ABOUT SCATTERING ON THE RING

V.B.Bogevolnov[1], A.B.Mikhailova[1], B.S.Pavlov[2], A.M.Yafyasov[1]

[1] *Department of Solid State Electronics, Institute for Physics, St.Petersburg State University, Ulianovskaya 1, Petrodvorets, St.Petersburg, 198904, Russia*
[2] *Department of Mathematics , University of Auckland, Private Bag 92019, Auckland, New Zealand*

Department of Mathematics report series 413, March 1999
The University of Auckland, New Zealand.

Abstract.

The mathematical model of a simplest quasi-one-dimensional quantum network constructed of relatively narrow waveguides (the width of the waveguide is less than the de Broghlie wavelengh of the electron in the material) is developed. This model allows to reduce the problem of calculating the current through the quantum network to the construction of scattered waves for some Schrödinger equation on the corresponding one-dimensional graph. We consider a graph consisting of a compact part and few semiinfinite rays attached to it via some boundary condition depending on a parameter β (analog of the inverse exponential hight e^{-bH} of a potential barrier H separating the rays from the compact part). This parameter regulates the connection between the rays and the compact part. Spectral properties of the Schrödinger operator on this graph are described with a special emphasis on the resonance case when the Fermi level in the rays coincides with one of eigenvalues of the nonperturbed Schrödinger operator on the ring.An explicit expression is obtained for the scattering matrix in the resonance case for weakening connection between the rays and the compact part.

1 Introduction.

The spectral properties of the Schrödinger equation on graphs,see [3] and the the most complete bibliography there, posess new interesting properties, which never appear for onedimensional Schrödinger Operator on the rel axis. For instance, the reflection coefficient on a homogeneous ring (length 2π) with one semi-infinite ray attached to it by "zero-current condition" (see section 2) reveals a periodic behaviour in momentum at infinity:

$$S(k) = \frac{i - tgk\pi}{i + tgk\pi}.$$

In particular it does not approach 1 when $k \to \infty$.

The most important of the characteristic features which sharply distinguish the Schrödinger equation on graph from the Schrödinger equation on real axis is absence of a global solution of Cauchy problem : the solution exists only on the edge containing the initial point, but generally can't be continued in a unique way across the neighbouring vertex with few edges adjacent to it. In this respect the Schrödinger equation on a graph takes an *intermediate position between ordinary and partial Schrödinged equations on a corresponding domain* , see also [4].

The modern interest to the investigation of spectral properties of the Schrödinger Operatprs on graph see [1],[2],[3] is partially motivated by the fact that despite the absence of "global" solutions of Cauchy problem we still may describe the whole set of solutions of the corresponding differential equation on graph as a spline of solutions of Cauchy problem for ordinary differential equations on edges with proper boundary conditions at the vertices.

On the other hand the onedimensional scalar Schrödinger equation on a graph is distinguished from a system of differential equations on a real axis because of *locality* of the corresponding potential: even if we assume that the solutions on different edges are different components of one vector function, we see, that the potential should be represented by some diagonal matrix. In this representation all essential information which permitts continuation the solution from one edge to ahother is encoded in the boundary conditions at the vertices, see[5]. Unfortunately this approach looks still too general to reveal the characteristic properties of differential equations on graphs approximating smooth manifolds.

In the present paper we consider the one-dimensional Schrödinger equation on a graph constructed of a compact part Γ_0 represented as a sum of oriented edges joined at the nodes with some self-adjoint boundary conditions connecting the boundary values of the wave function at the incident edges and with a finite number N of semi-infinite rays $\Gamma_s: \ 0 < x_s < \infty, \ s = 1, ..., N$ attached to the compact subgraph at the vertices $x = a_1, ..., a_N$ which are *inner points* of some edges of the compact part Γ_0

$$-u_0'' + q(x)u_0 = \lambda u_0,$$
$$-u_s'' = \lambda u_s, \ s = 1, 2, ...N$$

with proper boundary conditions at the vertices. We assume that only one ray is attached to each vertex. These boundary conditions correspond to selecting of Lagrangian planes of some simplectic boundary form (see for instance [2], [6], [7]). We assume that the potential $q(x)$ is a real bounded measurable function on the compact part L_0 and vanishes on the rays, $q(x_s) = 0, \ s = 1, 2, ...N$. We choose the boundary conditions such that the component u_0 of the total wave function on the compact part Γ_0 is a continuous function and the boundary conditions connect the values of it and the jump of its derivative at the inner point – vertex – a_s of the oriented edge (arc) in Γ_0 to the boundary values $u'_s(0), u_s(0)$ of the component of the wave function on the ray attached to a_s.

The object we get in this way

$$\Gamma_0 + \sum_{s=1}^{N} \Gamma_s$$

is a special sort of graph where the inner *nodes* of it with general self-adjoint boundary conditions and *vertices* a_s with special boundary conditions are in fact the elements of a similar nature. Still we prefer to distinguish them as nodes and vertices, because of the special role of boundary conditions assigned to vertices.

The simplest but still nontrivial graph which possesses the features mentioned above is just a ring with few rays attached to it. Further we call our graph just "ring" but in fact the whole analysis is valid for any compact graph with few semi-infinite rays attached to it *as described above.*

We consider below a one-parameter family of *special* boundary conditions (see section 2) which correspond to the weakening connection ($\beta \to 0$) between the rays and the compact part Γ_0.One can show that these boundary conditions simulate the interaction between real quantum wires when the rays are joined to the ring non directly but are connected to it via quantum tunnelling through the potential barrier with the height proportional to $\ln \frac{1}{|\beta|}$. Our analysis shows that the resonance properties of the corresponding scattering matrices are defined by the properties of eigenfunctions of the Schrödinger operator on the compact part Γ_0. In particular for the resonance case when $\lambda = k^2$ is a simple eigenvalue of the Schrödinger operator on the compact part Γ_0 the transmission coefficients $S_{st}(k)$ for pairs of rays attached at the points a_s, a_t are essentially defined by the products of values of the corresponding eigenfunction $\varphi_\lambda(a_s)\varphi_\lambda(a_t)$ at the vertices:

$$S_{st}(\lambda) = -\frac{2}{\sum_r |\varphi_\lambda(a_r)|^2} \varphi_\lambda(a_s)\varphi_\lambda(a_t) + O(|\beta|^2).$$

Of course the limit value of the transmission coefficient for $\beta \to 0$ is not a continuous function of energy near the point λ, so , though the limit value of it for $\beta = 0$ is finite, practically the average value of it over Fermi distribution tends to zero when $\beta \to 0$ for any (small) value of temperature.

The last formula shows that the quantum current from one ray to another in *the resonance situation* when Fermi level in the rays is equal to some eigenvalue of the Schrödinger operator of the compact part can be controlled by the classical electric field applied to the ring. The physical meaning and technical implementation of this phenomenon will be discussed in following publications.

2 Schrödinger operator on the graph.

In this section we collect several facts about graphs formulated in a convenient form. We use this fact in following sections.

Consider the Schrödinger operator defined by the differential expression

$$l_0 u_0 = -u_0'' - q(x) u_0$$

on the “ring” Γ_0 with real bounded measurable potential q and some general self-adjoint boundary conditions at the nodes of Γ_0. We assume that few semi-infinite rays $\Gamma_s, 0 \leq x_s < \infty$, are attached to the “ring” at the vertices $a_1, a_2, ..., a_s, ..a_N \subset \Gamma_0$, $x_s|_{a_s} = 0$, the vertices being the inner points of some oriented edges (arcs) of Γ_0 where the wave functions of the “nonperturbed” Schrödinger operator L_0 on the compact subgraph Γ_0 is a smooth function :

$$u_0(a_s - 0) = u_0(a_s + 0) \equiv u_0(a_s)\ , u_0'(a_s - 0) = u_0'(a_s + 0).$$

We relate the Schrödinger operator L_0 with the Schrödinger operators L_s on the rays defined by the differential expressions

$$l_s u_s = -u_s'',$$

restricting all of them onto the subspace of all smooth functions vanishing near the vertices a_s, $s = 1, ..., N$ and then extending them with the boundary conditions connecting the jump of the derivative $[u_0']|_{a_s}$ of the *continuous* function u_0 on the corresponding oriented edge of Γ_0 with the boundary values $u_s(0)$, $u_s'(0)$ of the component of the wave function u_s on the ray Γ_s at the corresponding vertex:

$$u_0(a_s - 0) = u_0(a_s + 0) \equiv u_0(a_s), \tag{1}$$

$$\begin{pmatrix} [u_0']|_{a_s} \\ u_s(0) \end{pmatrix} = B_s \begin{pmatrix} u_0(a_s) \\ -u_s'(0) \end{pmatrix} \tag{2}$$

generally by some Hermitian matrix

$$B_s = B_s^* = \begin{pmatrix} \beta_{00}^s & \beta_{01}^s \\ \beta_{10}^s & \beta_{11}^s \end{pmatrix}.$$

Further we assume that

$$B_s = B_s^* = \begin{pmatrix} \beta_{00}^s & \beta_{01}^s \\ \beta_{10}^s & \beta_{11}^s \end{pmatrix} \equiv \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix},$$

β is the same for all vertices a_s. We call these boundary conditions *special* boundary conditions. Choosing $\beta = 1$ we receive “zero-current condition”, but choosing $\beta \to 0$ we get the sequence of scattering problems with weakening connection between the compact part Γ_0 and the rays.

Theorem 1 *The operator $\mathcal{L}$ defined in $L_2(\Gamma_0) \oplus \sum_{s=1}^N L_2(\Gamma_s)$ by the differential expression $l_0 \oplus \sum_{s=1}^N l_s$ is essentially self-adjoint in the domain D_0 consisting of all smooth functions defined on the graph Γ which satisfy the boundary conditions* $(1, 2)$.

Proof One can easily check the symmetry of this operator just integrating by parts: for $u, v \in D_0$ due to the boundary conditions $(1, 2)$ we have

$$< \mathcal{L}u, v > - < u, \mathcal{L}v >= 0.$$

On the other hand the adjoint operator $\mathcal{L}^*$ is defined in $L_2(\Gamma_0) \oplus \sum_{s=1}^N L_2(\Gamma_s)$ by the same differential expression $l_0 \oplus \sum_{s=1}^N l_s$ in the domain consisting of L_2 – functions with square-integrable first and second derivatives which satisfy the same boundary conditions at the vertex. Really for $u \in D_0, v \in D_0^*$ we have zero *boundary form*

$$< \mathcal{L}u, v > - < u, \mathcal{L}^*v >= 0.$$

Denoting by $[f]|_a$ the jump $f(a+0)-f(a-0)$ of the function f at the vertex a and by $\{f\}|_a$ the mean value $\frac{f(a-0)+f(a+0)}{2}$ of it we can represent the boundary form as follows:

$$< \mathcal{L}u, v > - < u, \mathcal{L}^* v >=$$
$$\sum_{s=1}^{N}\left([u_0']|_{a_s}\overline{\{v_0\}}|_{a_s} - \{u_0\}|_{a_s}\overline{[v_0']}|_{a_s}\right) +$$
$$\sum_{s=1}^{N}\left(\{u_0'\}|_{a_s}\overline{[v_0]}|_{a_s} - [u_0]|_{a_s}\overline{\{v_0'\}}|_{a_s}\right) +$$
$$\sum_{s=1}^{N}\left(u_s'(0)\overline{v_s(s)} - u_s(0)\overline{v_s'(s)}\right).$$

Special choice of functions $u_0, u_1, ...u_N$ satisfying conditions $u_0(a_s) = u_0'(a_s) = u_s(0) = u_s'(0) = 0$ for each vertex except a_t and $u_0(a_t) = [u_0']|_{a_t} = u_s(0) = u_s'(0) = 0$, $\{u_0'\}|_{a_t} = 1$ permits to deduce from the vanishing boundary form that v_0 is to be continuous at the vertex $[v_0]|_{a_s} = 0$. Then expressing $[u_0']|_{a_t}$, $u_t(0)$ in terms of $u_0(a_t), u_t'(0)$ by the boundary conditions we deduce from the independence of the initial values $u_0(a_t), u_t'(0)$ on the ray Γ_s that the boundary values of the element $v \in D_0^*$ at the vertex a_s satisfy the same boundary condition as the boundary values of $u \in D_0$. Then using the smoothness of v, $v \in W_2^2(\Gamma)$ we deduce[1] from integration by parts that $\mathcal{L}^*$ is symmetric and hence it coincides with the closure of $\mathcal{L}$:

$$\mathcal{L}^* \subseteq (\mathcal{L}^*)^* = \overline{\mathcal{L}} = (\overline{\mathcal{L}}^*)^* \subseteq \mathcal{L}^*.$$

This implies the essential self-adjointness

$$\overline{\mathcal{L}} = \overline{\mathcal{L}}^*.$$

End of the proof

The resolvent kernel (Green function) of the operator $\mathcal{L}$ can be obtained as a solution of the corresponding inhomogeneous equation

$$\mathcal{L}g = \lambda g + \delta(x-\xi).$$

We shall represent it via the resolvent kernel (Green function) of the *nonperturbed* operator $\mathcal{L}^0 \equiv \mathcal{L}_0^0 \oplus \sum_{s=1}^N \mathcal{L}_s^0$ which is defined by the same differential expression in $L_2(\Gamma_0) \oplus \sum_{0<s\leq N} L_2(\Gamma_s)$ with the self-adjoint boundary conditions at the nodes of the ring Γ_0 and separating homogeneous boundary conditions at the vertices:

$$[u_0]|_{a_s} = 0, \ u_s(0) = 0, \ s = 1, 2, ...N.$$

This operator is a limit of operators corresponding to weakening family of boundary conditions (1,2) when $|\beta| \longrightarrow 0$. We assume that the eigenvalues, eigenfunctions and the resolvent kernel $g_0^0(x,\xi,\lambda)$ of the component $\mathcal{L}_0^0$ of the nonperturbed operator $\mathcal{L}^0$ on the ring Γ_0

$$-\frac{d^2 g_0^0(x,\xi,\lambda)}{dx^2} + q(x) g_0^0(x,\xi,\lambda) = \lambda g_0^0(x,\xi,\lambda) + \delta(x-\xi)$$

are known.

[1]Sobolev class W_2^2 is embedded into the class C_1 of all continuous and continuously differentiable functions on each component $\Gamma a_s, s = 0, 1, ...N$ of the graph Γ hence the integration by parts with elements $v \in D_0^*$ is possible.

Theorem 2 *The spectrum of the operator $\mathcal{L}_0^0$ is discrete and the resolvent kernel of it is represented as a sum of an absolutely and uniformly convergent series*

$$g_0^0(x,\xi,\lambda) = \sum_{s=1}^{\infty} \frac{\varphi_s(x)\varphi_s(\xi)}{\lambda_s - \lambda},$$

where $\{\varphi_s\}$ are the normalized eigenfunctions of $\mathcal{L}_0$

$$\mathcal{L}_0\varphi_s = \lambda_s\varphi_s, \quad |\varphi_s|_{L_2(\Gamma_0)} = 1.$$

The system $\{\varphi_s\}$ of all eigenfunctions is automatically orthogonal and complete if the specrtrum of $\mathcal{L}_0$ is simple. In the case of multiple spectrum a normalized orthogonal system of eigenfunctions may be chosen as well.

When constructing the Green function of the perturbed operator $\mathcal{L}$ we use the fact that the Green function $g_0^0(x,\xi\lambda)$ satisfies the homogeneous Schrödinger equation on Γ_0

$$l_0 g_0^0(x,\xi,\lambda) = \lambda g_0^0(x,\xi,\lambda),\ x \neq \xi$$

and the boundary condition at the point ξ:

$$[g_0']|_\xi = -1.$$

The esssential part of the proof of the Theorem 2 - the convergence of the spectral series for thr green function g_0^0 may be obtained from embedding theorems (see also [1]). It is worth to note here that the regular asymptotics of eigenvalues at infinity is generally absent in this case because of mixing terms corresponding to nonconmeasurable edges as the following simple example shows.

Example. Consider an ring $q = 0$, $0 \leq x, 2\pi$ with nodes at the points a_1, a_2, $a_1 - a_2 = \Delta$, and the boundary conditions $[u_0']|_{a_s} = \beta u_0|_{a_s}$. The resolvent kernel $g(x,\xi,\lambda)$ on the ring with these boundary conditions is represented as a linear combination of the Green functions $G(x,\xi,\lambda)$ on the "empty ring" with no boundary conditions:

$$G(x,s,\lambda) = -\frac{cos(x-\pi-s)\sqrt{\lambda}}{2\sqrt{\lambda}sin\pi\sqrt{\lambda}}$$

in form $g(x,\xi,\lambda) = G(x,\xi,\lambda) + u_1 G(x,a_1,\lambda) + u_2 G(x,a_2,\lambda)$ where $u_s, \quad s = 1,2$ may be found from the linear system

$$-u_1 = \beta[G(a_1,\xi,\lambda) + u_1 G(a_1,a_1,\lambda) + u_2 G(a_1,a_2,\lambda)],$$

$$-u_2 = \beta[G(a_2,\xi,\lambda) + u_1 G(a_2,a_1,\lambda) + u_2 G(a_2,a_2,\lambda)],$$

with the determinant

$$\det D_0 = -\begin{pmatrix} \frac{cos\pi\sqrt{\lambda}}{2\sqrt{\lambda}sin\pi\sqrt{\lambda}} - \beta^{-1} & \frac{cos(\Delta-\pi)\sqrt{\lambda}}{2\sqrt{\lambda}sin\pi\sqrt{\lambda}} \\ \frac{cos(\Delta-\pi)\sqrt{\lambda}}{2\sqrt{\lambda}sin\pi\sqrt{\lambda}} & \frac{cos\pi\sqrt{\lambda}}{2\sqrt{\lambda}sin\pi\sqrt{\lambda}} - \beta^{-1} \end{pmatrix}$$

which vanishes if

$$cos\pi\sqrt{\lambda} - 2\sqrt{\lambda}\beta^{-1}sin\pi\sqrt{\lambda} = \pm cos(\Delta-\pi)\sqrt{\lambda}.$$

If Δ and π are not conmeasurable then the set of zeroes of the determinant is "disordered" as a set of roots of a sum of two periodic functions with nonconmeasurable periods.

Further we use the fact that generally the component $g_0(x,\, \xi,\, \lambda)$ of the Green function of the "perturbed operator" $\mathcal{L}$ on the compact subgraph Γ_0 may be found as a linear combination of Green functions of the nonperturbed operator $\mathcal{L}_0^0$ attached to the pole ξ and the nodes. One can easily see that in the generic case when all eigenvalues of the operator $\mathcal{L}_0^0$ are simple we must distinguish the two cases (situations):

1. For a given eigenvalue λ_0 the corresponding eigenfunction φ_0 of $\mathcal{L}_0^0$ vanishes at *all vertices* a_s. In this case the function φ_0 being continued as identical zero onto all rays satisfies the boundary conditions 1, 2 hence the continued function is an eigenfunction of the perturbed operator. It is obviously orthogonal to the subspace of absolutely continuous spectrum of the perturbed operator in both cases when $\lambda_0 < 0$ or $\lambda_0 > 0$. In the second case λ_0 proves to be *imbedded* eigenvalue. The existence of imbedded eigenvalues (even for compactly-supported potentials) is a characteristic feature of Schrödinger operators on graphs.

2. For a given eigenvalue λ_0 of $\mathcal{L}_0^0$ there exists at least one vertex a_s such that the corresponding eigenfunction φ_0 does not vanish at a_s, $\varphi_0(a_s) \neq 0$. In this case the spectral point λ_0 will not be the eigenvalue of the perturbed operator at least for small values of β, i.e. in the case of weakly connected inner and outer channels. We shall give the proof of this statement in the next section as a corollary of more general statement on "compensation of singularities". We shall show also that for negative λ_0 and weakly connected channels there exists a negative eigenvalue of the perturbed operator close to it:

$$\lambda_\beta = \lambda_0 + O(|\beta|^2),$$

and for positive λ_0 there exists a resonance of the perturbed operator close to λ_0.

We finish this section with a general statement concerning the representation of the resolvent of the perturbed operator. We can assume now that neither of eigenfunctions of the nonperurbed operator $\mathcal{L}_0^0$ vanishes *at all vertices* thus neither of eigenfunctions of $\mathcal{L}_0^0$ remains an eigenfunction of the perturbed operator.

Theorem 3 *The component $g_0 \equiv g_0(x, \xi, \lambda)$, $x, \xi \in \Gamma_0$ of the Green function of perturbed operator $\mathcal{L}$ in Γ_0 is represented in terms of the Green function g_0^0 of the nonperturbed operator the following way:*

$$g_0(x,\xi,\lambda) = \sum_{s=1}^{N} u_s g_0^0(x, a_s, \lambda) + g_0^0(x,\xi,\lambda),$$

where $\{u_s = u_s(\xi,\, \lambda)\}|_{s=1}^N$ are defined as solutions of the following linear algebraic system

$$\sum_r \left[g_0^0(a_s, a_r, \lambda) + \delta_{sr} \left(-ik|\beta|^2\right)^{-1} \right] u_r + g_0^0(a_s, \xi) = 0,$$

where $\lambda = k^2$, $\Im k > 0$. The spectrum of the perturbed operator $\mathcal{L}$ consists of all singularities of the Green function in the complex plane of the spectral parameter λ. In particular the absolutely continuous spectrum of $\mathcal{L}$ fills the positive half-axis $\lambda \geq 0$ with the constant multiplicity N. The eigenvalues $\lambda_r = k_r^2$, $\Im k_r > 0$ and resonances $\lambda_r = k_r^2$, $\Im k_r < 0$ of the operator $\mathcal{L}$ are found as roots of the following dispersion equation in upper $\Im k > 0$ and lower $\Im k < 0$ half-planes respectively:

$$det \left(g_0^0(a_s, a_r, \lambda) + \delta_{sr} \left(-ik|\beta|^2\right)^{-1} \right) = 0. \tag{3}$$

Proof. Being solutions of the homogeneous equation $\mathcal{L}g = \lambda g$ the components of the complete Green function g of the perturbed operator on the rays Γ_s coincide generally with exponentials:

$$g_s(x_s, \xi, \lambda) = b_s e^{ikx_s}, \quad x_s > 0, k = \sqrt{\lambda}.$$

Then due to the boundary conditions at each vertex $a = a_1, a_2, , , , a_N$ we have

$$[g_0']|_{a_s} = -ik|\beta|^2 g_0(a_s) \tag{4}$$

this implies the announced linear algebraic system for the coefficients u_s:

$$-u_s = \left(\sum_{r=1}^{N} u_r g_0^0(a_s, a_r) + g_0^0(a_s, \xi)\right)\left(-ik|\beta|^2\right).$$

Thus we get for the vector $\vec{u} = (u_1, ..., u_N)$ the representation

$$\vec{u} = -\{G + i\frac{1}{k|\beta|^2} I\}^{-1} g(\vec{\xi}),$$

where $G = \{g_0^0(a_s, a_r)\}$ is an operator in corresponding auxillary *channel-space*[2] and

$$(g_0^0(a_1, \xi), g_0^0(a_2, \xi), ..., g_0^0(a_N, \xi)) = g(\vec{\xi}) \in E.$$

Hence we have the following expression for the component of the Green function of the perturbed operator

$$g(x, \xi, \lambda) = -g(\vec{x})\{G + \frac{i}{k|\beta|^2}\}^{-1} g(\vec{x}) + g_0^0(x, \xi, \lambda), \tag{5}$$

where $g(\vec{x}) = (g_0^0(x, a_1), .., g_0^0(x, a_N))$ and $g(\vec{\xi}) = (g_0^0(a_1, \xi), g_0^0(a_2, \xi), .., g_0^0(a_N, \xi))$.

We postpone the proof of the statement about zeroes of the determinant of the matrix $G + \frac{i}{k|\beta|^2}$ to the following section 3 where we prove that all singularities of the resolvent kernel of the perturbed operator appear from these zeroes of the determinant, if neither of eigenfunctions φ_l of $\mathcal{L}$ vanishes at all vertices, $\sum_{s=1}^{N} |\varphi_l(a_s)|^2 > 0$. Modulo this important statement this is the

End of the proof.

In the following section we continue the discussion of the properties of the resolvent of the perturbed operator beginning from the formula 5.

Note that all roots of the equation 3 in upper halfplane $\Im k > 0$ which corresponds to the *physical sheet* of the spectral variable λ are situated on the imaginary axis $k = i\kappa$ $(0 < \kappa < \infty)$ and correspond to the negative eigenvalues of $\mathcal{L}$. The roots of the dispersion equation 3 in the lower halfplane $\Im k_s < 0$ which correspond to the *nonphysical sheet* are called resonances because of the special role they play in the description of asymptotic properties of solutions of the corresponding nonstationary equation see [9]

$$\frac{1}{i}\frac{\psi}{dt} = \mathcal{L}\psi$$
$$\psi|_{t=0} = \psi_0.$$

[2]The precise meaning of this space will be clarified later when we discuss scattering matrix

The solution of this equation may be represented by the Riesz integral of the resolvent $R_\lambda f(x) = \int g(x,\xi,\lambda)f(\xi)d\xi$

$$e^{i\mathcal{L}t}\psi_0 = -\frac{1}{2\pi i}\int_{\Gamma_\mathcal{L}} e^{i\lambda t}R_\lambda d\lambda\psi_0$$

on some contour $\Gamma_\mathcal{L}$ on the physical sheet of the spectral variable around the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$. The resonances become involved if we may deform this contour to the lower halfplane see [9]. The spectral analysis of resonances is developed in [8] where the corresponding hyperbolic equation:

$$u_{tt} + \mathcal{L}u = 0.$$

is considered. In our situation the similar analysis can be developed as well.

For an asymptotic analysis of the Riesz integrals when $t \to \infty$ the description of the poles of the resolvent – the roots of the dispersion equation (3) both in the upper and the lower halfplane $\Im k > 0$, $\Im k < 0$ is required. We can perform the corresponding analysis for the family (sequence) of perturbed operators $\mathcal{L}_\beta$ which correspond to weakening connection between the ring and the rays, $\beta \to 0$. The limit operator coincides with the nonperturbed operator

$$\mathcal{L}^0 = \mathcal{L}_0^0 \oplus \sum_{s=1}^N \mathcal{L}_s^0.$$

One can prove (see Theorem 4 in the next section) that

$$\mathcal{L}_\beta \longrightarrow \mathcal{L}_0$$

in a sense of the uniform convergence of resolvents

$$(\mathcal{L}_\beta - \lambda I)^{-1} \longrightarrow (\mathcal{L}_0 - \lambda I)^{-1}$$

on each compact of the complement of the spectrum $\sigma(\mathcal{L}_0)$ of the limit operator $\mathcal{L}_0$ in the complex plane.

3 Weakening connection limit in resonance case.

We assume now that the nonperturbed operator has a simple spectrum and neither of its eigenfunctions φ_l vanishes at all vertices, $\sum_{s=1}^N |\varphi_l(a_s)||^2 \equiv |\varphi|_\beta^2 \neq 0$. In this section we investigate the asymptotic behaviour of the resolvent kernel $g_\beta(x,\,\xi,\,\lambda)$ and the scattering matrix $S_\beta(\lambda)$ for weakening connection $\beta \to 0$ in both nonresonance and resonance case:

$$\lambda \in \sigma(\mathcal{L}_0^0),\ \ \lambda \overline{\in} \sigma(\mathcal{L}_0^0)$$

i.e. when λ coincides with one of eigenvalues of $\mathcal{L}_0^0$ or not.

Theorem 4 *Consider a sequence of operators $\mathcal{L}_\beta$ which correspond to the vanishing connection between the ring and the rays: $\beta \to 0, \epsilon \to 0$. The resolvents of them*

$$(\mathcal{L}_\beta - \lambda I)^{-1}$$

converge uniformly to the resolvent of the nonperturbed operator $\mathcal{L}_0^0 \oplus \sum_{s=1}^N \mathcal{L}_s$ on each compact subset Ω of the complement of the spectrum of the nonperturbed operator. Besides, if λ_0 is an *eigenvalue of the nonperturbed operator $\mathcal{L}_0$, then for sufficiently small β itcan't be an eigenvalue of the perturbed operator $\mathcal{L}_\beta$ but there exist an eigenvalue of the perturbed operator (for $\lambda_0 < 0$) or resonance (for $\lambda_0 > 0$) in a $|\beta|^2$- neighborhood of it.*

Proof. Consider the case when $x, \xi \in \Gamma_0$. We use the representation of the Green function of the perturbed operator derived in Theorem 2:

$$g(x, \xi, \lambda) = -g(\vec{x})\{G + \frac{i}{k|\beta|^2}\}^{-1} g(\vec{\xi}) + g_0^0(x, \xi, \lambda). \tag{6}$$

The singularities of the resolvent of the nonperturbed operator $\mathcal{L}_0^0$ are present in both terms of the expression for the perturbed resolvent kernel. In fact they compensate each other. Let us consider the last representation for sufficiently values of β in a small neighborhood of the given eigenvalue λ_0 of the operator $\mathcal{L}_0^0$. At first sight leading terms of the operator

$$G(\lambda) + i\frac{1}{k|\beta|^2}$$

are

$$\frac{\vec{\varphi_0}\vec{\varphi_0}}{\lambda_0 - \lambda} + i\frac{1}{k|\beta|^2}.$$

Here $\vec{\varphi_0}\vec{\varphi_0}$ is a matrix combined of values of the eigenfunction φ_0 at the points $a_s,\ a_t$. It is proportional to the projection operator P_0 in the N-dimensional auxillary channel-space E:

$$\vec{\varphi_0}\vec{\varphi_0} = \sum_s |\varphi_0(a_s)|^2 P_0 = |\vec{\varphi_0}|^2 P_0.$$

In fact it is slightely more convenient to write down the leading terms as orthogonal decomposition in two orthogonal subspaces $P_0E + (I - P_E)E \equiv P_0E + P_0^\perp E$. Separating from the matrix $G_{st} = \sum_{l=0}^{\infty} \frac{\varphi_l(a_s)\varphi_l(a_t)}{\lambda_l - \lambda}$ the term singular at the point λ_0

$$G = \frac{|\vec{\varphi_0}|^2 P_0}{\lambda_0 - \lambda} + \sum_{\lambda_l \neq \lambda_0} \frac{|\vec{\varphi_l}|^2 P_l}{\lambda_l - \lambda} \equiv$$

$$\frac{|\vec{\varphi_0}|^2 P_0}{\lambda_0 - \lambda} + K_0,$$

and decomposing the expression in orthogonal sum we get the following formula for the denominator:

$$G(\lambda) + i\frac{1}{k|\beta|^2} =$$

$$P_0\left(\frac{|\vec{\varphi_0}|^2}{\lambda_0 - \lambda} + i\frac{1}{k|\beta|^2}\right)P_0 + P_0K_0P_0 + P_0K_0P_0^\perp +$$

$$P_0^\perp K_0 P_0 + P_0^\perp i\frac{1}{k|\beta|^2}P_0^\perp + P_0^\perp K_0 P_0^\perp. \tag{7}$$

Note that the leading term of the last expression – the diagonal matrix

$$\begin{pmatrix} P_0(\frac{|\vec{\varphi_0}|^2}{\lambda_0 - \lambda} + i\frac{1}{k|\beta|^2})P_0 & 0 \\ 0 & P_0^\perp i\frac{1}{k|\beta|^2}P_0^\perp \end{pmatrix} \equiv \Delta$$

is invertible

$$\Delta^{-1} = \begin{pmatrix} \frac{(\lambda_0-\lambda)k|\beta|^2}{k|\beta|^2|\vec{\varphi_0}|^2+i(\lambda_0-\lambda)} P_0 & 0 \\ 0 & -ik|\beta|^2 P_0^{\perp} \end{pmatrix} \equiv \begin{pmatrix} \Delta_{PP}^{-1} & 0 \\ 0 & \Delta_{P^{\perp}P^{\perp}}^{-1} \end{pmatrix}$$

and the inverse is holomorphic with respect to the variable k in a small neighborhood of $\lambda_0 = k_0^2$ for all sufficiently small β. Then the inverse of $G + \frac{i}{k|\beta|^2}$ can be calculated as

$$\begin{pmatrix} \frac{(\lambda_0-\lambda)k|\beta|^2}{k|\beta|^2|\vec{\varphi_0}|^2+i(\lambda_0-\lambda)} P_0 & 0 \\ 0 & -ik|\beta|^2 P_0^{\perp} \end{pmatrix} \times$$

$$\begin{pmatrix} P_0 + P_0 K_0 P_0 \Delta_{PP}^{-1} & -P_0 K_0 P_0^{\perp} k_0 |\beta|^2 \\ P_0^{\perp} K_0 P_0 \Delta_{pp}^{-1} & P_0^{\perp} - ik|\beta|^2 P_0^{\perp} K_0 P_0^{\perp} \end{pmatrix}^{-1}.$$

Consider the first term of the expression 6 for the Green function of the perturbed problem. The left and right factors of it

$$g(\vec{x}) = \sum_l \frac{\varphi_l(x)\vec{\varphi_l}}{\lambda_l - \lambda} = \frac{\varphi_0(x)\vec{\varphi_0}}{\lambda_0 - \lambda} + \sum_{\lambda_l \neq \lambda_0} \frac{\varphi_l(x)\vec{\varphi_l}}{\lambda_l - \lambda}$$

$$g(\vec{\xi}) = \sum_l \frac{\varphi_l(\xi)\vec{\varphi_l}}{\lambda_l - \lambda} = \frac{\varphi_0(\xi)\vec{\varphi_0}}{\lambda_0 - \lambda} + \sum_{\lambda_l \neq \lambda_0} \frac{\varphi_l(\xi)\vec{\varphi_l}}{\lambda_l - \lambda}$$

obviously have singularities at the einegvalue λ_0 with the factors $\vec{\varphi_0}$ in front of them. Then a direct calculation of singularities of the whole expression shows that only first order term remains, since $P_0^{\perp}\vec{\varphi_0} = 0$ and the coefficient in front of it is $-\varphi_0(x)\varphi_0(\xi)$. Combining this singularity $-\frac{\varphi_0(x)\varphi_0(\xi)}{\lambda_0-\lambda}$ with the corresponding term in $g_0^0(x, \xi, \lambda)$ we see that both singular terms compensate each other. Thus we see that in the case when $\vec{\varphi_0} \neq 0$ the inner component of the Green function of the perturbed operator is a holomorphic function at the eigenvalue λ_0 for sufficiently weak connection between the ring and the rays. On the other hand a new singularity from the denominator $G + i\frac{1}{k|\beta|^2}$ can appear. If $k_0^2 = \lambda_0 < 0$ then for small β the denominator has zero eigenvalue for some pure imaginary value of k close to k_0. This follows from the orthogonal decomposition (7)

$$[G(\lambda) + i\frac{1}{k|\beta|^2}]\vec{u} =$$

$$P_0 \left(\frac{|\vec{\varphi_0}|^2}{\lambda_0 - \lambda}\vec{u} + i\frac{1}{k|\beta|^2} \right) P_0\vec{u} + P_0 K_0 P_0 \vec{u} + P_0 K_0 P_0^{\perp}\vec{u} +$$

$$P_0^{\perp} K_0 P_0 \vec{u} + P_0^{\perp} i \frac{1}{k|\beta|^2} P_0^{\perp}\vec{u} + P_0^{\perp} K_0 P_0^{\perp}\vec{u} = 0. \qquad (8)$$

It follows from the operator version of Rouchet theorem [10] that the solution of the last equation (8) is close to the solution of the equation combined of leading terms and for small β

$$\lambda_\beta \equiv k_\beta^2 \approx \lambda_0 - i\sqrt{\lambda_0}|\vec{\varphi_0}|^2|\beta|^2,$$

$$\vec{u}_\beta \approx \vec{\varphi_0}.$$

Then the corresponding solutions of the Schrödinger equation are restored as

$$u(x) = < \vec{g}(x), \vec{u}_\beta > .$$

If $\lambda_0 < 0$, then $\lambda_\beta \equiv k_\beta^2 < 0,\ k_\beta = i\kappa,\ \kappa > 0$ hence the exponentials continuing the solution u from Γ_0 onto rays Γ_s are square integrable and the total solution of the Schrödinger equation $\mathcal{L}u = \lambda u$ on the whole graph is a square-integrable function, i.e. is an eigenfunction of the operator $\mathcal{L}$. The finitness of the total number of negative eigenvalues follows directly from the analyticity of the matrix $G + i\frac{1}{k|\beta|^2}$.

Vice versa if $\lambda_0 > 0$ then $\Im k_\beta < 0$ hence the corresponding solution u_β of the Schrödinger equation is exponentially growing at least on some rays. So it is not an eigenfunction but a resonance solution – "a resonance state". The total number of resonances is infinite which can be derived from the asymptotic behaviour of the matrix $G + i\frac{1}{k|\beta|^2}$ at infinity. The corresponding analysis will be done elsewhere.

End of the proof

If we take into account that $k = \sqrt{\lambda}$ we see that zero is a branching point of the operator - function $[G(\lambda) + i\frac{1}{k|\beta|^2}]^{-1}$ and the positive axis is a cut with different values of the resolvent kernel on different shores of it.

To accomplish the study of the non-resonance case we can now formulate the following statement concerning general spectral properties of the operator $\mathcal{L}$.

Theorem 5 *The spectrum of the operator $\mathcal{L}$ consists of an absolutely continuous branch* $(0, \infty)$ *multiplicity N and a finite number of negative eigenvalues. The eigenfunctions of an absolutely continuous spectrum are given by N families of scattered waves which serve as solutions of the homogeneous equation $\mathcal{L}\psi = \lambda\psi$. For the components on the rays we have*

$$\psi_s = \begin{cases} S_{st} e^{-ikx_s}, s \neq t \\ e^{ikx_t} + S_{tt} e^{-ikx_t},\ s = t, \end{cases}$$

and for the components on the ring we have

$$\mathcal{L}_0 \psi_0 = k^2 \psi_0$$

with the boundary conditions

$$[\psi_0']|_{a_s} = -\beta \psi_s'|_0$$
$$\psi_s(0) = \bar{\beta}\psi_0(a_s),$$

$$\psi_0 = \sum_{s=1}^{N} u_s g_0^0(x, a_s) = < \vec{g}(x), \vec{u} >$$

These eigenfunctions are orthogonal and normalized in $L_2(\Gamma)$. The scattering matrix S_{st} in the ansatz above as well as the components of the scattered waves on Γ_0 are defined from the boundary conditions as:

$$S = \frac{G - \frac{i}{k|\beta|^2}}{G + \frac{i}{k|\beta|^2}}$$

$$\vec{u} = \frac{\beta}{G + \frac{i}{k|\beta|^2}} \vec{\nu}_t,$$

where $\vec{\nu}_t = \{\delta_{st}\}$.

Proof of this statement can be obtained with use of the standard Riesz techniques of contour integration of the resolvent basing on the asymptotic formulae for solutions of the homogeneous equation see also [6]. Though it is not entirely equivalent to the techniques for one-dimensional Schrödinger operator we omit the essential part of the proof here and calculate only the expressions for transmission and reflection coefficients and the component of the scattered wave on the compact subgraph. Other properties of scattered waves and scattering matrix will be discussed elsewhere.

For the scattered wave initiated from the ray Γ_t we have the following anzatz for the component on the ring Γ_0

$$\psi_0 = \sum_{r=1}^{N} u_r g(x, a_r),$$

and the linear algebraic system for the coefficients u_r:

$$\begin{cases} -u_s = -\beta\psi_s'(0), \\ \psi_s(0) = \bar{\beta}\sum_{r=1}^{N} g(a_s, a_r)u_r, \ \ s = 1, 2, ..N, \end{cases}$$

where

$$\begin{cases} \psi_t = e^{ikx_t} + S_{tt}e^{-ikx_t} \\ \psi_s = S_{st}e^{-ikx_s}, \ \ s \neq t. \end{cases}$$

Eliminating the variables of exteriors channels we get

$$\begin{cases} -u_s = ik\beta(-\delta_{st} + S_{st}) \\ \delta_{st} + S_{st} = \bar{\beta}\sum_{r=1}^{N} g(a_s, a_r)u_r \end{cases}$$

or

$$\begin{cases} \vec{u} = i\beta k(I - S)\vec{\nu}_t \\ \bar{\beta}G\vec{u} = (I + S)\vec{\nu}_t. \end{cases} \tag{9}$$

It gives immediately

$$\left(ik|\beta|^2 G - I\right)\vec{u} = 2ik\beta\vec{\nu}_t.$$

Then from the system (9) we get the expression for the scattering matrix.

End of the proof

In the remaining part of our paper we analyze a special but practically important situation when the energy λ of the scattered wave coincides with some eigenvalue λ_0 of the nonperturbed operator $\mathcal{L}_0$. Following [3] we call this situation a *resonance case*. In this case we use the block-representation of the operator $G - \frac{i}{k|\beta|^2}$ with respect to the orthogonal decomposition of the auxilllary channel-space E used in the proof of the theorem 4

$$G(\lambda) - i\frac{1}{k|\beta|^2} =$$

$$P_0\left(\frac{|\vec{\varphi}_0|^2}{\lambda_0 - \lambda} - i\frac{1}{k|\beta|^2}\right)P_0 + P_0K_0P_0 + P_0K_0P_0^{\perp} +$$

$$P_0^{\perp}K_0P_0 - P_0^{\perp}i\frac{1}{k|\beta|^2}P_0^{\perp} + P_0^{\perp}K_0P_0^{\perp}. \tag{10}$$

Further we use the notations

$$P_0 K_0 P_0 \equiv K_{00};\ \ P_0 K_0 P^{\perp} \equiv K_{0\perp};\ \ P_0^{\perp} K_0 P_0 \equiv K_{\perp 0}$$

$$P_0^{\perp} K_0 P_0^{\perp} \equiv K_{\perp\perp};\ \ \left(\frac{|\vec{\varphi}_0|^2}{\lambda_0 - \lambda} \pm i \frac{1}{k|\beta|^2}\right) \equiv (\pm).$$

It is obvious that $(\pm) \approx \beta^{-2}$, $\beta \to 0$.

Theorem 6 *The scattering matrix of the ring with few rays attached to it via the weakening boundary condition $\beta \to 0$ has the asymptotics for simple resonance eigenvalue* [3]*:*

$$I - \frac{2i(\lambda_0 - \lambda)}{k|\beta|^2 |\varphi|^2 + i(\lambda_0 - \lambda)} P_0 + O(\beta^2).$$

Proof The leading terms of the denominator in the expression for the scattering matrix derived in the last theorem are represented near the resonance eigenvalue by the diagonal matrix in the orthogonal decomposition of the auxillary space $E = P_0 E + P_0^{\perp} E$.

$$\begin{pmatrix} (\frac{|\vec{\varphi}_0|^2}{\lambda_0 - \lambda} + i\frac{1}{k|\beta|^2}) P_0 & 0 \\ 0 & i\frac{1}{k|\beta|^2} P_0^{\perp} \end{pmatrix} = \begin{pmatrix} (+)P_0 & 0 \\ 0 & i\frac{1}{k|\beta|^2} P_0^{\perp} \end{pmatrix} \equiv \Delta$$

$$\Delta^{-1} = \begin{pmatrix} \frac{1}{(+)} P_0 & 0 \\ 0 & -ik|\beta|^2 P_0^{\perp} \end{pmatrix}.$$

Hence we can write the expression for the scattering matrix as

$$\begin{pmatrix} (-)P_0 + K_{00} & K_{0\perp} \\ K_{\perp 0} & -i\frac{1}{k|\beta|^2} P_0^{\perp} + K_{\perp\perp} \end{pmatrix} \times$$

$$\begin{pmatrix} \frac{1}{(+)} P_0 & 0 \\ 0 & -ik|\beta|^2 P_0^{\perp} \end{pmatrix} \times$$

$$\begin{pmatrix} P_0 + K_{00}\frac{1}{(+)} & -ik|\beta|^2 K_{0\perp} \\ K_{\perp 0}\frac{1}{(+)} & P_0^{\perp} - ik|\beta|^2 K_{\perp\perp} \end{pmatrix}^{-1}.$$

The product of the first and the second factors gives

$$\begin{pmatrix} \frac{(-)}{(+)} P_0 + \frac{1}{(+)} K_{00} & ik|\beta|^2 K_{0\perp} \\ ik|\beta|^2 K_{\perp 0} & P_0^{\perp} + ik|\beta|^2 K_{\perp\perp} \end{pmatrix}.$$

Then the expression for the scattering matrix can be transformed to

$$S = I + \begin{pmatrix} \frac{(-)}{(+)} P_0 - P_0 & 0 \\ 0 & -2P_0^{perp} \end{pmatrix} \times \begin{pmatrix} P_0 + K_{00}(+)^{-1} & ik|\beta|^2 K_{0\perp} \\ K_{\perp 0}(+)^{-1} & P_0^{\perp} - ik|\beta|^2 K_{\perp\perp} \end{pmatrix}^{-1}.$$

[3] Mr. M. Harmer noticed, that this statement remains true for multiple eigenvalues as well

The last factor is represented in form of convergent series for small values of β, $\lambda_0 - \lambda$:

$$I - \begin{pmatrix} K_{00} & K_{0\perp} \\ K_{\perp 0} & K_{\perp\perp} \end{pmatrix} \times \begin{pmatrix} \frac{1}{(+)} P_0 & 0 \\ 0 & -ik|\beta|^2 P_0^{\perp} \end{pmatrix} + ...,$$

which gives for the scattering matrix the approximate expression

$$S(k) = I - \frac{2i(\lambda_0 - \lambda)}{k|\beta|^2|\varphi|^2 + i(\lambda_0 - \lambda)} P_0 +$$

$$\begin{pmatrix} \frac{2i(\lambda_0-\lambda)}{k|\beta|^2|\varphi|^2+i(\lambda_0-\lambda)} & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} K_{00} & K_{0\perp} \\ K_{\perp 0} & K_{\perp\perp} \end{pmatrix} \times \begin{pmatrix} \frac{1}{(+)} P_0 & 0 \\ 0 & -ik|\beta|^2 P_0^{\perp} \end{pmatrix} + ... =$$

$$I - \frac{2i(\lambda_0 - \lambda)}{k|\beta|^2|\varphi|^2 + i(\lambda_0 - \lambda)} P_0 + O(\beta^2).$$

End of the proof

In particular for λ close to λ_0 and $\beta \to 0$ we have the following approximate expression for the transmission coefficient for weakly connected rays:

$$S_{s,t}(\lambda_0) = -\frac{2k|\beta|^2}{k|\beta|^2|\varphi|^2 + i(\lambda_0 - \lambda)} \varphi(a_s)\varphi(a_t) + O(\beta^2), \ s \neq t.$$

End of the proof

Remark. The last formula being applied formally to the case $\lambda = \lambda_0$ shows, that the transmission coefficient is approximately equal to

$$S_{s,t}(\lambda_0) = -\frac{2}{|\vec{\varphi}|^2} \varphi(a_s)\varphi(a_t) + O(\beta^2).$$

This looks surprizing for $\beta = 0$ since it gives a nonzero transmission coefficient for zero connection.Actually it means that the transmission coefficients are not continuous with respect to energy λ uniformly in β. The physically significant values of the transmission coefficient may be obtained via averaging with respect to Fermi distribution

$$\rho(\lambda, T) = \frac{1}{e^{\frac{\lambda - \lambda_f}{\kappa T}}} :$$

$$\overline{|S_{ij}(T)|^2} = \int |S_{ij}(\sqrt{\lambda})|^2 \left| \frac{d\rho(\lambda, T)}{d\lambda} \right| d\lambda$$

Here λ_f is the Fermi-level in the material used to produce the wires.

One may consider two different cases: $\frac{\kappa T}{\beta^2} << 1$ and $\frac{\kappa T}{\beta^2} >> 1$. In the first case

$$\overline{|S_{ij}(T)|^2} \approx 2\frac{|\varphi(a_s)\varphi(a_t)|^2}{|\vec{\varphi}|^4} + 2\frac{|\varphi(a_s)\varphi(a_t)|}{|\vec{\varphi}|^2} \ O(\beta^2) + O(\beta^4).$$

but in the second case, when β is small comparing with κT we have:

$$\overline{|S_{ij}(T)|^2} \approx 2\frac{|\varphi(a_s)\varphi(a_t)|^2}{|\vec{\varphi}|^4} 2\pi \frac{\beta^2}{\kappa T} +$$

$$2\frac{|\varphi(a_s)\varphi(a_t)|}{|\vec{\varphi}|} \frac{\sqrt{\lambda_0}\beta^2}{\kappa T} \ln \frac{\kappa T}{|\vec{\varphi}|\sqrt{\lambda_0}\beta^2} \ O(\beta^2) + O(\beta^4).$$

Hence for small β the averaged transmission coefficient is small, according to our physical expectations.

4 Acknowledgements

We are grateful to the Commission of the European Communities for financial support in the frame of EC-Russia Exploratory Collaborative Activity under EU ESPRIT Project 28890 NT-CONGS and partial support from the Russian Academy of Sciences (Grant RFFI 97 - 01 - 01149). One of us (B.P) proudly recognizes the support from the Royal Society of New Zealand (Marsden Fund Grant 3368152)

References

[1] N.I. Gerasimenko and B.S. Pavlov. Scattering problems on compact graphs. Theor. Math. Phys. 74, 230 (1988).

[2] S.P. Novikov, Schr"odinger operators on graphs and symplectic geometry, in: The Arnol'dfest (Proceedings of the Fields Institute Conference in Honour of the 60th Birthday of Vladimir I. Arnol'd), eds. E. Bierstone, B. Khesin, A. Khovanskii, and J. Marsden, to appear in the Fields Institute Communications Series.

[3] V. Kostrykin and R. Schrader Kirchhoff's rule for quantum wires J. Phys. A: Math. Gen. 32 (1999) 595-630;

[4] Y.Melnikov,B.Pavlov. Quantum Scattering on Graphs.(Manuscript)

[5] Carlson. Differential Operators on Graphs.(Manuscript)

[6] B.S. Pavlov. A model of zero-radius potential with internal structure. Theoret. and Math. Phys. 59 (3), 544 (1984).

[7] M. Faddeev, B. Pavlov. Scattering by resonator with the small opening. J. Sov. Math. 27, 2527 (1984).

[8] P.Lax and R. Phillips. Scattering theory for automorphic functions. Princeton University press and University of Tokyo press. Princeton, New Jersey. (1976)

[9] M.Reed and B.Simon. Methods of modern mathematical physics. Academic press. New York. London. (1972).

[10] I.S. Gohberg and E.I. Sigal. Operator extension of the theorem about logarithmis residue and Rouchet theorem. Mat. sbornik. 84, 607 (1971).