \not\cong A_6$. Then $G \cong H$ if and only if $\pi_e(G) = \pi_e(H)$.

The alternating group $A_6$ is not characterizable, in fact, infinitely many groups share the same element orders with $A_6$ (see [2, Lemma 2]).

In the following, we suppose a group is always a finite group and a simple group is always non-abelian. We denote by $\pi(G)$ the set of prime divisors of the order $|G|$ of $G$. The other notation are standard following from [3] and [13].

1. PRELIMINARY RESULTS AND LEMMAS

The results of the paper depends on the classification of finite simple groups and the following lemmas.

**Lemma 1.1.** Let $G$ be a simple group with no elements of order 6. Then $G$ is isomorphic to one of the following groups:

(a) $L_2(q)$ with $q \not\equiv \pm 1 \pmod{12}$;
(b) $Sz(2^{2n+1})$ with $n \geq 1$;
(c) $L_3(2^n)$ with $n \not\equiv 0 \pmod{6}$;
(d) $U_3(2^n)$ with $n \geq 2$ and $n \not\equiv 3 \pmod{6}$.

**Proof.** See [8] and [9].

**Remark.** From paper [9], a unitary group $U_3(2^n)$ has no elements of order 6 whenever $n \geq 2$ and $n \not\equiv 3, 5 \pmod{6}$. But, in fact, $U_3(2^n)$ with $n \equiv 5 \pmod{6}$ also has no elements of order 6. Indeed, suppose $n = 6k + y$ for some integers $k \geq 0$ and $0 \leq y \leq 5$. Then $2^n + 1 \equiv 2^y + 1 \equiv 0 \pmod{9}$ if and only if $y = 3$, that is $n = 6k + 3$.

From paper [6], $SU_3(2^n)$ has a single conjugacy class of involutions each with a centralizer of order $2^n (2^n + 1)$. If $U_3(2^n)$ with $n \equiv 5 \pmod{6}$ has elements of order 6, it follows
that 3 divides $2^n + 1$, so then 9 must divides $2^n + 1$ (note that $Z(SU_3(2^n))$ is cyclic of order $gcd(3, 2^n + 1)$), which is impossible.

The prime graph \( \Gamma(G) \) of a group \( G \) is the graph whose vertex set is the set \( \pi(G) \) and two vertices \( p, r \in \pi(G) \) are adjacent if and only if \( G \) contains an element of order \( pr \). Denote by \( t(G) \) the number of connected components of \( \Gamma(G) \), and by \( \pi_i = \pi_i(G) \) for \( i = 1, 2, \ldots, t(G) \) the connected components of \( \Gamma(G) \). If \( |G| \) is even, then suppose 2 is a vertex of \( \pi_1 \). We shall also use the following unpublished result of K.W. Gruenberg and O.H. Kegel, (see [19, Theorem A]).

**Lemma 1.2.** If \( G \) is a finite group whose prime graph has more than one component, then \( G \) has one of the following structures: (a) Frobenius or 2-Frobenius; (b) simple; (c) an extension of a \( \pi_1 \)-group by a simple group; (d) simple by \( \pi_1 \); or (e) \( \pi_1 \) by simple by \( \pi_1 \).

A group \( G \) is called 2-Frobenius if there exists a normal series \( 1 \leq H \leq K \leq G \) of \( G \) such that \( H \) is the Frobenius kernel of \( K \) and \( K/H \) is the Frobenius kernel of \( G/H \).

Suppose \( \epsilon = + \) or \( - \) and let \( L_3^\epsilon(q) = L_3(q) \) or \( U_3(q) \) according as \( \epsilon = + \) or \( - \). For simplicity, we always identify \( q - \epsilon \) with \( q - \epsilon L \).

**Lemma 1.3.** Suppose \( q = 2^n \) and \( \eta = \eta_\epsilon = \frac{1}{gcd(3, q-\epsilon)} \). Then
\[
\pi_\epsilon(L_3^\epsilon(q)) = \{1, 2, 4, s : s | (q - \epsilon), s | 2\theta(q - \epsilon), s | \eta(q^2 - 1) \text{ or } s | \eta(q^2 + \epsilon q + 1)\}.
\]

In addition, if \( (q, \epsilon) \in \{(2, +), (4, +)\} \), then \( \pi_1(L_3(q)) = \{2\}, \pi_2(L_3(q)) = \{3\} \) for \( q \in \{2, 4\} \), \( \pi_3(L_3(q)) = \{7\} \) or \( \{5\} \) according as \( q = 2 \) or \( 4 \) and \( \pi_4(L_3(4)) = \{7\} \); if \( (q, \epsilon) \not\in \{(2, +), (4, +)\} \), then \( \pi_1(L_3^\epsilon(q)) = \{r \text{ prime : } r | 2\theta(q - \epsilon) \text{ or } r | \eta(q + \epsilon)\} \) and \( \pi_2(L_3^\epsilon(q)) = \{r \text{ prime : } r | \eta(q^2 + \epsilon q + 1)\} \). In particular, \( t(L_3^\epsilon(q)) \geq 2 \).

**Proof.** The set \( \pi_\epsilon(L_3^\epsilon(q)) \) is determined by calculations from conjugacy \( SL_3^\epsilon(2^n) \)-classes given, say by [6, Tables VII-1 and IX-1], and \( \pi_i(L_3^\epsilon(q)) \) for all \( i \) are given by [12].

2. **THE PROOF OF THE THEOREM**

From Lemma 1.1, if \( H \) is a simple group with no elements of order 6, then \( H \) is isomorphic to \( L_2(q) \), \( Sz(2^{2n+1}) \) or \( L_5^\epsilon(2^n) \) for suitable \( q \) or \( n \). By [1] and [17], \( L_2(q) \) \( (q \neq 9 \) and \( L_2(9) \simeq A_6) \) and \( Sz(2^{2n+1}) \) \( (n \geq 1) \) are characterizable. It suffices to show that \( H = L_5^\epsilon(2^n) \) is characterizable.
Theorem 2.1. Let $G$ be a finite group and $H = L^\epsilon_3(2^n)$ with $n \not\equiv 0 \mod 6$ according as $\epsilon = +$ or $-$. Then $G \simeq H$ if and only if $\pi_e(G) = \pi_e(H)$.

Proof. It is clear that if $G \simeq H$, then $\pi_e(G) = \pi_e(H)$.

Let $q = 2^n$ with $n \not\equiv 0 \mod 6$ according as $\epsilon = +$ or $-$, and suppose $\pi_e(G) = \pi_e(H)$. By Lemma 1.3, $\pi_e(L^\epsilon_3(q)) = \{1, 2, 4, s : s | (q - \epsilon), s | 2q(q - \epsilon), s | q^2 - 1, s | q^2 + q + 1\}$, where $\eta = \frac{1}{\text{gcd}(3, q - \epsilon)}$. Thus $G$ has no elements of order $6$ and $t(G) \geq 2$.

(1). $G$ is non-solvable.

Suppose $G$ is solvable and $K$ is its $\{2, p, r\}$-Hall subgroup, where $p, r \in \pi(G)$ such that $p | q + \epsilon$ and $r | q^2 + q + 1$. Since $G$ has no elements of order $2p$, $2r$ and $pr$, it follows that $K$ is a solvable group all of whose elements are of prime power orders. By [11, Theorem 1], $|\pi(K)| \leq 2$, which is impossible.

(2). $G$ is neither Frobenius nor 2-Frobenius.

Suppose $G$ is a Frobenius group with complement $C$. By the structure of non-solvable Frobenius complements (see [15, Theorem 18.6]), we have that $C$ has a normal subgroup $C^*$ of index $\leq 2$ such that $C^* \simeq SL_2(5) \times Z$, where every Sylow subgroup of $Z$ is cyclic and $\pi(Z) \cap \{2, 3, 5\} = \emptyset$. Since $\pi_e(SL_2(5)) = \{1, 2, \ldots, 6, 10\}$, it contradicts with $6 \not\in \pi_e(G)$. Similarly, $G$ is not 2-Frobenius.

By Lemma 1.2, $G$ has a normal series $1 \leq N < G_1 \leq G$ such that (a) $N$ and $G/G_1$ are $\pi_1$-groups; (b) $\bar{G} = G_1/N$ is simple with $6 \not\in \pi_e(\bar{G})$. We claim that

(3). $\bar{G} \simeq H$.

By Lemma 1.1, $\bar{G}$ is isomorphic to one of the groups, $L_2(q')$ with $q' \neq \pm 1 \mod 12$, $S_2(2^{2m+1})$ with $m \geq 1$, $L^\beta_3(2^n)$ with $n \not\equiv 0 \mod 6$ according as $\beta = +$ or $-$.

Case (i). Suppose $\bar{G} \simeq L_2(q')$ and $q' = 2^m$ with $m \geq 2$. Since the vertexes set of $\pi_i$ is a subset of the set $\pi(G)$ and different vertexes sets are disjoint, each vertex of $\pi_i$ is coprime to any vertex of $\pi_j$ for $i \neq j$ and both $G/G_1$ and $N$ are $\pi_1$-groups, it follows that $\eta(q^2 + q + 1) \in \pi_e(\bar{G})$. Note that $\eta(q^2 + q + 1)$ is the largest or second largest number in the set

$$\{1, 2, 4, (q - \epsilon), 2q(q - \epsilon), \eta(q^2 - 1), \eta(q^2 + q + 1)\}$$

according as $\epsilon = +$ or $-$, and in addition, $G$ and hence $\bar{G}$ have no element $x$ such that $\eta(q^2 + q + 1)$ is a proper divisor of $|x|$. It follows that $\eta(q^2 + q + 1) = q' + 1$ and $\eta(q^2 - q + 1) \in \{q' + 1, q' - 1\}$, which is impossible as both $q$ and $q'$ are powers of 2.
Suppose $q' \equiv \gamma \pmod{4}$ and $q' = p^{m}$, where $p'$ is an odd prime and $\gamma = +$ or $-$. A similar proof to above shows that $\eta(q^2 + \epsilon q + 1) \in \{p', \frac{1}{2}(q' + \gamma)\}$, since $\eta(q^2 + \epsilon q + 1)$ is odd and $\frac{1}{2}(q' - \gamma)$ is even. If $\eta(q^2 + \epsilon q + 1) = p'$, then $m = 1$, since otherwise $\frac{1}{2}(q' + 1) > \frac{1}{2}(q' - 1) > p'$. Thus $\frac{1}{2}(q' + 1) = \frac{1}{2}(q' + 1) = \frac{1}{2}(\eta(q^2 + \epsilon q + 1) + 1) \in \pi_e(G)$.

If $\frac{1}{2}(\eta(q^2 + \epsilon q + 1) + 1) = 2$, then $\eta(q^2 + \epsilon q + 1) = 3$, which is impossible as $3$ is not a divisor of $\eta(q^2 + \epsilon q + 1)$. If $\frac{1}{2}(\eta(q^2 + \epsilon q + 1) + 1) = 4$, then $\eta(q^2 + \epsilon q + 1) = 7 = p'$, so that $\gamma = -$, and $(q, \epsilon) = (2, +)$ or $(4, +)$. In the former case, $L_2(p') = L_2(7)$, which is isomorphic to $H = L_3(2)$. In the latter case, $H = L_3(4)$, $\bar{G} = L_2(7)$ (note that $\pi_e(L_2(7)) = \{1, 2, 3, 4, 7\} \subseteq \pi_e(L_3(4)) = \{1, 2, 3, 4, 5, 7\}$), and by Proposition 2, $L_3(4)$ is characterizable, so that $\pi_e(G) \neq \pi_e(H)$, which is impossible.

If $r \geq 5$ is a common prime factor of $\frac{1}{2}(\eta(q^2 + \epsilon q + 1) + 1)$ and $q - \epsilon$, then $q \equiv \epsilon \pmod{r}$ and $\frac{1}{2}(\eta(q^2 + \epsilon q + 1) + 1) \equiv \frac{3m + 1}{2} \pmod{r}$. But $\frac{3m + 1}{2} \neq 0 \pmod{r}$, so $r$ is not a common factor of $\frac{1}{2}(\eta(q^2 + \epsilon q + 1) + 1)$ and $q - \epsilon$ (respectively, and $2\eta(q - \epsilon)$). Similarly, a prime $r \geq 5$ is not a common factor of $\frac{1}{2}(\eta(q^2 + \epsilon q + 1) + 1)$ and $q + \epsilon$. Thus $\frac{1}{2}(\eta(q^2 + \epsilon q + 1) + 1)$ is a power of $3$. Moreover, if $q > 2$ and $\eta = 1$, then $\frac{1}{2}(\eta(q^2 + \epsilon q + 1) + 1) = \frac{1}{2}(q^2 + 4) = \frac{1}{2}(q^2 + \epsilon) \equiv 2^{\eta(-1)t} \neq 0 \pmod{3}$, which is impossible. If $\eta = \frac{1}{2}$, then $\frac{1}{2}(\eta(q^2 + \epsilon q + 1) + 1) = \frac{1}{2}(q^2 + \epsilon q + 4) = 3^t$ has no solution for $n$ and $t$, which is a contradiction. If $q = 2$, then $\epsilon = +$ and $\eta = 1$, so $\eta(q^2 + \epsilon q + 1) = 7$, which is discussed above.

Suppose $\eta(q^2 + \epsilon q + 1) = \frac{1}{2}(q' + \gamma)$. Then $\frac{1}{2}(q' - \gamma) = \eta(q^2 + \epsilon q + 1) - \gamma \in \pi_e(G)$, which is impossible.

It follows that $\bar{G} \neq L_2(q')$ except when $q' = 7$ and $H = L_3(2)$, in which case $\bar{G} = L_2(7) \simeq H$.

Case (ii). Suppose $\bar{G} \simeq S_2(2^{m+1})$ with $m \geq 1$. A similar proof to that of Case (i) shows that $\eta(q^2 + q + 1) = 2^{m+1} + 2^m + 1$ and $\eta(q^2 + q + 1) \in \{2^{m+1} - 1, 2^{m+1} + 2^m + 1\}$, which are impossible. Thus $\bar{G} \neq S_2(2^{m+1})$.

Case (iii). Suppose $\bar{G} \simeq L_3^-\epsilon(2^m)$, where $m \geq 2$ when $\epsilon = +$. A similar proof to that of Case (i) shows that $\eta(q^2 + q + 1) = \mu(2^{2m} - 1)$ and

$$\eta(q^2 - q + 1) \in \{\mu(2^{2m} + 2^m + 1), \mu(2^{2m} - 1)\},$$

where $\mu = 1 / \gcd(3, 2^m + \epsilon)$. If $\eta(q^2 + \epsilon q + 1) = \mu(2^{2m} - 1)$ and $\mu(2^m + \epsilon) \neq 1$, then an odd prime divisor $r$ of $\mu(2^m + \epsilon)$ divides both $\eta(q^2 + \epsilon q + 1)$ and $2\mu(2^{2m} + \epsilon) \in \pi_e(\bar{G})$. In particular, $2r \in \pi_e(\bar{G})$ and so $2r \in \pi_e(H)$. A contradiction, since by Lemma 1.3, $2r \notin \pi_e(H)$ for $r|\eta(q^2 + \epsilon q + 1)$. If $\mu(2^m + \epsilon) = 1$, then $\eta(q^2 + \epsilon q + 1) = \mu(2^{2m} - 1) = 2^m - \epsilon$
has no solution. Similarly, \( \eta(q^2 - q + 1) = \mu(2^{2m} + 2^m + 1) \) can also induce a contradiction. It follows that \( \bar{G} \not\cong L_5^e(2^m) \).

Case (iv). Suppose \( \bar{G} \cong L_3^e(2^m) \). A similar proof to that of Case (i) shows that
\[
\eta(q^2 + q + 1) = \mu(2^{2m} + 2^m + 1)
\]
and
\[
\eta(q^2 - q + 1) \in \{ \mu(2^{2m} - 1), \mu(2^{2m} - 2^m + 1) \},
\]
where \( \mu = 1 / \gcd(3, 2^m - \epsilon) \). A similar proof to that of Case (iii) shows that \( \eta(q^2 - q + 1) \neq \mu(2^{2m} - 1) \), so \( \eta(q^2 + q + 1) = \mu(2^{2m} + \epsilon 2^m + 1) \) and \( m = n \). Thus \( \bar{G} \cong L_3^e(2^n) \cong H \).

(4). \( N = 1 \).

Suppose \( N \neq 1 \). Since \( \pi(N) \subseteq \pi_1(G) \), \( N \) is a normal \( \pi_1 \)-subgroup of \( G \). By \( t(G) \geq 2 \), a subgroup \( R/N \) of order \( d \in \pi_2(G/N) \) acts fixed-point freely on \( N \), so by a result of Thompson, [10, p.337], \( N \) is nilpotent. Replacing \( N \) by a factor group, we may suppose \( N \) is an elementary abelian \( r \)-subgroup, where \( r \) is a prime such that \( r = 2 \), \( r \mid q + \epsilon \) or \( r \mid q - \epsilon \).

Case (i). Suppose \( r \mid q + \epsilon \). Since \( 2r \not\in \pi_e(G) \), it follows that a Sylow 2-subgroup \( P_2 \) of \( \bar{G} \) acts fixed-point freely on \( N \), so that \( P_2 \) is cyclic or generalized quaternion, a contradiction since \( G_1 \) is non-solvable.

Case (ii). Suppose \( r = 2 \). Let \( Q \) be a cyclic subgroup of order \( \eta(q^2 + \epsilon q + 1) \) in \( G_1/N \), and \( N(Q) = N_{G_1/N}(Q) \). By [7, Theorem 3.1], \( N(Q) = Q; \mathbb{Z}_3 \) and suppose \( [N](Q; \mathbb{Z}_3) \) is its preimage in \( G_1 \). It follows by [10, Chapter 3, Theorem 4.4] that \( G_1 \) contains an element of order 6, which is impossible.

Case (iii). Suppose \( r \mid q - \epsilon \). If \( r = 3 \), then repeat the proof of Case (ii) above, so that by [1, Lemma 7], \( G_1 \) has an element of order 9 and \( 9 \in \pi_e(H) \). Since \( 3 \mid q - \epsilon \), it follows that \( 9 \mid q - \epsilon, \, n \equiv 0 \) or \( 3(\text{mod } 6) \) according as \( \epsilon = + \) or \( - \), and by Lemma 1.3, \( 6 \in \pi_e(H) \), which is also impossible. If \( r \neq 3 \), we denote \( S \in \text{Syl}_2(G_1) \). Let \( \bar{G} = G_1/N \cong L_3(q) \) (The case of \( U_3(q) \) is similar since the centralizer of every involution has a normal Sylow 2-subgroup in the two classes simple groups [18]). We have the following observations about \( \bar{G} \): The case of \( q = 4 \) is discussed, see Proposition 2.(c). \( Z(S) \) is an elementary abelian group of order \( q \), and for any \( 1 \neq x \in Z(S) \), \( C_{\bar{G}}(\bar{x}) \cong SK \), where \( K \) is of order \( (q - 1)/\gcd(3, q - 1) \) and acts on \( S \) as the set of diagonal matrices. And all elements of \( Z(S) \) are conjugate in \( G_1 \).

By co-prime action ([10, Chapter 5, Theorem 3.16]), \( N = \langle C_N(x) | 1 \neq x \in Z(S) \rangle \) and
$C_N(x) \neq 1$. Without lose any generality, we may assume that

$$x = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

put $U = C_N(x)$, $\tilde{C} = C_G(\bar{x})$, so $\tilde{C}$ acts on $U$. Assume $C_N(f) = 1$ for each element $f$ of order 4 in $S$. Then $\tilde{f}$ inverts each element of $U$, so $[\tilde{f}, \tilde{C}] \subseteq C_C(U)$. Put $D = \langle f \in S | f^2 = x \rangle$. As $q > 4$, $S/Z(S)$ is the direct sum of two non-isomorphic $K$-submodules of order $q$. As $D$ is $K$-invariant, $D = S$. Thus $[\bar{S}, \bar{C}] \subseteq C_C(U)$. But $\bar{S} = [\bar{S}, \bar{C}]$, so $[S, U] = 1$ which is a contradiction.

Therefore $N = 1$ and $G_1 \simeq L_3^s(q)$. Now $G_1$ is normal in $G$ and by $t(G) \geq 2$, $C_G(G_1) = 1$, so

$$G_1 \leq G \leq \text{Aut}(G_1),$$

since $N_G(G_1)/C_G(G_1) \leq \text{Aut}(G_1)$. Finally, we show that

(5). $G = G_1$.

Suppose $G = G_1:T$ and $q = 2^p$, where $T \leq \text{Out}(G_1)$. In the notation of Kleidman and Liebeck, [13, Propositions 2.2.3 and 2.3.5] and the fact that the automorphism group of $L_3^s(q)$ splits over $L_3^s(q)$,

$$\tilde{A} = \text{Out}(G_1) = \begin{cases} \langle \delta, \phi, i \rangle \simeq \mathbb{Z}_{4(3,q-1)}: \mathbb{Z}_n: \mathbb{Z}_2 \quad &\text{if } \epsilon = +, \\ \langle \delta, \phi \rangle \simeq \mathbb{Z}_{4(3,q+1)}: \mathbb{Z}_{2n} \quad &\text{if } \epsilon = -, \end{cases}$$

where $\delta = \text{diag} \{ \lambda, 1, 1 \} \in \text{GL}_3^s(q)$, $\phi((a_{ij})) = (a_{ij}^2)$ for each matrix $(a_{ij}) \in \text{GL}_3^s(q)$ and $\epsilon$ is the inverse-transpose of $\text{GL}_3^s(q)$. Let

$$L = \text{GL}_2^s(q) = \left\{ \begin{pmatrix} \det(X)^{-1} & 0 \\ 0 & X \end{pmatrix} : X \in \text{GL}_2^s(q) \right\} \leq K = \text{SL}_3^s(q).$$

Then $L \simeq \mathbb{Z}_{q-\epsilon} \times L_2(q)$ and $L \leq C_K(\delta)$, where $C_K(\delta)$ is the fixed-point set of $\delta$ in $K$. So

$$C_L(\tilde{A}) = C_{\mathbb{Z}_{q-\epsilon}}(\tilde{A}) \times C_{L_2(q)}(\tilde{A})$$

and $C_{L_2(q)}(\tilde{\phi}) = L_2(2) = \langle \sigma, \tau \rangle \simeq S_3 \leq G_1$, where $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus $|\sigma| = 3$, $|\tau| = 2$ and $\tau^{-1} \sigma \tau = \sigma^{-1}$. If $\epsilon = -$, then $T \leq \langle \bar{\delta}, \bar{\phi} \rangle$.

Suppose $\epsilon = +$ and $w = \hat{\delta}^k \hat{\phi}^\ell \iota \in T$ is a 2-element for some integers $k$ and $\ell$. Then $\sigma^w = \sigma$ and $6$ is a divisor of $|\sigma w \tau|$, which is impossible. Thus $T \leq \langle \bar{\delta}, \bar{\phi} \rangle$, since $w \in \mathbb{Z}_{q-\epsilon}$.
\langle w^2 \rangle \subseteq \langle \bar{\delta}, \bar{\phi} \rangle \text{ when } w \text{ is odd. So } L_2(2) \leq C_{G_1}(\langle \bar{\delta}, \bar{\phi} \rangle) \leq C_{G_1}(T) \text{ and neither } 2 \text{ nor } 3 \text{ is a factor of } |T|.

Since |\bar{\delta}| = 1 \text{ or } 3, \text{ it follows that } T \cap \langle \bar{\delta} \rangle = 1 \text{ and } T\langle \bar{\delta} \rangle / \langle \bar{\delta} \rangle \cong T. \text{ But } \langle \bar{\delta}, \bar{\phi} \rangle / \langle \bar{\delta} \rangle \cong \langle \bar{\phi} \rangle \text{ is cyclic, so is } T. \text{ If } T = \langle w \rangle, \text{ then } w \in \{ \bar{\delta}^k, \bar{\phi}^k, \bar{\delta}^{-1} \bar{\phi}^k \} \text{ for some } k.

If \( w = \bar{\delta} \bar{\phi}^k \) with \( k \) odd, then by [13, Propositions 2.2.3 and 2.3.5], \( w^2 = \bar{\delta} \bar{\phi}^k \bar{\delta} \bar{\phi}^{-k} \bar{\phi}^{2k} = \bar{\phi}^{2k} \). If \( w = \bar{\delta} \bar{\phi}^k \) with \( k \) even, then \( w^3 = \bar{\phi}^{3k} \). Since \( T = \langle w \rangle = 3D\langle w^2 \rangle = \langle w^3 \rangle \), we may suppose \( T = \langle w \rangle \) with \( w = \bar{\phi}^\ell \) for some \( \ell \). Similarly, if \( w = \bar{\delta}^{-1} \bar{\phi}^k \), then we may also suppose \( T = \langle w \rangle \) with \( w = \bar{\phi}^\ell \) for some \( \ell \).

If \( G_1 = L_3(q) \), then \( C_{G_1}(\bar{\phi}) = L_3(2) \leq C_{G_1}(\bar{\phi}^\ell) = C_{G_1}(T) \). If \( G_1 = U_3(q) \), then \( T = \langle w^2 \rangle = \langle \bar{\phi}^{2\ell} \rangle \) and \( C_{G_1}(\bar{\phi}^k) = U_3(2) \leq C_{G_1}(\bar{\phi}^{2\ell}) = C_{G_1}(T) \). Since \( L_3(5) \) has an element of order 4, it follows that \( 4|w| \in \pi_e(G) = \pi_e(H) \). But \( 4t \not\in \pi_e(H) \) for any \( t \geq 1 \), so \( T = 1 \) and \( G = G_1 \). This proves Theorem 2.1.

Since up to now, only finitely many non-abelian finite simple groups are found to be non-distinguishable, we raise the following conjecture, which is opposite to Proposition 1.

**Conjecture.** Almost all non-abelian finite simple groups are characterizable.

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