The p-local Rank of a Block

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Abstract

We generalize the p-local rank of a finite group, introduced by G. Robinson [7], to a p-block of a finite group and show that this has properties analagous to most of those given in [7].

1 Introduction

Let G be a finite group and p a prime. Given a chain of p-subgroups

$$\sigma : Q_0 < Q_1 < \cdots < Q_n \tag{1.1}$$

of G, define the length $|\sigma| = n$, the final subgroup $V^\sigma = Q_n$, the initial subgroup $V_\sigma = Q_0$, the k-th initial subchain

$$\sigma_k : Q_0 < Q_1 < \cdots < Q_k,$$

and the normalizer

$$G_\sigma = N_G(\sigma) = N_G(Q_0) \cap N_G(Q_1) \cap \cdots \cap N_G(Q_n). \tag{1.2}$$

Write $\mathcal{C}(G|Q)$ for the set of those chains with initial subgroup Q and $\mathrm{Blk}(G)$ for the set of all p-blocks of G.

Let $(K, \mathcal{O}, k)$ be a p-modular system which splits for G, and consider a p-block B of G with defect group $D(B)$. Given a subgroup H of G, write $\mathrm{Blk}(H|B) = \{b \in \mathrm{Blk}(H) : b^G = B\}$ (in the sense of Brauer).

The Knörr-Robinson reformulation of Alperin's weight conjecture (see [5] and [1]) states that if B has positive defect then $\sum_{\sigma \in \mathcal{C}(G|1)} l(G_\sigma, B) = 0$, where $l(G_\sigma, B)$ denotes the number of simple kG_σ-modules belonging to p-blocks of G_σ whose Brauer correspondent is B. Clearly if $l(G_\sigma, B) \neq 0$ then there is some p-block b of G_σ with $b^G = B$. It thus makes sense to consider the set

$$\mathcal{C}(G, B) = \{\sigma \in \mathcal{C}(G) : \mathrm{Blk}(G_\sigma|B) \neq \emptyset\}.$$

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We say that σ is a *p-chain for B*, or a *B-chain* (as we will see we may associate with σ a chain of B-subgroups in the sense of Alperin [2]) if $\sigma \in \mathcal{C}(G, B)$. This is in part the motivation for this paper.

As demonstrated in [5], when considering this and similar conjectures we may restrict our attention to certain subsets of $\mathcal{C}(G)$, such as the following. A p-subgroup R of G is *radical* if $O_p(N_G(R)) = R$, where $O_p(N_G(R))$ is the maximal normal p-subgroup of the normalizer $N_G(R)$. A p-chain σ given by (1.1) is said to be *radical* if $Q_i = O_p(N_G(\sigma_i))$ for each i. Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G and write

$$\mathcal{R}(G, B) = \{\sigma \in \mathcal{R}(G) : \mathrm{Blk}(N_G(\sigma)|B) \neq \emptyset\}.$$

By [7], the p-local rank $plr(G)$ of G is the number $plr(G) = \max\{|\sigma| : \sigma \in \mathcal{R}(G)\}$. We define the p-local rank $plr(B)$ of B to be the number

$$plr(B) = \max\{|\sigma| : \sigma \in \mathcal{R}(G, B)\}.$$

It is clear that $plr(B) \leq plr(G)$ for all $B \in \mathrm{Blk}(G)$. Also we can recover the p-local rank of G by considering the principal p-block. For suppose $plr(G) = n$ and $\sigma \in \mathcal{R}(G)$ with $|\sigma| = n$. If $b_0 = B_0(N_G(\sigma))$ is the principal block of $N_G(\sigma)$, then by [5, 3.2], b_0^G is well-defined and by Brauer's Third Main Theorem, $b_0^G = B_0 = B_0(G)$, so that $\sigma \in \mathcal{R}(G, B_0)$ and

$$plr(B_0) = plr(G).$$

The paper is organized as follows. In Section 2 we consider our definitions in terms of subpairs, and give the lemma which is the engineroom of this paper. In Sections 3 and 4 we prove that if B' is a p-block of a section of G corresponding to B under the Brauer correspondence and the block correspondence outlined in Section 4, then $plr(B') \leq plr(B)$, and give some conditions for equality/inequality. Finally, in Section 5 we give necessary and sufficient conditions for a p-block to have p-local rank one. We would like to draw the readers' attention to the fact that all proofs given here are direct, so the p-local structure is transparent(!) at each stage.

2 B-subgroups

Let R be a p-subgroup of G and $b_R \in \mathrm{Blk}(C_G(R)R)$. Then the pair (R, b_R) is called a *subpair* for G. If R is radical in G, then (R, b_R) is called a *radical subpair* for G. Following Alperin [2], for $B \in \mathrm{Blk}(G)$, the pair (R, b_R) is called a *B-subgroup* if $(1, B) \leq (R, b_R)$.

If (R, b_R) is a B-subgroup, then $b_R^G = B$ and $b_R^{N_G(R)}$ is a well-defined block of $N_G(R)$. In particular, $b_R^{N_G(R)} \in \mathrm{Blk}(N_G(R)|B)$ and $\mathrm{Blk}(N_G(R)|B) \neq \emptyset$. Conversely, if $\mathrm{Blk}(N_G(R)|B) \neq \emptyset$, then R is conjugate to a subgroup of $D(B)$ and there is a B-subgroup (R, b_R).

A B-subgroup (R, b_R) is called a *radical B-subgroup* if R is radical in G.

Suppose that $\sigma \in \mathcal{R}(G, B)$. Then there is a block $b \in \mathrm{Blk}(N_G(\sigma)|B)$ and we may suppose that $D(b) \leq D(B)$. Since $Q_0, \ldots, Q_n \trianglelefteq N_G(\sigma)$, it follows that $Q_i \leq D(b)$ and there is a B-subgroup $(Q_i, b_i) \leq (D(B), b_D)$ for each i, where $(D(B), b_D)$ is a fixed Sylow B-subgroup. Conversely if we can associate to $\sigma \in \mathcal{R}(G)$ such a chain of B-subgroups, then $\sigma \in \mathcal{R}(G, B)$. A radical p-chain σ is called a *radical B-chain* if $\sigma \in \mathcal{R}(G, B)$.

Lemma 2.1 *Let B be a p-block of G and (Q, b_Q) a B-subgroup. Then there is a radical B-subgroup (P, b_P) such that*

$$(Q, b_Q) \leq (P, b_P)$$

and $N_G(Q) \leq N_G(P)$. In particular, (Q, b_Q) and (P, b_P) are subpairs for $N_G(P)$. Moreover, suppose that $Q \leq H \leq G$ and Q is radical in H. Then $Q = P \cap N_H(Q) = P \cap H$ and $N_H(Q) = H \cap N_G(P)$.

PROOF: Let $M_1 = N_G(Q)$ and $M_i = N_G(O_p(M_{i-1}))$ for all $i \geq 2$. Then

$$N_G(Q) = M_1 \leq M_2 \leq \ldots \leq M_m \tag{2.1}$$

for all $m \geq 2$ and since $Q \leq O_p(M_1)$ and $O_p(M_{i-1}) \leq O_p(M_i)$, it follows that

$$Q \trianglelefteq O_p(M_1) \trianglelefteq O_p(M_2) \trianglelefteq \ldots \trianglelefteq O_p(M_m). \tag{2.2}$$

Thus there is a minimal $m \geq 1$ such that $M_m = M_{m+1} = N_G(O_p(M_m))$, so $O_p(M_m) = O_p(M_{m+1})$ and $O_p(M_m)$ is radical in G. Let $L = M_m$ and $P = O_p(L)$, so that $N_G(Q) \leq L = N_G(P)$. Since $C_G(Q) \leq N_G(Q) \leq L$, it follows that $C_G(Q) = C_L(Q)$, (Q, b_Q) is a subpair for L and b_Q^L is a well-defined block of L. Thus (Q, b_Q) is contained in a Sylow b_Q^L-subgroup (D_1, b_{D_1}), where $D_1 =_L D(b_Q^L)$. Since $P \leq D_1$, it follows that there is a unique b_Q^L-subgroup (P, b_P) such that

$$(Q, b_Q) \leq (P, b_P) \trianglelefteq (D_1, b_{D_1}).$$

But $C_G(P) \leq C_G(Q) \leq L$, so (P, b_P) is also a subpair for G and $(1, B) \leq (Q, b_Q) \leq (P, b_P)$. Thus (P, b_P) is a B-subgroup.

Suppose that $Q \leq H \leq G$ and $Q = O_p(K)$, where $K = N_H(Q) \leq L$. Then $Q \leq K \cap P \trianglelefteq K$ and $Q = K \cap P$. Since $Q \trianglelefteq N_{H \cap P}(Q) \leq K$ and $N_{H \cap P}(Q) \leq K \cap P = Q$, it follows that

$$Q = P \cap K = P \cap H.$$

If $h \in L \cap H$, then h normalizes $O_p(L) \cap H = Q$, so that $h \in K$, $K \leq L \cap H \leq K$ and $K = L \cap H$.

□

3 Subgoups and the p-local rank

Proposition 3.1 *Let H be a subgroup of G and $b \in \mathrm{Blk}(H|B)$. Suppose that*

$$\sigma : O_p(H) = Q_0 < Q_1 < \ldots < Q_n$$

is a radical b-chain with $n \geq 1$. Set $K_i = N_H(\sigma_i)$ for $i \geq 0$. Then there is a radical B-chain

$$\sigma^* : O_p(G) < P_1 < \ldots < P_n$$

of G such that for each $i \geq 1$, $K_i \leq N_{L_{i-1}}(Q_i) \leq L_i$, $K_i = L_i \cap H$, $C_G(Q_i) = C_{L_i}(Q_i)$ and $Q_i = P_i \cap K_i = P_i \cap H$, where $L_i = N_G(\sigma_i^*)$.

PROOF: We may suppose $D(b) \leq D(B)$. Since σ is a radical b-chain, it follows that $\mathrm{Blk}(N_H(\sigma)|b) \neq \emptyset$ and we may suppose that

$$O_p(H) = Q_0 < Q_1 < \ldots < Q_n \leq D(b) \leq D(B). \tag{3.1}$$

Let $(D(B), b_D)$ be a Sylow B-subgroup. For each $i \geq 0$, there is a B-subgroup (Q_i, b_i) such that $(Q_i, b_i) \leq (D(B), b_D)$.

By Lemma 2.1, there is a radical B-subgroup (P_1, B_1) such that $(Q_1, b_1) \leq (P_1, B_1)$, $K_1 \leq N_G(Q_1) \leq L_1 = N_G(P_1)$, $C_G(Q_1) = C_{L_1}(Q_1)$, $Q_1 = P_1 \cap K_1 = P_1 \cap H$ and $K_1 = H \cap L_1$. Thus $O_p(G) \leq P_1$ and by (3.1),

$$O_p(G) \cap H \leq O_p(H) < Q_1 \leq P_1 \cap H.$$

In particular, $P_1 \neq O_p(G)$,

$$\sigma_1^* : O_p(G) < P_1$$

is a radical B-chain and $L_1 = N_G(\sigma_1^*)$.

Suppose that $n \geq 2$ and there is a radical B-chain

$$\sigma_t^* : O_p(G) < P_1 < \ldots < P_t$$

with $1 \leq t < n$ satisfying the conditions of the Proposition. Since σ is radical, it follows that $Q_{t+1} \leq Q_n \leq K_n \leq K_t \leq L_t$, $C_G(Q_{t+1}) \leq C_G(Q_t) = C_{L_t}(Q_t)$ and (Q_{t+1}, b_{t+1}) is a subpair for L_t.

Apply Lemma 2.1 for $Q = Q_{t+1}$, $G = L_t$ and $H = K_t$. There is a radical subpair (P_{t+1}, B_{t+1}) for L_t such that $(Q_{t+1}, b_{t+1}) \leq (P_{t+1}, B_{t+1})$, $K_t \leq N_{L_t}(Q_{t+1}) \leq L_{t+1} = N_{L_t}(P_{t+1})$, $Q_{t+1} = P_{t+1} \cap K_t = P_{t+1} \cap K_{t+1}$, $K_{t+1} = L_{t+1} \cap K_t$ and $C_{L_t}(Q_{t+1}) = C_{L_{t+1}}(Q_{t+1})$.

Since $K_t = L_t \cap H$, it follows that $K_{t+1} = L_{t+1} \cap L_t \cap H = L_{t+1} \cap H$ and $Q_{t+1} = P_{t+1} \cap K_t = P_{t+1} \cap H$. Similarly, since $C_G(P_{t+1}) \leq C_G(Q_t) = C_{L_t}(Q_t) \leq L_t$, it follows by Lemma 2.1 that

$$C_G(P_{t+1}) = C_{L_t}(P_{t+1}) = C_{L_{t+1}}(P_{t+1}),$$

so that (P_{t+1}, B_{t+1}) is a subpair for G. But $(1, B) \leq (Q_{t+1}, b_{t+1}) \leq (P_{t+1}, B_{t+1})$, so (P_{t+1}, B_{t+1}) is a B-subgroup. Since $Q_t = P_t \cap H < Q_{t+1} = P_{t+1} \cap H$, it follows that $P_t < P_{t+1}$,

$$\sigma_{t+1}^* : O_p(G) < P_1 < \ldots < P_t < P_{t+1}$$

is a radical B-chain of G and $L_{t+1} = N_G(\sigma_{t+1}^*)$. The result now follows by induction on n.

$\square$

Proposition 3.2 *Let H be a subgroup of G and $b \in \mathrm{Blk}(H|B)$. Then*

$$plr(b) \leq plr(B).$$

PROOF: We may suppose $plr(b) \geq 1$. Suppose that σ is a radical b-chain given by (1.1) such that $|\sigma| = plr(b) = n$. Since Q_0 is radical in H, it follows that $O_p(H) \leq Q_0$. If $O_p(H) < Q_0$, then

$$\sigma' : O_p(H) < Q_0 < Q_1 < \ldots < Q_n$$

is also a radical b-chain of H and so $plr(b) \geq |\sigma'| = plr(b)+1$, which is impossible. Thus $O_p(H) = Q_0$. By Proposition 3.1, there is a radical B-chain σ^* of length n, so that $plr(b) \leq plr(B)$.

□

We examine the situation where H contains $O^{p'}(G)$, the minimal normal subgroup with p-regular index in G. We first need the following in order to locate b-subgroups when moving from G to H:

Lemma 3.3 *Let $N \trianglelefteq G$ and $N \leq H \leq G$. Suppose that $B \in \mathrm{Blk}(G)$ and $b \in \mathrm{Blk}(H|B)$. Then*

(i) there is $B' \in \mathrm{Blk}(N)$ such that B' is covered by both B and b,

(ii) B and b share a defect group D.

PROOF: (i) By [3, 58.17] $b|B_{H\times H}$, and by [3, 61.4] B covers $B' \in \mathrm{Blk}(N)$ if and only if $B'|B_{N\times N}$ (and similarly for b). Suppose that B covers $B' \in \mathrm{Blk}(N)$ and b covers $B'' \in \mathrm{Blk}(N)$. Then $B''|b_{N\times N}$ and $b|B_{H\times H}$, so $B''|B_{N\times N}$, and B' is conjugate in G to B''. Hence we can take $B' = B''$.

(ii) Let $B' \in \mathrm{Blk}(N)$ be covered by both B and b as in part (i), and let $D(G)$ be a defect group for B, $D(H)$ a defect group for b such that $D(G) \cap N =_G D'$ for some defect group D' of B' and $D(H) \cap N =_H D''$ for some other defect group D'' of B' (see [3, 61.2]). We may assume that $D'' = D'$. But $D(G) \cap N = D(G)$ and $D(H) \cap N = D(H)$, so $D(G) =_G D' =_H D(H)$, and we may take $D(G) = D(H)$.

□

Proposition 3.4 *Suppose that $O^{p'}(G) \leq H \leq G$ and $b \in \mathrm{Blk}(H|B)$. Then $\mathcal{R}(G,B) = \mathcal{R}(H,b)$, and in particular $plr(B) = plr(b)$.*

PROOF: Write $N = O^{p'}(G)$, and let D be a defect group for B and b as in Lemma 3.3.

Note that if X and Y are finite groups with $X \trianglelefteq Y$ and $[Y:X]_p = 1$, then $O_p(X) = O_p(Y)$, since $O_p(X)$ is the *unique* maximal normal p-subgroup of X. Also note that every p-subgroup of Y is contained in X.

Suppose that $\sigma \in \mathcal{R}(G,B)$. Then we may take $V^\sigma \leq D \leq H$. Now

$$O^{p'}(N_G(\sigma)) \leq N_H(\sigma) \leq N_G(\sigma),$$

so $O_p(N_H(\sigma)) = O_p(O^{p'}(N_G(\sigma))) = O_p(N_G(\sigma)) = V^\sigma$. Hence $\sigma \in \mathcal{R}(H)$. But $V^\sigma \leq D$, a defect group for b, so there is a b-subgroup (V^σ, b_{V^σ}). But $b_{V^\sigma}^{N_H(\sigma)}$ is a well-defined block, as is $(b_{V^\sigma}^{N_H(\sigma)})^H$, and $b_{V^\sigma}^H = b$, so $\sigma \in \mathcal{R}(H,b)$.

Ther argument that $\mathcal{R}(H,b) \subseteq \mathcal{R}(G,B)$ is similar.

□

Proposition 3.5 *Let G_1 and G_2 be finite groups and $B = B_1 \times B_2$ a p-block of $G = G_1 \times G_2$, where each $B_i \in \mathrm{Blk}(G_i)$. Then*

$$plr(B) = plr(B_1) + plr(B_2).$$

PROOF: Let π_i be the natural projection of G onto G_i for $i = 1, 2$. If $\sigma \in \mathcal{R}(G)$ is given by (1.1), then let $\Omega_i = \{\pi_i(Q_j) : 0 \le j \le n\}$ and $|\Omega_i| = u_i$. Relabel the subgroups in Ω_i such that

$$\Omega_i = \{W_0 < W_1 < \ldots < W_{u_i}\}.$$

It follows by [7, 4.6] that each radical subgroup R of G is of the form $R = R_1 \times R_2$, where $R_1 = R \cap G_1$ and $R_2 = R \cap G_2$. Thus $N_G(R) = N_{G_1}(R_1) \times N_{G_2}(R_2)$, and it follows that $\pi_i(\sigma) : W_0 < W_1 < \ldots < W_{u_i}$ is a radical chain of G_i and

$$N_G(\sigma) = N_{G_1}(\pi_1(\sigma)) \times N_{G_2}(\pi_2(\sigma)).$$

In addition, if σ is a radical B-chain, then $\pi_i(\sigma)$ is a radical B_i-chain, so that $plr(B) \le plr(B_1) + plr(B_2)$.

Conversely, suppose that $\sigma_1 : W_0 < W_1 < \ldots < W_{u_1}$ and $\sigma_2 : P_0 < P_1 < \ldots < P_{u_2}$ are radical B_1-chain and B_2-chain, respectively. Define

$$\sigma : Q_0 < Q_1 < \ldots < Q_{u_1+u_2},$$

such that $Q_i = W_i \times P_0$ for $0 \le i \le u_1$ and $Q_i = W_{u_1} \times P_{i-u_1}$ for $u_1 \le i \le u_1 + u_2$. Then σ is radical B-chain and $|\sigma| = |\sigma_1| + |\sigma_2|$. Thus $plr(B) \ge plr(B_1) + plr(B_2)$.

□

4 Quotient groups and the p-local rank

Let N be a normal subgroup of G and V an indecomposable OG-module which is indecomposable as an ON-module V_N, where O is $\mathcal{O}$ or k. Following [6], a block B of G *dominates* a block $\overline{B}$ of $\overline{G} = G/N$ through V if there is an $O\overline{G}$-module X such that $V \otimes \mathrm{Inf}(X)$ is a B-module, where $\mathrm{Inf}(X)$ is the inflation of X to G. This is equivalent, by [6], to the fact that B covers the block b of N containing V_N. In addition, by [6], $\overline{B}$ has a defect group QN/N such that

$$D \cap N \le Q \le D,$$

where $D = D(B)$.

Let $\overline{H} \le \overline{G} = G/N$ and $\overline{b} \in \mathrm{Blk}(\overline{H}|\overline{B})$. We may suppose that $D(\overline{b}) \le D(\overline{B}) = QN/N$. Since $QN/N \simeq Q/(Q \cap N)$, there is a subgroup W of Q such that $WN/N = D(\overline{b})$ and

$$D \cap N \le Q \cap N \le W \le Q \le D. \tag{4.1}$$

Moreover, suppose that W_1 and W_2 are subgroups of Q such that $Q \cap N \le W_1 \le W_2 \le Q$ and $W_1N/N \trianglelefteq W_2N/N$. Then $W_1N/N \simeq \eta(W_1) \trianglelefteq \eta(W_2) \simeq W_2N/N$, where η is the natural group homomorphism from Q onto $Q/(Q \cap N)$. Since $Q \cap N = \ker(\eta) \le W_1$, it follows that $W_1 \trianglelefteq W_2$.

Proposition 4.1 *Let N be a normal subgroup of G and $\overline{B} \in \mathrm{Blk}(G/N)$ which is dominated by a block B of G through V. Then*

$$plr(\overline{B}) \leq plr(B).$$

Moreovr, suppose that b is the block of N containing V_N and $plr(b) \geq 1$. Then $plr(\overline{B}) < plr(B)$.

PROOF: Let $\overline{G} = G/N$ and $\sigma \in \mathcal{R}(\overline{G}, \overline{B})$ with $|\sigma| = n \geq 1$. Then there is a block $\overline{b} \in \mathrm{Blk}(N_{\overline{G}}(\sigma)|\overline{B})$, so we may suppose that $WN/N = D(\overline{b})$ for some subgroup $W \leq Q$ satisfying (4.1). Thus we may suppose that there are subgroups $Q_0, Q_1, \ldots, Q_n$ of Q such that

$$D \cap N \leq Q \cap N \leq Q_0 < Q_1 < \ldots < Q_n \leq Q \leq D \qquad (4.2)$$

and $\sigma : Q_0N/N < Q_1N/N < \ldots < Q_nN/N$. Thus for each i with $0 \leq i \leq n$ there is a B-subgroup (Q_i, b_i) for G. Since σ is radical, it follows by the remark above that $Q_i \trianglelefteq Q_n$ for all $0 \leq i \leq n$.

By Lemma 2.1, there is a radical B-subgroup (P_1, b_{P_1}) containing (Q_1, b_1) and $N_G(Q_1) \leq L_1 = N_G(P_1)$, so that $O_p(G) \leq P_1$ and $Q_n \leq L_1$ as $Q_1 \trianglelefteq Q_n$. Since $\overline{O_p(G)} = O_p(G)N/N \trianglelefteq \overline{G}$ and $\overline{Q}_0$ is radical in $\overline{G}$, it follows that

$$\overline{O_p(G)} \leq \overline{Q}_0 < \overline{Q}_1 \leq \overline{P}_1,$$

and in particular, $O_p(G) < P_1$. Thus $\sigma_1^* : O_p(G) < P_1$ is a radical B-chain. In addition, since $N_G(Q_1) \leq L_1$, it follows that $\overline{N_G(Q_1)} = N_{\overline{G}}(\overline{Q}_1) \leq \overline{L}_1$, $\overline{Q}_1$ is a radical subgroup of $\overline{L}_1$, $\overline{P}_1 \leq \overline{Q}_1$ and $\overline{P}_1 = \overline{Q}_1$. Thus $N_{\overline{G}}(\overline{P}_1) = N_{\overline{G}}(\overline{Q}_1)$ and $\overline{L}_1 = N_{\overline{G}}(\overline{Q}_1)$.

Suppose that there is a radical B-chain

$$\sigma_t^* : O_p(G) < P_1 < \ldots < P_t$$

with $1 \leq t < n$ such that $Q_n \leq N_{L_{t-1}}(Q_t) \leq L_t$, $Q_t \leq P_t$, $\overline{Q}_t = \overline{P}_t$, $C_G(Q_t) = C_{L_t}(Q_t)$ and $N_{\overline{G}}(\sigma_t) = \overline{L}_t$, where $L_0 = G$. Thus $Q_{t+1} \leq L_t$, $C_G(Q_{t+1}) \leq C_G(Q_t) = C_{L_t}(Q_t) \leq L_t$ and $C_G(Q_{t+1}) = C_{L_t}(Q_{t+1})$. So (Q_{t+1}, b_{t+1}) is a subpair for L_t and by Lemma 2.1, $(Q_{t+1}, b_{t+1}) \leq (P_{t+1}, B_{t+1})$ for some radical subpair (P_{t+1}, B_{t+1}) for L_t and $N_{L_t}(Q_{t+1}) \leq L_{t+1} = N_{L_t}(P_{t+1})$. Since $C_G(P_{t+1}) \leq C_G(Q_{t+1}) \leq L_t$, it follows that $C_G(P_{t+1}) = C_{L_t}(P_{t+1}) \leq L_{t+1}$, so that $C_G(P_{t+1}) = C_{L_{t+1}}(P_{t+1})$ and (P_{t+1}, B_{t+1}) is a subpair for G. Since $(1, B) \leq (Q_{t+1}, b_{t+1}) \leq (P_{t+1}, B_{t+1})$, it follows that (P_{t+1}, B_{t+1}) is a B-subgroup. Since $Q_n \leq L_t$ and $Q_{t+1} \trianglelefteq Q_n$, it follows that $Q_n \leq L_{t+1}$.

Since P_{t+1} is radical in L_t and $N_{L_t}(Q_{t+1}) \leq L_{t+1}$, it follows that $P_t = O_p(L_t) \leq P_{t+1}$,

$$\overline{N_{L_t}(Q_{t+1})} = N_{\overline{L}_t}(\overline{Q}_{t+1}) = N_{\overline{G}}(\sigma_{t+1}) \leq \overline{L}_{t+1},$$

$\overline{Q}_{t+1}$ is radical in $\overline{L}_{t+1}$ and $\overline{P}_{t+1} \leq \overline{Q}_{t+1}$. Thus $\overline{P}_{t+1} = \overline{Q}_{t+1}$ and $N_{\overline{G}}(\sigma_{t+1}) = \overline{L}_{t+1}$. In particular, $\overline{Q}_t = \overline{P}_t < \overline{Q}_{t+1} = \overline{P}_{t+1}$ and $P_t \neq P_{t+1}$. Thus

$$\sigma_{t+1}^* : O_p(G) < P_1 < \ldots < P_t < P_{t+1}$$

is a radical B-chain and by induction on n, $plr(\overline{B}) \leq plr(B)$.

Suppose that $plr(b) > 0$, so that B covers b by [6]. We may choose above D so that $D(b) = D \cap N$. Thus

$$O_p(G) \cap N \leq O_p(N) < D \cap N \leq Q_0$$

as $plr(b) \geq 1$. By Lemma 2.1, there is a radical B-subgroup (W, b_W) containing the B-subgroup (Q_0, b_0) and $N_G(Q_0) \leq N_G(W)$. Thus $O_p(G) \cap N < D \cap N \leq Q_0 \leq W \cap N$ and $O_p(G) < W$. Repeat the proof above with G replaced by $N_G(W)$. Then there is a radical B-chain $O_p(Q) < W < P_1 < \ldots < P_n$ of length $n+1$, so that $plr(\overline{B}) < plr(B)$. □

5 Blocks of p-local rank one

We say that a subgroup H of G is trivial intersection (TI) if $H^g \cap H = 1$ for every $g \in G - N_G(H)$. By [7, 7.1] if $plr(G) > 0$, then $plr(G) = 1$ if and only if $G/O_p(G)$ has TI Sylow p-subgroups. Hence if B is the principal p-block, then $plr(B) = 1$ if and only if B has defect groups intersecting pairwise in $O_p(G)$. We show that this may be generalized to any p-block.

We then highlight the consequence of this to conjectures such as those of Alperin [1], Dade [4] and of Robinson [7], although the consequence to Alperin's conjecture has been proved in [8].

Proposition 5.1 *Let $B \in \mathrm{Blk}(G)$ have defect group $D \neq O_p(G)$. Then $plr(B) = 1$ if and only if $D/O_p(G)$ is a TI set in $G/O_p(G)$. Further, in this case $O_p(G)$ is the only radical p-subgroup of G contained in D.*

PROOF: Suppose first that $D/O_p(G)$ is a TI subgroup of $G/O_p(G)$, and that $Q \neq O_p(G)$ is a radical p-subgroup of G such that $\mathrm{Blk}(N_G(Q)|B) \neq \emptyset$. Then Q is conjugate to a subgroup of D, and we may suppose that $Q \leq D$. Suppose that $Q \neq D$. Since $N_G(Q) \leq N_G(D)$, it follows that $N_D(Q) \trianglelefteq N_G(Q)$, and so $Q < N_D(Q) \leq O_p(N_G(Q))$, contradicting our assumption that Q is radical. Hence a radical B-chain σ of maximal length must satisfy $\sigma_1 : O_p(G) < D$, since D is a radical p-subgroup such that $\mathrm{Blk}(N_G(D)|B) \neq \emptyset$. But V^σ is G-conjugate to a subgroup of D, so $O_p(G) < D$ is a radical p-chain for B of maximal length, and $plr(B) = 1$.

Now suppose that $plr(B) = 1$, but that $D/O_p(G)$ is not TI. We show first that D cannot contain any radical p-subgroups of G other than $O_p(G)$ and itself.

Suppose that R is a radical p-subgroup of G with $O_p(G) < R < D$, and let (D, b_D) be a Sylow B-subgroup. Then there is a unique radical B-subgroup (R, b_R) such that $(1, B) < (R, b_R) < (D, b_D)$. Now $b_R^{N_G(R)}$ is a well-defined block of $N_G(R)$ and has defect group R, since otherwise $O_p(G) < R < D(b_R^{N_G(R)})$ is a radical B-chain and $plr(B) > 1$, contradicting our original assumption. But then $(b_R^{N_G(R)})^G$ must have defect group $R \neq D$ by Brauer's First Main Theorem, a contradiction since $(b_R^{N_G(R)})^G = B$ and $D(B) = D$.

Since $D/O_p(G)$ is not TI, we can choose a defect group $D_1 \neq D$ for B with $|D_1 \cap D|$ maximized. Write $Q = D_1 \cap D > O_p(G)$. As in the above argument, there is a

B-subgroup (Q, b_Q) such that $(Q, b_Q) < (D, b_D)$. By Lemma 2.1, there is radical B-subgroup (P, b_P) strictly containing (Q, b_Q) with $N_G(Q) \leq N_G(P)$ (since Q is not itself a radical B-subgroup).

Now we have seen that the only radical p-subgroups for B are either $O_p(G)$ or conjugate to D. By maximality, $Q = P \cap D = P \cap D_1$, hence we may take $P = D$, and $N_G(Q) \leq N_G(D)$.

Let (D_1, b_{D_1}) be a Sylow B-subgroup and $R = N_{D_1}(Q)$. Then $Q < R \leq D_1$, so there is a B-subgroup (R, b_R). Since

$$C_G(R) \leq C_G(Q) \leq N_G(Q) \leq N_G(D),$$

(R, b_R) is a subpair for $N_G(D)$, and $b_R^{N_G(D)}$ is defined.

Let $D(b_R^{N_G(D)})$ be a defect group for $b_R^{N_G(D)}$ containing R. Then $D \leq D(b_R^{N_G(D)})$ as $D \trianglelefteq N_G(D)$. Since $(b_R^{N_G(D)})^G = b_R^G = B$, it follows that $D(b_R^{N_G(D)})$ is G-conjugate to a subgroup of D, so $D = D(b_R^{N_G(D)})$. Hence $Q < R = N_{D_1}(Q) \leq D(b_R^{N_G(D)}) = D$, and $Q < |N_{D_1}(Q)| < |D_1 \cap D|$, contradicting our choice of Q.

□

Remark We note that there is already a notion of a *TI block* (see [1]), where B is TI if every nontrivial B-subgroup (Q, b_Q) is contained in a unique Sylow B-subgroup. However, it is not clear whether this is equivalent to B having TI defect groups. To distinguish between the two definitions, we say that a block with TI defect groups is a *TI defect block*.

Corollary 5.2 *Suppose that B is a p-block of G. Let $f : \mathcal{R}(G, B) \to A$ be a G-stable function dependent only on $N_G(\sigma)$ and V_σ, where A is an abelian group. If $D \neq O_p(G)$ is a defect group for B, $plr(G) > 0$ and any two defect groups for B intersect in $O_p(G)$, then*

$$\sum_{\sigma \in \mathcal{R}(G,B)/G} (-1)^{|\sigma|} f(N_G(\sigma), V_\sigma) = f(G, O_p(G)) + f(N_G(D), D) - f(N_G(D), O_p(G)).$$

Remark With analogy to the remarks in [5] we may replace $\mathcal{R}(G, B)$ by $\mathcal{D}_{max}$ in any alternating sum defined by such a function f, where $\mathcal{D}_{max}$ is the set of p-chains of G which are intersections of defect groups of B (here we do not make any assumptions on $plr(B)$. The proof is similar to that given in [5], so we do not include it here.

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