

# Dade's Invariant Conjecture for General Linear and Unitary Groups in Non-defining Characteristics

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ABSTRACT. This paper is part of a program to study the conjecture of E. C. Dade on counting characters in blocks for several finite groups. The invariant conjecture of Dade is proved for general linear and unitary groups when the characteristic of the modular representation is distinct from the defining characteristic of the groups.

## INTRODUCTION

Let  $G$  be a finite group,  $r$  a prime and  $B$  an  $r$ -block. In [7] Dade introduced several conjectures on counting the numbers of ordinary irreducible characters in  $B$  with a fixed defect and announced that his final conjecture can be proved by verifying it for all non-abelian finite simple groups. The final conjecture has been verified for 15 sporadic simple groups, the simple Tits group,  $L_2(q)$  and  $Sz(2^{2m+1})$ , and for  ${}^2G_2(3^{2m+1})$ ,  $G_2(q)$  (with  $q \neq 3, 4$ ) and  ${}^3D_4(q)$  in non-defining characteristics. The ordinary conjecture has been verified by Olsson, Uno and the author for  $S_n$ ,  ${}^2F_4(2^{2m+1})$  in non-defining characteristics, and general linear groups in the defining characteristic. In this paper, we prove the invariant conjecture for a general linear or unitary group  $G$  in non-defining characteristics. The reductions used in this paper can also be applied to other classical groups.

In Section 1, we fix some notation, state the invariant conjecture and prove three lemmas. In Section 2, we reduce the family of radical chains to a simple  $G$ -invariant subfamily, central radical chains  $\mathcal{CR}(G)$ . Given an  $r$ -block  $B$ , in Section 3, we first reduce the family  $\mathcal{CR}(G)$  to its subfamily consisting of chains whose fixed-point subspace of each non-trivial subgroup has the same dimension as that of a defect group of  $B$ , then using the perfect isometries given by Broué, we reduce the final calculation for an  $r$ -block to that of the principal block. In the last section, we prove the invariant conjecture for the principal block using results of Fong, Srinivasan [10] and Olsson [12].

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AMS subject classification: 20C20 and 20G40

## 1. DADE'S CONJECTURE AND LEMMAS

Let  $R$  be an  $r$ -subgroup of a finite group  $G$ . Then  $R$  is *radical* if  $O_r(N(R)) = R$ , where  $O_r(N(R))$  is the largest normal  $r$ -subgroup of the normalizer  $N(R) = N_G(R)$ . Let  $\text{Blk}(G)$  be the set of  $r$ -blocks  $B$  of  $G$ ,  $D(B)$  a defect group of  $B$  and  $\text{Irr}(G)$  the set of all irreducible ordinary characters of  $G$ . If  $H \leq G$ , then denote  $\text{Blk}(H|B) = \{b \in \text{Blk}(H) : b^G = B\}$  (in the sense of Brauer).

Given an  $r$ -subgroup chain

$$(1.1) \quad C : P_0 < P_1 < \cdots < P_w$$

of  $G$ , define the *length*  $|C| = w$ ,  $C_k : P_0 < P_1 < \cdots < P_k$ ,  $C(C) = C_G(P_w)$ , and

$$N(C) = N_G(C) = N(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_w).$$

The chain  $C$  is said to be *radical* if it satisfies the following two conditions:

$$(a) \ P_0 = O_r(G) \quad \text{and} \quad (b) \ P_k = O_r(N(C_k)) \text{ for } 1 \leq k \leq w.$$

Denote by  $\mathcal{R} = \mathcal{R}(G)$  the set of all radical  $r$ -chains of  $G$ .

Suppose  $1 \rightarrow G \rightarrow E \rightarrow \bar{E} \rightarrow 1$  is an exact sequence, so that  $E$  is an extension of  $G$  by  $\bar{E}$ . Then  $E$  acts on  $\mathcal{R}$  by conjugation. Given  $C \in \mathcal{R}(G)$  and  $\xi \in \text{Irr}(N_G(C))$ , let  $N_E(C)$  and  $N_E(C, \xi)$  be the stabilizers of  $C$  and the pair  $(C, \xi)$  in  $E$ , respectively, so that

$$N_E(C, \xi)/N_G(C) \simeq N_{\bar{E}}(C, \xi) = N_E(C, \xi)G/G.$$

For  $B \in \text{Blk}(G)$ , an integer  $d \geq 0$  and  $U \leq \bar{E}$ , let  $k(N(C), B, d, U)$  be the number of characters in the set

$$\text{Irr}(N(C), B, d, U) = \{\xi \in \text{Irr}(N(C), B, d) : N_{\bar{E}}(C, \xi) = U\}.$$

Dade in [7] gives the following conjecture.

**Dade's Invariant Conjecture.** *If  $O_r(G) = 1$  and  $B$  is an  $r$ -block of  $G$  with defect  $d(B) > 0$ , then for any integer  $d \geq 0$ ,*

$$(1.2) \quad \sum_{C \in \mathcal{R}/G} (-1)^{|C|} k(N(C), B, d, U) = 0.$$

Let  $\text{Aut}(G)$  and  $\text{Out}(G)$  be the automorphism and outer automorphism groups of  $G$ , respectively. Then we may suppose  $\overline{E} \leq \text{Out}(G)$ .

Let  $m$  be a positive integer and  $a(m)$  the integer such that  $r^{a(m)}$  is the exact power of  $r$  dividing  $m$ . Set  $a(G) = a(|G|)$  when  $G$  is a finite group and  $a(\chi) = a(\chi(1))$  when  $\chi$  is a character of  $G$ .

Let  $H = H_1 \times H_2$  be a direct product of two finite groups  $H_1$  and  $H_2$ , and let  $\pi_i$  be the natural projection from  $H$  onto  $H_i$  for  $i = 1, 2$ . If  $C \in \mathcal{R}(H)$  is given by (1.1), then each  $\pi_i(P_t)$  is a radical subgroup of  $\cap_{\ell=0}^{t-1} N_{H_i}(\pi_i(P_\ell))$ . In general,  $O_r(H_i) \leq \pi_i(P_1) \leq \pi_i(P_2) \leq \dots \leq \pi_i(P_w)$  is not a radical chain of  $H_i$ . Let  $\mathcal{Y} = \{\pi_i(P_\ell) : 0 \leq \ell \leq w\}$  and suppose  $|\mathcal{Y}| = u$ . Relabel the subgroups in  $\mathcal{Y}$

$$\mathcal{Y} = \{Q_0, Q_1, \dots, Q_u\}$$

such that  $Q_i < Q_{i+1}$  for  $0 \leq i \leq u-1$ . Thus  $Q_0 < Q_1 < \dots < Q_u$  is a radical chain of  $H_i$ , which is denoted by  $\pi_i(C)$ .

**(1A).** Let  $H_1$  and  $H_2$  be two finite groups,  $n$  a nonnegative integer,  $B_i \in \text{Blk}(H_i)$  and  $\mathcal{R}(B_i)$  the subfamily of  $\mathcal{R}(H_i)$  consisting of chains  $C$  such that  $\text{Blk}(N_{H_i}(C)|B_i) \neq \emptyset$ .

(a) Suppose  $\mathcal{X}_i \subseteq \text{Irr}(H_i)$  for  $i = 1, 2$ , and  $\psi$  is a defect preserving bijection between  $\mathcal{X}_1$  and  $\mathcal{X}_2$ . In addition, let  $K_i = H_i \wr \mathbf{S}(n)$ , and let  $\mathcal{X}_i \wr \mathbf{S}(n)$  be the subset of  $\text{Irr}(K_i)$  consisting of all characters covering a character of  $(\mathcal{X}_i)^n$ , where  $\mathbf{S}(n)$  is the symmetric group on  $n$  letters. Then  $\psi$  can be extended to a defect preserving bijection  $\Psi$  between  $\mathcal{X}_1 \wr \mathbf{S}(n)$  and  $\mathcal{X}_2 \wr \mathbf{S}(n)$ . Moreover, if  $\tau$  is an automorphism of  $K_i$  for  $i = 1, 2$  such that  $(H_i)^\tau = H_i$ ,  $\psi(\eta)^\tau = \psi(\eta^\tau)$  and  $y^\tau = y$  for each  $\eta \in \mathcal{X}_1$  and  $y \in \mathbf{S}(n)$ . Then  $\tau$  is compatible with  $\Psi$ , that is,  $\Psi(\chi)^\tau = \Psi(\chi^\tau)$  for  $\chi \in \mathcal{X}_1 \wr \mathbf{S}(n)$ .

(b) Let  $H = H_1 \times H_2$  and  $B = B_1 \times B_2 \in \text{Blk}(H)$ . In addition, let  $\mathcal{L}(B_i)$  be an  $H_i$ -invariant subset of  $\mathcal{R}(B_i)$ , and  $\mathcal{L}(B)$  a subset of  $\mathcal{R}(B)$  consisting of all chains  $C$  such that  $\pi_i(C) \in \mathcal{L}(B_i)$  for  $i = 1, 2$ . Suppose  $U$  is a subgroup of  $\text{Out}(H)$  generated by  $\tau_1, \dots, \tau_t$  and suppose  $\tau_j = \pi_1(\tau_j) \times \pi_2(\tau_j)$ , where each  $\pi_i(\tau_j)$  is an automorphism of  $H_i$ . Let  $U_i = \langle \pi_i(\tau_1), \dots, \pi_i(\tau_t) \rangle$  for  $i = 1, 2$ . Then

$$\sum_{C \in \mathcal{L}(B)/H} (-1)^{|C|} \mathbf{k}(N_H(C), B, d, U) = \sum_{d_1 + d_2 = d} \left( \prod_{i=1}^2 \left( \sum_{C \in \mathcal{L}(B_i)/H_i} (-1)^{|C|} \mathbf{k}(N_{H_i}(C), B_i, d_i, U_i) \right) \right).$$

*Proof.* (a) Let  $(H_i)^n$  be the base subgroup of  $K_i = H_i \wr \mathbf{S}(n)$ , and  $\mathcal{X}_1 = \{\xi_1, \dots, \xi_k\}$ . Then  $\mathcal{X}_2 = \{\psi(\xi_1), \dots, \psi(\xi_k)\}$ . If  $\chi \in \mathcal{X}_1 \wr \mathbf{S}(n)$ , then the restriction  $\chi|_{(H_1)^n}$  of  $\chi$  to  $(H_1)^n$  is a character in  $(\mathcal{X}_1)^n$ . If  $\xi \in (\mathcal{X}_1)^n$  and  $m_i$  is the multiplicity of  $\xi_i$  in  $\xi$ , then  $\mathbf{m} = (m_1, m_2, \dots, m_k)$  is called the *type* of  $\xi$  and  $\xi$  has an extension  $\tilde{\xi}$  to the stabilizer  $(H_1)^n \mathbf{S}(\mathbf{m})$  of  $\xi$  in  $H_1 \wr \mathbf{S}(n)$ , where  $\mathbf{S}(\mathbf{m})$  is the Young subgroup of  $\mathbf{S}(n)$  of type  $\mathbf{m}$ . If  $\zeta \in \text{Irr}(\mathbf{S}(\mathbf{m}))$ , then  $\chi = \text{Ind}_{(H_1)^n \mathbf{S}(\mathbf{m})}^{H_1 \wr \mathbf{S}(n)}(\tilde{\xi}\zeta) \in \text{Irr}(K_1)$ , and each character of  $\mathcal{X}_1 \wr \mathbf{S}(n)$  is of this form. Define

$$\Psi(\chi) = \text{Ind}_{(H_2)^n \mathbf{S}(\mathbf{m})}^{H_2 \wr \mathbf{S}(n)}(\psi(\tilde{\xi})\zeta),$$

where  $\psi(\tilde{\xi})$  is an extension of the character  $\psi(\xi) \in (\mathcal{X}_2)^n$  whose multiplicity of  $\psi(\xi_i)$  is  $m_i$ . Then  $\Psi(\chi) \in \mathcal{X}_2 \wr \mathbf{S}(n)$  and  $\Psi$  is a bijection. Moreover,  $d(\Psi(\chi)) = a((H_2)^n \mathbf{S}(\mathbf{m})) - a(\psi(\xi)) - a(\zeta)$  and  $d(\chi) = a((H_1)^n \mathbf{S}(\mathbf{m})) - a(\xi) - a(\zeta)$ . Thus  $d(\chi) = d(\Psi(\chi))$  if and only if

$$na(H_2) - a(\psi(\xi)) = na(H_1) - a(\xi).$$

But  $na(H_2) - a(\psi(\xi)) = \sum_{j=1}^k m_j d(\psi(\xi_j))$  and  $na(H_1) - a(\xi) = \sum_{j=1}^k m_j d(\xi_j)$ , so  $\Psi$  is defect preserving. Since  $\psi(\eta)^\tau = \psi(\eta^\tau)$  for  $\eta \in \mathcal{X}_1$  and  $y^\tau = y$  for  $y \in \mathbf{S}(n)$ , it follows that  $\Psi(\chi)^\tau = \Psi(\chi^\tau)$ .

(b) Given  $C(0) \in \mathcal{L}(B)$  with length  $s \geq 0$ , let  $X = X_1 \times X_2$  be the final subgroup of  $C(0)$ , where  $X_i \leq H_i$ . If  $s \geq 1$ , then  $X$  is a radical subgroup of  $N_H(C(0)_{s-1}) = N_1 \times N_2$  with  $X \neq O_r(N_H(C(0)_{s-1}))$ , where  $X_1 \leq N_1 \leq H_1$  and  $X_2 \leq N_2 \leq H_2$ . If  $s = 0$ , then  $X_i = O_r(H_i)$  and we set  $N_i = H_i$  for  $i = 1, 2$ . Let  $\mathcal{L}(C(0))$  be the subset of  $\mathcal{L}(B)$  consisting of all chains  $C$  such that  $C_s = C(0)$ . If  $C \in \mathcal{L}(C(0))$  is given by (1.1), then  $P_t = R_1(t) \times R_2(t)$  with  $X_1 \leq R_1(t) \leq N_1$  and  $X_2 \leq R_2(t) \leq N_2$  for  $s \leq t \leq w$ . Fix a radical subgroup  $Y_1$  of  $N_{N_1}(X_1)$  with  $X_1 \neq Y_1$  and fix an integer  $u \geq s + 1$ . Let  $\mathcal{M}(C(0), Y_1, u)$  be the subset of  $\mathcal{L}(C(0))$  consisting of all chains  $C$  given above such that  $R_1(u) = Y_1$  and  $R_1(t) = X_1$  for  $s \leq t \leq u - 1$  up to conjugacy in  $N_H(C(0))$ , so that we may suppose  $P_u = Y_1 \times R_2(u)$  and  $X_2 \leq R_2(u - 1) \leq R_2(u)$ . Let  $\mathcal{M}^0(C(0), Y_1, u)$  and  $\mathcal{M}^+(C(0), Y_1, u)$  be the subsets of  $\mathcal{M}(C(0), Y_1, u)$  consisting of chains  $C$  such that  $R_2(u - 1) = R_2(u)$  and  $R_2(u - 1) \neq R_2(u)$ , respectively. If  $C \in \mathcal{M}^0(C(0), Y_1, u)$  is given by (1.1) with  $u \geq s + 2$ , then

$$\varphi(C) : O_r(H) < P_1 < \dots < P_{u-2} < P_u < \dots < P_w$$

is a chain of  $\mathcal{R}(B)$  and

$$(1.3) \quad N_{N_H(C(0))}(C) = N_{N_H(C(0))}(\varphi(C)).$$

Since  $\pi_i(C) = \pi_i(\varphi(C))$ , it follows that  $\varphi(C) \in \mathcal{M}^+(C(0), Y_1, u-1)$ . Conversely, if  $C$  is a chain of  $\mathcal{M}^+(C(0), Y_1, u)$  given by (1.1) with  $u \geq s+1$ , then

$$\varphi(C) : O_r(H) < P_1 < \dots < P_{u-1} < P < P_u < \dots < P_w$$

is a chain of  $\mathcal{M}^0(C(0), Y_1, u+1)$  and (1.3) still holds, where  $P = X_1 \times R_2(u)$ . Let

$$(1.4) \quad \mathcal{S}(C(0), Y_1) = \left( \bigcup_{u \geq s+2} \mathcal{M}^0(C(0), Y_1, u) \right) \cup \left( \bigcup_{u \geq s+1} \mathcal{M}^+(C(0), Y_1, u) \right).$$

Then  $\varphi(C)$  is a map from  $\mathcal{S} = \mathcal{S}(C(0), Y_1, u)$  to itself such that  $\varphi(\varphi(C)) = C$  and  $|\varphi(C)| = |C| \pm 1$ .

Suppose  $\chi \in \text{Irr}(N_H(C), B, d)$  and  $\chi^\tau = \chi$  for some  $\tau \in U$ . Then  $\chi = \chi_1 \times \chi_2$  and  $\tau = \pi_1(\tau) \times \pi_2(\tau)$ , where  $\chi_i \in \text{Irr}(H_i)$  and  $\pi_i(\tau) \in U_i$  for  $i = 1, 2$ . Thus  $\chi_i^{\pi_i(\tau)} = \chi_i$  and  $\chi \in \text{Irr}(N_{H_1}(\pi_1(C)), B_1, d_1, U_1) \times \text{Irr}(N_{H_2}(\pi_2(C)), B_2, d_2, U_2)$ , where  $d_i = d(\chi_i)$  for  $i = 1, 2$ . Conversely, if  $\chi_i \in \text{Irr}(N_{H_i}(\pi_i(C)), B_i, d_i, U_i)$  with  $d_1 + d_2 = d$  and if  $\chi = \chi_1 \times \chi_2$ , then  $\chi \in \text{Irr}(N_H(C), B, d, U)$ , so that

$$(1.5) \quad \begin{aligned} & \text{Irr}(N_H(C), B, d, U) \\ &= \bigcup_{d_1+d_2=d} \text{Irr}(N_{H_1}(\pi_1(C)), B_1, d_1, U_1) \times \text{Irr}(N_{H_2}(\pi_2(C)), B_2, d_2, U_2). \end{aligned}$$

Since  $\pi_i(C) = \pi_i(\varphi(C))$  for  $i = 1, 2$ ,  $k(N_H(C), B, d, U) = k(N_H(\varphi(C)), B, d, U)$  and

$$\sum_{C \in \mathcal{S}/N_H(C(0))} (-1)^{|C|} k(N_H(C), B, d, U) = 0$$

for all  $d \geq 0$ .

Let  $C \in \mathcal{L}(C(0)) \setminus (\bigcup_{Y_1} \mathcal{S}(C(0), Y_1))$  be a chain given by (1.1), where  $Y_1$  runs over all radical subgroups of  $N_{N_1}(X_1)$  with  $Y_1 \neq X_1$ . Then either  $P_t = X_1 \times R_2(t)$  for all  $s \leq t \leq w$  or  $P_{s+1} = Z_1 \times X_2$  for some radical subgroup  $Z_1$  of  $N_{N_1}(X_1)$  with  $Z_1 \neq X_1$ . Let

$$\mathcal{L}^*(B) = \mathcal{R}(B) \setminus \left( \bigcup_{C(0), Y_1} \mathcal{S}(C(0), Y_1) \right),$$

where  $C(0)$  runs over  $\mathcal{R}(B)$  and  $Y_1$  runs over all radical subgroups of  $N_{N_1}(X_1)$  with  $Y_1 \neq X_1$ .

Suppose  $C \in \mathcal{L}^*(B)$  is given by (1.1) with  $P_t = R_1(t) \times R_2(t)$ . If  $R_1(1) = O_r(H_1)$ , then  $R_1(w) = O_r(H_1)$ , and we set  $u = u(C) = 0$ . If  $R_1(1) \neq O_r(H_1)$ , then  $R_2(1) = O_r(H_2)$ , and there exists the largest integer  $u = u(C) \geq 1$  such that  $R_1(u-1) < R_1(u) = R_1(u+1)$ . Thus  $R_2(u) = O_r(H_2)$  and  $R_2(t) \neq R_2(t-1)$  for all  $u+1 \leq t \leq w$ , so that  $R_1(t) \neq R_1(t-1)$  for all  $1 \leq t \leq u$  and  $R_1(t) = R_1(t-1)$  for all  $u+1 \leq t \leq w$ . Define  $\varphi_1(C) : O_r(H_1) < R_1(1) < \dots < R_1(u)$  and  $\varphi_2(C) : O_r(H_2) < R_2(u+1) < \dots < R_2(w)$ . Then  $\varphi_i(C) = \pi_i(C) \in \mathcal{L}(B_i)$ .

Let  $C(0) \in \mathcal{L}^*(B)$  such that  $u(C(0)) = |C(0)|$ , and let  $\mathcal{L}^*(C(0)) = \mathcal{L}^*(B) \cap \mathcal{L}(C(0))$ . Then  $N_H(C(0)) = N_1 \times H_2$  and  $N_H(C) = N_1 \times N_{H_2}(\varphi_2(C))$ , where  $N_1 = N_{H_1}(\varphi_1(C(0)))$  and  $C \in \mathcal{L}^*(C(0))$ . Moreover,  $\varphi_2$  induces a bijection between  $\mathcal{L}^*(C(0))$  and  $\mathcal{L}(B_2)$ , and  $|\varphi_2(C)| = |C| - |C(0)|$ . A similar proof to that of (1.5) shows that

$$k(N_H(C), B, d, U) = \sum_{s+t=d} k(N_1, B_1, s, U_1) k(N_{H_2}(\varphi_2(C)), B_2, t, U_2)$$

and moreover,

$$\begin{aligned} \sum_{C \in \mathcal{L}^*(C(0))/N_H(C(0))} (-1)^{|C|} k(N_H(C), B, d, U) &= \sum_{t \geq 0} \sum_{s+t=d} (-1)^{|C(0)|} k(N_1, B_1, s, U_1) \\ &\times \sum_{C \in \mathcal{L}(B_2)/H_2} (-1)^{|C|} k(N_{H_2}(C), B_2, t, U_2). \end{aligned}$$

Let  $\mathcal{L}^+(B) = \mathcal{L}^*(B) \setminus (\cup_{C(0)} \mathcal{L}^*(C(0)))$ , where  $C(0)$  runs over all chains of  $\mathcal{L}(B)$  such that  $u(C(0)) = |C(0)|$ . Then  $\varphi_1$  induces a bijection from  $\mathcal{L}^+(B)$  to  $\mathcal{L}(B_1)$  such that  $|\varphi_1(C)| = |C|$  and  $N_H(C) = N_{H_1}(\varphi_1(C)) \times H_2$ . Thus (b) follows by  $\mathcal{L}^*(B)/H = \cup_{C \in \mathcal{L}^+(B)/H} \mathcal{L}^*(B)/N_H(C)$ .

**Hypothesis (1B).** For a finite group  $X$ , suppose an  $X$ -invariant subfamily  $\mathcal{L}(X)$  of  $\mathcal{R}(X)$  has an  $X$ -invariant decomposition

$$(1.6) \quad \mathcal{L}(X) = \mathcal{L}_1(X) \cup \mathcal{L}_2(X) \quad (\text{disjoint})$$

satisfying the following condition:

There exists a bijection  $\varphi$  between the  $X$ -invariant subfamilies  $\mathcal{L}_1(X)$  and  $\mathcal{L}_2(X)$  such that for each  $C \in \mathcal{L}_1(X)$  given by (1.1),  $\varphi(C)$  is given by

$$(1.7) \quad \varphi(C) : O_r(X) < P_1 < \dots < P_v < Q < P_{v+1} < \dots < P_w$$

for an integer  $v = v(C)$  and a radical subgroup  $Q = Q(C)$  of  $N(C_v)$ .

Set  $v(\varphi(C)) = v(C) = v$ . Let  $n$  be a positive integer,  $H = X \wr \mathbf{S}(n)$ , and let  $Y = X_1 \times X_2 \times \dots \times X_n$  be the base subgroup of  $H$ , where  $X_i$  is a copy of  $X$ . In addition, let  $\pi_i$  be the natural projection of  $Y$  onto  $X_i$  for  $1 \leq i \leq n$  and  $\mathcal{L}(H)$  the subfamily of  $\mathcal{R}(Y)$  consisting of chains  $C$  such that  $\pi_i(C) \in \mathcal{L}(X_i) = \mathcal{L}(X)$  for all  $i$ . Then  $\mathcal{L}(H)$  is  $H$ -invariant.

Let  $\mathcal{M}(H)/H$  be the subset of chains  $C \in \mathcal{L}(H)$  given by (1.1) such that each  $P_i = Q_i^n$  for some subgroup  $Q_i$  of  $X$ , and let  $\mathcal{M}(H)$  be the subfamily of  $\mathcal{L}(H)$  consisting of  $H$ -orbits of  $\mathcal{M}(H)/H$ . For  $C \in \mathcal{M}(H)/H$ ,  $\pi_i(C) = \pi_j(C)$  for all  $i, j$ ,  $|C| = |\pi_i(C)|$  and

$$N_H(C) = N_X(\pi(C)) \wr \mathbf{S}(n),$$

where  $\pi(C) = \pi_1(C)$ , viewed as a chain of  $X$ . In particular,  $\pi$  is a bijection between  $\mathcal{M}(H)/H$  and  $\mathcal{L}(X)/X$ .

If  $\mathbf{m} = (n_1, \dots, n_t)$  is a sequence of positive integers such that  $|\mathbf{m}| = n_1 + n_2 + \dots + n_t = n$ , then  $X \wr \mathbf{S}(\mathbf{m}) = H_1 \times H_2 \times \dots \times H_t$  is a subgroup of  $H$ , where each  $H_i = X \wr \mathbf{S}(n_i)$ . Let  $\pi_{n_i}$  be the projection from  $X \wr \mathbf{S}(\mathbf{m})$  onto  $H_i$  for all  $i$ .

(1C). *In the notation above, suppose  $C \in \mathcal{L}(H)$ . Then there is a sequence  $\mathbf{m} = \mathbf{m}(C) = (n_1, \dots, n_t)$  of positive integers such that  $\pi_{n_i}(C) \in \mathcal{M}(H_i)$ ,  $|\mathbf{m}| = n$  and*

$$(1.8) \quad N_H(C) = N_X(C(1)) \wr \mathbf{S}(n_1) \times N_X(C(2)) \wr \mathbf{S}(n_2) \times \dots \times N_X(C(t)) \wr \mathbf{S}(n_t),$$

where  $C(i) = \pi(\pi_{n_i}(C)) \in \mathcal{L}(X)$ . Moreover,  $\mathcal{L}(H)$  satisfies Hypothesis (1B) and if  $C \in \mathcal{L}_1(H)$  with  $N_H(C)$  given by (1.8), then there is an integer  $s = s(C) \geq 1$  such that

$$(1.9) \quad N_H(\Phi(C)) = N_X(C(1)) \wr \mathbf{S}(n_1) \times \dots \times N_X(C(s-1)) \wr \mathbf{S}(n_{s-1}) \\ \times N_X(\varphi(C(s))) \wr \mathbf{S}(n_s) \times N_X(C(s+1)) \wr \mathbf{S}(n_{s+1}) \times \dots \times N_X(C(t)) \wr \mathbf{S}(n_t),$$

where  $\Phi(C) \in \mathcal{L}_2(H)$ . In particular,  $C(s) \in \mathcal{L}_1(X)$ .

*Proof.* Suppose  $R$  is a radical subgroup of the base subgroup  $N_X(\pi(C))^n$  of  $N_H(C)$ , where  $C$  is a chain of  $\mathcal{M}(H)/H$ . Then

$$(1.10) \quad R = R_1^{n_1} \times R_2^{n_2} \times \dots \times R_t^{n_t}$$

for some sequence  $\mathbf{m} = (n_1, \dots, n_t)$  with  $|\mathbf{m}| = n$ , where  $R_i \neq R_j$  for  $i \neq j$  viewed as subgroups of  $X$ . Thus

$$(1.11) \quad N_{N_H(C)}(R) = N_{N_X(\pi(C))}(R_1) \wr \mathbf{S}(n_1) \times \dots \times N_{N_X(\pi(C))}(R_t) \wr \mathbf{S}(n_t)$$

and  $N_{N_X(\pi(C))}(R_i) \wr \mathbf{S}(n_i) = N_{N_{H_i}(\pi_{n_i}(C))}(R_i^{n_i})$  for all  $i \geq 1$ .

If  $C$  is given by (1.1) and  $R \neq O_r(N_H(C))$  such that  $C' : O_r(H) < P_1 < \dots < P_w < R$  is a chain of  $\mathcal{L}(H)$ , then  $\pi_{n_i}(C') \in \mathcal{M}(H_i)/H_i$  and

$$N_{N_X(\pi(C))}(R_i) \wr \mathbf{S}(n_i) = N_X(C'(i)) \wr \mathbf{S}(n_i) = N_{H_i}(\pi_{n_i}(C'))$$

for all  $i$ , where  $C'(i) = \pi(\pi_{n_i}(C')) \in \mathcal{L}(X)$ . It follows by induction on  $n$  that for each  $C \in \mathcal{L}(H)$ ,  $N_H(C)$  is of the form (1.8) for some sequence  $\mathbf{m} = \mathbf{m}(C)$ .

Fix a sequence  $\mathbf{m} = (n_1, \dots, n_t)$ , let  $\mathcal{L}(\mathbf{m})$  be the subfamily of  $\mathcal{L}(H)$  consisting of chains  $C$  such that  $\mathbf{m}(C) = \mathbf{m}$ , so that  $\mathcal{L}(H)$  is a disjoint union of  $\mathcal{L}(\mathbf{m})$ 's,  $N_H(C)$  satisfies (1.8) for each  $C \in \mathcal{L}(\mathbf{m})$ ,  $\mathcal{L}(\mathbf{m})$  is  $X \wr \mathbf{S}(\mathbf{m})$ -invariant and  $N_H(C) = N_{X \wr \mathbf{S}(\mathbf{m})}(C)$ .

Suppose  $C \in \mathcal{L}(\mathbf{m})$  is given by (1.1),  $C(i) = \pi(\pi_{n_i}(C))$  for all  $i$ , and for some fixed  $\ell$ ,

$$(1.12) \quad C(\ell) : O_r(X) < Q_1 < \dots < Q_v < \dots < Q_u,$$

where  $v = v(C(\ell))$ . Then each  $P_j$  has a decomposition

$$(1.13) \quad P_j = W_{j,1}^{n_1} \times W_{j,2}^{n_2} \times \dots \times W_{j,t}^{n_t},$$

where  $W_{j,\ell} \leq \{O_r(X), Q_1, \dots, Q_u\}$  for all  $j$ . Since  $C(\ell) = \pi(\pi_{n_\ell}(C))$ , there is an integer  $m_\ell = m_\ell(C)$  such that  $W_{m_\ell, \ell} = Q_v$  and  $W_{m_\ell+1, \ell} \neq Q_v$ , so that  $W_{m_\ell+1, \ell} = Q_{v+1}$ . If  $Q = Q(C(\ell))$ , then by definition,  $Q = Q_{v+1}$  or  $Q < Q_{v+1}$  according as  $C(\ell) \in \mathcal{L}_2(X)$  or  $C(\ell) \in \mathcal{L}_1(X)$ . Thus

$$(1.14) \quad Q(n_\ell) = W_{m_\ell, 1}^{n_1} \times \dots \times W_{m_\ell, \ell-1}^{n_{\ell-1}} \times Q^{n_\ell} \times W_{m_\ell, \ell+1}^{n_{\ell+1}} \times \dots \times W_{m_\ell, t}^{n_t} \leq P_{m_\ell+1}$$

and define

$$(1.15) \quad \Phi_{n_\ell}(C) : P_0 < \dots < P_{m_\ell} < Q(n_\ell) \leq P_{m_\ell+1} < \dots < P_w.$$

It follows that  $\Phi_{n_\ell}(C) \in \mathcal{L}(\mathbf{m})$  and  $N_H(\Phi_{n_\ell}(C))$  has the form (1.9) with  $s = \ell$  when  $\Phi_{n_\ell}(C) \neq C$ .

Let  $\mathcal{L}_1(\mathbf{m})$  be a subfamily of  $\mathcal{L}(\mathbf{m})$  consisting of chains  $C \in \mathcal{L}(\mathbf{m})$  such that

$$C = \Phi_{n_1}(C) = \dots = \Phi_{n_{s-1}}(C), \quad \Phi_{n_s}(C) \neq C,$$

and define  $s(C) = s$  for  $C \in \mathcal{L}_1(\mathbf{m})$ . In addition, let  $\mathcal{L}_2(\mathbf{m})$  be the subfamily of  $\mathcal{L}(\mathbf{m})$  consisting of the chains  $C \in \mathcal{L}(\mathbf{m})$  such that  $C = \Phi_{n_s}(C')$  for some  $C' \in \mathcal{L}_1(\mathbf{m})$  with  $s = s(C')$ . Then  $\mathcal{L}_1(\mathbf{m})$  and  $\mathcal{L}_2(\mathbf{m})$  are  $X \wr \mathbf{S}(\mathbf{m})$ -invariant.



Suppose  $C \in \mathcal{L}(\mathbf{m}) \setminus (\mathcal{L}_1(\mathbf{m}) \cup \mathcal{L}_2(\mathbf{m})) \neq \emptyset$  and suppose  $N_H(C)$  is given by (1.8). If  $C(1)$  is given by (1.12) with  $v = v(C(1))$ . By (1.15),  $\Phi_{n_1}(C) \in \mathcal{L}(\mathbf{m})$ , and since  $C \notin \mathcal{L}_1(\mathbf{m})$ , it follows that  $Q(n_1) = P_{m_1+1}$ , where  $Q(n_1)$  is given by (1.14). Let

$$(1.16) \quad \alpha(C) : P_0 < \dots < P_{m_1} < P_{m_1+2} < \dots < P_w,$$

so that  $\alpha(C) \in \mathcal{L}(H)$ . If  $\alpha(C) \in \mathcal{L}(\mathbf{m})$ , then  $\Phi_{n_1}(\alpha(C)) = C$  and so  $C \in \mathcal{L}_2(\mathbf{m})$ , which is impossible. Thus  $\mathbf{m}(\alpha(C)) \neq \mathbf{m}(C)$  and  $N_H(C)$  is a proper subgroup of  $N_H(\alpha(C))$ . Since  $\pi_{n_i}(\alpha(C)) \in \mathcal{M}(H_i)/H_i$  for all  $i$  and since  $N_H(\alpha(C))$  is of the form (1.8), it follows that there is some  $n_i$ , say  $n_2$ , such that

$$(1.17) \quad \pi_{n_1+n_2}(\alpha(C)) \in \mathcal{M}(X \wr \mathbf{S}(n_1 + n_2))/X \wr \mathbf{S}(n_1 + n_2).$$

In particular,  $\pi_{n_1}(\alpha(C)) = \pi_{n_2}(\alpha(C))$ . If  $P_j^+ = \prod_{k=3}^t W_{j,k}^{n_k}$  for all  $j \geq 0$ , then  $P_{m_1} = Q_v^{n_1} \times Q_v^{n_2} \times P_{m_1}^+$ ,  $P_{m_1+1} = Q^{n_1} \times Q_v^{n_2} \times P_{m_1}^+$  and  $P_{m_1+2} = Z^{n_1} \times Z^{n_2} \times P_{m_1+2}^+$  for some  $r$ -subgroup  $Z$  of  $X$ . In particular,  $Q_v < Q \leq Z$ ,

$$Q(n_2) = Q^{n_1} \times Q^{n_2} \times P_{m_1}^+$$

and  $\Phi_{n_2}(C)$  is the chain

$$\Phi_{n_2}(C) : P_0 < \dots < P_{m_1} < P_{m_1+1} < Q(n_2) \leq P_{m_1+2} < \dots < P_w.$$

If  $\Phi_{n_2}(C) = C$ , then  $Z = Q$ ,  $P_{m_1}^+ = P_{m_1+2}^+$ ,  $m_2 = m_1 + 1$ , and  $C \in \mathcal{L}_2(\mathbf{m})$ . If  $\Phi_{n_2}(C) \neq C$ , then  $C \in \mathcal{L}_1(\mathbf{m})$ . This is impossible, so that  $\mathcal{L}(\mathbf{m}) = \mathcal{L}_1(\mathbf{m}) \cup \mathcal{L}_2(\mathbf{m})$  and (1C) follows.

Suppose  $b$  is a block of  $X$  with positive defect. For  $C \in \mathcal{R}(X)$ , let

$$\text{Irr}(N_X(C), b) = \bigcup_{b_1 \in \text{Blk}(N_X(C)|b)} \text{Irr}(b_1)$$

and if  $\mathcal{X} \subseteq \text{Irr}(N_X(C))$ , then denote  $\text{Irr}(\mathcal{X}, b) = \mathcal{X} \cap \text{Irr}(N_X(C), b)$ .

**Hypothesis (1D).** *Suppose an  $X$ -invariant subfamily  $\mathcal{Q}(X)$  of  $\mathcal{R}(X)$  has an  $X$ -invariant decomposition*

$$(1.18) \quad \mathcal{Q}(X) = \mathcal{Q}_1(X) \cup \mathcal{Q}_2(X) \cup \mathcal{S}(X) \quad (\text{disjoint})$$

*satisfying the following two conditions:*

(a) The family  $\mathcal{Q}_1(X) \cup \mathcal{Q}_2(X)$  satisfies Hypothesis (1B) and for each  $C \in \mathcal{Q}_1(X)$ , there is a defect preserving bijection  $\psi$  between  $\text{Irr}(N_X(C), b)$  and  $\text{Irr}(N_X(\varphi(C)), b)$ .

(b) For each  $C \in \mathcal{S}(X)$  given by (1.1), there are some maps  $\varphi$  such that  $\varphi(C) \in \mathcal{S}(X)$  is given by (1.7) with  $Q \leq P_{v+1}$ , and

$$\text{Irr}(N_H(C), b) = \Omega_C(0) \bigsqcup_{\varphi} \Omega_C(\varphi), \quad (\text{disjoint})$$

where  $\varphi$  runs over the maps with  $C \neq \varphi(C)$ . Moreover, there is a defect preserving bijection  $\psi$  between  $\Omega_C(\varphi)$  and  $\Omega_{\varphi(C)}(0)$ , and if  $|C| \neq 0$ , then  $\varphi(C') = C$  for a unique chain  $C' \in \mathcal{S}(X)$  and map  $\varphi$ .

For  $\tau \in \text{Aut}(X)$  the family  $\mathcal{Q}(X)$  is called  $\tau$ -invariant if  $\tau$  stabilizes  $C \in \mathcal{Q}_1(X) \cup \mathcal{S}(X)$  if and only if  $\tau$  stabilizes  $\varphi(C)$ , and each defect preserving bijection  $\psi$  is compatible with  $\tau$ .

Let  $b(Y) = b^n$  be the block of the base subgroup  $Y = X^n$  of  $H = X \wr \mathbf{S}(n)$ . Since  $C_H(D(b(Y))) \leq Y$ ,  $b(Y)^H$  is a regular block  $B$  of  $H$ . Denote by  $\mathcal{Q}(H)$  the subfamily of  $\mathcal{R}(Y)$  consisting of the chains  $C$  such that  $\pi_i(C) \in \mathcal{Q}(X)$  for each  $i \geq 1$ , where  $\pi_i$  is the natural projection of  $Y$  onto its  $i$ -th component  $X_i = X$ .

**(1E).** In the notation above, suppose  $\mathcal{Q}(X)$  satisfies Hypothesis (1D) for  $b$ .

(a) Then  $\mathcal{Q}(H)$  also satisfies Hypothesis (1D) for the block  $B = b(Y)^H$  and if  $C_0(X) \in \mathcal{S}(X)$ , then  $\Omega_{C_0(H)}(0) = \Omega_{C_0(X)}(0) \wr \mathbf{S}(n)$ , where  $C_0(K)$  is the radical chain of a finite group  $K$  with length 0. Moreover, if  $\tau \in \text{Aut}(H)$  such that  $X_i^\tau = X_i$  and  $y^\tau = y$  for all  $y \in \mathbf{S}(n)$  and if  $\mathcal{Q}(X)$  is compatible with  $\tau$ , then  $\mathcal{Q}(H)$  is compatible with  $\tau$ .

(b) Given  $i = 1, 2$ , suppose each  $M_i$  is a finite group and  $\mathcal{Q}(M_i)$  is an  $M_i$ -invariant subfamily of  $\mathcal{R}(M_i)$  satisfies Hypothesis (1D) for  $B_i \in \text{Blk}(M_i)$ . Let  $M = M_1 \times M_2$  and let  $\mathcal{Q}(M)$  be the subfamily of  $\mathcal{R}(M)$  consisting of chains  $C$  such that  $\pi_i(C) \in \mathcal{Q}(M_i)$ . Then  $\mathcal{Q}(M)$  satisfies Hypothesis (1D) for the block  $B_1 \times B_2$  and if each  $C_0(M_i) \in \mathcal{S}(M_i)$ , then  $C_0(M) \in \mathcal{S}(M)$  and  $\Omega_{C_0(M)}(0) = \Omega_{C_0(M_1)}(0) \times \Omega_{C_0(M_2)}(0)$ . In addition, if  $\tau \in \text{Aut}(M)$  stabilizes each  $M_i$  and each  $\mathcal{Q}(M_i)$  is  $\tau$ -invariant, then  $\mathcal{Q}(M)$  is  $\tau$ -invariant.

*Proof.* (a) Suppose  $C \in \mathcal{Q}(H)$  and  $b(C) \in \text{Blk}(N_H(C))$ . Then  $N_H(C)$  is of the form (1.8) for some sequence  $\mathbf{m}$ , and  $K = \prod_{i=1}^t N_X(C(i))^{n_i}$  is a normal subgroup of  $N_H(C)$ . We claim that  $b(C) \in \text{Blk}(N_H(C)|B)$  if and only if  $b(C)$  covers a block  $b_K \in \text{Blk}(K|b(Y))$ . By induction, we may suppose  $t = 1$ , so that  $n = n_1$ . Thus  $N_H(C) = N_X(C(1)) \wr \mathbf{S}(n)$ ,  $K$  is the base subgroup of  $N_H(C)$  and  $b(C)$  is regular. If  $(b_K)^Y = b(Y)$  and  $b(C)$  covers

$b_K$ , then  $(b_K)^{N_H(C)} = b(C)$ ,  $(b_K)^H = B$  and  $b(C)^H = B$ . Conversely, if  $b(C)^H = B$  and  $b(C)$  covers  $b_K$ , then  $((b_K)^Y)^H = B = b(Y)^H$ , so that  $B$  covers both  $(b_K)^Y$  and  $b(Y)$ , and hence  $(b_K)^Y$  is conjugate to  $b(Y)$  in  $H$ . But  $b(Y) = b^n$  is  $H$ -invariant, so  $(b_K)^Y = b(Y)$  and the claim holds. In particular,

$$(1.19) \quad \text{Irr}(N_H(C), B) = \prod_{i=1}^t \text{Irr}(N_X(C(i)), b) \wr \mathbf{S}(n_i).$$

Let  $\mathcal{Q}_0(H)$  be the subfamily of  $\mathcal{Q}(H)$  consisting of chains  $C$  such that in the decomposition of  $N_H(C)$  given by (1.8),  $C(i) \in \mathcal{Q}_1(X) \cup \mathcal{Q}_2(X)$  for some  $i$ . Then  $\mathcal{Q}_0(H)$  is  $H$ -invariant and the same proof as that of (1C) with some obvious modifications shows that  $\mathcal{Q}_0(H) = \mathcal{Q}_1(H) \cup \mathcal{Q}_2(H)$  satisfies Hypothesis (1B).

Suppose  $C \in \mathcal{Q}_1(H)$  with  $N_H(C)$  given by (1.8) and  $\Phi(C) \in \mathcal{Q}_2(H)$  with  $N_H(\Phi(C))$  given by (1.9). Then  $C(s) \in \mathcal{Q}_1(X)$  and by Hypothesis (1D), there is a defect preserving bijection  $\psi$  between  $\text{Irr}(N_X(C(s)), b)$  and  $\text{Irr}(N_X(\varphi(C(s))), b)$ . By (1A) (a) and (1.19),  $\psi$  can be extended to a defect preserving bijection  $\Psi$  between  $\text{Irr}(N_H(C), B)$  and  $\text{Irr}(N_H(\Phi(C)), B)$ . Thus  $\mathcal{Q}_0(H)$  satisfies Hypothesis (1D) for  $B$ , and moreover, if  $\mathcal{Q}(X)$  is  $\tau$ -invariant, then so is  $\mathcal{Q}_0(H)$ . We may suppose  $\mathcal{Q}_1(X) = \mathcal{Q}_2(X) = \emptyset$ , so that  $\mathcal{Q}(H) = \mathcal{S}(H)$ .

Suppose  $C \in \mathcal{Q}(H)$  with  $N_H(C)$  given by (1.8). Let  $\Sigma_C(i) = \text{Irr}(N_X(C(i)), b) \wr \mathbf{S}(n_i)$ ,  $\Sigma_C(i)^0 = \Omega_{C(i)}(0) \wr \mathbf{S}(n_i)$  and  $\Sigma_C(i)^+ = \Sigma_C(i) \setminus \Sigma_C(i)^0$ . If  $C = C_0(H)$ , then  $n = n_1$  and we define  $\Omega_C(0) = \Omega_{C_0(X)}(0) \wr \mathbf{S}(n)$ . Suppose

$$(1.20) \quad \Omega_C(0) = \prod_{i \in I_0} \Sigma_C(i)^0 \times \prod_{j \in I_1} \Sigma_C(j),$$

where  $I_0 \cup I_1$  is a partition of  $\{1, 2, \dots, t\}$ . By reordering, we may suppose  $I_0 = \{1, 2, \dots, k\}$  and  $I_1 = \{k+1, \dots, t\}$ . In addition, if  $\varphi(C(i)) = C(i)$  for all maps  $\varphi$ , then by definition,  $\Omega_{C(i)}(0) = \text{Irr}(N_X(C(i)), b)$  and so  $\Sigma_C(i)^0 = \Sigma_C(i)$ . Thus we may suppose for each  $i \in I_0$ , there is some map  $\varphi$  such that  $\varphi(C(i)) \neq C(i)$ , and suppose  $I_0 \neq \emptyset$ . Let  $\Sigma_C^+ = \prod_{i=k+1}^t \Sigma_C(i)$  and  $\Omega_C(+) = \text{Irr}(N_H(C), B) \setminus \Omega_C(0)$ . Then  $\Omega_C(+)$  is a disjoint union

$$(1.21) \quad \Omega_C(+) = \bigcup_{\ell=1}^k \left( \prod_{i=1}^{\ell-1} \Sigma_C(i) \times \Sigma_C(\ell)^+ \times \prod_{i=\ell+1}^k \Sigma_C(i)^0 \times \Sigma_C^+ \right).$$

For simplicity of notation, we suppose each  $\text{Irr}(N_X(C(\ell)), b) = \Omega_{C(\ell)}(0) \cup \Omega_{C(\ell)}(\varphi_\ell)$ . Since  $\Omega_{C(\ell)}(0) \cap \Omega_{C(\ell)}(\varphi_\ell) = \emptyset$ , it follows that there is a defect preserving bijection between

$\Sigma_C(\ell)^+$  and the disjoint union

$$\bigcup_{1 \leq n_{\ell,2} \leq n_{\ell}} \bigcup_{n_{\ell,1} + n_{\ell,2} = n_{\ell}} \Omega_{C(\ell)}(0) \wr \mathbf{S}(n_{\ell,1}) \times \Omega_{C(\ell)}(\varphi_{\ell}) \wr \mathbf{S}(n_{\ell,2}),$$

and we may identify these two sets. Fix nonnegative integers  $n_{\ell,1}, n_{\ell,2}$  such that  $n_{\ell,1} + n_{\ell,2} = n_{\ell}$  and  $n_{\ell,2} \geq 1$ . Define a chain  $\Phi(C)$  as (1.15) such that

$$N_H(\Phi(C)) = \prod_{i \neq \ell} N_X(C(i)) \wr \mathbf{S}(n_i) \times N_X(C(\ell)) \wr \mathbf{S}(n_{\ell,1}) \times N_X(\varphi_{\ell}(C(\ell))) \wr \mathbf{S}(n_{\ell,2}).$$

This is possible since  $\varphi_{\ell}(C(\ell)) \neq C(\ell)$ . Thus  $|\Phi(C)| = |C| + 1$ . Let

$$\Omega_{\Phi(C)}(0) = \prod_{i=1}^{\ell-1} \Sigma_C(i) \times \Omega_{C(\ell)}(0) \wr \mathbf{S}(n_{\ell,1}) \times \Omega_{\varphi_{\ell}(C(\ell))}(0) \wr \mathbf{S}(n_{\ell,2}) \times \prod_{i=\ell+1}^k \Sigma_C(i)^0 \times \Sigma_C^+$$

and  $\Omega_C(\Phi) = \prod_{i=1}^{\ell-1} \Sigma_C(i) \times \Omega_{C(\ell)}(0) \wr \mathbf{S}(n_{\ell,1}) \times \Omega_{C(\ell)}(\varphi_{\ell}) \wr \mathbf{S}(n_{\ell,2}) \times \prod_{i=\ell+1}^k \Sigma_C(i)^0 \times \Sigma_C^+$ .

Then

$$\Omega_C(+) = \Omega_C(0) \bigcup_{\Phi} \Omega_C(\Phi)$$

and by (1A) (a), the bijection  $\psi$  between  $\Omega_{C(\ell)}(\varphi_{\ell})$  and  $\Omega_{\varphi_{\ell}(C(\ell))}(0)$  can be extended to a defect preserving bijection  $\Psi$  between  $\Omega_C(\Phi)$  and  $\Omega_{\Phi(C)}(0)$ . Since  $\Omega_{\Phi(C)}(0)$  is also of the form (1.20), it follows by induction on  $|C|$  that for each chain  $C' \in \mathcal{S}(H)$ ,  $\Omega_{C'}(0)$  is of the form (1.20) and  $\text{Irr}(N_H(C'), B)$  has a required decomposition.

If  $I_0 = \emptyset$ , then  $\Omega_C(0) = \text{Irr}(N_H(C), B)$  and  $\Phi(C') = C$  for some  $C' \in \mathcal{S}(H)$ . In addition, if  $\mathcal{Q}(X)$  is  $\tau$ -invariant, then so is  $\mathcal{Q}(H)$ . This completes the proof.

(b) Then proof of part (b) follows easily by that of (1A) (b) and the part (a) above.

## 2. CENTRAL RADICAL CHAINS

Let  $q = p^f$  be a power of a prime  $p$  distinct from the odd prime  $r$ ,  $\epsilon = +$  or  $-$ , and let  $e$  be the multiplicative order of  $\epsilon q$  modulo  $r$  and  $a = a(q^e - \epsilon)$ . In addition, let  $\text{GL}^{\epsilon}(n, q) = \text{GL}(n, q)$  or  $\text{U}(n, q)$  according as  $\epsilon = +$  or  $-$ . The radical subgroups of  $G$  are classified by [2] and [3]. We shall follow the notation of [2].

Given integers  $\alpha \geq 0$  and  $\gamma \geq 0$ , let  $Z_{\alpha}$  be the cyclic group of order  $r^{\alpha}$ ,  $E_{\gamma}$  an extraspecial group of order  $r^{2\gamma+1}$  and  $Z_{\alpha}E_{\gamma}$  the central product of  $Z_{\alpha}$  and  $E_{\gamma}$  over  $\Omega_1(Z_{\alpha}) = Z(E_{\gamma})$ , where  $Z(H)$  denotes the center of a finite group  $H$ . Then  $Z_{\alpha}E_{\gamma}$  can be embedded as a subgroup of  $\text{GL}^{\epsilon}(er^{\alpha+\gamma}, q)$ , and its image  $R_{\alpha,\gamma}$  is determined uniquely by  $Z_{\alpha}E_{\gamma}$  up to conjugacy in  $\text{GL}^{\epsilon}(er^{\alpha+\gamma}, q)$ .

Given an integer  $m \geq 1$ , the image  $R_{m, \alpha, \gamma}$  of  $R_{\alpha, \gamma}$  under the  $m$ -fold diagonal mapping in  $\text{GL}^\epsilon(\text{mer}^{\alpha+\gamma}, q)$  given by

$$g \mapsto \begin{pmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix},$$

is also determined up to conjugacy. The center  $Z(R_{m, \alpha, \gamma})$  is cyclic of order  $r^{\alpha+\alpha}$ , so that  $\Omega_a(Z(R_{m, \alpha, \gamma})) = \langle z \rangle$ , where  $z$  is an element of order  $r^\alpha$  in  $\text{GL}^\epsilon(\text{mer}^{\alpha+\gamma}, q)$ . Moreover,

$$C_{\text{GL}^\epsilon(\text{mer}^{\alpha+\gamma}, q)}(z) \simeq \text{GL}^\epsilon(\text{mer}^{\alpha+\gamma}, q^e),$$

so that  $z$  is a primary element of  $\text{GL}^\epsilon(\text{mer}^{\alpha+\gamma}, q)$ , where a semisimple element is primary if it has a unique elementary divisor.

For each nonnegative integer  $c$ , let  $A_c$  denote the elementary abelian group of order  $r^c$  represented by its regular permutation representation. For any sequence  $\mathbf{c} = (c_1, c_2, \dots, c_\ell)$  of nonnegative integers, let  $A_{\mathbf{c}}$  be the wreath product  $A_{c_1} \wr A_{c_2} \wr \dots \wr A_{c_\ell}$ ,  $|\mathbf{c}| = c_1 + c_2 + \dots + c_\ell$ , and let

$$R_{m, \alpha, \gamma, \mathbf{c}} = R_{m, \alpha, \gamma} \wr A_{\mathbf{c}}$$

be the wreath product in  $\text{GL}^\epsilon(d, q)$ , where  $d = \text{mer}^{\alpha+\gamma+|\mathbf{c}|}$ . Then  $R_{m, \alpha, \gamma, \mathbf{c}}$  is uniquely determined up to conjugacy in  $\text{GL}^\epsilon(d, q)$ , which is called a *basic* subgroup of  $\text{GL}^\epsilon(d, q)$ .

Let  $A(R_{m, \alpha, \gamma, \mathbf{c}})$  be the intersection of all maximal normal abelian subgroups of a basic subgroup  $R_{m, \alpha, \gamma, \mathbf{c}}$  and  $Q = \Omega_a(A(R_{m, \alpha, \gamma, \mathbf{c}}))$ . As shown in the proof of [2, (4.1)]  $A(R_{m, \alpha, \gamma, \mathbf{c}}) = Z(R_{m, \alpha, \gamma})^{r^{|\mathbf{c}|}}$ , so that  $Q = \langle z \rangle^{r^{|\mathbf{c}|}}$ , where  $z \in \text{GL}^\epsilon(\text{mer}^{\alpha+\gamma}, q)$  is a primary element of order  $r^\alpha$ . Thus  $Q$  is a characteristic subgroup of  $R_{m, \alpha, \gamma, \mathbf{c}}$  and will be called the *primary subgroup* of  $R_{m, \alpha, \gamma, \mathbf{c}}$ .

**(2A).** *Let  $G = \text{GL}^\epsilon(n, q)$ , and let  $V$  be the underlying space of  $G$  and  $R$  a radical subgroup of  $G$ . Then there exists a corresponding decomposition*

$$(2.1) \quad \begin{aligned} V &= V_0 \perp V_1 \perp \dots \perp V_t, \\ R &= R_0 \times R_1 \times \dots \times R_t \end{aligned}$$

such that  $R_0 = \langle 1_{V_0} \rangle$  and  $R_i$  is a basic subgroup of  $\text{GL}^\epsilon(V_i)$  for  $i \geq 1$ , where  $\text{GL}^+(V_i) = \text{GL}(V_i)$  or  $\text{GL}^-(V_i) = \text{U}(V_i)$  according as  $V_i$  is a linear or unitary space, and  $V_i \perp V_j = V_i \oplus V_j$  when  $V$  is linear. Moreover, the extraspecial components of  $R_i$  have exponent  $r$  for  $i \geq 1$ .

*Proof.* The proof is given by [2, (4A)] and [3, (2B)].

Given  $H \leq G = GL^\epsilon(n, q) = GL^\epsilon(V)$ , let  $C_V(H)$  and  $[V, H]$  be the subspaces of  $V$  generated by the vectors of  $V$  fixed and moved by  $H$ , respectively.

Let  $A(R)$  be the intersections of all the maximal normal abelian subgroups of an  $r$ -subgroup  $R \leq G$ ,  $P(R) = \Omega_a(A(R))$  and  $M_G(R) = O_r(N_G(P(R)))$ . Then  $A(R)$  and  $P(R)$  are characteristic subgroups of  $R$ , and

$$N_G(R) \leq N_G(P(R)) \leq N_G(M_G(R)).$$

If  $R$  has a decomposition (2.1), then  $P(R)$  will be called the *primary subgroup* of  $R$  and a *primary* subgroup of  $G$  is a primary subgroup of a subgroup  $R$  with decomposition (2.1). If  $Q$  is a primary subgroup of  $G$ , then there exists a corresponding decomposition

$$(2.2) \quad \begin{aligned} V &= M_0 \perp M_1 \perp \cdots \perp M_s, \\ Q &= X_0 \times X_1 \times \cdots \times X_s, \end{aligned}$$

where  $X_0 = \langle 1_{M_0} \rangle$ ,  $X_i = \langle z_i \rangle$  for  $i \geq 1$  such that  $[M_i, \langle z_i \rangle] = M_i$  and  $z_i$  is a primary element of order  $r^a$  in  $GL^\epsilon(M_i)$ . In particular,  $X_i = R_{u_i, 0, 0}$ , where  $u_i$  is an integer such that  $u_i e = \dim M_i$  for  $i \geq 1$ . So  $Q$  has a decomposition (2.1) and

$$(2.3) \quad C_G(Q) = GL^\epsilon(M_0) \times \prod_{i=1}^s GL^\epsilon(u_i, q^e).$$

Let  $M(u) = \bigoplus_i M_i$  and  $X(u) = \prod_i X_i$ , where  $i$  runs over the indices such that  $X_i = R_{u, 0, 0}$ . Then

$$(2.4) \quad \begin{aligned} N_G(Q) &= GL^\epsilon(M_0) \times \prod_{u \geq 1} N_{GL^\epsilon(M(u))}(X(u)), \\ N_{GL^\epsilon(M(u))}(X(u)) &= N_{GL^\epsilon(ue, q)}(R_{u, 0, 0}) \wr \mathbf{S}(t_u), \end{aligned}$$

where  $t_u$  is the number of basic components  $R_{u, 0, 0}$  in  $X(u)$ . Moreover,

$$(2.5) \quad N_{GL^\epsilon(ue, q)}(R_{u, 0, 0}) \simeq \langle GL^\epsilon(u, q^e), \tau_u \rangle,$$

where  $\tau_u \in GL^\epsilon(ue, q)$  has order  $e$  acting as a field automorphism on  $GL^\epsilon(u, q^e)$ .

In the rest of this paper we suppose  $\gcd(r, q - \epsilon) = 1$ , so that  $O_r(G) = \langle 1_V \rangle$ . In particular,  $r$  is odd and  $e \geq 2$ .

**(2B).** Let  $GL^\epsilon(u, q^e) \wr \mathbf{S}(v) \leq GL^\epsilon(uve, q)$  and  $W = O_r(GL^\epsilon(u, q^e) \wr \mathbf{S}(v))$ . If  $(q, \epsilon, r) \neq (2, +, 3)$ , then  $W = (R_{u, 0, 0})^v$ . If  $(q, \epsilon, r) = (2, +, 3)$ , then  $W = (R_{u, 0, 0})^v$  or  $\mathbb{Z}_3 \wr \mathbb{Z}_3$

according as  $(u, v) \neq (1, 3)$  or  $(u, v) = (1, 3)$ . In particular,  $P(W) = (R_{u,0,0})^v$  and  $N_{\text{GL}^\epsilon(uv, q^e)}(W) = N_{\text{GL}^\epsilon(uv, q^e)}((R_{u,0,0})^v)$ .

*Proof.* For a finite group  $K$ ,  $O_r(K \wr \mathbf{S}(v)) = O_r(K)^v$  except when  $r = 3$ ,  $O_3(K) = K$  and  $v = 3$ , in which case,  $O_3(K \wr \mathbf{S}(3)) = K \wr \mathbb{Z}_3$ . Thus (2B) follows when  $r \geq 5$ .

Suppose  $r = 3$  and  $O_3(\text{GL}^\epsilon(u, q^e)) = \text{GL}^\epsilon(u, q^e)$ . Then  $u = 1$ ,  $3^a = p^{ef} - \epsilon$ ,  $p^{ef} = 3^a + \epsilon$  is even and  $p = 2$ . If  $\epsilon = +$ , then  $ef = 2e_1$  is even,  $3^a = (2^{e_1} - 1)(2^{e_1} + 1)$  and so  $e_1 = 1$ . Thus  $e = 2$ ,  $f = 1$  and  $q = 2$ . If  $\epsilon = -$  and  $a = 2a_1 + 1$  is odd, then  $ef = 2e_1 + 1$  is odd and

$$(3^{a_1} - 1)(3^{a_1} + 1) = 3^{2a_1} - 1 = 2^{ef} - 2 \cdot 3^{2a_1} = 2(2^{2e_1} - 3^{2a_1}),$$

which is impossible, since  $2 \mid (3^{a_1} - 1)$  and  $2 \mid (3^{a_1} + 1)$  when  $a_1 \geq 1$ . Similarly, if  $a = 2a_1$ , then  $(3^{a_1} - 1)(3^{a_1} + 1) = 2^{ef}$ , and since 4 is not a common factor of  $3^{a_1} - 1$  and  $3^{a_1} + 1$ , either  $3^{a_1} - 1 = 2$  or  $3^{a_1} + 1 = 2$ , which is impossible as  $e \geq 2$  and 3 is not a divisor of  $q + 1$ . This implies (2B).

We say that a radical chain

$$(2.6) \quad C : 1 < P_1 < \cdots < P_w$$

of  $G = \text{GL}^\epsilon(n, q) = \text{GL}^\epsilon(V)$  is *central radical* if  $P_k = M_{N(C_{k-1})}(P_k)$  for  $1 \leq k \leq w$ . By (2.4) and (2.5),  $P_1$  is a primary subgroup of  $G$  and if  $(q, \epsilon, r) \neq (2, +, 3)$ , then by (2B),  $P_k = P(P_k)$  is a primary subgroup of  $C_G(C_{k-1})$  for  $k \geq 1$ . A central radical chain  $C$  has the following properties:

(a) There exists a decomposition

$$(2.7) \quad V = U_0 \perp U_1 \perp \cdots \perp U_v$$

such that  $P_j = Q_{j,0} \times Q_{j,1} \times \cdots \times Q_{j,v}$  for all  $j \geq 1$ , where either  $Q_{j,i}$  is a primary subgroup of  $\text{GL}^\epsilon(U_i)$  or  $Q_{j,i}$  is a direct product of a primary subgroup and some  $(\mathbb{Z}_3 \wr \mathbb{Z}_3)^\beta$  for all  $j \geq 1$  and  $i \geq 0$ . In the latter case,  $(q, \epsilon, r) = (2, +, 3)$ . Moreover, for each  $k$  with  $1 \leq k \leq v$ , there exists  $j$  such that  $Q_{j,k} \neq 1$ .

(b)  $Q_{1,1} \neq \langle 1_{U_1} \rangle$ .

(c)  $Q_{w,0} = \langle 1_{U_0} \rangle$  and  $[U_i, Q_{w,i}] = U_i$  for all  $i \geq 1$ .

(d) If  $Q_{j,i} = \langle 1_{U_i} \rangle$ , then  $Q_{k,\ell} = \langle 1_{U_\ell} \rangle$  for all  $k \leq j$  and  $\ell \geq i$ .

(e) If  $Q_{j,i} \neq \langle 1_{U_i} \rangle$ , then  $[U_i, Q_{j,i}] = U_i$ .

(f) Given  $i \geq 1$ , if  $j$  is the smallest index such that  $Q_{j,i} \neq \langle 1_{U_i} \rangle$ , then  $Q_{j,k} = \langle 1_{U_k} \rangle$  for all  $k \geq i + 1$ .

We shall call  $v = z(C)$  the *size* of  $C$ . Denote by  $\mathcal{CR} = \mathcal{CR}(G)$  or  $\mathcal{MR} = \mathcal{MR}(G)$  the set of all central radical chains of  $G$  according as  $(q, \epsilon, r) \neq (2, +, 3)$  or  $(q, \epsilon, r) = (2, +, 3)$ . Then  $\mathcal{CR}$  and  $\mathcal{MR}$  are  $G$ -invariant subfamilies of  $\mathcal{R}$ .

Suppose  $C$  is central radical with decomposition (2.7). In order to compute  $N_E(C)$  for an extension  $E$  of  $G$ , we always suppose  $\sigma = \sigma_0 \times \sigma_1 \times \dots \times \sigma_v$  is a field automorphism of  $G$  such that each  $\sigma_i$  is a field automorphism of  $GL^\epsilon(U_i)$  of order  $f$  or  $2f$  according as  $\epsilon = +$  or  $-$ . In addition, suppose  $\iota = \iota_0 \times \iota_1 \times \dots \times \iota_v$  such that each  $\iota_i$  is the inverse-transpose of  $GL^\epsilon(U_i)$ , so that  $[\sigma_i, \iota_i] = 1$  for all  $i$  and  $\iota$  is a field automorphism of order 2 when  $\epsilon = -$ . We may always suppose  $E \leq G \rtimes \langle \sigma, \iota \rangle$  and  $\text{Out}(G) = \langle \sigma, \iota \rangle$ .

For  $1 \leq i \leq t$ , let  $I_i$  be a finite subset of positive integers,  $\mathbf{n}_i = \{n_{i,j} : j \in I_i\}$  a set of positive integers,  $\mathbf{n} = \{\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_t\}$ ,  $\mathbf{X}_i = \{X_{i,j} : j \in I_i\}$  and  $\mathbf{X} = \{\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_t\}$ , where  $X_{i,j}$  is a subgroup of  $\mathbf{S}(n_{i,j})$ . In addition, let  $K$  be a subgroup of  $GL^\epsilon(m, q)$ . Then  $K \wr X_{1,j}$  for any  $j \in I_1$  is a subgroup of  $GL^\epsilon(mn_{1,j}, q)$ . Let  $K \wr \mathbf{X}_1 = \prod_{j \in I_1} K \wr X_{1,j}$  and

$$K \wr \mathbf{X} = K \wr \mathbf{X}_1 \wr \mathbf{X}_2 \wr \dots \wr \mathbf{X}_t$$

be the subgroups of  $GL^\epsilon(m|\mathbf{n}_1|, q)$  and  $GL^\epsilon(m|\mathbf{n}|, q)$ , respectively, where  $|\mathbf{n}_i| = \sum_{j \in I_i} n_{i,j}$  and  $|\mathbf{n}| = \sum_{i=1}^t |\mathbf{n}_i|$ . If  $X_{i,j} = \mathbf{S}(n_{i,j})$  for all  $i, j$ , then we set

$$(2.8) \quad K \wr \mathbf{S}(\mathbf{n}) = K \wr \mathbf{X}.$$

Given  $1 \leq i \leq t$  and  $j \in I_i$ , let  $\mathbf{c}_{i,j}$  be a sequence of nonnegative integers,  $n_{i,j} = r^{|\mathbf{c}_{i,j}|}$  and  $X_{i,j}$  a basic subgroup  $A_{\mathbf{c}_{i,j}}$  of  $\mathbf{S}(r^{|\mathbf{c}_{i,j}|})$ . We set

$$(2.9) \quad K \wr A_{\mathbf{z}} = K \wr \mathbf{X},$$

where  $\mathbf{z} = \{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_t\}$  such that  $\mathbf{z}_i = \{\mathbf{c}_{i,j} : j \in I_i\}$ . In particular,  $R_{m,\alpha,\gamma,\mathbf{c}} = R_{m,\alpha,\gamma} \wr A_{\mathbf{z}}$  with  $\mathbf{z} = \{\mathbf{c}\}$ , and if  $K$  is an  $r$ -subgroup, then so is  $K \wr A_{\mathbf{z}}$ .

Suppose the chain  $C$  given by (2.6) is central radical with decomposition (2.7). Then for  $1 \leq j \leq w$

$$P_j = P_{j,0} \times Q_{j,1} \times \dots \times Q_{j,i-1} \times Q_{j,i},$$



where  $P_{j,0}$  is the trivial subgroup of  $GL^\epsilon(C_V(P_j))$  and  $Q_{j,k} \neq 1$  for  $1 \leq k \leq i$ . By the property (f),  $i \leq j$  and by (d),  $C_V(P_j) = U_0 + \sum_{k=i+1}^v U_k$ . Thus  $N(C_j) = GL^\epsilon(C_V(P_j)) \times N_{j,1} \times \cdots \times N_{j,i}$ , where  $N_{j,k} = \cap_{\ell=1}^j N_{GL^\epsilon(U_k)}(Q_{\ell,k})$  for  $1 \leq k \leq i$ . In particular,

$$(2.10) \quad N(C) = GL^\epsilon(U_0) \times N_1 \times N_2 \times \cdots \times N_v,$$

where  $N_k = \cap_{\ell=1}^w N_{GL^\epsilon(U_i)}(Q_{\ell,k})$  for all  $k \geq 1$ . Moreover, if  $w = 1$ , then  $v = w = 1$  and  $N_1 = N_{GL^\epsilon(U_1)}(Q_{1,1})$  is given by (2.4). If  $w \geq 2$ , then by (2.4) (with  $e = 1$ ),

$$(2.11) \quad N_k = \prod_{i \in I'_0} \left\langle \tau_i, \left( \prod_{j \in I_0} GL^\epsilon(m_j, q^e) \wr \mathbf{S}(\mathbf{w}_j) \right) \right\rangle \wr \mathbf{S}(i),$$

where  $|\tau_i| = e$ ,  $I'_0$  and  $I_0$  are sets of some positive integers, and  $GL^\epsilon(m_j, q^e) \wr \mathbf{S}(\mathbf{w}_j)$  is defined by (2.8) with  $K = GL^\epsilon(m_j, q^e)$  and  $\mathbf{w}_j = \mathbf{n}$ .

An  $r$ -subgroup  $D$  is called a *quasi-radical* subgroup of  $G = GL^\epsilon(n, q)$  if  $D = \prod_{i=0}^t D_i$  such that  $D_0$  is the trivial subgroup of  $GL^\epsilon(C_V(D))$  and  $D_i = R_i \wr A_{\mathbf{z}_i}$  for  $i \geq 1$ , where  $R_i$  is a radical subgroup of  $GL^\epsilon(m_i, q^e)$  and  $R_i \wr A_{\mathbf{z}_i}$  is defined by (2.9). By [1, Theorem 2] and induction,

$$A(D) = D_0 \times \prod_{i=1}^t (A(R_i))^{|z_i|}$$

and  $P(D)$  is a primary subgroup of  $GL^\epsilon(n, q)$ , called the *primary* subgroup of  $D$ . Since  $P(M_G(D)) = P(D)$  by (2B), it follows that

$$(2.12) \quad N_G(D) \leq N_G(P(D)) = N_G(M_G(D)).$$

**(2C).** Let  $G = GL^\epsilon(n, q) = GL^\epsilon(V)$ , and let  $C$  be the central radical chain (2.6) with  $w \geq 1$  and decomposition (2.7).

(a) Fix  $1 \leq i \leq v$ . Let  $N_i = \cap_{\ell=1}^w N_{GL^\epsilon(U_i)}(Q_{\ell,i})$ , and let  $W$  be a radical subgroup of  $N_i$ . Then  $W$  is quasi-radical,  $S = M_{N_i}(W)$  is radical in  $N_i$  and  $S \leq W$ . In particular,  $Q_{w,i} \trianglelefteq S$  and  $N_{N_i}(W) \leq N_{N_i}(S)$ . Moreover, if  $S = Q_{w,i}$ , then  $W$  is radical in  $N'_i = \cap_{\ell=1}^{w-1} N_{GL^\epsilon(U_i)}(Q_{\ell,i})$  and  $N_{N_i}(W) = N_{N'_i}(W)$ , where  $N'_i = GL^\epsilon(U_1)$  when  $w = 1$ .

(b) If  $R$  is a radical subgroup of  $N(C)$ , then  $R$  is quasi-radical. If  $D = M_{N(C)}(R)$ , then  $D$  is radical in  $N(C)$ ,  $P_w \trianglelefteq D \leq R$ , and  $N_{N(C)}(R) \leq N_{N(C)}(D)$ . In addition, if  $P_w \neq D$ , then

$$C' : P_0 < P_1 < \cdots < P_w < D$$

is a central radical chain of  $G$ ,  $R$  is radical in  $N(C')$ , and  $N_{N(C')}(R) = N_{N(C)}(R)$ . If  $P_w = D$ , then  $R$  is radical in  $N(C_{w-1})$  and  $N_{N(C_{w-1})}(R) = N_{N(C)}(R)$ .

*Proof.* (a). Let  $H = N_i$  decompose as (2.11). By [14, 2.2, 2.3 and 2.5] and induction, we may suppose  $H = \langle \tau, \text{GL}^\epsilon(m, q^\epsilon) \wr \mathbf{S}(\mathbf{z}) \rangle$ . Since  $\gcd(e, r) = 1$  and  $\text{GL}^\epsilon(m, q^\epsilon) \wr \mathbf{S}(\mathbf{z}) \trianglelefteq H$  and since  $W \leq \text{GL}^\epsilon(m, q^\epsilon) \wr \mathbf{S}(\mathbf{z})$ , it follows by [14, 2.1] that we may suppose  $H = \text{GL}^\epsilon(m, q^\epsilon) \wr \mathbf{S}(\mathbf{z})$ . By [14, 2.3 and 2.5] and induction, we may suppose  $H = \text{GL}^\epsilon(m, q^\epsilon)$ . Thus  $W$  is quasi-radical. By (2.12),  $O_r(N_{N_i}(S)) = S$  and  $S$  is radical in  $N_i$ , so that  $Q_{w,i} \leq S$ . Since  $N_{N_i}(W) \leq N_{N_i}(S)$ , it follows that  $W$  is radical in  $N_{N_i}(S)$ , and so  $S = O_r(N_{N_i}(S)) \leq W$ .

If  $S = Q_{w,i}$ , then  $N_i = N_{N'_i}(S)$  and by (2.12),

$$N_{N'_i}(W) = N_{\text{GL}^\epsilon(U_i)}(W) \cap N'_i \leq N_{\text{GL}^\epsilon(U_i)}(S) \cap N'_i = N_i,$$

so that  $N_{N'_i}(W) = N_{N_i}(W)$  and  $W$  is radical in  $N'_i$ .

(b). Suppose  $R$  is a radical subgroup of  $N(C)$ . By (2.10) and [14, Lemma 2.2],  $R = R(0) \times \prod_{i=1}^v R(i)$ , where  $R(0)$  and  $R(i)$  are radical subgroups of  $\text{GL}^\epsilon(U_0)$  and  $N_i$ , respectively for all  $i \geq 1$ . By (2A) and part (a) above,  $R(0)$  and  $R(i)$  are quasi-radical, so is  $R$ . Let  $D(0) = M_{\text{GL}^\epsilon(U_0)}(R(0))$  and  $D(i) = M_{N_i}(R(i))$ . Then  $D = D(0) \times \prod_{i=1}^v D(i) = M_{N(C)}(R)$  and  $D(0)$  and  $D(i)$  are radical in  $\text{GL}^\epsilon(U_0)$  and  $N_i$ , respectively. So  $D$  is radical in  $N(C)$ , and the chain  $C'$  defined in (b) is central radical. Since  $N_{N(C)}(R) \leq N_{N(C)}(D)$ , it follows that  $N_{N(C)}(R) = N_{N_{N(C)}(D)}(R) = N_{N(C')}(R)$ .

Finally, if  $D = P_w$ , then  $R(0) = D(0) = \langle 1_{U_0} \rangle$  and  $D(i) = Q_{w,i}$ . By part (a), each  $R(i)$  is radical in  $N'_i = \cap_{\ell=1}^{w-1} N_{\text{GL}^\epsilon(U_i)}(Q_{\ell,i})$  for  $i \geq 1$ , so that  $R$  is radical in  $N(C_{w-1})$ ,  $N_{N(C_{w-1})}(R) \leq N_{N(C_{w-1})}(D) = N(C)$  and  $N_{N(C_{w-1})}(R) = N_{N(C)}(R)$ . This proves (2C).

*Remark.* Suppose  $(q, \epsilon, r) = (2, +, 3)$  and  $C \in \mathcal{MR}(G)$  given by (2.6). Let

$$P(C) : 1 < P(P_1) < P(P_2) < \dots < P(P_w)$$

and  $\mathcal{CR} = \mathcal{CR}(G) = \{P(C) : C \in \mathcal{MR}(G)\}$ . Then each subgroup of a chain in  $\mathcal{CR}$  is a primary subgroup of  $G$ , the map  $C \mapsto P(C)$  is a bijection between  $\mathcal{MR}(G)$  and  $\mathcal{CR}$  and by (2.12),  $N_G(C) = N_G(P(C))$ . Moreover, if  $E$  is an extension of  $G$ , then

$$N_E(C) = N_E(P(C)).$$

It follows that we can identify  $C$  with  $P(C)$  and view  $\mathcal{CR}$  as a subfamily of  $\mathcal{R}$ .

(2D). Let  $G = \mathrm{GL}^\epsilon(n, q) = \mathrm{GL}^\epsilon(V)$ ,  $B \in \mathrm{Blk}(G)$  with defect  $d(B) \geq 1$ ,  $d \geq 0$  an integer and  $U \leq \mathrm{Out}(G)$ . Then

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} k(N(C), B, d, U) = \sum_{C \in \mathcal{CR}/G} (-1)^{|C|} k(N(C), B, d, U).$$

*Proof.* Let  $\mathcal{M}$  be the complementary  $G$ -invariant subfamily  $\mathcal{R} \setminus \mathcal{CR}$  of  $\mathcal{CR}$  in  $\mathcal{R}$ . It suffices to show that

$$\sum_{C \in \mathcal{M}/G} (-1)^{|C|} k(N(C), B, d, U) = 0.$$

Suppose  $C \in \mathcal{M}$  is given by (1.1). Let  $m = m(C) \geq 0$  be the largest integer such that  $C_m \in \mathcal{CR}$  and  $C_{m+1} \notin \mathcal{CR}$ , so that  $0 \leq m \leq w - 1$ . Since  $P_{m+1}$  is radical in  $N(C_m)$ , it follows that  $P_{m+1}$  is quasi-radical,  $D = M_{N(C_m)}(P_{m+1})$  is radical in  $N(C_m)$ , and  $P_m \trianglelefteq D$ . Define a map  $\varphi$  from  $\mathcal{M}$  to itself such that

$$\varphi(C) : \begin{cases} 1 < P_1 < \dots < P_m < D < P_{m+1} < \dots < P_w & \text{if } P_m \neq D, \\ 1 < P_1 < \dots < P_{m-1} < P_{m+1} < \dots < P_w & \text{if } P_m = D. \end{cases}$$

By (2C),  $N(C) = N(\varphi(C))$ ,  $\varphi(\varphi(C)) = C$  and  $|\varphi(C)| = |C| \pm 1$ . Since  $N_E(C) = N_E(\varphi(C))$  for an extension  $E$  of  $G$ , (2D) follows.

### 3. MORE REDUCTIONS

Let  $\mathcal{F} = \mathcal{F}_q$  be the set of polynomials serving as elementary divisors for all semisimple elements of  $G = \mathrm{GL}^\epsilon(n, q)$ . If  $G = \mathrm{GL}(n, q)$ , then  $\mathcal{F}$  consists of all monic irreducible polynomials over the field  $\mathbb{F}_q$  of  $q$  elements with non-zero roots. Suppose  $G = \mathrm{U}(n, q)$  and let  $\Delta(T) = T^m + a_{m-1}T^{m-1} + \dots + a_1T + a_0$  be a monic polynomial of  $\mathbb{F}_{q^2}[T]$  with  $a_0 \neq 0$ , and  $\tilde{\Delta}(T) = (a_0^{-1})^q T^m \Delta^q(T^{-1})$ . Then

$$\mathcal{F}_1 = \{\Delta : \Delta \text{ is monic, irreducible, } \Delta \neq T, \Delta = \tilde{\Delta}\}$$

$$\mathcal{F}_2 = \{\Delta \tilde{\Delta} : \Delta \text{ is monic, irreducible, } \Delta \neq T, \Delta \neq \tilde{\Delta}\}$$

and  $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ .

A semisimple element  $s \in G$  has a primary decomposition  $V = \sum_{\Gamma \in \mathcal{F}} V_\Gamma$ ,  $s = \prod_{\Gamma \in \mathcal{F}} s_\Gamma$ , where each  $s_\Gamma \in \mathrm{GL}^\epsilon(V_\Gamma)$  has a unique elementary divisor  $\Gamma \in \mathcal{F}$  of multiplicity  $m_\Gamma(s)$ . Let  $d_\Gamma$  be the degree of  $\Gamma \in \mathcal{F}$ ,  $\delta_\Gamma = d_\Gamma$  (respectively,  $\epsilon_\Gamma = +$ ) except when  $\epsilon = -$  and  $\Gamma \in \mathcal{F}_2$  (respectively,  $\Gamma \in \mathcal{F}_1$ ), in which case  $\delta_\Gamma = \frac{1}{2}d_\Gamma$  (respectively,  $\epsilon_\Gamma = -$ ), and let  $e_\Gamma$  be the multiplicative order of  $\epsilon_\Gamma q^{\delta_\Gamma}$  modulo  $r$ . The Brauer pairs  $(R, b)$  of  $G$

have been labelled by Broué [4, (3.1)]. This labelling is by ordered triples  $(R, s, \kappa)$ , where  $s$  is a semisimple  $r'$ -element of the dual group  $G^*$  of  $G$ , and  $\kappa = \prod_{\Gamma \in \mathcal{F}} \kappa_\Gamma$  is a product of partitions  $\kappa_\Gamma$  such that each  $\kappa_\Gamma$  is an  $e_\Gamma$ -core. This labelling extends the labelling [10, (5D)], by Fong and Srinivasan for block  $B \in \text{Blk}(G)$  by ordered pair  $(s, \kappa)$ . Since  $G \simeq G^*$ , we may identify  $G^*$  with  $G$ , so that  $s \in G$ .

Given  $C \in \mathcal{CR}$  with a final subgroup  $P_w$ , let  $C_V(C)$  be the fixed-point subspace  $C_V(P_w)$  of  $P_w$  in  $V$ . Fix an integer  $m \geq 1$ , suppose  $V$  decomposes as (2.7) with  $\dim U_1 = me$ , and let

$$V(m) = U_1, \quad V(m)' = U_0 \perp U_2 \perp \cdots \perp U_v,$$

so that  $V = V(m) \perp V(m)'$ . Set  $G(m) = \text{GL}^\epsilon(V(m))$  and  $G(m)' = \text{GL}^\epsilon(V(m)')$ .

Let  $X$  be a primary radical subgroup of  $G(m)$  with  $[V(m), X] = V(m)$ , and  $\mathcal{S}(X)$  the subset of  $\mathcal{CR}(G)$  consisting of all chains  $C$  such that its first non-trivial subgroup is  $P_X = X \times \langle 1_{V(m)'} \rangle$ . Then

$$N(C_1) = N(P_X) = N_X \times G(m)',$$

where  $N_X = N_{G(m)}(X)$ .

**(3A).** *In the notation above, suppose  $B \in \text{Blk}(G)$  with defect group  $D(B) \neq 1$  such that  $m \neq \dim [V, D(B)]$ . Then*

$$(3.1) \quad \sum_{C \in \mathcal{S}(X)/N(P_X)} (-1)^{|C|} k(N(C), B, d, U) = 0$$

for all integers  $d \geq 0$  and  $U \leq \text{Out}(G)$ . If  $m = \dim [V, D(B)]$ , then we may suppose  $C_V(C) = C_V(D(B)) = V(m)'$  for  $C \in \mathcal{S}(X)$ .

*Proof.* Let  $\mathcal{CR}(N_X \times G(m)')$  and  $\mathcal{CR}(N_X)$  be the subfamilies of  $\mathcal{R}(N_X \times G(m)')$  and  $\mathcal{R}(N_X)$ , respectively consisting of chains  $C$  given by (1.1) with  $P(P_i) = P_i$  for all  $i \geq 1$ . Here we have a similar identification to that in the remark after (2C) when  $(q, \epsilon, r) = (2, +, 3)$ . If  $C \in \mathcal{S}(X)$ , then  $C : 1 < P_X < P_2 < \cdots < P_w$  and  $\varphi(C) : P_X < P_2 < \cdots < P_w$  is a chain of  $\mathcal{CR}(N_X \times G(m)')$ ,  $|C| = |\varphi(C)| + 1$ ,  $N(C) = N_{N_X \times G(m)'}(\varphi(C))$  and  $\varphi$  is a bijection between  $\mathcal{S}(X)$  and  $\mathcal{CR}(N_X \times G(m)')$ .

Let  $\pi_m$  and  $\pi'$  be the projections of  $N(P_X)$  onto  $N_X$  and  $G(m)'$ , respectively. Suppose  $C \in \mathcal{CR}(N_X \times G(m)')$  is given by (1.1). Then  $\pi_m(C) \in \mathcal{CR}(N_X)$ ,  $\pi'(C) \in \mathcal{CR}(G(m)')$

and  $N(C) = N_{N_X}(\pi_m(C)) \times N_{G(m)'}(\pi'(C))$ , where  $\pi_m(C)$  and  $\pi'(C)$  are defined as (1A). Conversely, if  $C \in \mathcal{R}(N_X \times G(m)')$  such that  $\pi_m(C) \in \mathcal{CR}(N_X)$  and  $\pi'(C) \in \mathcal{CR}(G(m)'),$  then by definition,  $C \in \mathcal{CR}(N_X \times G(m)').$

Given  $C \in \mathcal{S}(X)$ , let  $C_X = \pi_m(\varphi(C))$  and  $C' = \pi'(\varphi(C))$ , so  $C(C) = C_{N_X}(C_X) \times C_{G(m)'}(C')$  and

$$N(C) = N_{N_X \times G(m)'}(\varphi(C)) = N_{N_X}(C_X) \times N_{G(m)'}(C').$$

Let  $(s, \kappa)$  be the label of  $B$ . If  $b(C) \in \text{Blk}(N(C)|B)$ , then  $b(C) = b(C)_X \times b(C)'$ , where  $b(C)_X$  and  $b(C)'$  are blocks of  $N_{N_X}(C_X)$  and  $N_{G(m)'}(C')$ , respectively. Let  $b_X \in \text{Blk}(C_{N_X}(C_X))$  and  $b' \in \text{Blk}(C_{G(m)'}(C'))$  such that  $b(C)_X = b_X^{N_{N_X}(C_X)}$  and  $b(C)' = b'^{N_{G(m)'}(C')}$ . If  $b = b_X \times b'$ , then  $b^G = B$ , so that  $(P_w, b)$  is a Brauer  $B$ -pair. Let  $(P_w, s_m \times s'_m, \kappa_m \times \kappa'_m)$  be the label of the pair  $(P_w, b)$ , where  $s_m \in C_{N_X}(C_X)$  and  $s'_m \in C_{G(m)'}(C')$ . By the Broué-Puig theorem, [4, (3.2)],  $s$  and  $s_m \times s'_m$  are conjugate in  $\text{GL}^\epsilon(V)$ . Since  $X \leq O_r(C_{N_X}(C_X)) \leq D(b_X)$ , it follows that  $\kappa_m = -$  is the empty set and  $\kappa = \kappa'_m$ . Let  $B_m$  and  $B'_m$  be the blocks of  $G(m)$  and  $G(m)'$  labelled by  $(s_m, -)$  and  $(s'_m, \kappa)$ , respectively. Then  $b_X^{G(m)} = b(C)_X^{G(m)} = B_m$  and  $b'^{G(m)'} = b(C)'^{G(m)'} = B'_m$ . The blocks  $B_m$  and  $B'_m$  depend only on the decomposition  $s_m \times s'_m$  of  $s$  in  $G(m) \times G(m)'$  and not on the choice of  $C \in \mathcal{S}(X)$ .

Let  $\sigma' = \sigma_0 \times \sigma_2 \times \sigma_3 \times \dots \times \sigma_v$  and  $\iota' = \iota_0 \times \iota_2 \times \iota_3 \times \dots \times \iota_v$ , so that  $\sigma = \sigma_1 \times \sigma'$  and  $\iota = \iota_1 \times \iota'$ . Suppose  $U = \langle \gamma, \rho \rangle \leq \text{Out}(G) = \langle \sigma, \iota \rangle$ , where  $\gamma = \sigma^\ell$  and  $\rho = \iota^i$  for some integers  $\ell$  and  $i$ . Let  $\gamma_X = \sigma_1^\ell$ ,  $\gamma' = \sigma'^\ell$ ,  $\rho_X = \iota_1^i$ ,  $\rho' = \iota'^i$ ,  $U_X = \langle \gamma_X, \rho_X \rangle$  and  $U' = \langle \gamma', \rho' \rangle$ . Since  $\sigma_1$  and  $\iota_1$  are field and the inverse-transpose maps of  $\text{GL}^\epsilon(V(m))$ , respectively, we may suppose they stabilize  $X$ , so that  $\sigma_1, \iota_1 \in \text{Aut}(N_X)$ . It follows by (1A) (b) that

$$(3.2) \quad \sum_{C \in \mathcal{S}(X)/N(P_X)} (-1)^{|C|} k(N(C), B, d, U) = - \sum_{u+t=d} \sum_{C_X} (-1)^{|C_X|} k(N_{N_X}(C_X), B_m, u, U_X) \\ \times \sum_{C'} (-1)^{|C'|} k(N_{G(m)'}(C'), B'_m, t, U'),$$

where  $C_X$  and  $C'$  run over  $\mathcal{CR}(N_X)/N_X$  and  $\mathcal{CR}(G(m)')/G(m)'$ , respectively.

Since  $D(B_m)$  and  $D(B'_m)$  can be viewed as Sylow  $r$ -subgroups of  $C_{G(m)}(s_m)$  and  $C_{G(m)'}(s'_m)$ , respectively, we may suppose

$$P_X \leq D(B_m) \times D(B'_m) \leq D(B).$$

In addition, we may suppose  $Z(D(B)) \leq Z(D(b_X)) \times Z(D(b'))$  and  $D(b') \leq D(B'_m)$ . Thus  $B'_m$  has defect 0 if and only if  $m = \dim [V, D(B)]$ .

If  $m \neq \dim [V, D(B)]$ , then  $m < \dim [V, D(B)]$  and  $\dim V(m)' < \dim V$ . If  $\dim V(m)' \leq e$ , then  $\text{GL}^\epsilon(V(m)')$  has a cyclic Sylow  $r$ -subgroup and Dade's invariant conjecture for  $B'_m$  follows by [8, Theorem 7.11]. By induction on  $\dim V$ ,

$$\sum_{C \in \mathcal{CR}(G(m)')/G(m)'} (-1)^{|C|} k(N_{G(m)'}(C), B'_m, t, U') = 0$$

for all integers  $t \geq 0$ . Thus (3.1) follows by (3.2).

If  $m = \dim [V, D(B)]$ , then  $B'_m$  has defect 0,  $D(B'_m) = 1$  and  $V(m)' = C_V(P_X) = C_V(D(B))$ . We may suppose each non-trivial subgroup of a chain  $C \in \mathcal{S}(X)$  is a subgroup of  $D(B)$ , so  $C_V(C) = C_V(D(B)) = V(m)'$ . This proves (3A).

Let  $\mathcal{CR}(B) = \{C \in \mathcal{CR} : \text{Blk}(N(C)|B) \neq \emptyset\}$ , and let

$$\mathcal{CR}^*(B) = \mathcal{CR}(B) \setminus \left( \bigcup_X \mathcal{S}(X) \right)$$

and  $\mathcal{CR}^*(G) = \cup_B \mathcal{CR}^*(B)$ , where  $P_X = X \times \langle 1_{C_V(P_X)} \rangle$  runs over primary subgroups of  $G$  such that  $\dim C_V(P_X) \neq \dim C_V(D(B))$ , and  $B$  runs over all blocks of  $G$  with positive defect. If  $C \in \mathcal{CR}^*(B)$ , then by (3A), we may suppose  $C_V(C) = C_V(D(B))$ .

Let  $\chi$  be an irreducible unipotent character of  $G = \text{GL}^\epsilon(V) = \text{GL}^\epsilon(n, q)$ . If  $\epsilon = +$ , then  $\chi$  is rational and  $\iota$  induces the dual map on  $\text{Irr}(G)$ , so that  $\iota$  stabilizes  $\chi$ . Similarly, the field automorphism  $\sigma$  also stabilizes  $\chi$ . If  $\epsilon = -$ , then  $\chi$  is a linear combination of some  $R_T^G(1)$  with rational coefficients, where each  $T$  is a  $\sigma$ -invariant maximal torus of  $G$ . So  $\sigma$  also stabilizes  $\chi$ . In both cases,  $\chi^\tau = \chi$  for  $\tau \in G \rtimes \langle \sigma, \iota \rangle$ .

Suppose  $\chi_{s, \mu}$  is an irreducible character of  $G$  labelled by  $(s, \chi_\mu)$ , where  $\chi_\mu$  is an irreducible unipotent character of  $C_G(s)$ . If  $\tau \in \langle \sigma, \iota \rangle$ , then we may suppose  $\tau$  stabilizes  $\chi_\mu$ ,  $\chi_{s, \mu}^\tau$  is labelled by  $(s^\tau, \chi_\mu)$  and so

$$(3.3) \quad \chi_{s, \mu}^\tau = \chi_{s^\tau, \mu}.$$

In addition,  $d(\chi_{s, \mu}) = a(G) - a(\chi_{s, \mu})$  and  $a(\chi_{s, \mu}) = a(|G : C_G(s)|) + a(\chi_\mu)$ . Thus

$$(3.4) \quad d(\chi_{s, \mu}) = a(C_G(s)) - a(\chi_\mu) = d(\chi_\mu).$$

Let  $B$  be a block of  $G = \text{GL}^\epsilon(V)$  labelled by  $(s, \kappa)$ ,  $D$  a defect group of  $B$ ,  $V_0 = C_V(D)$  and  $V_+ = [V, D]$ . Then  $D = D_0 \times D_+$  and we may suppose  $s \in C_G(D) = \text{GL}^\epsilon(V_0) \times C_+$ ,

where  $D_0 = \langle 1_{V_0} \rangle$ ,  $D_+ \leq G_+ = \text{GL}^\epsilon(V_+)$  and  $C_+ = C_{\text{GL}^\epsilon(V_+)}(D_+)$ . Thus  $s = s_0 \times s_+$  and we may suppose  $\sigma = \sigma_0 \times \sigma_+$ ,  $\iota = \iota_0 \times \iota_+$  such that  $s^\sigma = s_0^{\sigma_0} \times s_+^{\sigma_+}$  and  $s^\iota = s_0^{\iota_0} \times s_+^{\iota_+}$ , where  $s_0 \in \text{GL}^\epsilon(V_0)$  and  $s_+ \in C_+$ . Suppose  $U \leq \text{Out}(G) = \langle \sigma, \iota \rangle$ , so that  $U = \langle x, \rho \rangle$ , where  $x = \sigma^\ell$  and  $\rho = \iota^i$  for some integers  $\ell, i \geq 0$ . Let  $x_0 = \sigma_0^\ell$ ,  $\rho_0 = \iota_0^i$ ,  $x_+ = \sigma_+^\ell$ ,  $\rho_+ = \iota_+^i$ ,  $U_0 = \langle x_0, \rho_0 \rangle$  and  $U_+ = \langle x_+, \rho_+ \rangle$ . Then  $U_0 \leq \text{Out}(\text{GL}^\epsilon(V_0)) = \langle \sigma_0, \iota_0 \rangle$  and  $U_+ \leq \text{Out}(G_+) = \langle \sigma_+, \iota_+ \rangle$ .

**(3B).** *In the notation above, let  $B(0)$  and  $B(+)$  be blocks of  $G_0 = \text{GL}^\epsilon(V_0)$  and  $G_+$  labelled by  $(s_0, \kappa)$  and  $(s_+, -)$ , respectively. Then there exists a defect preserving bijection between  $\text{Irr}(B)$  and  $\text{Irr}(B(+))$ . Moreover, there exists a bijection  $\varphi$  between  $\mathcal{CR}^*(B)$  and  $\mathcal{CR}^*(B(+))$  such that  $N(C) = G_0 \times N_{G_+}(\varphi(C))$  for  $C \in \mathcal{CR}^*(B)$  with  $|C| \geq 1$ . In addition,*

$$(3.5) \quad k(N(C), B, d, U) = k(N_{G_+}(\varphi(C)), B(+), d, U_+)$$

for any integers  $d \geq 0$  and  $U \leq \text{Out}(G)$ .

*Proof.* Let  $\chi_{sy, \mu}$  be an irreducible character of  $\text{Irr}(B)$  labelled by  $(sy, \chi_\mu)$ , where  $y$  is an  $r$ -element of  $C_G(s)$ . Then  $y = 1_{V_0} \times y_+$  for some element  $y_+ \in C_{G_+}(s_+)$ , and  $sy = \prod_{\Gamma} (sy)_{\Gamma}$  and  $\mu = \prod_{\Gamma} \mu_{\Gamma}$ , where  $\mu_{\Gamma}$  is a partition of  $m_{\Gamma}(sy)$ . Suppose  $\kappa_{\Gamma}$  and  $Q(\mu_{\Gamma})$  are the  $e_{\Gamma}$ -core and  $e_{\Gamma}$ -quotient of  $\mu_{\Gamma}$ , respectively. Then  $\kappa = \prod_{\Gamma} \kappa_{\Gamma}$  and  $\mu_{\Gamma}$  is uniquely determined by  $\kappa_{\Gamma}$  and  $Q(\mu_{\Gamma})$ . Let  $(\mu_{\Gamma})_+$  be a partition of  $m_{\Gamma}(s_+y_+)$  with  $Q(\mu_{\Gamma})$  as its  $e_{\Gamma}$ -quotient and  $-$  as its  $e_{\Gamma}$ -core, and let  $\mu_+ = \prod_{\Gamma} (\mu_{\Gamma})_+$ . Then the character  $\xi_{s_+y_+, \mu_+}$  of  $G_+$  labelled by  $(s_+y_+, \chi_{\mu_+})$  is an character of  $\text{Irr}(B(+))$  and each character of  $\text{Irr}(B(+))$  is of this form. Define  $\Psi(\xi_{s_+y_+, \mu_+}) = \chi_{sy, \mu}$ . By [10, Theorem (7A)],  $\Psi$  is a bijection between  $\text{Irr}(B(+))$  and  $\text{Irr}(B)$ . In addition, by (3.4) and the hook-length formula [10, (1.15)],  $\Psi$  is defect preserving. If  $\tau = \tau_0 \times \tau_+ \in \text{Out}(G)$  for some  $\tau_0 \in \text{Out}(\text{GL}^\epsilon(V_0))$  and  $\tau_+ \in \text{Out}(G_+)$ , then  $\chi_{sy, \mu}^{\tau} = \chi_{(sy)^{\tau}, \mu}$  and  $\xi_{s_+y_+, \mu_+}^{\tau_+} = \xi_{(s_+y_+)^{\tau_+}, \mu_+}$ . It follows that

$$\Psi(\xi)^{\tau} = \Psi(\xi^{\tau_+})$$

for any  $\tau \in \langle \sigma, \iota \rangle$ , so that  $k(G, B, d, U) = k(G_+, B(+), d, U_+)$ .

Suppose  $C \in \mathcal{CR}^*(B)$  is given by (1.1) with  $|C| \geq 1$ . We may suppose  $P_t = \langle 1_{V_0} \rangle \times P_+(t)$  for all  $t \geq 1$ , where  $P_+(t) \leq G_+$ . Define  $\varphi(C) : 1 < P_+(1) < P_+(2) < \dots < P_+(w)$ . Then  $\varphi(C) \in \mathcal{CR}^*(B(+))$  and  $N(C) = G_0 \times N_{G_+}(\varphi(C))$ . Since  $\text{Irr}(B(0))$  has exactly one character of defect 0, it follows that  $k(G_0, B(0), t) = 1$  or 0 according as  $t = 0$  or  $t \neq 0$ . This and a similar proof to that of (1.5) imply (3.5).

Let  $B$  be a block of  $G = \mathrm{GL}^\epsilon(n, q) = \mathrm{GL}^\epsilon(V)$  with defect group  $D$ . By (3B), we may suppose  $[V, D] = V$ , so that  $n = ue$  and  $B$  is labelled by  $(s, -)$ . Let  $V = \sum_{\Gamma} V_{\Gamma}$ ,  $s = \prod_{\Gamma} s_{\Gamma}$  be the primary decomposition of  $s$  and  $L = C_G(s) = \prod_{\Gamma} L_{\Gamma}$ , where  $L_{\Gamma} = \mathrm{GL}^{\epsilon_{\Gamma}}(m_{\Gamma}(s), q^{\delta_{\Gamma}}) = C_{\mathrm{GL}^{\epsilon}(V_{\Gamma})}(s_{\Gamma})$ . We may choose the decomposition such that  $\sigma$  and  $\iota$  stabilizes each  $V_{\Gamma}$ . Thus  $\sigma = \prod_{\Gamma} \sigma_{\Gamma}$  and  $\iota = \prod_{\Gamma} \iota_{\Gamma}$ , where  $\sigma_{\Gamma}$  and  $\iota_{\Gamma}$  induce field and the inverse-transpose maps on  $\mathrm{GL}^{\epsilon}(V_{\Gamma})$ , respectively. Suppose  $U$  is a subgroup of  $\mathrm{Out}(G) = \langle \sigma, \iota \rangle$ , so that  $U = \langle x, \rho \rangle$  with  $x = \sigma^{\ell}$  and  $\rho = \iota^i$  for some integers  $\ell, i \geq 0$ . Let  $x_{\Gamma} = \sigma_{\Gamma}^{\ell}$ ,  $\rho_{\Gamma} = \iota_{\Gamma}^i$  and  $U_{\Gamma} = \langle x_{\Gamma}, \rho_{\Gamma} \rangle$ . Then  $U_{\Gamma} \leq \mathrm{Out}(\mathrm{GL}^{\epsilon}(V_{\Gamma})) = \langle \sigma_{\Gamma}, \iota_{\Gamma} \rangle$ .

Let  $\pi_{\Gamma}$  be the natural projection onto  $V_{\Gamma}$  and  $C \in \mathcal{CR}^*(B)$ . We may suppose  $D$  is a Sylow  $r$ -subgroup of  $L$  and each subgroup of  $C$  is a subgroup of  $D$ . Thus  $C$  is an  $r$ -subgroup chain of  $L$ . Let  $\mathcal{Y}_{\Gamma} = \{\pi_{\Gamma}(P_{\ell}) : 0 \leq \ell \leq w\}$  and relabel the subgroups of  $\mathcal{Y}_{\Gamma}$  such that  $\mathcal{Y}_{\Gamma} = \{1 = W_0 < W_1 < \dots < W_{w'}\}$ . Define  $\pi_{\Gamma}(C) : 1 = W_0 < W_1 < \dots < W_{w'}$ , so that  $\pi_{\Gamma}(C)$  is an  $r$ -subgroup chain of  $L_{\Gamma}$ .

Choose a representative set  $\mathcal{CR}^*(B)/G$  for the  $G$ -orbits in  $\mathcal{CR}^*(B)$  such that each chain  $C \in \mathcal{CR}^*(B)/G$  is also an  $r$ -subgroup chain of  $D$ . Let  $B_L$  be the block of  $L$  labelled by  $(s, -)$ . Then  $B_L = \prod_{\Gamma} B_{\Gamma}$  for some  $B_{\Gamma} \in \mathrm{Blk}(L_{\Gamma})$  labelled by  $(s_{\Gamma}, -)$ .

**(3C).** *In the notation above, suppose  $\mathcal{L}(B_{\Gamma})$  is the set of  $r$ -subgroup chains of  $L_{\Gamma}$  consisting of all  $L_{\Gamma}$ -conjugates of  $\pi_{\Gamma}(C)$  for all  $C \in \mathcal{CR}^*(B)/G$ . Then*

$$\begin{aligned} & \sum_{C \in \mathcal{CR}^*(B)/G} (-1)^{|C|} k(N(C), B, d, U) \\ &= \sum_{u_{\Gamma}} \left( \prod_{\Gamma'} \left( \sum_{C \in \mathcal{L}(B_{\Gamma'})/L_{\Gamma'}} (-1)^{|C|} k(N_{L_{\Gamma'}}(C), B_{\Gamma'}, u_{\Gamma}, U_{\Gamma'}) \right) \right), \end{aligned}$$

where both  $\Gamma$  and  $\Gamma'$  run over elementary divisors of  $s$ , and  $u_{\Gamma}$  runs over nonnegative integers such that  $\sum_{\Gamma} u_{\Gamma} = d$ .

*Proof.* Suppose  $C \in \mathcal{CR}^*(B)/G$  has a non-trivial final subgroup  $Q$ . Then  $Q \leq D \leq L$ ,  $Q$  is a primary subgroup of  $G$  and  $[V, Q] = V$ . In addition,  $C_G(Q)$  and  $N(Q)$  are given by (2.3) and (2.4), respectively, with  $M_0 = C_V(Q) = 0$ . Thus  $C_G(Q) = C(C)$  and  $N(C) \leq N(Q)$ . Let  $H = C_G(Q)$  and  $K = C_H(s)$ , so that  $K = C_L(Q)$ .

Suppose  $B_H \in \mathrm{Blk}(H|B)$  and  $B_K \in \mathrm{Blk}(K|B)$ . Then both blocks are labelled by  $(s, -)$ , and  $R_K^H$  is a perfect isometry between  $\mathrm{Irr}(B_K)$  and  $\mathrm{Irr}(B_H)$ . Let  $\psi_{sy, \mu} \in \mathrm{Irr}(B_K)$ ,  $\varphi_{sy, \mu'} = R_K^H(\psi_{sy, \mu}) \in \mathrm{Irr}(B_H)$ , and let  $N_L(\psi_{sy, \mu})$  and  $N_G(\varphi_{sy, \mu'})$  be the stabilizers of  $\psi_{sy, \mu}$  and  $\varphi_{sy, \mu'}$  in  $N_L(C)$  and  $N(C)$ , respectively, where  $y \in D$ . If  $z \in N_L(C)$  stabilizes



$\psi_{sy, \mu}$ , then  $s^z = s$ ,  $K^z = K$ ,  $(R_K^H)^z = R_K^H$ , and  $(\varphi_{sy, \mu'})^z = \varphi_{sy, \mu'}$ . Conversely, if  $z \in N(C)$  stabilizes  $\varphi_{sy, \mu'}$ , then  $(sy)^z$  is  $H$ -conjugate to  $sy$ , so that  $s^{zh} = s$  for some  $h \in H$ . Thus  $zh \in N_L(C)$ ,  $(R_K^H)^{zh} = R_K^H$  and  $(\psi_{sy, \mu})^{zh} = \psi_{sy, \mu}$ . It follows that

$$N_G(\varphi_{sy, \mu'})/H \simeq N_L(\psi_{sy, \mu})/K.$$

By (2.3) and (2.4),  $\varphi_{sy, \mu'}$  and  $\psi_{sy, \mu}$  have extensions to  $N_{N(Q)}(\varphi_{sy, \mu'})$  and  $N_{N_L(Q)}(\psi_{sy, \mu})$ , and so they have extensions to  $N_G(\varphi_{sy, \mu'})$  and  $N_L(\psi_{sy, \mu})$ , respectively. Since  $\sigma$  is a field automorphism and  $\iota$  is the inverse-transpose map, we may suppose  $s^\tau \in \langle s \rangle$  and  $Q^\tau = Q$  for  $\tau = \sigma$  or  $\iota$ , so  $K^\tau = K$  and  $(R_K^H)^\tau = R_K^H$ . Moreover, since we only consider the action of  $\tau$  on  $\text{Irr}(B_H)$  and  $\text{Irr}(B_K)$ , it follows by (2.4) and (2.5) that we may suppose  $\tau$  commutes with each element of  $N(Q)/H$  and  $N_L(Q)/L$ .

Since a defect group of  $B_H$  is conjugate to a Sylow  $r$ -subgroup of  $C_H(s) = K$ , it follows that  $B_H$  and  $B_K$  have the same defect. Now  $R_K^H$ ,  $\text{Ind}_{N_G(\varphi_{sy, \mu'})}^{N(C)}$  and  $\text{Ind}_{N_L(\psi_{sy, \mu})}^{N_L(C)}$  are defect preserving. By Clifford theory, there is a defect preserving bijection  $\Psi$  between  $\text{Irr}(N(C), B)$  and  $\text{Irr}(N_L(C), B_L)$  such that  $\Psi(\chi^\tau) = \Psi(\chi)^\tau$  for each  $\chi \in \text{Irr}(N(C), B)$  and  $\tau \in \text{Out}(G)$ , so

$$(3.6) \quad k(N(C), B, d, U) = k(N_L(C), B_L, d, U).$$

Since  $L$  is a regular subgroup of  $G$ ,  $R_L^G$  is a perfect isometry and  $(R_L^G)^\tau = R_L^G$  for  $\tau \in \text{Out}(G)$ , so that (3.6) still holds when  $|C| = 0$ .

Suppose  $C \in \mathcal{CR}^*(B)/G$  is given by (2.6). For  $1 \leq i \leq w$ , let  $(P_i)_\Gamma = \pi_\Gamma(P_i)$  be the subgroup of  $L_\Gamma$ . Then  $P_i \leq \prod_\Gamma (P_i)_\Gamma \leq \prod_\Gamma (P_{i+1})_\Gamma$  and  $N_L(P_i) \leq \prod_\Gamma N_{L_\Gamma}((P_i)_\Gamma)$ .

Let  $\mathcal{M}$  be the subfamily of  $\mathcal{CR}^*(B)$  consisting of all  $G$ -conjugates of chains  $C \in \mathcal{CR}^*(B)/G$  such that  $P_i \neq \prod_\Gamma (P_i)_\Gamma$  for some  $i$ . Given  $C \in \mathcal{M}$ , let  $v = v(C)$  be the largest integer such that  $P_v \neq \prod_\Gamma (P_v)_\Gamma$ . Let

$$(3.7) \quad C : 1 < P_1 < \dots < P_v < P_{v+1} < \dots < P_w$$

be a chain of  $\mathcal{M}$  with  $v = v(C)$ , so that  $P_{v+1} = \prod_\Gamma (P_{v+1})_\Gamma$  and  $\prod_\Gamma (P_v)_\Gamma \leq P_{v+1}$ . Let  $\mathcal{M}^0$  and  $\mathcal{M}^+$  be the subfamilies of  $\mathcal{M}$  consisting of chains  $C$  such that  $\prod_\Gamma (P_v)_\Gamma = P_{v+1}$  and  $\prod_\Gamma (P_v)_\Gamma < P_{v+1}$ , respectively. If  $C \in \mathcal{M}^0$  is given by (3.7), then define

$$g(C) : 1 < P_1 < \dots < P_v < P_{v+2} < \dots < P_w,$$

so that  $g(C) \in \mathcal{M}^+$  and  $N_L(C) = N_L(g(C))$ . If  $C \in \mathcal{M}^+$  is given by (3.7), then define

$$h(C) : 1 < P_1 < \dots < P_v < \prod_{\Gamma} (P_v)_{\Gamma} < P_{v+1} < \dots < P_w,$$

so that  $h(C) \in \mathcal{M}^0$  and  $N_L(C) = N_L(h(C))$ . In addition, since  $\tau = \prod_{\Gamma} \tau_{\Gamma}$  for each  $\tau \in \langle \sigma, \iota \rangle$ , it follows that  $N_{\langle \sigma, \iota \rangle}(C) = N_{\langle \sigma, \iota \rangle}(g(C))$  or  $N_{\langle \sigma, \iota \rangle}(h(C))$  according as  $C \in \mathcal{M}^0$  or  $\mathcal{M}^+$ . Since  $gh$  and  $hg$  are identities, it follows that

$$\sum_{C \in \mathcal{M}/L} (-1)^{|C|} k(N_L(C), B, d, U) = 0$$

for all  $d$  and  $U \leq \langle \sigma, \iota \rangle$ . We may suppose  $C \notin \mathcal{M}$ , so that  $P_i = \prod_{\Gamma} (P_i)_{\Gamma}$  for all  $i$ . Thus (3C) follows by (3.6) and the same proof as that of (1A) (b) with some obvious modifications.

#### 4. THE PROOF OF THE INVARIANT CONJECTURE

Let  $\mathcal{F}_q^k = \mathcal{F}_q^k(r)$  be the subset of  $\mathcal{F} = \mathcal{F}_q$  consisting of all polynomials whose roots have multiplicative order  $r^k$ . In addition, let  $\mathcal{F}_q(r, a) = \mathcal{F}_q^0 \cup \mathcal{F}_q^1 \cup \dots \cup \mathcal{F}_q^a$  and  $\mathcal{F}_q(r) = \cup_{k \geq 0} \mathcal{F}_q^k(r)$ . Then  $\mathcal{F}_q^0 = \{T - 1\}$  and by [12, (1) and (2)],  $|\mathcal{F}_q(r, a)| = 1 + (r^a - 1)/e$  and  $|\mathcal{F}_q^{a+i}| = (r^a - r^{a-1})/e$  for  $i \geq 1$  (note that the number  $e$  here is the  $e'$  in [11] and [12]). Moreover, by [11, (1.5)],  $d_{\Gamma} = e$  or  $er^i$  according as  $\Gamma \in \mathcal{F}_q(r, a) \setminus \{T - 1\}$  or  $\mathcal{F}_q^{a+i}$ , and each elementary divisor of an  $r$ -element is an element of  $\mathcal{F}_q(r)$ . Following the notation of [12], we write

$$\mathcal{F}_q(r, a) = \{\Delta(0, j) : 0 \leq j \leq (r^a - 1)/e\}, \quad \Delta(0, 0) = T - 1$$

and for  $i \geq 1$

$$\mathcal{F}_q^{a+i}(r) = \{\Delta(i, j) : 1 \leq j \leq (r^a - r^{a-1})/e\}.$$

A sequence  $(w_i) = (w_0, w_1, w_2, \dots, w_{\ell})$  of nonnegative integers is called an  $r$ -weight sequence of  $u$  if

$$\sum_{i=0}^{\ell} w_i r^i = u.$$

Given such a sequence  $(w_i)$ , let  $\Omega_G((w_i))$  be the subset of  $\text{Irr}(B_0)$  consisting of irreducible characters  $\chi_{y, \mu}$  of  $G = \text{GL}^{\epsilon}(ue, q)$  such that

$$\sum_{j=0}^{1+(r^a-1)/e} m_{0,j} = w_0, \quad \sum_{j=1}^{(r^a-r^{a-1})/e} m_{i,j} = w_i$$

for  $i \geq 1$  and  $\mu = \prod_{\ell, j} \mu_{\ell, j}$ , where  $m_{\ell, j} = m_{\Delta(\ell, j)}(y)$  and  $\mu_{\ell, j}$  is a partition of  $m_{\ell, j}$  for all  $\ell$  and  $j$ . As shown in the proof of [12, Proposition 6]

$$(4.1) \quad |\Omega_G((w_i))| = k(e + (r^a - 1)/e, w_0) \prod_{i \geq 1} k((r^a - r^{a-1})/e, w_i),$$

where  $k(s, t)$  is the number of  $s$ -tuples  $(\mu_1, \dots, \mu_s)$  of partitions such that  $\sum_{j=1}^s |\mu_j| = t$ . Moreover,

$$(4.2) \quad \text{Irr}(B_0) = \bigcup_{(w_i)} \Omega_G((w_i)) \quad (\text{disjoint}),$$

where  $(w_i)$  runs over all  $r$ -weight sequences of  $u$ . An  $r$ -weight sequence  $(w_i)$  determines a unique partition  $\lambda = (\lambda_0^{\beta_0}, \dots, \lambda_{\ell}^{\beta_{\ell}})$  of  $u$ , called an  $r$ -weight partition, where  $\lambda_i = r^i$  and  $\beta_i = w_i$  for all  $i \geq 0$ .

**(4A).** Let  $K = \text{GL}^{\epsilon}(r^j, q^e)$ ,  $H = \text{GL}^{\epsilon}(tr^j, q^e)$  and  $K \wr \mathbf{S}(t) \leq H$ , where  $j \geq 0$  and  $t \geq 1$ . In addition, let  $\mathcal{X} = \{\xi_{\Delta} : \Delta \in (\mathcal{F}_{q^e}^{a+j})^*\}$ , where  $(\mathcal{F}_{q^e}^{a+j})^* = \mathcal{F}_{q^e}^{a+j}$  or  $\mathcal{F}_{q^e}(r, a)$  according as  $j \geq 1$  or  $j = 0$ , and  $\xi_{\Delta}$  is the irreducible character of the principal block  $B_0(K)$  labelled by  $(\Delta, 1)$ . Thus  $\mathcal{X} = \text{Irr}(B_0(K))$  when  $j = 0$ . Then there is a defect preserving bijection  $\Psi$  between  $\Omega_H((w_i))$  and  $\mathcal{X} \wr \mathbf{S}(t)$ , where  $w_i = \delta_{ij}t$  and  $\mathcal{X} \wr \mathbf{S}(t)$  is defined by (1A). Moreover,  $\Psi$  is compatible with  $\tau \in \text{Out}(H)$ .

*Proof.* If  $\chi = \chi_{y, \mu} \in \Omega_H((w_i))$ , then  $\sum_{\Delta \in (\mathcal{F}_{q^e}^{a+j})^*} m_{\Delta}(y) = t$  and  $\mu = \prod_{\Delta} \mu_{\Delta}$ , where  $\mu_{\Delta} \vdash m_{\Delta}(y)$ . Let  $\xi \in \text{Irr}(K^n)$  such that  $m_{\xi_{\Delta}}(\xi) = m_{\Delta}(y)$ , where  $m_{\xi_{\Delta}}(\xi)$  is the multiplicity of  $\xi_{\Delta}$  in  $\xi$ . Then the stabilizer of  $\xi$  in  $K \wr \mathbf{S}(t)$  is  $K^t \mathbf{S}(\mathbf{m})$  and  $\xi$  has an extension  $\tilde{\xi}$  to  $K^t \mathbf{S}(\mathbf{m})$ , where

$$\mathbf{m} = (m_{\Delta(j, 1)}(y), \dots, m_{\Delta(j, r^a - r^{a-1})}(y))$$

or  $(m_{\Delta(0, 0)}(y), \dots, m_{\Delta(0, r^a)}(y))$  according as  $j \geq 1$  or  $j = 0$ . Define

$$\Psi(\chi) = \text{Ind}_{K^t \mathbf{S}(\mathbf{m})}^{K \wr \mathbf{S}(t)}(\tilde{\xi} \phi_{\mu}),$$

where  $\phi_{\mu} = \prod_{\Delta} \phi_{\mu_{\Delta}}$  with  $\phi_{\mu_{\Delta}}$  the irreducible character of  $\mathbf{S}(m_{\Delta}(y))$  labelled by  $\mu_{\Delta}$ . Since  $d(\chi) = d(\chi_{\mu})$ , it follows that

$$d(\chi) = \sum_{\Delta} (m_{\Delta}(y)(a + j) + a(\mathbf{S}(m_{\Delta}(y)))) - a(\chi_{\mu}) = t(a + j) + a(\mathbf{S}(\mathbf{m})) - a(\chi_{\mu}).$$

Similarly,  $d(\Psi(\chi)) = a(K^t \mathbf{S}(\mathbf{m})) - a(\xi \phi_{\mu})$ . But  $a(\xi) = \sum_{\Delta} m_{\Delta}(y) a(\xi_{\Delta})$  and

$$a(\xi_{\Delta}) = a(|\text{GL}^{\epsilon}(r^j, q^e) : \text{GL}^{\epsilon}(1, q^{er^j})|) = r^j a + a(\mathbf{S}(r^j)) - (a + j),$$

so by [10, (8.5)],  $d(\Psi(\chi)) = t(a + j) + a(\mathbf{S}(\mathbf{m})) - a(\phi_\mu) = d(\chi)$ . Thus  $\Psi$  is a defect preserving bijection between  $\Omega_H((w_i))$  and  $\mathcal{X} \wr \mathbf{S}(t)$ .

If  $\tau \in \text{Out}(H)$ , then  $\chi_{y,\mu}^\tau = \chi_{y^{\tau},\mu}$ . The base group  $K^t$  is uniquely determined up to conjugacy in  $H$ , we may suppose  $K^\tau = K$  and moreover, we may suppose  $[\tau, x] = 1$  for  $x \in \mathbf{S}(t)$ , so that  $\Psi(\chi)^\tau = \Psi(\chi^\tau)$ . This proves (4A).

Let  $B \in \text{Blk}(G)$  with defect group  $D$ . By (3B), we may suppose  $[V, D] = V$  and  $G = \text{GL}^\epsilon(eu, q)$ . Let  $C \in \mathcal{CR}^*(B)$  be a chain given by (2.6) such that the Sylow subgroups of  $C(C)$  are abelian. Then by (2.10) and (2.11),  $N(C) = N_1 \times \dots \times N_v$  and each

$$N_k = \prod_{i \in I'_0} \left\langle \tau_i, \left( \prod_{j \in I_0} \text{GL}^\epsilon(m_j, q^e) \wr \mathbf{S}(\mathbf{w}_j) \right) \right\rangle \wr \mathbf{S}(i),$$

so that  $m_j \leq r - 1$ . Let  $Q$  be a Sylow  $r$ -subgroup of  $C(C)$  and

$$(4.3) \quad C^* : 1 < P_1 < \dots < P_w \leq Q.$$

Then  $Q = (R_{1,0,0,0})^u$  and  $N(C^*) = N_1^* \times \dots \times N_v^*$  such that for  $k \geq 1$

$$N_k^* = \prod_{i \in I'_0} \left\langle \tau_i, \left( \prod_{j \in I_0} \text{GL}^\epsilon(1, q^e) \wr \mathbf{S}(m_j) \wr \mathbf{S}(\mathbf{w}_j) \right) \right\rangle \wr \mathbf{S}(i).$$

**(4B).** *In the notation above, suppose  $C(C)$  has an abelian Sylow  $r$ -subgroup  $Q$  and  $B_0 = B_0(G)$ . Then*

$$k(N(C), B_0, d, U) = k(N(C^*), B_0, d, U)$$

*for all integers  $d \geq 0$  and  $U \leq \text{Out}(G)$ . Moreover, Dade's invariant conjecture holds for the principal block with an abelian defect group.*

*Proof.* We may suppose  $C \neq C^*$ . Let  $K = \text{GL}^\epsilon(m_j, q^e)$  and  $K^* = \text{GL}^\epsilon(1, q^e) \wr \mathbf{S}(m_j)$ . By (4A), there is a defect preserving bijection  $\psi$  between  $\text{Irr}(B_0(K))$  and  $\text{Irr}(B_0(K^*))$ . Let  $W = K \wr \mathbf{S}(\mathbf{w}_j)$  and  $W^* = K^* \wr \mathbf{S}(\mathbf{w}_j)$ . By (1A) (a), there is a defect preserving bijection  $\Psi$  between  $\text{Irr}(B_0(W))$  and  $\text{Irr}(B_0(W^*))$ . Since  $\tau_i$  is a field automorphism of  $\text{GL}^\epsilon(U_i)$ , it follows that we may suppose  $[\tau_i, y] = 1$  for each permutation matrix  $y \in \text{GL}^\epsilon(U_i)$ , so that  $\Psi(\chi)^{\tau_i} = \Psi(\chi^{\tau_i})$ . Thus  $\Psi$  can be extended to a defect preserving bijection  $\Psi^*$  between  $\text{Irr}(B_0(N(C)))$  and  $\text{Irr}(B_0(N(C^*)))$ . Similarly, if  $\tau \in \text{Out}(G)$ , then we may suppose  $[\tau, y] = 1$  for each permutation matrix  $y \in G$  and  $[\tau, \tau_i] = 1$  (see the remark after (2.7)). If  $\tau$  stabilizes  $C^*$ , then  $C^\tau = C$ . If  $C^\tau = C$ , then  $Q^{\tau h} = Q$  for some  $h \in C(C)$ . It follows that  $\Psi^*(\chi)^\tau = \Psi^*(\chi^\tau)$  for any  $\chi \in \text{Irr}(B_0(N(C)))$ .

If  $B_0$  has an abelian defect group, then  $C(C)$  has an abelian Sylow subgroup for each  $C \in \mathcal{CR}^*$ . Suppose  $C \in \mathcal{CR}^*(B_0)$  is given by (2.6). Define  $\varphi(C) = C_{w-1}$  or  $C^*$  according as  $C = C^*$  or  $C \neq C^*$ . Then  $\varphi(C) \in \mathcal{CR}^*(B_0)$ ,  $\varphi(\varphi(C)) = C$  and  $|\varphi(C)| = |C| \pm 1$ . This implies that

$$\sum_{C \in \mathcal{CR}^*(B_0)/G} (-1)^{|C|} k(N(C), B_0, d, U) = 0.$$

This proves (4B).

Let  $R_u = R_{u,0,0,0}$  be a basic subgroup of  $G = \text{GL}^\epsilon(eu, q)$ , so that  $H = C_G(R_u) = \text{GL}^\epsilon(u, q^e)$ . If  $\lambda = (\lambda_1^{\beta_1}, \lambda_2^{\beta_2}, \dots, \lambda_\ell^{\beta_\ell})$  is a partition of  $u$  with  $\lambda_1 < \lambda_2 < \dots < \lambda_\ell$ , then

$$(4.4) \quad R_\lambda = \prod_{j=1}^{\ell} (R_{\lambda_j})^{\beta_j}$$

is a primary subgroup of  $G$  and  $H$ ,  $C(R_\lambda) = \prod_{j=1}^{\ell} \text{GL}^\epsilon(\lambda_j, q^e)^{\beta_j}$  and

$$N_H(R_\lambda) = \prod_{j=1}^{\ell} \text{GL}^\epsilon(\lambda_j, q^e) \wr \mathbf{S}(\beta_j).$$

Conversely, each primary subgroup of  $G$  and  $H$  with no nonzero fixed-point on the underlying space is determined by a partition of  $u$ .

For  $K \leq G$ , we denote by  $\mathcal{CR}^*(K)$  the subfamily of  $\mathcal{R}(C_K(P(O_r(K))))$  consisting of the chains  $C$  such that each subgroup of  $C$  is primary in  $G$ . Here we have the same identification as that of the remark of (2C) when  $(q, \epsilon, r) = (2, +, 3)$ . Let  $\mathcal{CR}_H^*(\lambda)$  be the subfamily of  $\mathcal{CR}^*(H)$  consisting of all chains  $C$  whose first subgroup is equal to  $R_\lambda$ . If  $C \in \mathcal{CR}_H^*(\lambda)$  is given by (1.1) with  $R_\lambda \neq O_r(H)$ , then  $g(C) : R_\lambda < P_2 < \dots < P_w$  is a chain of  $\mathcal{CR}^*(N_H(R_\lambda))$ ,  $|g(C)| = |C| - 1$ ,  $N_H(C) = N_{N_H(R_\lambda)}(g(C))$  and  $g$  is a bijection between  $\mathcal{CR}_H^*(\lambda)$  and  $\mathcal{CR}^*(N_H(R_\lambda))$ . We can identify  $\mathcal{CR}_H^*(\lambda)$  with  $\mathcal{CR}^*(N_H(R_\lambda))$ .

**(4C).** *Let  $H = \text{GL}^\epsilon(u, q^e)$  for  $u \geq 1$ . Then  $\mathcal{Q}(H) = \mathcal{CR}^*(H)$  satisfies Hypothesis (1D) for the principal block  $B_0 = B(H)$  and it is  $\tau$ -invariant for each  $\tau \in \text{Out}(H)$ . In addition,  $C_0(H) \in \mathcal{S}(H)$  such that  $\Omega_{C_0(H)}(0) = \emptyset$  or  $\Omega_H((\delta_{it}))$  according as  $u$  is not a power of  $r$  or  $u = r^t$ .*

*Proof.* If  $u = 1$ , then  $\mathcal{Q}(H) = \{C_0(H)\}$  and by definition,  $\text{Irr}(H, B_0) = \Omega_H((1, 0)) = \Omega_{C_0(H)}(0)$ . Thus  $\mathcal{Q}_1(H) = \mathcal{Q}_2(H) = \emptyset$  and  $\mathcal{S}(H) = \mathcal{Q}(H)$ .

If  $1 < u < r$ , then  $H$  has an abelian Sylow  $r$ -subgroup. Let  $\mathcal{Q}_1(H) = \{C \in \mathcal{Q}(H) : C \neq C^*, |C| \neq 0\}$ ,  $\mathcal{Q}_2(H) = \{C \in \mathcal{Q}(H) : C = C^*, |C| \neq 1\}$  and  $\mathcal{S}(H) =$

$\{C_0(H), C_0(H)^*\}$ , where  $C^*$  is defined by (4.3). Set  $\Omega_{C_0(H)}(0) = \emptyset$ ,  $\varphi(C_0(H)) = C_0(H)^*$ ,  $\Omega_{C_0(H)}(\varphi) = \Omega_H((u, 0)) = \text{Irr}(H, B_0)$  and  $\Omega_{C_0(H)^*}(0) = \text{Irr}(N_H(C_0(H)^*), B_0)$ . It follows by (4A) and (4B) that  $\mathcal{Q}(H) = \mathcal{Q}_1(H) \cup \mathcal{Q}_2(H) \cup \mathcal{S}(H)$  satisfies Hypothesis (1D) and is  $\tau$ -invariant for  $\tau \in \text{Out}(H)$ .

Let  $\lambda = (\lambda_0^{\beta_0}, \lambda_1^{\beta_1}, \dots, \lambda_\ell^{\beta_\ell})$  be a partition of  $u$  such that  $R_\lambda \neq O_r(H)$ , and let  $K_i = \text{GL}^\epsilon(\lambda_i, q^e)$  and  $W_i = \text{GL}^\epsilon(\lambda_i, q^e) \wr \mathbf{S}(\beta_i)$  for  $1 \leq i \leq \ell$ . Thus  $\lambda_i < u$  for all  $i$ , and by induction,  $\mathcal{Q}(K_i) = \mathcal{CR}^*(K_i)$  satisfies Hypothesis (1D) for  $B_0(K_i)$  and is  $\tau$ -invariant for  $\tau \in \text{Out}(K_i)$ . By (1E),  $\mathcal{Q}(W_i) = \mathcal{CR}^*(W_i)$  and  $\mathcal{Q}(N_H(R_\lambda)) = \mathcal{CR}_H^*(\lambda)$  both satisfy Hypothesis (1D) for  $B_0(W_i)$  and  $B_0(N_H(R_\lambda))$  and they are  $\tau$ -invariant for  $\tau \in \text{Out}(W_i)$  and  $\tau \in \text{Out}(N_H(R_\lambda))$ , respectively.

Suppose  $\lambda_i$  is not a power of  $r$  for some  $i$ . Then  $\Omega_{C_0(K_i)}(0) = \emptyset$ ,  $\Omega_{C_0(W_i)}(0) = \Omega_{C_0(K_i)}(0) \wr \mathbf{S}(\beta_i) = \emptyset$  and so  $\Omega_{C_0(N_H(R_\lambda))}(0) = \prod_{j=0}^\ell \Omega_{C_0(W_j)}(0) = \emptyset$ . In particular, by (1A) (b),

$$\sum_{C \in \mathcal{CR}_H^*(\lambda)/N_H(R_\lambda)} (-1)^{|C|} k(N_H(C), B_0, d, U) = 0.$$

Suppose each  $\lambda_i$  is a power of  $r$ , and we may suppose  $\lambda_i = r^i$  for all  $i \geq 0$ . Thus  $(\beta_0, \beta_1, \dots, \beta_\ell)$  is an  $r$ -weight sequence,  $\Omega_{C_0(K_i)}(0) = \Omega_{K_i}((\delta_{ji}))$  and

$$\Omega_{C_0(N_H(R_\lambda))}(0) = \prod_{i=0}^\ell \Omega_{K_i}((\delta_{ji})) \wr \mathbf{S}(\beta_i).$$

It follows by (4A) that there is a defect preserving bijection between  $\Omega_H((\beta_0, \beta_1, \dots, \beta_\ell))$  and  $\Omega_{C_0(N_H(R_\lambda))}(0)$ , which is compatible with  $\tau \in \text{Out}(H)$ .

Let  $\varphi_\lambda(C_0(H))$  be the chain  $O_r(H) < R_\lambda$ ,  $\Omega_{C_0(H)}(0) = \emptyset$  or  $\Omega_H((\delta_{jt}))$  according as  $u$  is not a power of  $r$  or  $u = r^t$ , and  $\Omega_{C_0(H)}(\varphi_\lambda) = \Omega_H((\beta_0, \beta_1, \dots, \beta_\ell))$ . Then

$$\text{Irr}(H, B_0) = \Omega_{C_0(H)}(0) \bigcup_{\lambda} \Omega_{C_0(H)}(\varphi_\lambda), \quad (\text{disjoint})$$

where  $\lambda$  runs over the  $r$ -weight partitions of  $u$ . Now

$$\mathcal{CR}^*(H) = \bigcup_{\mu} \mathcal{CR}_H^*(\mu), \quad (\text{disjoint})$$

where  $\mu$  runs over the partitions of  $u$  such that  $\mu$  is not the partition  $(u)$ . Thus  $\mathcal{Q}(H) = \mathcal{CR}^*(H)$  satisfies Hypothesis (1D) for  $B_0(H)$  and  $\Omega_{C_0(H)}(0)$  given above, and  $\mathcal{Q}(H)$  is  $\tau$ -invariant for  $\tau \in \text{Out}(H)$ . This completes the proof.

(4D). Let  $G = \mathrm{GL}^\epsilon(n, q) = \mathrm{GL}^\epsilon(V)$  such that  $O_r(G) = 1$ .

(a) If  $n = me$  for some integer  $m$ , then

$$\sum_{C \in \mathcal{CR}^*(G)/G} (-1)^{|C|} k(N_G(C), B_0, d, U) = 0$$

for any integer  $d \geq 0$  and any  $U \leq \mathrm{Out}(G)$ .

(b) Dade's invariant conjecture holds for each  $r$ -block  $B$  of  $G$  with defect  $d(B) \geq 1$ .

*Proof.* (a) Suppose  $G = \mathrm{GL}^\epsilon(me, q)$  and  $C \in \mathcal{CR}_G^*(\lambda)$ , where  $\lambda = (\lambda_0^{\beta_0}, \lambda_1^{\beta_1}, \dots, \lambda_u^{\beta_u})$  is a partition of  $m$ . If  $\lambda_i$  is not a power of  $r$  and  $K = \mathrm{GL}^\epsilon(\lambda_i, q^e)$ , then by (4C),  $\mathcal{CR}^*(K)$  satisfies Hypothesis (1D) and is  $\tau$ -invariant for each  $\tau \in \mathrm{Out}(K)$ , so that

$$\sum_{C \in \mathcal{CR}^*(K)/K} (-1)^{|C|} k(N_K(C), B_0, d, U_K) = 0,$$

where  $U_K \leq \mathrm{Out}(K)$ . If  $L = \langle \tau_i, \mathrm{GL}^\epsilon(\lambda_i, q^e) \rangle$ , then

$$\sum_{C \in \mathcal{CR}^*(L)/L} (-1)^{|C|} k(N_L(C), B_0, d, U_L) = 0,$$

for  $U_L \leq \mathrm{Out}(\mathrm{GL}^\epsilon(e\lambda_i, q))$ , since  $\mathcal{CR}^*(L) = \mathcal{CR}^*(K)$  and  $\mathcal{CR}^*(K)$  is  $\tau$ -invariant for each  $\tau \in \mathrm{Out}(\mathrm{GL}^\epsilon(e\lambda_i, q))$ . Thus by (1E),

$$\sum_{C \in \mathcal{CR}^*(W)/W} (-1)^{|C|} k(N_W(C), B_0, d, U_W) = 0,$$

where  $W = \langle \tau_i, \mathrm{GL}^\epsilon(\lambda_i, q^e) \rangle \wr S(\beta_i)$ ,  $U_W \leq \mathrm{Out}(\mathrm{GL}^\epsilon(e\beta_i\lambda_i, q))$ . By (1A) (b),

$$\sum_{C \in \mathcal{CR}_G^*(\lambda)/N_G(R_\lambda)} (-1)^{|C|} k(N_{N_G(R_\lambda)}(C), B_0, d, U) = 0.$$

We may suppose  $\lambda_i = r^i$  for each  $i$ , so that  $\beta = (\beta_0, \beta_1, \dots, \beta_u)$  is an  $r$ -weight sequence.

Thus

$$|\Omega_G((\beta_0, \beta_1, \dots, \beta_u))| = k(e + (r^a - 1)/e, \beta_0) \prod_{i=1}^u k((r^a - r^{a-1})/e, \beta_i)$$

and  $\mathrm{Irr}(B_0(G)) = \cup_{\beta} \Omega_G(\beta)$ , where  $\beta$  runs over all  $r$ -weight sequences. Let  $\Omega_{K_i}^2 = \mathrm{Irr}(B_0(K_i)) \setminus \Omega_{K_i}((w_\ell^i))$ , where  $w_\ell^i = \delta_{\ell i}$  and  $(w_\ell^i) = (w_0^i, w_1^i, \dots)$ . We may suppose  $[y, \tau_i] = 1$  for  $y \in \mathrm{Out}(\mathrm{GL}^\epsilon(er^i, q))$ . Then by (4C),

$$\sum_{C \in \mathcal{CR}^*(L)/L, |C| \neq 0} (-1)^{|C|} k(N_L(C), B_0, d, U_L) + k(\langle \tau_i, \Omega_{K_i}^2 \rangle, d, U_L) = 0,$$

where  $L = \langle \tau_i, K_i \rangle$ ,  $U_L \leq \text{Out}(\text{GL}^\epsilon(er^i, q))$  and  $\langle \tau_i, \Omega_{K_i}^2 \rangle$  denotes the characters of  $\text{Irr}(L)$  covering  $\Omega_{K_i}^2$ . By (1A) and (1E),

$$\sum_{C \in \mathcal{CR}_G^*(\lambda)/N_G(R_\lambda), |C| \neq 1} (-1)^{|C|} k(N_{N_G(R_\lambda)}(C), B_0, d, U) = k\left(\prod_{i=0}^u (\langle \tau_i, \Omega_{K_i}^2 \rangle \wr \mathbf{S}(\beta_i)), d, U\right).$$

Therefore

$$\sum_{C \in \mathcal{CR}_G^*(\lambda)/N_G(R_\lambda)} (-1)^{|C|} k(N_{N_G(R_\lambda)}(C), B_0, d, U) = -k\left(\prod_{i=1}^u (\langle \tau_i, \Omega_{K_i}((w_\ell^i)) \rangle \wr \mathbf{S}(\beta_i)), d, U\right).$$

Now  $|\Omega_{K_i}((w_\ell^i))| = r^a - r^{a-1}$  or  $r^a$  according as  $i \geq 1$  or  $i = 0$ , and each character of  $\Omega_{K_i}((w_\ell^i))$  is labelled by  $(\Delta, 1)$  for some  $\Delta \in (\mathcal{F}_{q^e}^{a+i})^*$ , where  $(\mathcal{F}_{q^e}^{a+i})^* = \mathcal{F}_{q^e}^{a+i}$  or  $\mathcal{F}_{q^e}(r, a)$  according as  $i \geq 1$  or  $i = 0$ . Since  $\tau_i$  induces a field automorphism of order  $e$  on  $K_i = \text{GL}^\epsilon(r^i, q^e)$ , it follows that

$$|\langle \tau_i, \Omega_{K_i}((w_\ell^i)) \rangle| = (r^a - r^{a-1})/e \quad \text{or} \quad e + (r^a - 1)/e$$

according as  $i \geq 1$  or  $i = 0$ . If a character  $\chi \in \langle \tau_i, \Omega_{K_i}((w_\ell^i)) \rangle$  is not an extension of the trivial character, then  $\chi$  is labelled by  $(\Delta', 1)$  for some  $\Delta' \in (\mathcal{F}_q^{a+i})^*$ , where  $(\mathcal{F}_q^{a+i})^*$  is defined similar as before. It follows from (1A) (a) that there exists a defect preserving bijection, compatible with elements of  $\text{Out}(G)$ , between

$$\prod_{i=0}^u \langle \tau_i, \Omega_{K_i}((w_\ell^i)) \rangle \wr \mathbf{S}(\beta_i)$$

and  $\Omega_G((\beta_0, \beta_1, \dots, \beta_u))$ . Thus

$$\sum_{C \in \mathcal{CR}_G^*(\lambda)/N_G(R_\lambda)} (-1)^{|C|} k(N_{N_G(R_\lambda)}(C), B_0, d, U) + k(\Omega_G((\beta_0, \beta_1, \dots, \beta_u)), d, U) = 0$$

for all  $r$ -weight partitions  $\lambda = (\lambda_0^{\beta_0}, \dots, \lambda_u^{\beta_u})$  of  $m$ . This proves part (a).

(b) If  $B = B_0 = B_0(G)$  is the principal block, then Dade's invariant conjecture for  $B$  follows by (3B) and part (a) above.

Suppose  $B \neq B_0$  and by (3B), we may suppose  $B$  is labelled by  $(s, -)$ . By (3C) and (1A), we may suppose  $L = C_G(s) = L_\Gamma = \text{GL}^{\epsilon_\Gamma}(m_\Gamma(s), q^{\delta_\Gamma})$ , so that  $s = s_\Gamma$ . Let  $R = O_r(Z(L_\Gamma))$  and let  $C : 1 < P_1 < \dots < P_w$  be a chain of  $\mathcal{L}(B_\Gamma)$  (see (3C)).

Suppose  $R = 1$ , so that  $r$  and  $q^{\delta_\Gamma} - \epsilon_\Gamma$  are coprime. Let  $Q_i = \Omega_{a+\alpha_\Gamma}(O_r(C_{L_\Gamma}(P_i)))$  for all  $i \geq 1$ , where  $\alpha_\Gamma = a(\delta_\Gamma)$ . Then

$$\varphi(C) : 1 < Q_1 < \dots < Q_w$$



is a chain of  $\mathcal{CR}^*(L_\Gamma)$  and  $N_{L_\Gamma}(C) = N_{L_\Gamma}(\varphi(C))$ . Since  $P_i = \Omega_a(Q_i)$ , it follows that  $N_{\text{Out}(L_\Gamma)}(C, \xi) = N_{\text{Out}(L_\Gamma)}(\varphi(C), \xi)$  for any  $\xi \in \text{Irr}(N_{L_\Gamma}(C))$ , and  $\mathcal{CR}^*(B_\Gamma) = \{\varphi(C) : C \in \mathcal{L}(B_\Gamma)\}$ . But  $s \in Z(L_\Gamma)$ , so

$$k(N_{L_\Gamma}(C), B_\Gamma, d, U) = k(N_{L_\Gamma}(C), B_0(L_\Gamma), d, U)$$

for all integers  $d \geq 0$  and  $U \leq \text{Out}(L_\Gamma)$ . Thus Dade's invariant conjecture for  $B_\Gamma$  follows by part (a).

Suppose  $R \neq 1$ , so that  $\Omega_a(R) \leq P_1$ . Define

$$\varphi(C) : \begin{cases} 1 < \Omega_a(R) < P_1 < \dots < P_w & \text{if } \Omega_a(R) \neq P_1, \\ 1 < P_2 < \dots < P_w & \text{if } \Omega_a(R) = P_1. \end{cases}$$

Then  $\varphi : \mathcal{L}(B_\Gamma) \rightarrow \mathcal{L}(B_\Gamma)$ ,  $N_{L_\Gamma}(C) = N_{L_\Gamma}(\varphi(C))$ ,  $N_{\text{Out}(L_\Gamma)}(C, \xi) = N_{\text{Out}(L_\Gamma)}(\varphi(C), \xi)$  for any  $\xi \in \text{Irr}(N_{L_\Gamma}(C))$ ,  $|\varphi(C)| = |C| \pm 1$  and  $\varphi(\varphi(C)) = C$ . It follows that

$$\sum_{C \in \mathcal{L}(B_\Gamma)/L_\Gamma} (-1)^{|C|} k(N_{L_\Gamma}(C), B_\Gamma, d, U) = 0$$

for all integers  $d \geq 0$  and  $U \leq \text{Out}(L_\Gamma)$ . This completes the proof.

#### ACKNOWLEDGMENTS

Part of the results of the paper was obtained during the author visiting Singapore, Beijing, Chongqing and Shanghai. The author would like to thank Dr. Yuqing You, Professors Jiping Zhang, Yingbo Zhang, Shengming Shi, Wujie Shi and Jianpan Wang for their supporting and great hospitality. The part of the invariant conjecture was obtained during the author visiting the University of Illinois at Chicago. The author would like to thank Professors Paul Fong and Bhama Srinivasan for their sincere help and support, and thank the Department of Mathematics of the University of Illinois at Chicago for the support and hospitality. The author would also like to thank Professor Geoffrey Robinson for helpful conversations.

#### REFERENCES

1. J. L. Alperin, *Large abelian subgroups of  $p$ -groups*, Trans. Amer. Math. Soc. **117** (1965), 10-20. MR **30**# 1180
2. J. L. Alperin and P. Fong, *Weights for symmetric and general linear groups*,

- J. Algebra **131** (1990), 2-22. MR **91h**:20014
3. Jianbei An, *Weights for classical groups*, Trans. Amer. Math. Soc. **342** (1994), 1-42. MR **94e**:20015
  4. M. Broué, *Les  $\ell$ -blocs des groupes  $GL(n, q)$  et  $U(n, q^2)$  et leurs structures locales*, Séminaire Bourbaki Astérisque **133-134** (1986), 159-188. MR **87e**:20021
  5. M. Broué, *Isométries parfaites, types de blocs, catégories dérivées*. Astérisque **181-182** (1990), 61-92. MR **91i**:20006
  6. E. Dade, *Counting characters in blocks*, I, Invent. Math. **109** (1992), 187-210. MR **93g**:20021
  7. E. Dade, *Counting characters in blocks*, II.9. in Representation Theory of Finite Groups (R. Solomon, editor), Ohio State University Math. Res. Inst. Publ. 6, de Gruyter, Berlin 1997. MR **99b**:20016
  8. E. Dade, *Counting characters in blocks with cyclic defect groups*, I, J. Algebra **186** (1996), 934-969. MR **98b**:20013
  9. W. Feit, *The representation theory of finite groups*, North Holland, 1982. MR **83g**:20001
  10. P. Fong and B. Srinivasan, *The blocks of finite general linear and unitary groups*, Invent. Math. **69** (1982), 109-153. MR **83k**:20013
  11. G. O. Michler and J. B. Olsson, *Character correspondences in finite general linear, unitary and symmetric groups*, Math. Z. **184** (1983), 203-233. MR **85e**:20014
  12. J. B. Olsson, *On the number of characters in blocks of finite general linear, unitary and symmetric groups*, Math. Z. **186** (1984), 41-47. MR **85d**:20008
  13. J. B. Olsson and K. Uno, *Dade's conjecture for general linear groups in the defining characteristic*, Proc. London Math. Soc. **72** (1996), 359-384. MR **97b**:20010
  14. J. B. Olsson and K. Uno, *Dade's conjecture for symmetric groups*, J. Algebra **176** (1995), 534-560. MR **96h**:20025

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