Scattering on a compact domain with few semiinfinite wires attached: resonance case.

A.Mikhailova[1], B.Pavlov[2],I.Popov[3],T.Rudakova[1],A.M.Yafyasov[1]

[1]*Institute for Physics, St.Petersburg State University, Ulianovskaya 1, Petrodvorets, St.Petersburg, 198904, Russia*
[2]*Department of Mathematics,the University of Auckland, New Zealand, Solvay Institutes, Brussels, Belgium,*
[3]*Institute for Precizion Mechanics and Optics, St. Petersburg, Russia*

ABSTRACT

Scattering problem for Neumann Laplacean with a continuous potential on a domain with a smooth boundary and few semiinfinite wires attached to it is studied. In resonance case when the Fermi level in the wires coincides with some *resonance* energy level in the domain the approximate formula for the transmission coefficient from one wire to another is derived : in the case of weak interaction between the domain and the wires the transmission coefficient is proportional to the product of values of the corresponding resonance eigenfunction of inner problem at the points of contact.

1 Introduction.

In our previous papers [1],[2] we discussed the scattering problem motivated by attempt of designing quantum electronic devices for triadic logic. Both devices are based on specific properties of scattering for a second order differential operator on a compact graph or domain with several semi-infinte quasi-onedimensional wires attached to it. The corresponding multy-channel scattering matrix may be represented in explicite form via Green's functions of the differential operator on the compact. In our paper [1] we investigated the resonance case when the Fermi level of electrons in the wires is equal to some *resonance* eigenvalue (multiplicity one) of the Schrödinger operator on the ring. We found that in the case of the weak connection between the wires and the ring the transmission coefficient from one wire to another *is approximately proportional to the product of values of the corresponding resonance eigenfunction at the contact points.*

The similar problem of resonance scattering may be investigated in a domain with few semiinfinite onedimensional waveguides (wires) attached at the points a_1, a_2... The techniques developed in [1] for onedimensional case can't be directly transfered to this case since the Green's function on the domain is discontinuous and it's spectral series is divergent. Using the iterated Hilbert identity we regularize the values of the Green's function at the poles and extend the analysis of the onedimensional resonance situation developed in [1] to the case of a domain. In particular we show in the case of the weak connection between the domain and the wires the transmission coefficient S_{ij} from one wire to another is approximately proportional to the product of values of the resonance eigenfunction at the contact points. One may show that this statement remains true also in the case of the third boundary condition on the boundary $\partial\Omega$: $\frac{\partial u}{\partial n} - \sigma u\Big|_{\partial\Omega} = 0$ or even for Schrödinger operators with some singular potentials (see an example below in the last section) provided the resonance eigenvalue is simple.

We consider here the case of the three-dimensional domain. The analysis of the corresponding two-dimensional problem oriented to the technological applications of the effect discussed will be done in the following publication.

Here is a plan of our paper: in the second section we describe the symplectic extension procedure used for attaching the wires; in the third and fourth sections we calculate the scattering matrix and the resolvent and investigate the resonance behaviour of them at the resonance eigenvalue.

2 Extension procedure.

We consider the spectral problem for Schrödinger operator $\mathcal{A}$ with an uniformly - continuous real potential $q(x)$ on a bounded domain Ω in R_3 with smooth boundary $\partial\Omega$:

$$-\triangle u + qu = \lambda u,$$

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0.$$

The operator extension scheme we use to attach the semiinfinite wires at contact points $a_1,\ a_2,\ ...a_N \subset \partial\Omega$ or perturb the operator at inner points $a_{N+1},\ a_{N+2},\ ...a_{N+M}$ is the following

1) We restrict the operator $\mathcal{A}* = -\Delta * + q*$ to the operator $\mathcal{A}_0$ defined by the same differential expression on the class of all smooth functions vanishing near the points $a_1, a_2, ...a_N$ and/or $a_{N+1},\ a_{N+2},\ ,, a_{n+M}$. Then the deficiency elements of $\mathcal{A}$ for complex values of the spectral parameter $\overline{\lambda}$ coincide with Green's functions $G_\lambda(x, a_s)$ of which are square-integrable but have non-square-integrable gradients:

$$G_\lambda(x,y) = \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} + g(x,y,\lambda),\ \ x,y \in \Omega. \tag{1}$$

Here the non-singular term $g(x, y, \lambda)$ is constructed via the solution of the corresponding Lippmann-Schwinger equation, see the remark below, section 3. For Neumann Laplacean ($q(x) \equiv 0$) the potential-theory approach combined with the reflection principle gives the asymptotics of the Green's function near the boundary point a_s in form

$$\lim_{x\to a_s} G_\lambda(x, a_s) = \frac{A_s}{2\pi|x-a_s|} + B_s + o(1) + A_s L \tag{2}$$

where by L some logarithmic term is denoted, see [3]. Similar asymptotics with the same leading singular term remains true also for Schrödinger operator with continuous potential. Planning to use the symplectic KLASSICHESKIJ O KOTOROM DIFFERENCIAL'NYH OPERATOROV SHCREDINGERA MY IMEEM VESCH' NAZYVAETSA SVJAZAT' u, u' na kotoroj formami) granicnymi uslovijami zadajut ermitovy geometricheskaja Ja dumaju, tebe ee vkljuchit' version of the operator extension procedure, see [9] we introduce the asymptotis boundary values:*singular amplitudes* $A_s(u)$ and the *regularized values of* u $B_s(u)$ for elements u of the domain of the adjoint operator $\mathcal{A}$ at the points a_s. We may assume that the nonperturbed operator $\mathcal{A}$ satisfies the condition $\mathcal{A} + I > 0$. Then its resolvent $[\mathcal{A} - \lambda I]^{-1}$ is a bounded integral operator with the kernel $G_{-1}(x, y)$ which may be used as an etalon of the growing rate of elements of the domain of adjoint operator at the poles:

$$u(x) = A_s G_{-1}(x, a_s) + B_s + o(1), \; x \to a_s. \tag{3}$$

The next statement shows that both A_s, B_s exist for deficiency elements $G_\lambda(x, a_s)$.

Lemma 1 *For any regular point* $\lambda \bar{\in} \sigma(\mathcal{A})$ *of the operator* $\mathcal{A}$ *and any* $a \in \{a_s\}_{s=1}^{N+M}$ *the following representation is true:*

$$G_\lambda(x, a) = G_{-1}(x, a) + (\lambda + 1) G_{-1} * G_\lambda(x, a),$$

where the second addend $(\lambda + 1) G_{-1} * G_\lambda(x, a) \equiv g_\lambda(x, a)$ *is a continuous function of* x *and the spectral series of it on eigenfunctions* φ_l *of the nonperturbed operator* $\mathcal{A}$

$$\mathcal{A} \varphi_l = \lambda_l \varphi_l$$

is absolutely and uniformly convergent in Ω. *The separation of the singularity at each eigenvalue* λ_0 *is possible:*

$$g_\lambda(x, a) = (\lambda + 1) \sum_l \frac{\varphi_l(x) \varphi_l(a)}{(\lambda_l + 1)(\lambda_l - \lambda)} =$$

$$\frac{(\lambda + 1) \varphi_0(x) \varphi_0(a)}{(\lambda_0 + 1)(\lambda_0 - \lambda)} + \sum_{l \neq 0} \frac{\varphi_l(x) \varphi_l(a)}{(\lambda_l + 1)(\lambda_l - \lambda)} \equiv$$

$$\frac{\varphi_0(x) \varphi_0(a)}{\lambda_0 - \lambda} + g^0_\lambda(x, a) \tag{4}$$

with uniformly and absolutely convergent series for $g^0_\lambda(x, a)$ *in a neighbourhood of* λ_0.

Proof of this statement is based on the classical Mercer theorem. The analysis of the Lippmann-Schwinger equation [6] shows that the Green's-function $G_\lambda(x, y)$ of the operator $\mathcal{A}$ admitts a representation in form (1) which implies that the positive integral operator $G_{-1} * G_{-1}$ –the convolution of resolvents at the spectral points $\lambda = -1$ – has a continuous kernel on the closed domain $\Omega = \overline{\Omega}$. Then using Mercer theorem we EST' - may check the uniform convergence in Ω of the spectral series for it's kernel

$$(\lambda+1)G_{-1} * G_{-1}(x,y) = (\lambda+1)\sum_l \frac{\varphi_l(x)\varphi_l(y)}{(\lambda_l+1)^2}.$$

The last statement implies the absolute and uniform convergence of the spectral series for the kernel on the diagonal $x = y$. The uniformly convergence of the spectral series for the kernel

$$G_{-1} * G_\lambda(x,y) = \sum_l \frac{\varphi_l(x)\varphi_l(y)}{(\lambda_l+1)(\lambda_l-\lambda)}$$

on the domain $\overline{\Omega}$ and each compact subset of the complement of the spectrum $\sigma(\mathcal{A})$ may be derived using Cauchy inequality for the remainder of the spectral series. Together with the obvious fact of continuity of eigenfunctions of the operator $\mathcal{A}$ this implies the continuity of $G_{-1} * G_\lambda(x,y)$ on the complement of the spectrum and the continuity of the difference

$$g^0_\lambda(x,y) \equiv (\lambda+1)G_{-1} * G_\lambda(x,y) - \frac{\varphi_0(x)\varphi_0(y)}{\lambda_0-\lambda} =$$

$$\sum_{l\neq 0} \frac{(\lambda+1)\varphi_l(x)\varphi_l(y)}{(\lambda_l+1)(\lambda_l-\lambda)} - \frac{\varphi_0(x)\varphi_0(y)}{\lambda_0+1}$$

in some neighbourhood of the eigenvalue λ_0.

□

Corollary For any element u of the domain of the adjoint operator $\mathcal{A}_0)^+$ the *asymptotic boundary values* $A_s(u), B_s(u)$ are defined at each point $a_s \in \overline{\Omega}$ as coefficients of the asymptotics of it at the point a_s:

$$u(x) = A_s(u)G_{-1}(x,a_s) + B_s + o(1), \;\; x \to a_s.$$

The existence of these asymptotic boundary values will be proved below. The commonly used asymptotic boundary values $A'_s(u), B'_s(u)$, see [10] which are defined by the asymptotics

$$u(x) = \frac{A'_s(u)}{4\pi|x-a|} + B'_s + o(1), \;\; x \to a_s$$

for inner points a_s are connected with $A_s(u), B_s(u)$ by real linear transformation, but still can't be defined at the boundary contact points according to the result [3] quoted above.

Note that the iterated Hilbert Identity was used in a similar way in [4]for separation the spectral and spacial singularities of Green's functions.

2). Integrating by parts we get the following expression for the boundary form of $\mathcal{A}_0^+$

$$< (\mathcal{A}_0)^+ u, v > - < u, (\mathcal{A}_0)^+ v >= \sum_s B_s(u)\bar{A}_s(v) - A_s(u)\bar{B}_s(v) \equiv \mathcal{J}(u,v) \qquad (5)$$

This complex symplectic form is an analog of Wronskian for partial differential equations, see [9], [4],[7],[8]. In particular this form calculated for two solutions of the homogeneous adjoint equations with the spectral parameters λ, $\bar{\lambda}$ vanish, which is an analog of the independence of the Wronskian $W(x)$ on x for corresponding ordinary differential equations.

Any Lagrangian plane $\mathcal{L}$ of this form defines a selgadjoint extension of the operator $\mathcal{A}_0$. For instance the corresponding Schrödinger operators with zero-range potentials at the points $a_s,\ s = N+1,\ N+2,\ ...N+M$ may be defined by generalized boundary conditions

$$B_s = \gamma_s A_s,\ \gamma_s = \bar{\gamma}_s.\ s = N+1,\ N+2,\ ...N+M, \tag{6}$$
$$A_s = 0,\ \ s = 1,\ 2,\ ...N.$$

The corresponding selfadjoint operator $\mathcal{A}_{0,\gamma}$ in $L_2(\Omega)$ has purely discrete spectrum.We consider the common extension of the operator $\mathcal{A}_0$ and Schrödinger operators in $l_2(\Gamma_s)$ defined by the differential expressions $l_s = -\dfrac{d^2}{dx^2} + q_s(x)$ with real rapidly decreasing potentials on the wires, $\Gamma_s,\ s = 1, 2, ...N$. The boundary form of each of them is $\mathcal{J}_s(u_s, v_s) = [u_s'\bar{v}_s - u_s\bar{v}_s']\Big|_{x=0}$. The total boundary form

$$\mathcal{J}(u,v) + \sum_{s=1}^{N} \mathcal{J}_s(u_s, v_s) \tag{7}$$

vanishes on the Lagrangian plane defined by the corresponding boundary condition, for instance by the combination of the boundary condition (6) at the inner points $a_s,\ s = N+1, ...N+M$ and the boundary conditions

$$\begin{pmatrix} A_s \\ u_s(0) \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} -B_s \\ -u_s'(0) \end{pmatrix},\ \ \beta > 0,\ \ s = 1,\ 2,\ ..N. \tag{8}$$

at the contact points $a_s,\ s = 1, 2, ...N$. The corresponding selfadjoint operator will be denoted by $\mathcal{A}_{\beta,\gamma}$.

We consider below also the families of operators in $L_2(\Omega) \oplus \sum_s L_2(\Gamma_s)$

$$\mathcal{A}_{\beta,0},\ \mathcal{A}_{\beta,\gamma},\ \beta \to 0$$

which correspond to the weakening connection ($\beta \to 0$) between the wires and the domain Ω. One can show that these boundary conditions simulate the interaction when the wires are joined to the domain non directly but are connected to it via quantum tunnelling through the potential barrier with the height proportional to $\ln \frac{1}{\beta}$.

3). For components of eigenfunctions of these extensions we have the adjoint homogeneous equations:

$$(\mathcal{A})^+\psi = \lambda\psi \text{ inside the domain } \Omega,$$
$$-u_s'' = \lambda u_s \text{ on the wires.}$$

with the selfadjoint boundary conditions (8) (or (6) or both of them) which annihilate the total boundary from (7). The formal proof of the selfadjointness of corresponding operatots $\mathcal{A}_{\beta,0}$, $\mathcal{A}_{0,\gamma}$, $\mathcal{A}_{\beta,\gamma}$ may be reduced to the proof of the symmetricity of the corresponding adjoint operators similarly to the corresponding fact in [1].

The components of eigenfunctions of the operators $\mathcal{A}_{\beta,\gamma}$ inside the domain may be represented by Ansatz

$$u = \sum_s \bar{A}_s G_\lambda(x, a_s), \tag{9}$$

predydyshih formulah. We use further the notations $\{A_1,\ A_2,\ ...\} \equiv \vec{A};\ \ \{G_\lambda(x,\ a_1),\ G_\lambda(x,\ a_2), ...\} \equiv \vec{G}_\lambda(x)$, which permitt to represent the above Ansatz (9) as $< \vec{A}, \vec{G}_\lambda(x) >$ One can see from the Lemma 1 that the singular amplitudes $A_1,\ A_2,\ A_{N+M}$ just coincide with the coefficients of the Ansatz and the regularized boundary values $\{B_1,\ B_2,\ ...B_N\} = \vec{B}$ of u are defined as

$$\vec{B} = Q\vec{A},$$

where

$$Q(\lambda) = \begin{pmatrix} g_\lambda(a_1, a_1) & G_\lambda(a_1, a_2) & .. & ... & G_\lambda(a_1, a_{N+M}) \\ G_\lambda(a_2, a_1)) & g_\lambda(a_2, a_2) & .. & ... & G_\lambda(a_2, a_{N+M}) \\ .. & .. & g_\lambda(a_3, a_3) & ... & .. \\ .. & .. & .. & ... & .. \\ G_\lambda(a_{N+M}, a_1)) & .. & .. & ... & g_\lambda(a_{N+M}, a_{N+M}) \end{pmatrix}. \tag{10}$$

and $g_\lambda(y, y)$ – the regularized value of the Greens function $G_\lambda(x, y)$ at the pole y – is defined in Lemma 1. The matrix Q defined by (10) is actually the Krein's Q-matrix, see [11] which accumulates the spectral information on some selected (underlying) selfadjoint extension of the considered Hermitian operator $\mathcal{A}_0$. In our case this selected extension is just the nonperturbed Schrödinger operator $\mathcal{A}$ on the domain. ¿From the symmetry of the Green's functions of the Schrödinger operators with real continuous potential follows that the Q-matrix is symmetric and has the positive imaginary part in upper halfplane

$$\left[\frac{Q - Q^+}{2i}\right] \Im\lambda \geq 0.$$

The expression (10) for Q-matrix is just a version of the general formula connecting the boundary values of abstract Hermitian operators, see for instance [11, 9]: in general case the Q - matrix has the form

$$P_{N_i} \frac{1 + \lambda\mathcal{A}}{\mathcal{A} - \lambda I}\Big|_{N_i}.$$

3 The Resolvent and the Scattering Matrix.

In this section we assume that the potentials $q_s,\ s = 1, 2, ...N$ on the wires Γ_s are trivial $q_s(x) \equiv 0$ and $\gamma_s = 0,\ \ s = N + 1,\ N + 2,\ ...N + M$, hence the zero-range potentials at the iner points $a_s,\ s = N + 1,\ N + 2,\ ...N + M$ are eliminated.

The next statement gives the explicite formulae and description of properties of the component of the resolvent kernels (Green's functions) $G_\lambda^{\beta,0}(x, y)$ of operators $\mathcal{A}_{\beta,0}$ on the domain Ω and the scattered waves of the operator $\mathcal{A}_{\beta,0}$.

We denote the Green's function of the nonperturbed operator $\mathcal{A}$ by $G(x, y, \lambda)$ and consider the *generic case when neither of eigenfunctions of the nonperturbed operator vanishes at all contact points* a_s, $s = 1, 2, ...N$ *simultaneously.* Note that the eigenfunctions of the perturbed operator vanishing at all contact points automatically satisfy the boundary conditions (8) if continued by the identical zero on the wires.In generic case this situation is eliminated. In particular we do not have *embedded* eigenvalues in generic case, since all nonzero solutions e^{ikx_s} of the adjoint homogeneous equation of the wires for positive $\lambda = k^2$ are non square integrable.

In what follows we call the auxilliary finitedimensional Hilbert space E, dim$E = N$, *thechannelspace*, having in mind the role of this space in scattering problem. The elements $\vec{A}$ of E are complex vectors $\{A_1, A_2, ...A_N\}$. In particular we use vectors combined of the values of the nonperturbed Green's functions attached to the points $\{a_s\} : \{G_\lambda(x, a_s)\} \equiv \vec{G}_\lambda(x)$. We assume that the metric form of E is trivial, for instance the dot product of two vectors above is given by the formula:

$$< \vec{A}, \vec{G}_\lambda^{\beta,0}(x) >= \sum_s \bar{A}_s G^{\beta,0}(x, a_s, \lambda)$$

Theorem 1 *The component of the the resolvent kernel of the operator* $\mathcal{A}_{\beta,0}$ *on the domain* Ω *is represented in terms of Green's functions of the nonperturbed operator*$\mathcal{A}$ *the following way:*

$$< \overline{\vec{G}_\lambda(y)}, \left[\frac{1}{ik\beta^2} - Q\right]^{-1} \vec{G}_\lambda(x) >, \; k^2 = \lambda, \; \Im k \geq 0. \tag{11}$$

The spectrum $\sigma(\mathcal{A}_{\beta,0})$ *of the perturbed operator* $\mathcal{A}_{\beta,0}$ *consists of all singularities of the matrix*

$$\left[\frac{1}{ik\beta^2} - Q\right]^{-1}$$

in the complex plane of the spectral parameter λ. *In particular the absolutely continuous spectrum of* $\mathcal{A}_{\beta,0}$ *fills the positive half-axis* $\lambda \geq 0$ *with the constant multiplicity* N. *The eigenvalues* $\lambda_r = k_r^2$, $\Im k_r > 0$ *and resonances* $\lambda_r = k_r^2$, $\Im k_r < 0$ *of the operator* $\mathcal{A}_{\beta,0}$ *are defined, counting multiplicity, respectively: by the poles of the matrix* $\left[\frac{1}{ik\beta^2} - Q\right]^{-1}$ *on the positive imaginary half-axis (for eigenvalues* $\lambda = k^2 < 0$*) or in lower half-plane* $\Im k < 0$ *(for resonances). They may be found as roots of the corresponding dispersion equation in upper* $\Im k > 0$ *and lower* $\Im k < 0$ *half-planes respectively:*

$$det \left[\frac{1}{ik\beta^2} - Q\right] = 0. \tag{12}$$

The eigenfunctions of the absolutely- continuous spectrum of the operator $\mathcal{A}_{\beta,0}$ *are presented by the scattered waves which form complete orthogonal system of eigenfunctions in the absolutely-continuous subspace. In particular the scattered wave iniciated by the plane wave in the first channel (on the wire attached to* a_1*) has the form:*

$$\psi_s^1(x_s) = \delta_{1,s} e^{-ikx_s} + S_{s,1} e^{ikx_s}, \; x_s \in \Gamma_s,$$

$$\Psi^1(x) = \beta < \vec{\bar{G}}_\lambda(x), \left[\frac{1}{ik\beta^2} - Q\right]^{-1} \vec{\delta}_1, \; x \; \in \Omega.$$

Proof. Being solutions of the homogeneous equation $\mathcal{A}_0^+ U = \lambda U$ the components of the resolvent kernel of the perturbed operator on the wires coincide with exponentials $u_s e^{ikx_s}$ and the component of the resolvent kernel inside the domain is represented by the linear combination of Green's functions on the nonperturbed operator:

$$G_\lambda^{\beta,0}(x,y) = G_\lambda(x,y) + \sum_s \bar{A}_s G_\lambda(x, a_s).$$

Then due to the boundary conditions (8) at the points $a_1, a_2, ..., a_N$ we have

$$\begin{pmatrix} \bar{A}_s \\ u_s \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} -\bar{A}_s g_\lambda(a_s) - \sum_{t\neq s} \bar{A}_t G(a_s, a_t, \lambda) \\ -iku_s \end{pmatrix}, \quad s = 1, 2, 3, ...N. \qquad (13)$$

Due to the formula (10) for Q-matrix we see that $\bar{A}_s g_\lambda(a_s, a_s) + \sum_{t\neq s} \bar{A}_t G(a_s, a_t, \lambda) = \left[Q\vec{\bar{A}}\right]_s$ the solution $\vec{\bar{A}}$ of the last system is represented as

$$\vec{\bar{A}} = \left[\frac{1}{ik\beta^2} - Q\right]^{-1} \vec{G}_\lambda(y),$$

provided the matrix $\left[\frac{1}{ik\beta^2} - Q\right]$ is invertible. This gives the announced representation (11) for the component of the resolvent kernel of the perturbed operator inside the domain

$$G_\lambda^{\beta,0}(x,y) = G_\lambda(x,y) + \sum_s G_\lambda(x, a_s) \left[\frac{1}{ik\beta^2} - Q\right]^{-1}_{s,r} G_\lambda(y, a_r), \; k^2 = \lambda, \; \Im k \geq 0.$$

To prove the second statement of the theorem notice first that regular points $\lambda = k^2$ of the matrix $\left[\frac{1}{ik\beta^2} - Q\right]$

$$\det \left[\frac{1}{ik\beta^2} - Q\right] \neq 0,$$

can't be eigenvalues of the perturbed operator since the homogeneous equation with the boundary condition (8) is reduced to the homogeneous equation for $\vec{A}$

$$\left[\frac{1}{ik\beta^2} - Q\right] \vec{\bar{A}} = 0, \qquad (14)$$

which may have in this case trivial solutions only. Then the corresponding solution of the adjoint homogeneous equation is a smooth function on the domain and may be the eigenfunction of the perturbed operator only if it vanishes at the contact points which is prohibited in generic case. On the other hand each nontrivial solution $\vec{A}$ of the equation (14) with the *symmertic* matrix $\left[\dfrac{1}{ik\beta^2} - Q\right]$ at the negative spectral point $\lambda = k^2$ defines the corresponding square-integrable solution of the adjoint homogeneous equation satisfying the boundary conditions at the contact points, which is obviously an eigenfunction of $\mathcal{A}_{\beta,0}$. One may check, using the symmerty and the special form of the matrix, that the dimension of the nul-space of it coincides with the dimension of the corresponding eigenspace of the perturbed operator. Similar reasoning may be used to acomplish the proof of the statement about resonances if the resonances are definad just as the poles of an analytical continuation of the resolvent kernel across the interval $[0, \infty)$ filled with absolutely continuous spectrum.

We derive now the expression for the scattering matrix $S_{il},\ i, l = 1, 2, 3...N$ constructing the scattered waves - the solution of the spectral problem for the extension $\mathcal{A}_{\beta,0}$ which fulfills the boundary conditions (8) and the asymptotic condition at infinity. Thus for the scattered wave generated by the incoming wave from the wire attached to the point a_1 we have for $x \longrightarrow \infty$

$$u^s(x_s) = \delta_{s1} e^{-ikx_s} + S_{s1} e^{ikx_s}.$$

Inserting this Ansatz into (8) we obtain the linear system:

$$\begin{cases} B_s & = -ik\beta(\delta_{s1} - S_1) \\ (\delta_{s1} + S_1) & = A_s. \end{cases}$$

Then using the connection (10) between $\vec{A}$, $\vec{B}$ via Q- matrix, and the notations

$$\vec{S}_1 = (S_{11}, S_{21}, ...S_{N1}),\ \ \vec{\delta}_1 = (1, 0, ...0)$$

we get the equations for $\vec{S}_1,\ \vec{A}$ in vector form:

$$\vec{A}_1 = ik\beta(\vec{\delta}_1 - \vec{S}_1),\ \vec{\delta}_1 + \vec{S}_1 = -\beta Q \vec{\xi}_1. \tag{15}$$

¿From the last equation one may find the first row $\vec{S}_1$ of the scattering matrix. In a similar way we may derive the explicit expression for the whole of scattering matrix :

$$S = S^{\beta,0}(\lambda) = \frac{Q + \frac{I}{ik\beta^2}}{Q - \frac{I}{ik\beta^2}} \tag{16}$$

One can easily see that the poles of the scattering matrix in the upper halfplane $\Im k > 0$ coincide with eigenvalues of the operator $\mathcal{A}_{\beta,0}$. POLOVINOK¿ The expression (16) is similar to the expression for the scattering matrix in [1]. We shall use this expression to investigate the resonanse transmission.

If we take into account that $k = \sqrt{\lambda}$ we see that zero is a branching point of the operator - function $[Q(\lambda) + \dfrac{1}{ik\beta^2}]^{-1}$ and the positive axis is a cut with different values of the resolvent kernel on different shores of it. This cut is actually the only branch of the absolutely - continuous spectrum multiplisity N of the operator $\mathcal{A}_{\beta,0}$.The eigenfunctions of the absolutely-continuous spectrum may be constructed as linear combination of exoonentials on the wires, for instance

$$\psi_s^1 = \begin{cases} \psi_t = e^{-ikx_t} + S_{tt}e^{ikx_t} \\ \psi_s = S_{st}e^{ikx_s}, \ s \neq 1. \end{cases}$$

for scattered wave iniciated by plane wave on the first wire attached to a_1 and linear combinations of green's functions inside the domain Ω:

$$\Psi_\lambda = < \vec{A}, \vec{G}_\lambda(x) >,$$

where $\vec{A} = \dfrac{\beta}{Q - \frac{I}{ik\beta^2}}\vec{\delta_1}$. The constructed system of eigenfunctions of discrete and absolutely-continuous spectrum is complete and orthogonal in the total Hilbert space $L_2(\Omega) \oplus \sum_s L_2(\Gamma_s)$. $\square$

We did not discuss here neither completeness of the resonance states nor other special properties of resonances. The delicate analysis of resonances in terms of Lax-Phillips approach [?] requires considering the wave equations in the domain and on the wires. It will be done in the following publication.

4 Weakening connection limit in resonance case.

We assume now that the nonperturbed operator has a simple spectrum and neither of its eigenfunctions φ_l vanishes at *all* contact points. In terms of the channel space E it means that $\sum_{s=1}^N |\varphi_l(a_s)|^2 \equiv |\vec{\varphi}_l|^2 > 0$. In this section we investigate the asymptotic behaviour of the component of the resolvent kernel of the perturbed operator $G^{\beta,0}(x,\, y,\, \lambda)$ and the scattering matrix $S^{\beta,0}(\lambda)$ for weakening connection $\beta \to 0$ in both nonresonance and resonance case The next statemant similar to the paralel statement proven in [1] has a general meaning and is valid for any resolvent-like matrices. It is based on an observation concerning the inverse matrix near the pole. Other important facts concerning Operator Matrices of this type see for instance in [5].

Theorem 2 *Consider a sequence of operators $\mathcal{A}_{\beta,0}$ which corresponds to the vanishing connection between the domain and the wires: $\beta \to 0$. The components of resolvent's of them on the domain Ω*

$$P_{L_2(\Omega)} \left[\mathcal{A}_{\beta,0} - \lambda I\right]^{-1}\Big|_{L_2(\Omega)}$$

converges uniformly to the resolvent of the nonperturbed operator $\mathcal{A}$ on each compact subset of the complement of the spectrum $\sigma(\mathcal{A})$ of the nonperturbed operator. Besides, if λ_0 is an eigenvalue of the nonperturbed operator $\mathcal{A}$,then for sufficiently small β it can't be an eigenvalue of the perturbed operator $\mathcal{A}_\beta$ but there exist an eigenvalue of the perturbed operator (for $\lambda_0 < 0$) or resonance (for $\lambda_0 > 0$) in a β^2- neighborhood of it.

Proof is based on the formula (11) derived in the previous section. Note that the singularities of the resolvent of the nonperturbed operator $\mathcal{A}$ are present in both terms of the expression for the perturbed resolvent kernel. We shall prove that in generic case for sufficiently small values of β in a small neighborhood of given eigenvalue λ_0 of the nonperturbed operator they compensate each other and the only singularity of the Gree's function of the perturbed operator appeares from the denominator of the second term

$$Q(\lambda) - \frac{1}{ik\beta^2}.$$

According to the Theorem 1 the leading term of the denominator near the eigenvalue λ_0 for small β is equal to

$$\frac{\vec{\varphi}_0 \vec{\varphi}_0}{\lambda_0 - \lambda} - \frac{1}{ik\beta^2}.$$

Here $\vec{\varphi}_0 \vec{\varphi}_0$ is an onedimensional operator in the auxilliary space E. Generally the matrix $\vec{\varphi}_l \vec{\varphi}_l$ is proportional to the projection operator P_l in the E onto the Span$\{\vec{\varphi}_l\}$

$$\vec{\varphi}_l \vec{\varphi}_l = \sum_s |\varphi_l(a_s)|^2 P_l = |\vec{\varphi}_l|^2 P_l.$$

To construct the inverse $\left[Q(\lambda) - \frac{1}{ik\beta^2}\right]^{-1}$ for small β in a small neignbourhood of λ_0 we use the orthogonal decomposition of E into two orthogonal subspaces $P_0 E + (I - P_0)E \equiv P_0 E + P_0^{\perp} E$. Using (10) we may separate from the denominator $Q - \frac{1}{ik\beta^2}$ the singular term at the point λ_0

$$Q - \frac{1}{ik\beta^2} = \frac{|\vec{\varphi}_0|^2 P_0}{\lambda_0 - \lambda} - \frac{1}{ik\beta^2} + g_\lambda^0 + [Q - \mathrm{diag} g_\lambda] \equiv$$

$$\frac{|\vec{\varphi}_0|^2 P_0}{\lambda_0 - \lambda} - \frac{1}{ik\beta^2} + K_0(\lambda)$$

where $g^0(\lambda) = \mathrm{diag}\,\{g_\lambda\}$ is an analytic near the spectral point λ_0 diagonal matrix-function defined in the previous section, see Lemma 1, and $K_0 = g^0(\lambda) + [Q - \mathrm{diag}\,\{g_\lambda\}]$ is an analytic function near λ_0. Decomposing the leading terms of the last expression into orthogonal sum we get the following formula for the denominator:

$$Q(\lambda) - \frac{1}{ik\beta^2} =$$

$$\left(\frac{|\vec{\varphi}|_0^2}{\lambda_0 - \lambda} - \frac{1}{ik\beta^2}\right) P_0 - \frac{1}{ik\beta^2} P_0^{\perp} + K_0(\lambda). \qquad (17)$$

Note that the leading term of the last expression - the diagonal matrix

$$\begin{pmatrix} (\frac{|\vec{\varphi}_0|^2}{\lambda_0 - \lambda} - \frac{1}{ik\beta^2}) P_0 & 0 \\ 0 & \frac{1}{ik\beta^2} P_0^{\perp} \end{pmatrix} := \Delta$$

is invertible

$$\Delta^{-1} = ik\beta^2 \begin{pmatrix} \frac{\lambda_0-\lambda}{ik\beta^2|\vec{\varphi}_0|^2-(\lambda_0-\lambda)}P_0 & 0 \\ 0 & P_0^{\perp} \end{pmatrix}$$

and the inverse of it is holomorphic with respect to the variable $k = \sqrt{\lambda}$ in a small neighborhood of k_0, $\lambda_0 = k_0^2$ for all positive β. Then the inverse of $Q - \dfrac{1}{ik\beta^2}$ can be calculated as

$$ik\beta^2 \left(\frac{\lambda_0 - \lambda}{ik\beta^2|\vec{\varphi}_0|^2 - (\lambda_0 - \lambda)} P_0 - P_0^{\perp} \right) \times (I + K_0 \Delta^{-1})^{-1}$$

for positive β. Consider the second term of the expression (11) for the Green's function of the perturbed problem. The left and right factors of it have the form

$$\vec{G}_\lambda(y) = \vec{G}_{-1}(y) + \frac{\varphi_0(y)\vec{\varphi}_0}{\lambda_0 - \lambda} + \vec{g}_\lambda^0(y)$$

$$\vec{G}_\lambda(x) = \vec{G}_{-1}(x) + \frac{\varphi_0(x)\vec{\varphi}_0}{\lambda_0 - \lambda} + \vec{g}_\lambda^0(x)$$

where $\vec{g}_\lambda^0(x) = \{g_\lambda^0(x, a_1),\ g_\lambda^0(x, a_2),\ ...g_\lambda^0(x, a_N)\}$. These expressions obviously have singularities at the einegvalue λ_0 containing the factors $\vec{\varphi}_0$. Then the direct calculation of singularities of the second term shows that only first order term in $(\lambda_0 - \lambda)$ remains, since $P_0^{\perp}\vec{\varphi}_0 = 0$ and the coefficient in front of it is $-\varphi_0(x)\varphi_0(y)$. Combining this singularity $-\frac{\varphi_0(x)\varphi_0(y)}{\lambda_0-\lambda}$ with the corresponding term in $G_\lambda(x, y)$ we see that both singular terms at λ_0 compensate each other. Thus we see that in the case when $\vec{\varphi}_0 \neq 0$ the inner component of the Green function of the perturbed operator is a holomorphic function at the eigenvalue λ_0 for sufficiently weak connection between the ring and the wires.

On the other hand a new singularity caused by the denominator $Q - \frac{1}{ik\beta^2}$ appeares. If $k_0^2 = \lambda_0 < 0$ then for small β the denominator has zero eigenvalue for some pure imaginary value of k close to k_0. This follows from the orthogonal decomposition (16)

$$[Q(\lambda) - \frac{1}{ik\beta^2}]\vec{u} =$$

$$P_0 \left(\frac{|\vec{\varphi}_0|^2}{\lambda_0 - \lambda} - \frac{1}{ik\beta^2} \right) P_0\vec{u} + P_0K_0P_0\vec{u} + P_0K_0P_0^{\perp}\vec{u} +$$

$$P_0^{\perp}K_0P_0\vec{u} - P_0^{\perp}\frac{1}{ik\beta^2}P_0^{\perp}\vec{u} + P_0^{\perp}K_0P_0^{\perp}\vec{u} = 0. \qquad (18)$$

Now the operator version of Rouchet theorem [12] may be used to conclude that the solution of the last equation (19) is close to the solution of the equation combined of leading terms and for small β

$$\lambda_\beta \equiv k_\beta^2 \approx \lambda_0 - i\sqrt{\lambda_0}|\vec{\varphi}_0|^2\beta^2,$$

$$\vec{u}_\beta \approx \vec{\varphi}_0.$$

The corresponding solutions of the Schrödinger equation on the domain are restored from $\vec{u}_\beta$ as

$$u(x) = < \bar{\vec{u}}_\beta, \vec{G}_\lambda(x) > .$$

If $\lambda_0 < 0$, then $\lambda_\beta \equiv k_\beta^2 < 0$, $k_\beta = i\kappa$, $\kappa > 0$ hence the exponentials continuing the solution u from Ω onto wires are square integrable and the total solution of the Schrödinger equation is a square-integrable function, i.e. is an eigenfunction of the operator $\mathcal{A}_{\beta,0}$. The finitness of the total number of negative eigenvalues follows directly from the analyticity of the matrix $Q - \dfrac{1}{ik\beta^2}$.

Vice versa if $\lambda_0 > 0$ then $\Im k_\beta < 0$ hence the corresponding solution u_β of the Schrödinger equation is exponentially growing at least on some wires. So it is not an eigenfunction but a resonance solution – "a resonance state". The total number of resonances is infinite which can be derived from the asymptotic behaviour of the matrix $Q - \frac{1}{ik\beta^2}$ at infinity. The corresponding analysis will be done elsewhere.
□

Now following ([1]) we analyse the situation when the energy λ of the scattered wave coincides with some eigenvalue λ_0 of the nonperturbed operator $\mathcal{A}$. Following [8] we call this situation a *resonance case*. In this case we use the block-representation of the operator $Q - \dfrac{1}{ik\beta^2}$ with respect to the orthogonal decomposition of the auxilllary channel-space E used in the proof of the previous theorem:

$$Q(\lambda) \mp \frac{1}{ik\beta^2} =$$

$$\left(\frac{|\vec{\varphi}_0|^2}{\lambda_0 - \lambda} \mp \frac{1}{ik\beta^2} \right) P_0 \mp \frac{1}{ik\beta^2} P_0^\perp + K_0 \equiv \Delta_\mp K_0. \tag{19}$$

Further we use also the notations

$$\left(\frac{|\vec{\varphi}_0|^2}{\lambda_0 - \lambda} \mp \frac{1}{ik\beta^2} \right) \equiv (\mp).$$

It is obvious that $(+) \approx \beta^{-2}$, $\beta \to 0$ provided $\lambda \neq \lambda_0$.

Theorem 3 *The scattering matrix $S^{\beta,0}$ of the operator $\mathcal{A}_{\beta,0}$ for the weakening boundary condition $\beta \to 0$ has in generic case the following asymptotics at the simple resonance eigenvalue:* [1]*:*

$$-I - 2ik\beta^2 K_0 + (I + ik\beta^2 K_0) \frac{2k\beta^2 |\vec{\varphi}_0|^2}{k\beta^2 |\vec{\varphi}_0|^2 + i(\lambda_0 - \lambda)} P_0 + O(|\beta|^4).$$

[1] Mr M.Harmer found recently a similar statement for multiple eigenvalues.

Proof. The leading terms of the denominator in the expression for the scattering matrix derived in the last theorem are represented near the resonance eigenvalue by the diagonal matrix in the orthogonal decomposition of the auxillary space $E = P_0 E + P_0^{\perp} E$.

$$\begin{pmatrix} (\frac{|\vec{\varphi}_0|^2}{\lambda_0-\lambda} - \frac{1}{ik\beta^2})P_0 & 0 \\ 0 & -\frac{1}{ik\beta^2}P_0^{\perp} \end{pmatrix} = \begin{pmatrix} (-)P_0 & 0 \\ 0 & -\frac{1}{ik\beta^2}P_0^{\perp} \end{pmatrix} \equiv \Delta_- .$$

$$\Delta_-^{-1} = \begin{pmatrix} \frac{1}{(-)}P_0 & 0 \\ 0 & -ik\beta^2 P_0^{\perp} \end{pmatrix}.$$

Hence we can write the expression for the scattering matrix as

$$(\Delta_+ + K_0)\Delta_-^{-1}(I + K_0\Delta_-^{-1})^{-1}.$$

The product of the first and the second factors gives

$$\Delta_+\Delta^{-1} + K_0\Delta_-^{-1} = -I + \frac{2k\beta^2|\vec{\varphi}_0|^2}{k\beta^2|\vec{\varphi}_0|^2 + i(\lambda_0-\lambda)}P_0 + K_0\Delta_-^{-1}.$$

The last factor is represented in form of convergent series for small values of β, $\lambda_0 - \lambda$:

$$I - K_0\Delta^{-1} + O(\beta^4).$$

which gives for the scattering matrix the approximate expression

$$S(k) = -I + \frac{2k\beta^2|\vec{\varphi}_0|^2}{k\beta^2|\vec{\varphi}_0|^2 + i(\lambda_0-\lambda)}P_0 +$$

$$2k\beta^2 K_0\left(\frac{\lambda_0-\lambda}{k\beta^2|\vec{\varphi}_0|^2 + i(\lambda_0-\lambda)}P_0 - iP_0^{\perp}\right) + O(\beta^4) =$$

$$I - 2ik\beta^2 K_0 + (I + ik\beta^2 K_0)\frac{2k\beta^2|\vec{\varphi}_0|^2}{k\beta^2|\vec{\varphi}_0|^2 + i(\lambda_0-\lambda)}P_0 + O(|\beta|^4).$$

□

In particular for λ close to λ_0 and $\beta \to 0$ we have the following approximate expression for the transmission coefficient for weakly connected wires:

$$S_{s,t}(\lambda_0) = \frac{2k\beta^2}{k\beta^2|\vec{\varphi}|^2 + i(\lambda_0-\lambda)}\varphi(a_s)\varphi(a_t) + O(\beta^2), \;\; s \neq t.$$

where the second term is uniformly small when $\beta \to 0$, but the first one exhibits a nonuniform behaviour in dependence on ratio $(\lambda_0-\lambda)/\beta^2$, see below.

Remark. The last formula being applied formally to the case $\lambda = \lambda_0$ shows, that the transmission coefficient is approximately equal to

$$S_{s,t}(\lambda_0) = \frac{2}{|\vec{\varphi}|^2}\varphi(a_s)\varphi(a_t) + O(\beta^2).$$

This looks surprising for $\beta = 0$ since it gives a nonzero transmission coefficient for zero connection.Actually it means that the transmission coefficients are not continuous with respect to the energy λ uniformly in β. The physically significant values of the transmission coefficient may be obtained via averaging with respect to Fermi distribution

$$\rho(\lambda, T) = \frac{1}{1 + e^{\frac{\lambda - \lambda_f}{\kappa T}}}.$$

Here λ_f is the Fermi-level in the wires.

One may consider two different cases: $\kappa T << \lambda_0 |\vec{\varphi}|^2 \beta^2$ and $\kappa T >> \lambda_0 |\vec{\varphi}|^2 \beta^2$ which correspond to averaging on intervals $|\lambda - \lambda_0| < \kappa T$ for $\kappa T << \lambda_0 |\vec{\varphi}|^2 \beta^2$ and $\kappa T >> \lambda_0 |\vec{\varphi}|^2 \beta^2$. In the first case we still have:

$$\overline{|S_{ij}(T)|^2} \approx \frac{2|\varphi(a_s)\varphi(a_t)|^2}{|\vec{\varphi}|^4}$$

but in the second case, when $\lambda_0 |\vec{\varphi}|^2 \beta^2$ is small comparing with κT :

$$\overline{|S_{ij}(T)|^2} \approx 4 \frac{|\varphi(a_s)\varphi(a_t)|^2}{|\vec{\varphi}|^4} \frac{1}{1 + \frac{\kappa^2 T^2}{\lambda_0 \beta^4 |\vec{\varphi}|^4}}.$$

Hence for small β and non-zero temperatures the averaged transmission coefficient is small, according to natural physical expectations.

Example. Consider the case when the nonperturbed operator is just a Neumann Laplacean $\mathcal{A}$ with a zero-range potentials defined by the boundary conditions (6). Then the spectrum $\sigma(\mathcal{A})$ of it is discrete and the eigenfunctions φ may be found in form defined by the Ansatz

$$\varphi(x) = \sum_{s=N+1}^{N+M} A_s G_\lambda(x, a_s)$$

with coefficients A_s satisfying the equations:

$$A_s g_\lambda(a_s, a_s) + \sum_{t \neq s} G_\lambda(a_s,\, a_t) A_t = \gamma_s A_s, \;\; s = N+1,\, N+2,\, ...N+M, \tag{20}$$

under the conditions $\det[Q(\lambda) + \text{ diag}\,(g_\lambda(a_s, a_s) - \gamma_s)] = 0, \;\; s = N+1,\, N+2,\, ...N+M$. In particular consider a single zero-range potential at the point $a \in \Omega$ with the intensity γ. Then the role of the equation 20 is played now by the equation

$$g_\lambda(a, a) = \gamma, \tag{21}$$

and the eigenfunctions are just $G_\lambda(x, a)$ for λ satisfying (21). If few wires are weakly attached to the domain Ω now, $0 < \beta << 1$ then the corresponding transmision coefficients $S_{j,k}$ of the operator $\mathcal{A}_{\beta,\gamma}$ are approximately equal for resonance energy λ to

$$S_{j,l} = -2 \frac{G_\lambda(a, a_1) G_\lambda(a, a_j)}{\sum_j |G_\lambda(a, a_j)|^2} + O(\beta^2).$$

Hence

$$\frac{S_{j,1}}{S_{s,1}} \approx \frac{G_\lambda(a, a_j)}{G_\lambda(a, a_s)}.$$

If the total number of contact points is greater then 4 in three - dimensional domain, then the position of the point a may be found from scattering data $S_{j,l}$ on the intersection of the level surfaces of ratios of Green's functions. It two-dimensional domain to localise the zero-range potential by scattering data one need only four contact points.

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