HILBERT THEOREM ON LEMNISCATE AND THE SPECTRUM OF THE PERTURBED SHIFT

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Abstract.

The spectrum of the perturbed shift operator $T: f(n) \to \alpha f(n+1) + a(n) f(n)$ in $l^2(\mathbb{Z})$ is considered for periodic $a(n)$ and fixed constant $\alpha > 0$. It is proven that the spectrum is continuous and fills a lemniscate. Some isospectral deformations of the sequence $a(n)$ are described. Similar facts for the perturbed shift in the spaces of sequences of some hypercomplex numbers is derived.

1 Introduction

1.1 Lemniscate

Following [8], by a *lemniscate* we mean the locus of a point $z \in \mathbb{C}$ for which the product of distances to a finite number of fixed points $\{z_1, z_2, ..., z_N\} \in \mathbb{C}^N$ is equal to the fixed positive constant r^N. The condition may be written in the form

$$|(z - z_1)(z - z_2)...(z - z_N)| = r^N > 0. \tag{1}$$

The lemniscate with equation $|z^2 - 1| = r^2$ is called the *Cassini oval*, or, in the case $r = 1$, the *Bernoulli lemniscate* (which looks like the "infinity"-sign ∞).

Some properties of the lemniscate (1) can be deduced from the maximum modulus principle for an analytic function:
1) the lemniscate (1) separetes each point z_k from infinity;
2) no point of the lemniscate (1) can lie interior to Jordan curve consisting wholly of the points of (1) ;
3) each such Jordan curve must contain inside at least one point z_k;
4) therefore the lemniscate (1) consists of a finite number of bounded Jordan curves.
For other basic properties of lemniscates the reader is referred to [5], [8], [9].

1.2 Hilbert theorem on lemniscate

For the larger number N, we may obtain lemniscates (1) of most varied appearance. In fact one can approximate the boundary of any bounded simply connected domain by a lemniscate (see [3], [5] p. 379).

Theorem (D. Hilbert, 1897) *For any bounded simply connected domain D and for any $\varepsilon > 0$ there exists a lemniscate $l \subset D$ which consists of the single continuum*[1] *such that the boundary ∂D of the domain D lies in the ε-neighborhood of l and the lemniscate l lies in the ε-neighborhood of the boundary ∂D, i.e.*

$$\partial D \subset \{z \in \mathbb{C} : \operatorname{dist}(z, l) < \varepsilon\},$$

$$l \subset \{z \in \mathbb{C} : \operatorname{dist}(z, \partial D) < \varepsilon\}.$$

One can find in [8], [9]. the extention of Hilbert's theorem to the general case of an unbounded multiply-connected domains with a compact complement.

1.3 Shift operator

We denote by S the shift operator on the doubly infinite sequences of the complex numbers, i.e.

$$(Sf)(n) = f(n + 1), \quad n \in \mathbb{Z}.$$

[1] This important condition is omitted by the author of the book [5].

The operator S is a unitary operator on $l^2(\mathbb{Z})$ and the unit circle is its continuous spectrum. We consider the scaled operator αS with a positive number α, and perturbed by adding a multiplication operator by some function $a : \mathbb{Z} \to \mathbb{C}$. So, we consider the operator $T = \alpha S + a$. One can see that the description of the spectrum $\sigma(T)$ of T in $l^2(\mathbb{Z})$ may be reduced to finding all complex numbers z such that the homogeneous difference equation

$$\alpha f(n+1) + a(n)f(n) = zf(n), \quad n \in \mathbb{Z},$$

has a nontrivial solution[2] and each solution is bounded. We denote the set of all such numbers z by $s(T)$, $\sigma(T) = \overline{s(T)}$.

Our nearest aim is to show that for every $\alpha > 0$ and for every periodic function a the set $s(T)$ is a lemniscate and each lemniscate (1) coincides with the set $s(T)$ for some periodic function a with a period $N_0 \leq N$. On the other hand, due to Hilbert's theorem, for every set Γ which is the boundary of a simply connected domain D (i.e. $\Gamma = \partial D$) there exist a periodic function a and a positive number α such that the analytic curve $s(T)$ approximates Γ. In fact using the extention of Hilbert's theorem to the case of unbounded multiply connected domains with a bounded complement (see [6], [7], [8], [9]) we can approximate the boundary of every closed bounded set with connected complement by a lemnicsate and therefore by the set $s(T)$ for some periodical function a.

The main theorem is formulated at the end of section 2 and it is proved in the section 3. In the section 4 we consider the perturbed shift on sequences of quaternions and Cayley numbers.

2 Main Theorem

Let f be a two-sided sequence of the complex numbers, i.e. $f : \mathbb{Z} \to \mathbb{C}$. Put $|f| := \sup_n |f(n)|$. By S we denote the *shift operator*

$$(Sf)(n) = f(n+1), \quad n \in \mathbb{Z}.$$

The shift S is a *unitary* operator on

$$l^2(\mathbb{Z}) := \{f : \sum_{n\in\mathbb{Z}} |f(n)|^2 < \infty\}.$$

It means that $\|Sf\|_{l^2(\mathbb{Z})} = \|f\|_{l^2(\mathbb{Z})}$ for each $f \in l^2(\mathbb{Z})$.

For a given sequence $a : \mathbb{Z} \to \mathbb{C}$ and a positive number $\alpha > 0$ we consider the *perturbed shift*

$$T_{\alpha,a} := \alpha S + a.$$

Note that the operator $T_{\alpha,a}$ is bounded on $l^2(\mathbb{Z})$ if and only if the function $a(n)$ is bounded, i.e. $|a| < \infty$.

[2]It means that $\inf\{n : a(n) = z\} > -\infty$.

We say that a complex number z belongs to the *spectrum* $s(T_{\alpha,a})$ of the operator $T_{\alpha,a}$ if and only if the following difference equation

$$\alpha Sf + af = zf,$$

$$\alpha f(n+1) + a(n)f(n) = zf(n), \quad n \in \mathbb{Z}, \tag{2}$$

has a nontrivial solution and every solution f is bounded, i.e. $|f| < \infty$.

If we put $f(0) = 1$ then for every $z \notin \{a(n) : n < 0\}$ the corresponding solution of the equation (2) has the following explicit form

$$f(n) = \begin{cases} \alpha^{-n} \prod_{k=0}^{n-1}(z - a(k)), & n \geq 1, \\ 1, & n = 0, \\ \alpha^{|n|} \prod_{k=1}^{|n|}(z - a(-k))^{-1}, & n \leq -1. \end{cases} \tag{3}$$

On the other hand, if $z = a(n_0)$ then for every solution f of the equation (2) we get $f(n) \equiv 0$, $n > n_0$.

Remarks. 1. For every $\alpha > 0$ the set $s(T_{\alpha,a})$ is a *translation-invariant* one, that is for every $m \in \mathbb{Z}$ we have $s(T_{\alpha,a}) = s(T_{\alpha,a'})$ with $a'(n) = a(n+m)$, $n \in \mathbb{Z}$. Really, if $T_{\alpha,a}f = zf$ then $T_{\alpha,a'}f' = zf'$ with $f'(n) = f(n+m)$. Therefore we can suppose that $a(n) \notin s(T_{\alpha,a})$ for $n < 0$ (see the footnote[2]).

2. The spectrum $s(T_{\alpha,a})$ coincides with the set of the *stability* of the equation (2) (see [1], theorem 5.4.1).

3. Let $a(n)$ be a *stationary* sequence, $a_0 := a(0) = a(n)$, $n \in \mathbb{Z}$, then the set $s(T_{\alpha,a})$ is a circle of radius α centered at the point a_0. Indeed, let f be a solution of the equation (2) then by (3) $f(n) = f(0)(z - a_0)^n/\alpha^n$, $n \in \mathbb{Z}$, is a geometric progression and the solution f is bounded if and only if $|z - a_0|/\alpha = 1$. In particular $s(S) = \{z \in \mathbb{C}, \ |z| = 1\}$ is the unit circle.

For a fixed positive integer N let us consider a finite ordered set of complex numbers $\vec{a} = \{a_1, a_2, ..., a_N\} \in \mathbb{C}^N$ and a sequence π of *permutations* π_n of the set $\vec{N} = \{1, 2, ..., N\}$, $\pi_n \in \mathcal{S}(\vec{N})$, $n \in \mathbb{Z}$, i.e. $\pi : \mathbb{Z} \to \mathcal{S}(\vec{N})$ (see [4], p.77). For given $\vec{a}$ and π we construct a function $a_{\pi,\vec{a}} : \mathbb{Z} \to \mathbb{C}$ such that for every $n = kN + m$ with $k \in \mathbb{Z}$ and m in the set $\{0, 1, ..., N-1\}$

$$a_{\pi,\vec{a}}(n) = a_{\pi,\vec{a}}(kN + m) = a_{\pi_k(m+1)}. \tag{4}$$

Note that for a stationary sequece $\pi = \{\pi_0\} : \ \pi_n = \pi_0$, $n \in \mathbb{Z}$, we get the periodic function $a(n) = a_{\pi,\vec{a}}(n)$ with the period $N_0 \leq N$, i.e. $a(n + N_0) = a(n)$, $n \in \mathbb{Z}$. For instance, if all numbers of the set $\vec{a}$ are different from each other then $N_0 = N$ for every $\pi_0 \in \mathcal{S}(\vec{N})$. Denoting by e the unit element of the permutation group $\mathcal{S}(\vec{N})$ and choosing $\pi = \{e\} : \ \pi_n = e$, $n \in \mathbb{Z}$, we have got the following periodic sequence $a_{\vec{a}} := a_{\{e\},\vec{a}}$, that is

$$a_{\vec{a}}(n + kN - 1) = a_n, \quad n \in \vec{N}, \quad k \in \mathbb{Z}.$$

For given $\alpha > 0$ and a set $\vec{a}$ let us consider the polynomial

$$P_{\vec{a}}(z) := (z - a_1)(z - a_2)...(z - a_N).$$

The locus of a point for which the product of distances from points $\vec{a}$ is the positive constant α^N is a *lemniscate* $l_{\alpha,\vec{a}}$. Thus, the lemniscate $l_{\alpha,\vec{a}}$ is defined by the equation

$$|P_{\vec{a}}(z)| = \alpha^N.$$

We now proceed to formulate our main

Theorem. *For each fixed positive number α, each set $\vec{a}$ of complex numbers, and every sequence $\pi = \{\pi_n\}_{n\in\mathbb{Z}}$ of permutations π_n of the set $\vec{N} = \{1, 2, ..., N\}$ we have*

$$s(T_{\alpha,a}) = s(T_{\alpha,a'}) = l_{\alpha,\vec{a}},$$

where $a = a_{\pi,\vec{a}}$ and $a' = a_{\vec{a}}$.

3 Proof of the Main Theorem

Proof. Fix a permutation $\pi_0 \in \mathcal{S}(\vec{N})$ and put

$$\vec{a}_0 := \{a_{\pi_0(1)}, a_{\pi_0(2)}, ..., a_{\pi_0(N)}\}.$$

Then from the equation

$$P_{\vec{a}_0}(z) \equiv P_{\vec{a}}(z) = (z - a_1)(z - a_2)...(z - a_N) \tag{5}$$

we see that the lemniscates generated by $\vec{a}$ and $\vec{a}_0$ coincide for every $\alpha > 0$, i.e.

$$l_{\alpha,\vec{a}} = l_{\alpha,\vec{a}_0}.$$

The set $l_{\alpha,\vec{a}}$ is bounded and no point of the set $\vec{a}$ can lie on the lemniscate $l_{\alpha,\vec{a}}$. Therefore there are two positive constants $d = d(\vec{a}, \alpha)$ and $D = D(\vec{a}, \alpha)$ such that

$$d \leq |z - a_n| \leq D \tag{6}$$

for every $z \in l_{\alpha,\vec{a}}$ and $n \in \vec{N}$.

Fix α, $\vec{a}$, π and z. Let f be a solution of the equation (2) with $a = a_{\pi,\vec{a}}$. Then from the last equation, the definition (4) of the function a, and the property (5) we get

$$f(N) = \frac{1}{\alpha}(z - a_{\pi,\vec{a}}(N-1))f(N-1) = \frac{1}{\alpha}(z - a_{\pi_0(N)})f(N-1) =$$
$$\frac{1}{\alpha^2}(z - a_{\pi_0(N)})(z - a_{\pi_0(N-1)})f(N-2) = ... = \frac{1}{\alpha^N}P_{\vec{a}}(z)f(0). \tag{7}$$

We let $p(z) := \alpha^{-N}P_{\vec{a}}(z)$, and for every positive integer k using the equation (7) we obtain $f(kN) = p(z)^k f(0)$. Hence the condition $z \in s(T_{\alpha,\vec{a}})$ implies $\sup_{k>0} |f(kN)| < \infty$, or $|p(z)| \leq 1$. On the other hand by the equation (2) we have ($z \notin \vec{a}$)

$$f(-1) = \frac{\alpha}{z - a(-1)}f(0) = \frac{\alpha}{z - a_{\pi_{-1}(N)}}f(0),$$

$$f(-2) = \frac{\alpha^2}{(z - a_{\pi_{-1}(N-1)})(z - a_{\pi_{-1}(N)})} f(0),$$

and finally

$$f(-N) = \frac{f(0)}{p(z)}.$$

Therefore for each positive integer k we obtain

$$f(-kN) = \frac{f(0)}{p(z)^k} \,. \tag{8}$$

So, $z \in s(T_{\alpha,a})$ implies $z \notin \vec{a}$ and $|p(z)| \geq 1$. At last we have

$$|p(z)| = 1. \tag{9}$$

It means that

$$s(T_{\alpha,a}) \subset l_{\alpha,\vec{a}} = \{z \in \mathbb{C} : |p(z)| = 1\}.$$

To complete the proof of the theorem note that for each $z \in l_{\alpha,\vec{a}}$ and $n = kN + m$ with m in $\{1, ..., N-1\}$, $k \in \mathbb{Z}$, using (6), (9) we have the following estimates

$$|f(n)| \leq (D/\alpha)^m |f(0)| \leq (cD)^{N-1} |f(0)| \quad \text{if } \ k \geq 0$$

and

$$|f(n)| \leq (\alpha/d)^m |f(0)| \leq (c/d)^{N-1} |f(0)| \quad \text{if } \ k < 0$$

with $c := \max\{\alpha, 1/\alpha\}$. So, the condition $z \in l_{\alpha,\vec{a}}$ implies $z \in s(T_{\alpha,\vec{a}})$, and

$$s(T_{\alpha,a}) = l_{\alpha,\vec{a}} = s(T_{\alpha,a'}).$$

End of the proof.

Remarks. **1**. Every lemniscate (1) is the spectrum of some operator $T_{r,a}$. For example, we can choose $a = a_{\{e\},\vec{a}}$ with $\vec{a} = \{z_1, z_2, ..., z_N\}$. Then $a(n)$ is a periodic function. Therefore

2. every lemniscate (1) is the spectrum of some operator $T_{r,a}$ with a periodic function $a(n)$.

3. For every lemniscate (1) such that $z_1 \neq z_2$ and for every positive integer k there exists a periodical function $a(n)$ with the period kN such that the spectrum $s(T_{r,a})$ coincides with the lemniscate (1). For example, we can construct the desired function as $a = a_{\{e\},\vec{a}}$ with the ordered set $\vec{a} = \{a_1, a_2, ..., a_{kN}\}$ which consists of the $(k-1)$ identical blocks: $[z_1, z_2, ..., z_N]$, but the last one is $[z_2, z_1, ..., z_N]$.

4. Let $a = a_{\pi,\vec{a}}$ then for every *finite* function $b : \mathbb{Z} \to \mathbb{C}$, i.e. $b(n) \equiv 0$ for $n : \ n(n - N_0) > 0$, we have

$$s(T_{\alpha,a}) = s(T_{\alpha,a+b})$$

and no point of the complex plain $\mathbb{C}$ is an *eigenvalue* of the operator $T_{\alpha,a+b}$ on the space $l^2(\mathbb{Z})$. The latter statement means that the corresponding solution f of the equation

$$(\alpha S + a + b) f = zf$$

belongs to $l^2(\mathbb{Z})$ if and only if $f \equiv 0$.

Really, the embedding $s(T_{\alpha,a+b}) \subset l_{\alpha,a}$ is obviuos. The polynomial

$$(z - a(0) - b(0))(z - a(1) - b(1))...(z - a(N_0) - b(N_0))$$

is bounded on the lemniscate $l_{\alpha,a}$, hence $z \in l_{\alpha,a}$ implies $z \in s(T_{\alpha,a+b})$. On the other hand any $l^2(\mathbb{Z})$-solution f is bounded, $|f| < \infty$. Therefore by (8), (9) we get $|f(-kN)| = |f(0)|$ for every positive integer k. So, $f(0) = 0 = f(n)$, $n \in \mathbb{Z}$.

5. Let $a = a_{\pi,\vec{a}}$ then by the same way as above we can prove that for every function $b : \{0, 1, 2, ..., N_0\} \to \mathbb{C}$ we have

$$s(T_{\alpha,a}) = s(T_{\alpha,a'})$$

with

$$a'(n) = \begin{cases} a(n) & \text{if } n < 0, \\ b(n) & \text{if } 0 \leq n \leq N_0, \\ a(n - N_0 - 1) & \text{if } n > N_0. \end{cases}$$

No point of the complex plain $\mathbb{C}$ is an eigenvalue of the operator $T_{\alpha,a'}$ on the space $l^2(\mathbb{Z})$.

Example. Let m_0, m_1 be nonnegative integer, $m_0 + m_1 = N \geq 1$. We consider a lemniscate generated by the following polynomial

$$P(z) = P_{(m_0,m_1)}(z) := z^{m_0}(z - 1)^{m_1},$$

namely

$$l = l_{(m_0,m_1)} := \{z \in \mathbb{C} : |P(z)| = 1\}.$$

Due to $l_{(km_0,km_1)} = l_{(m_0,m_1)}$ we can suppose that the numbers m_0, m_1 are *mutually prime*. Note that for $N = 1$ the lemniscates $l_{(1,0)}$, $l_{(0,1)}$ are the unit circle centered at the point 0 or 1 correspondently. If $N \geq 2$ then at the unique *critical point* z_0, $z_0 \neq 0, 1$, of the polymomial $P(z)$, i.e. $P'(z_0) = 0$ and $P(z_0) \neq 0$, we have $|P(z_0)| < 1$. Therefore the lemniscate l consists of the single continuum (see [8]).

Let $\mathbf{a} = \mathbf{a}(m_0, m_1)$ be a set of all functions $a : \mathbb{Z} \to \{0, 1\}$, such that $s(T_{1,a}) = l$. With each $a \in \mathbf{a}$ we can associate the following real number from the inteval $[0, 1]$:

$$q_a = \sum_{n=0}^{\infty} \frac{a(n)}{2^{n+1}}.$$

We denote by $Q_{(m_0,m_1)}$ the union of all q_a when a is scanning over the whole set $\mathbf{a}$. If $N > 1$ then the set of all sequences $\pi_+ = \{\pi_n\}_{n\geq 0}$ of permutations of $\{\pi_n\} \subset \mathcal{S}(\vec{N})$ is uncountable hence the set $Q_{(m_0,m_1)}$ is uncountable if $m_j \neq 0$, $j = 0, 1$. Consider any rational number $r \in (0, 1)$ with a finite number of units in its binary reprezentation, i.e.

$$r = \sum_{n=0}^{N_0} \frac{r_n}{2^{n+1}}, \quad r_{N_0} = 1.$$

Then choosing $b(n) = r_n - a(n)$, $0 \le n \le N_0$ we derive from the remark 4 that the number

$$q = r + \sum_{n=N_0+1}^{\infty} \frac{a(n)}{2^{n+1}}$$

belongs to $Q_{(m_0,m_1)}$ if $a = a_{\vec{a}} \in \mathbf{a}(m_0, m_1)$. This means that the set $Q_{(m_0,m_1)}$ is everywhere dence in $[0,1]$ for each (m_0, m_1). On the other hand

$$Q_{(m_0,m_1)} \bigcap Q_{(m_0',m_1')} = \emptyset$$

if and only if $(m_0, m_1) \neq (m_0', m_1')$ since $l_{(m_0,m_1)} \neq l_{(m_0',m_1')}$.

Here are few questions which are connected with the problem of description of spectra of the perturbed shifts

a) Description of properties of the set $Q_{(m_0,m_1)}$. In particular is it measurable, or just zero-measure ?

b) Are $Q_{(1,0)}$ and $Q_{(0,1)}$ countable ?

c) What is the complement of the union of all sets $Q_{(m_0,m_1)}$ on the interval $[0,1]$?.

4 Shift on Sequences of Quaternions and Cayley Numbers

The proof of our main theorem depends crucially on the product rule for moduli : $|zw| = |z||w|$ which holds for all $z, w \in \mathbb{C}$. One can expect that once this condition is fulfilled, we may consider the shift on generalized complex numbers: the quaternions $\mathbb{H}$ and the Carley numbers $\mathbb{K}$ (see [2], [10]). Note that for these numbers the commutative law for multiplication does no hold and for the Cayley numbers even the associative law for multiplication is lost. But still each non-zero element of $\mathbb{H}$ or $\mathbb{K}$ has an inverse.

We denote by $\mathcal{A}$ one of the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}$. These four algebras are the only non-isomorphic algebras over the real field of finite dimension with a unit and the product rule for moduli, i.e.

$$|ab| = |a||b| \text{ for all } a, b \in \mathcal{A}.$$

Here $|\ |$ stays for the Euclidean length in $\mathbb{R}^m$, $m = 1, 2, 4, 8$.

Let f be a two sided sequence of elements of the algebra $\mathcal{A}$, $f : \mathbb{Z} \to \mathcal{A}$. Consider the shift operator S and define the perturbed shift (as above) for a positive number α and given function $a : \mathbb{Z} \to \mathcal{A}$

$$T_{\alpha,a} := \alpha S + a.$$

We say that an element ζ of the algebra $\mathcal{A}$ belongs to the $\mathcal{A}$-*spectrum* $s_{\mathcal{A}}(T_{\alpha,a})$ of the operator $T_{\alpha,a}$ with $a \in \mathcal{A}$ if and only if the following difference equation

$$\alpha f(n+1) + a(n) f(n) = \zeta f(n), \quad n \in \mathbb{Z}, \tag{10}$$

has a nontrivial solution and every solution $f \in \mathcal{A}$ is bounded, i.e. $|f| := \sup_n |f(n)| < \infty$.

Remarks. **1**. We put $\zeta \in \mathcal{A}$ in the equation (10) instead of $z \in \mathbb{C}$ as in the equation (2) because every finite dimensional complex division algebra with unit element is isomorphic to $\mathbb{C}$.

2. $s_{\mathbb{C}}(T_{\alpha,a}) = s(T_{\alpha,a})$.

3. Let us consider the quaternions $\mathbb{H}$ as a four-dimensional real vector subspase of the matrix space $\mathrm{Mat}(2, \mathbb{C})$ with the matrix multiplication (see [2]), i.e.

$$\mathbb{H} := \left\{ \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix} : w, z \in \mathbb{C} \right\}.$$

Then $s_{\mathbb{H}}(T_{\alpha,a})$ coincides with the set of the stability of the difference system (10) (see[1]) with

$$f = \begin{pmatrix} f_1 & -f_2 \\ \bar{f}_2 & \bar{f}_1 \end{pmatrix}, \quad a = \begin{pmatrix} a_1 & -a_2 \\ \bar{a}_2 & \bar{a}_1 \end{pmatrix}, \quad \zeta = \begin{pmatrix} \zeta_1 & -\zeta_2 \\ \bar{\zeta}_2 & \bar{\zeta}_1 \end{pmatrix},$$

$$f_j, \ \ a_j, \ \ \zeta_j \in \mathbb{C}, \quad j = 1, 2.$$

In this case

$$|f|^2 := \det f = |f_1|^2 + |f_2|^2 = \sup_{n \in \mathbb{Z}} (|f_1(n)|^2 + |f_2(n)|^2).$$

4. Let $a(n) \equiv a(0)$ be a stationary sequence. Then the set $s_{\mathcal{A}}(T_{\alpha,a})$ is a $(m-1)$-dimensional sphere in $\mathbb{R}^m$, $m = \dim \mathcal{A}$ with the radius α centered at the point $a(0) = (x_1, x_2, ..., x_m) \in \mathbb{R}^m$. In particular $s_{\mathcal{A}}(S) = \{a \in \mathcal{A}, \ \ |a| = 1\}$ - the unit sphere.

For given $\vec{a} = \{a_1, a_2, ..., a_N\} \in \mathcal{A}^N$ and a sequence π of permutations of the set $\vec{N}$ we define a function $a_{\pi,\vec{a}} : \mathbb{Z} \to \mathcal{A}$ and a function $a_{\vec{a}}$ by the same way as in the case $\mathcal{A} = \mathbb{C}$.

The locus of a point for which the product of distances from points of the set $\vec{a} \in \mathcal{A}^N$ is the constant α^N is a $\mathcal{A}$-*lemniscate* $l_{\alpha,\vec{a}}$. Thus, the $\mathcal{A}$-lemniscate $l_{\alpha,\vec{a}}$ is defined by the equation

$$l_{\alpha,\vec{a}} = \{\zeta \in \mathcal{A} : \ |\zeta - a_1|\,|\zeta - a_2|...|\zeta - a_N| = \alpha^N\}.$$

Therefore $\dim l_{\alpha,\vec{a}} = \dim \mathcal{A} - 1$.

The next statement may be proved word for word as the Main Theorem above (the case $\mathcal{A} = \mathbb{C}$).

Theorem. *For any positive number α,any set $\vec{a} \in \mathcal{A}^N$, and any sequence $\pi = \{\pi_n\}_{n \in \mathbb{Z}}$ of permutations π_n of the set $\vec{N} = \{1, 2, ..., N\}$ we have*

$$s_{\mathcal{A}}(T_{\alpha,a}) = s_{\mathcal{A}}(T_{\alpha,a'}) = l_{\alpha,\vec{a}},$$

where $a = a_{\pi,\vec{a}}$ and $a' = a_{\vec{a}}$.

Note that all remarks 1 - 5 of the section 3 can be easily formulated in terms of the algebra $\mathcal{A}$. Any multi-dimensional version of Hilbert Theorem would help description spectra of general perturbed shift operator.

5 Conclusion

The problem of description of spectra of periodic and quasiperiodic differential operators belongs to the most challenging problems of the spectral analysis. Our results show that beyond the frames of selfadjoint and weakly perturbed selfadjoint operators some new patterns of spectra appear, in particular the spectrum of periodically perturbed shift operator may fill the boundary of almost any open domain with a compact complement. On the other hand the problem of description of isospectral deformations of the coefficients admits very natural sollution. The main facts remain true for shifts in all spaces of sequences of hypercomplex numbers for which the norm of product is equal to the product of norms (real, complex, Cayley numbers and quaternions).

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