ISOCHRONES AND BRACHISTOCHRONES

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Abstract

Christiaan Huygens proved in 1659 that a particle sliding smoothly (under uniform gravity) on a cycloid with axis vertically down reaches the base in a period independent of the starting point. He built very accurate pendulum clocks, with cycloidal pendulums. Mark Denny has constructed another curve purported to give descent to the base in a period independent of the starting point; but the cycloid is the only smooth plane curve with that property. Johann Bernoulli 1st proved in 1696 that, for any pair of fixed points, the brachistochrone (the curve of quickest descent) under uniform gravity is an arc of a cycloid. In 1976, Ian Stewart asked, what is the brachistochrone for central gravity under the inverse square law? The solution is found explicitly, in terms of elliptic integrals.

Keywords: brachistochrone, quickest descent, constrained motion, central forces, inverse square gravity, elliptic integrals

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Epigraph

"In this enormous volume, the author never succeeds in proving that the shortest distance between two points is a straight line."

G. H. Hardy [15], on Forsyth's treatise **Calculus of Variations** [8].

1 Cycloids as Isochrones and Brachistochrones

In 1582, when Galileo Galilei (1564-1642) was a student of medicine in Pisa, he observed a lamp swinging on a long chain in Pisa Cathedral. Timing the swings against his own pulse, he found that the oscillation period remained effectively constant as the amplitude of the oscillations decayed. Soon afterwards he applied that observation to design his *pulsilogium* for measuring pulse rates. That consisted of a board to which a pendulum was attached, so that a thumb could be pressed against the board to stop the string to give any desired length from the string to the swinging bob. The physician synchronized the swinging of the bob with the patient's pulse, and the pulse rate could then be read from the position of the thumb against calibrations marked on the board. That pulsilogium was the <u>first</u> instrument ever to be used to measure medical data.

The device was elaborated by others, and Galileo never claimed the invention of the pulsilogium for himself (Bedini [1], page 257).

Galileo, to the end of his life in 1642, insisted that he had proved that the period of a simple pendulum is independent of the amplitude of the oscillation (Galileo [9], page 95). When he was old and blind, a prisoner of the Holy Inquisition in his own home, in 1641 he discussed with his son Vincenzio Galilei his ideas on pendulums, and attempted to produce a clock with a pendulum regulator. Vincenzio died in 1649, leaving an incomplete pendulum clock based on Galileo's design (Bedini [1], pages 287–288).

Some working pendulum clocks were then made by Christiaan Huygens and others, and by 1656 some of those makers had found that a simple pendulum is not strictly isochronous. Small swings have periods that are closely equal, but increasing the amplitude of swing does increase somewhat the period of the swing. In 1659, Huygens discovered that a particle sliding on a smooth cycloid (generated by a circle of radius α), with uniform gravitational acceleration g in the direction of the axis of symmetry, reaches the bottom of the cycloid arch after the period $\pi\sqrt{\alpha/g}$, wherever on the arch the particle starts from rest. In view of that remarkable property, the cycloid is called an *isochrone* (or a *tautochrone*).

Huygens built highly accurate pendulum clocks, for which purpose he invented the theory of evolutes and involutes of curves [10]. The pendulum rod was supported by a flexible strip, which wrapped around two opposing cycloidal cheeks, which constrained the pendulum bob to move on an equal cycloid.

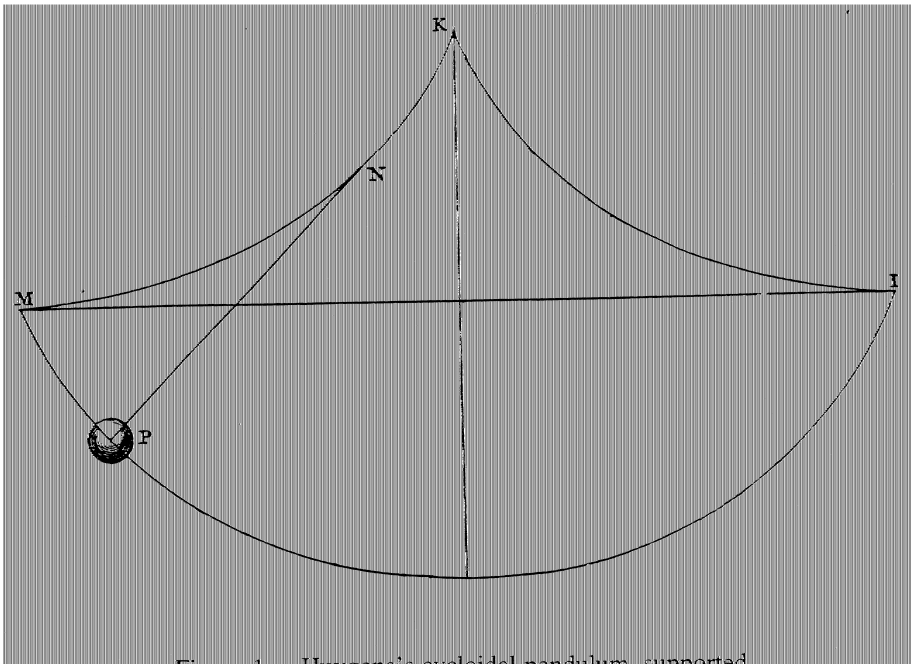


Figure 1: Huygens's cycloidal pendulum, supported by flexible strip wrapping around cycloidal cheeks

Huygens's illustration of a cycloidal pendulum is reproduced here as Figure 1, from Fig. 19 in the collected edition (1934) of **Horologium Oscillatorium** [10].

1.1 Cycloid

The cycloid is defined kinematically, as the plane curve traced by a point on a circle rolling along a straight line. Take the line as $y = 2\alpha$, with y–axis orthogonal to that line. The cycloid generated by a point (initially at the origin O) on a circle of radius α rolling under that line has the equation in parametric form:

$$x \;=\; \pm\alpha(\eta + \sin\eta), \qquad y \;=\; \alpha(1 - \cos\eta), \tag{1}$$

where η is the angle though which the circle has rolled from O. Eliminating η, we obtain the non–parametric equation of the cycloid for $y \in [0, 2\alpha]$ as:

$$x \;=\; \pm\Big(\alpha\ \arccos\left(1 - \tfrac{y}{\alpha}\right) + \sqrt{y(2\alpha - y)}\Big). \tag{2}$$

1.1.1 Incorrect Diagrams of Cycloids

At the ends of a cycloid arch, the tangents are parallel to the axis of symmetry. But, many published diagrams of cycloids incorrectly represent them with the ends of the arch having tangents which make misleading nonzero angles to the axis of symmetry. As examples, there are the diagrams (apart from Figure 18) in the collected edition (1934) of **Horologium Oscillatorium** which are reproduced in the 1968 edition [10] (with Huygens's Fig. 19 reproduced above as Figure 1), Fig. 13 in Munem & Foulis [12] (page 729), and Figures 9.11.1 & 9.11.2 in Salas & Hille ([14], 7th edition, pages 630 & 631). There are also the diagrams on page 135 and page 207 in Edwards [5]—but the diagram there on page 250 correctly depicts an end of the cycloid half–arch with tangent parallel to the axis of symmetry.

1.2 All Plane Isochrones Are (Piecewise) Cycloids

Mark Denny [3] has constructed another curve purported to have the property of isochronism under uniform gravity. He asserts that, in the frictionless case, "the only force acting on the particle is gravity". But there is also the normal reaction force on the particle, which constrains the particle to move on that curve! [4]

In fact, it is easy to prove (cf. (Routh [13], §209, pages 125–126) that the half-arch of a cycloid is the only smooth plane curve such that particles released from rest anywhere on the curve move (under uniform gravity) to the base of the curve in the same time. (Huygens also constructed clocks [10] (Part 5), with conical pendulum revolving isochronously in horizontal circles on a paraboloid of revolution.)

1.2.1 Uniqueness of the Cycloid

Take the base of any differentiable-everywhere curve in a vertical plane as origin, with horizontal x–axis and uniform gravitational acceleration $-g$ in the y–direction. Denote by s the arc length of the curve from the origin, represent the curve by $s = f(y)$, and denote the speed of the particle on the curve by $v = |\mathrm{d}s/\mathrm{d}t|$. Release the particle at height h at time $t = 0$, so that at height y the kinetic energy equals the loss of potential energy, and hence $g(h - y) = \frac{1}{2}v^2$

and $v = \sqrt{2g(h-y)} = -\mathrm{d}s/\mathrm{d}t$. Therefore, the time T for reaching the lowest point is:

$$T = \int \mathrm{d}t = \int_h^0 \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}y}\,\mathrm{d}y = \int_0^h \frac{f'(y)}{v}\,\mathrm{d}y = \frac{1}{\sqrt{2g}}\int_0^h \frac{f'(y)}{\sqrt{h-y}}\,\mathrm{d}y\ . \tag{3}$$

Substitute $y = hz$, so that

$$T\sqrt{2g} = \int_0^1 \frac{f'(hz)\sqrt{h}}{\sqrt{1-z}}\,\mathrm{d}z = \int_0^1 f'(hz)\sqrt{hz}\frac{\mathrm{d}z}{\sqrt{z}\sqrt{1-z}}\ . \tag{4}$$

If the curve is an isochrone, the time T for descent to the base is to be the same for all positive values of h, and so $\mathrm{d}T/\mathrm{d}h = 0$. Therefore,

$$\int_0^1 \frac{\partial}{\partial h}\Big(f'(hz)\sqrt{hz}\Big)\frac{\mathrm{d}z}{\sqrt{z(1-z)}} = 0\ . \tag{5}$$

If that derivative in the integrand were non-zero for any values of hz, then by taking h sufficiently small we could ensure that the derivative does not change sign for $0 < z < 1$, but the derivative is positive (or else negative) for part of that range of z. The other factor in the integrand is $1/\sqrt{z(1-z)}$, which is positive for all $z \in (0,1)$, and hence the integral would be positive (or else negative). But the integral equals 0, and hence that derivative must equal 0 for all z and h.

Therefore $f'(hz)\sqrt{hz} = \pm\sqrt{2\alpha}$, where α is some positive constant, independent of h and z, and so $f'(y) = \pm\sqrt{2\alpha/y}$. Thus,

$$\left(\frac{\mathrm{d}s}{\mathrm{d}y}\right)^2 = \frac{2\alpha}{y}\ , \tag{6}$$

and hence the curve has vertical tangent at height $y = 2\alpha$.

Substitute $y = 2\alpha\,\sin^2(\eta/2) = \alpha(1-\cos\eta)$, so that $\mathrm{d}y = \alpha\,\sin\eta\,\mathrm{d}\eta$. Now,

$$\begin{aligned}(\mathrm{d}x)^2 &= (\mathrm{d}s)^2 - (\mathrm{d}y)^2 = \left[\left(\frac{\mathrm{d}s}{\mathrm{d}y}\right)^2 - 1\right](\mathrm{d}y)^2 \\ &= \left[\frac{2\alpha}{y} - 1\right](\mathrm{d}y)^2 = \left[\frac{2\alpha}{2\alpha\sin^2(\eta/2)} - 1\right]\alpha^2\sin^2\eta\ (\mathrm{d}\eta)^2 \\ &= \frac{1-\sin^2(\eta/2)}{\sin^2(\eta/2)}\ 4\alpha^2\sin^2(\eta/2)\ \cos^2(\eta/2)(\mathrm{d}\eta)^2 \\ &= 4\alpha^2\cos^4(\eta/2)(\mathrm{d}\eta)^2.\end{aligned} \tag{7}$$

Therefore $\mathrm{d}x = 2\alpha\cos^2(\eta/2)\,\mathrm{d}\eta = \alpha(1+\cos\eta)\,\mathrm{d}\eta$, and hence

$$x = \alpha\int(1+\cos\eta)\mathrm{d}\eta = \alpha(\eta + \sin\eta), \tag{8}$$

since the base of the curve is taken as the origin $(x,y) = (0,0)$.

Hence, every smooth isochrone is given in the parametric form as in (1).

Thus, every smooth curve with equal descent times (for uniform gravity) is an half–arch of a cycloid, with vertical axis of symmetry and concave upwards.

Since (cf. (6))

$$\frac{\mathrm{d}s}{\mathrm{d}y} \;=\; \pm\sqrt{\frac{2\alpha}{y}}\;, \tag{9}$$

the arc length s of a cycloid from the base to height y is:

$$\pm s \;=\; \sqrt{2\alpha}\int_0^y \frac{\mathrm{d}z}{\sqrt{z}} \;=\; 2\sqrt{2\alpha}\sqrt{y}\;, \tag{10}$$

so that

$$y \;=\; \frac{s^2}{8\alpha}\;. \tag{11}$$

In 1659, Huygens discovered [10] that an isochronous curve must have this property, which he recognized as being a property of the cycloid generated by a circle of radius α. He also found that the tangential force on a particle sliding smoothly on a cycloid is proportional to s. Thereupon he developed the theory of simple harmonic motion, and applied that theory (and Hooke's Law for springs) by inventing (simultaneously with Hooke) the clock balance-wheel controlled by a spiral elastic spring.

1.2.2 Alternatives To Complete Cycloid

If two opposed half–arches of cycloids (with parameters α and β) in the same plane are joined at their bases, a particle sliding smoothly on the combined half–arches will oscillate isochronously, with period $2\pi\big(\sqrt{\alpha}+\sqrt{\beta}\big)/\sqrt{g}$.

Figure 2 shews the bob on an half-arch with parameter of 1, which is joined to an half-arch with parameter of $\frac{1}{2}$.

However, it would be difficult to construct an accurate pendulum of that form with unequal half–arches, since Huygens' flexible strip would slap abruptly against the smaller cycloidal cheek. Figure 3 shews the bob on the smaller half-arch.

Furthermore, at any point C on a cycloid AB with vertical axis, the cycloid could be reflected (within its plane) from a vertical line through C, with the arc CB getting reflected into an arc CD, as is shewn in Figure 4. Then, a particle sliding under uniform gravity from A to B could bounce off the vertical line through C, reversing the horizontal component of its velocity but leaving its vertical component unchanged, so that the particle oscillates isochronously on the joined cycloid arcs AC and CD. And similarly for 2 or more such reflections of arcs of a cycloid. But it would be difficult to construct an accurate pendulum of that form, since infinite force would have to be applied to the particle when it bounces at C, unlike the smoothly–varying reactive force elsewhere on the curve.

Figure 2: Asymmetric Isochrone: Bob on Larger Half-Arch

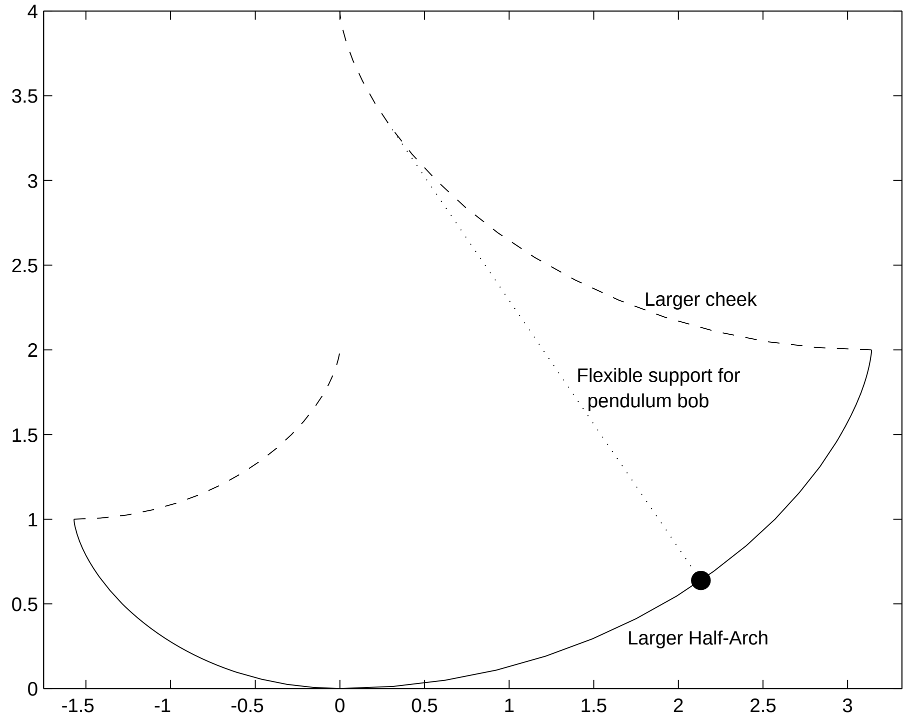


Figure 3: Asymmetric Isochrone: Bob on Smaller Half-Arch

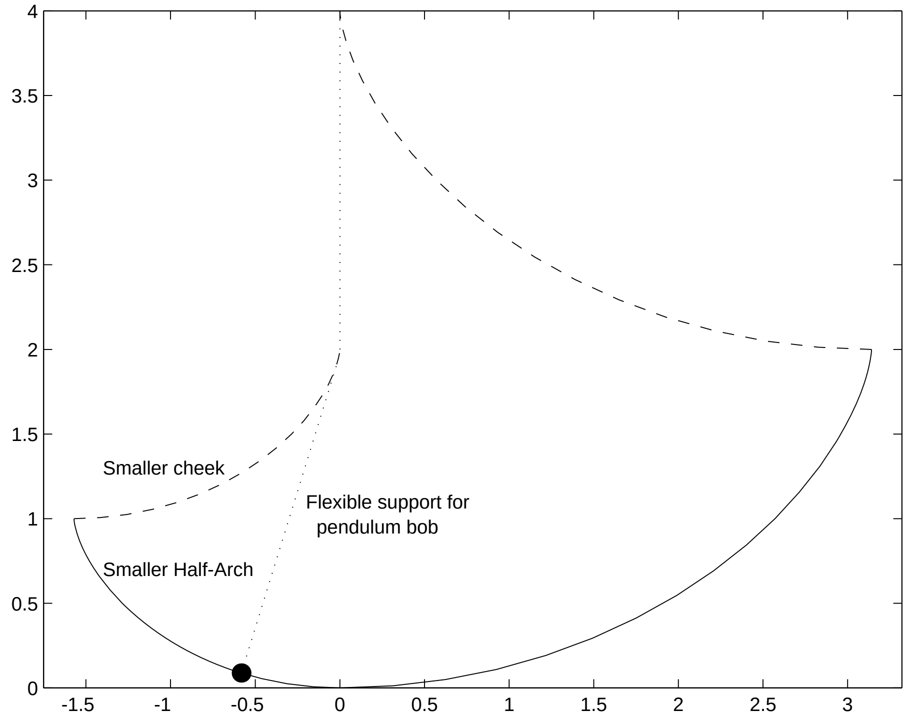
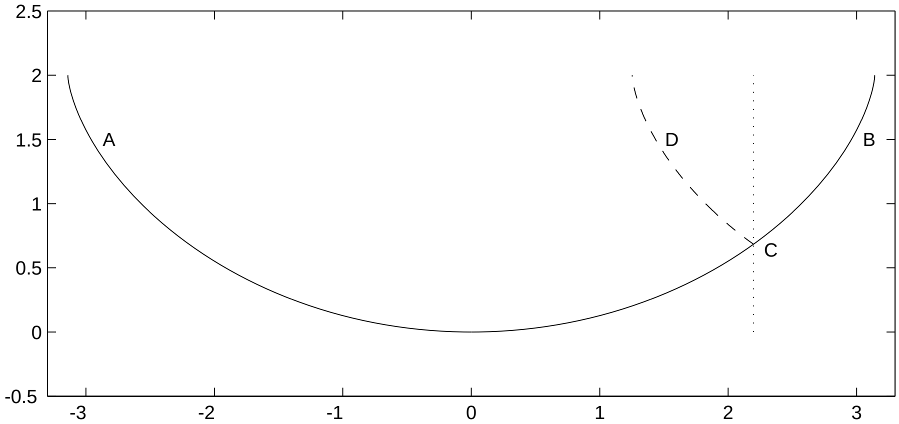


Figure 4: Reflected Cycloid Arch as Isochrone



Hence, for practical pendulums the cycloid is the only plane curve that gives isochronous oscillations under uniform gravity.

1.3 Brachistochrone

The Calculus of Variations was founded in 1696 by Johann Bernoulli 1st, when he found that, for any pair of fixed points in a uniform gravitational field, the curve of quickest descent for a particle released at the higher point to reach the lower is an arc of a cycloid generated by a point on a circle, rolling under the horizontal line through the higher point, with a vertical cusp at the higher point. Johann Bernoulli 1st challenged other mathematicians to find the curve of quickest descent, which he called the *brachistochrone*. The cycloid solution was found by his brother Jakob Bernoulli 1st, by Newton, by Leibniz and by l'Hôpital, using diverse methods that significantly advanced the calculus of variations (Westfall [19], pages 581–583).

1.3.1 Conservative Force Fields

For a particle moving with speed v under a conservative force field with potential φ, the kinetic energy plus the potential energy is constant:

$$\tfrac{1}{2}v^2 + \varphi \;=\; C. \tag{12}$$

This conservation of energy holds for a free particle, and also for a particle constrained to move on a smooth curve or surface.

For each point C on a brachistochrone (under a conservative force field) between A and B the arc AC is also a brachistochrone; since if some arc ADC gave a shorter time for descent from A to C than did the arc AC, then the path $ADCB$ would take less passage time than the path ACB. (The speed of the particle at C is independent of the path from A to C).

In any smooth force field, as the entire arc AB converges to the startpoint A (so that the force field over the arc AB converges to a uniform force field), the brachistochrone AB approaches the shape of an arc of a cycloid. In particular, if $\varphi_A = \varphi_B$, so that the particle ends at rest at B, then the curve AB approaches the shape of a complete cycloid arch.

2 Brachistochrones Under Central Forces

For a spherically symmetric potential $\varphi(r)$, where the radius r is the distance from O, the force (per unit mass) on the particle is directed to the centre O, with magnitude

$$F \;=\; \frac{\mathrm{d}\varphi}{\mathrm{d}r}. \tag{13}$$

For a particle P (either free, or sliding smoothly) released from rest at radius a, with speed v at radius r,

$$0+\varphi(a) \;=\; \tfrac{1}{2}v^2+\varphi(r). \qquad \therefore \quad v^2 \;=\; 2(\varphi(a)-\varphi(r)). \tag{14}$$

Ian Stewart published in 1976 a query [16], in which he remarked that "it is well known that in a uniform gravitational field, the path of quickest descent between two given points is a cycloid. Reformulate this question for a central field, and in particular for the case of an inverse square force". In response, Henry George Forder pointed out [7] that the problem was dealt with by Edward John Routh ([13], page 373).

2.1 Reduction of Brachistochrones to Free Orbits

In 1850, John Hewitt Jellett shewed [11] that each brachistochrone between a pair of points in a conservative force field is also the path in space (but not in time) of a free particle, moving under a transformed potential (Routh [13], page 368). That transformed potential must be such that, at each point where the constrained particle has speed v, the associated free particle has speed $w = 1/v$. In particular, for a central force field where the constrained particle has speed $v = v(r)$ at radius r, the brachistochrone is the path of a free particle that moves with speed $w = w(r) = 1/v(r)$ at each radius r. Thus the extensively-developed theory of orbits of a free particle under a central force can be applied to brachistochrones under central forces.

In particular, each orbit of a free particle lies in a plane through the centre of force. If an orbit under central force has any point at which the radius vector is a maximum or minimum then that point is called an *apse*, and that extreme length is called an *apsidal distance*. The tangent at an apse is orthogonal to the radius vector, and each orbit is symmetrical about each apsidal line. An orbit might not have any apse, or it might have any number of apses; and a circular orbit (under any attractive central force) has an apse at each point. But, however many apses an orbit might have, there can only be either one apsidal distance (e.g. parabola or hyperbola under inverse square force), or two apsidal distances (e.g. ellipse under inverse square attraction) (Tait & Steele [17], pages 123–124). In view of Jellett's Theorem, each of these geometrical properties of orbits under general central forces also applies to brachistochrones under central forces.

Routh gave only a sketchy and obscure hint about brachistochrones under central forces ([13], page 373). Jellett's theorem is applied here to answer Ian Stewart's question in detail, with equations for the brachistochrone "for the case of an inverse square force" being constructed in terms of elliptic integrals.

2.2 Construction of Brachistochrone

Let a particle be released from rest at A and move under central force directed to O with potential $\varphi(r)$, along a smooth curve to B (where $\varphi_A \geq \varphi_B$), reaching B in minimum passage time. If $\varphi_A = \varphi_B$, so that the particle ends at rest at B, then the curve AB is called a *complete brachistochrone*.

If B lies on the line OA then the brachistochrone must lie in every plane through OA, and hence the brachistochrone then is the interval AB.

Otherwise, consider polar coordinates (r, ϑ) in the plane AOB, with the initial radius vector OA as the angle origin $\vartheta = 0$. Take the initial radius OA as the unit of length. Denote by s the arclength, increasing from $s = 0$ at A to B, so that at radius r the constrained particle has speed $v(r)$, as given by (14). Hence, the associated free particle has speed

$$\frac{\mathrm{d}s}{\mathrm{d}t} \;=\; w(r) \;=\; \frac{1}{v(r)}\;. \tag{15}$$

The angular momentum of the free particle is constant:

$$r^2\frac{\mathrm{d}\vartheta}{\mathrm{d}t} \;=\; K, \tag{16}$$

so that

$$wr^2\frac{\mathrm{d}\vartheta}{\mathrm{d}s} \;=\; K. \tag{17}$$

Hence,

$$\frac{r\ \mathrm{d}\vartheta}{\mathrm{d}s} \;=\; \frac{K}{wr} \;=\; \frac{Kv}{r}\;. \tag{18}$$

At the starting point A the speed $v = 0$, so that $\mathrm{d}\vartheta/\mathrm{d}s\ = 0$, and hence the brachistochrone is tangential to the radius vector at A (and likewise at B, for a complete brachistochrone).

Now,

$$(\mathrm{d}s)^2 \;=\; (\mathrm{d}r)^2 + (r\ \mathrm{d}\vartheta)^2, \tag{19}$$

and so

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 \;=\; 1 - \left(\frac{r\ \mathrm{d}\vartheta}{\mathrm{d}s}\right)^2 \;=\; 1 - \left(\frac{Kv}{r}\right)^2, \tag{20}$$

and hence

$$\frac{\mathrm{d}r}{\mathrm{d}s} \;=\; \mp\sqrt{1 - \frac{K^2v^2}{r^2}}\;. \tag{21}$$

The positive square root is to be taken where r increases with time t (and hence with arclength s), and the negative square root is to be taken where r decreases with time t.

2.3 Angle as a Function of Radius

Dividing (21) by (18), we get the differential equation:

$$\frac{\mathrm{d}r}{\mathrm{d}\vartheta} \;=\; \frac{\mp r^2\ \sqrt{1 - \frac{K^2v^2}{r^2}}}{Kv} \;=\; \mp r\ \sqrt{\frac{r^2}{K^2v^2} - 1}\;. \tag{22}$$

Therefore, the polar equation of the free particle's orbit (and hence of the brachistochrone for the constrained particle) is given explicitly as

$$\vartheta \;=\; \int_1^r \frac{\mathrm{d}x}{\mp x\,\sqrt{\frac{x^2}{K^2 v(x)^2} - 1}} \;. \tag{23}$$

The brachistochrones starting at a fixed point A form a one–parameter family parametrized by K, the angular momentum of the associated free particle. The direction for increasing ϑ can be chosen so that $K \geq 0$.

2.3.1 Brachistochrone With Apse

The brachistochrone has an apse at apsidal distance R, if and only if R is a stationary value of r; and it follows from (21) that that is equivalent to

$$0 \;=\; \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 \;=\; 1 - \left(\frac{K v(R)}{R}\right)^2 . \tag{24}$$

<u>If</u> such an apsidal distance R exists, then it follows that

$$K \;=\; \frac{R}{v(R)} \;, \tag{25}$$

and hence

$$\vartheta \;=\; \int_1^r \frac{\mathrm{d}x}{\mp x\,\sqrt{\left(\frac{v(R)}{R}\right)^2 \frac{x^2}{v(x)^2} - 1}} \;. \tag{26}$$

Attractive Force For central force which is attractive everywhere the apsidal radius is R, with $0 < R \leq r \leq 1$, and initially r is a decreasing function of s (and of t). Thus, on the first half–arch of the brachistochrone the negative square root is to be used in (26), and the polar equation is:

$$\vartheta \;=\; \Theta(r,R) \;\stackrel{\text{def}}{=}\; \int_r^1 \frac{\mathrm{d}x}{x\,\sqrt{\left(\frac{v(R)}{R}\right)^2 \frac{x^2}{v(x)^2} - 1}} \qquad (R < 1) \;. \tag{27}$$

Repulsive Force For central force which is repulsive everywhere the apsidal radius is R, with $R \geq r \geq 1$, and initially r is an increasing function of s (and of t). Thus, on the first half–arch of the brachistochrone the positive square root is to be used in (26), and the polar equation is:

$$\vartheta \;=\; \Theta(r,R) \;\stackrel{\text{def}}{=}\; \int_1^r \frac{\mathrm{d}x}{x\,\sqrt{\left(\frac{v(R)}{R}\right)^2 \frac{x^2}{v(x)^2} - 1}} \qquad (R > 1) \;. \tag{28}$$

For both attractive and repulsive force, the first half–arch of the brachistochrone ends at the apse with apsidal distance R, where the angle is $\vartheta =$

$\Theta(R,R)$. The orbit is symmetrical about the the apsidal line, so that on the second half–arch the angle is

$$\vartheta \;=\; 2\Theta(R,R) - \Theta(r,R), \tag{29}$$

and the complete brachistochrone has angle $\vartheta = 2\Theta(R,R)$.

Or, if angle ω is measured from the apse, then the complete brachistochrone has the polar equation

$$\omega \;=\; \pm\int_R^r \frac{\mathrm{d}x}{x\sqrt{\left(\frac{v(R)}{R}\right)^2 \frac{x^2}{v(x)^2} - 1}}\;, \tag{30}$$

for both attractive and repulsive force.

2.4 Arclength as a Function of Radius

It follows from (21) that the arclength at radius r is given by

$$s(r) \;=\; \int_1^r \frac{\mathrm{d}x}{\mp\sqrt{1 - \frac{K^2 v(x)^2}{x^2}}}\;, \tag{31}$$

with the sign of the square root chosen as for ϑ.

2.4.1 Brachistochrone With Apse

If an apsidal distance R exists, then it follows from (25) and (31) that the arclength at radius r is:

$$s \;=\; \int_1^r \frac{\mathrm{d}x}{\mp\sqrt{1 - \left(\frac{R}{v(R)}\right)^2 \frac{v(x)^2}{x^2}}}\;. \tag{32}$$

Attractive Force Similarly to the treatment of ϑ as a function of r, we get that on the first half–arch of the brachistochrone the arclength s is given as a function of radius r by

$$s \;=\; S(r,R) \;\stackrel{\text{def}}{=}\; \int_r^1 \frac{\mathrm{d}x}{\sqrt{1 - \left(\frac{R}{v(R)}\right)^2 \frac{v(x)^2}{x^2}}} \qquad (R<1)\;. \tag{33}$$

Repulsive Force Similarly to the treatment of ϑ as a function of r, we get that on the first half–arch of the brachistochrone the arclength s is given as a function of radius r by

$$s \;=\; S(r,R) \;\stackrel{\text{def}}{=}\; \int_1^r \frac{\mathrm{d}x}{\sqrt{1 - \left(\frac{R}{v(R)}\right)^2 \frac{v(x)^2}{x^2}}} \qquad (R>1)\;. \tag{34}$$

For both attractive and repulsive force, the first half–arch of the brachistochrone ends at the apse with apsidal distance R, where the arclength is

$s = S(R,R)$. On the second half–arch the arclength is $s = 2S(R,R) - S(r,R)$, and the complete brachistochrone has arclength $s = 2S(R,R)$.

Or, if arclength σ is measured from the apse, then the complete brachistochrone is given by

$$\sigma = \pm \int_R^r \frac{\mathrm{d}x}{\sqrt{1 - \left(\frac{R}{v(R)}\right)^2 \frac{v(x)^2}{x^2}}} , \qquad (35)$$

for both attractive and repulsive force.

2.5 Time as a Function of Radius

The time required for the constrained particle to reach radius r is given by

$$t = \int \mathrm{d}t = \int \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}r}\mathrm{d}r = \int \frac{\mathrm{d}s}{\mathrm{d}r}\,\frac{\mathrm{d}r}{v(r)} , \qquad (36)$$

where $\mathrm{d}s/\mathrm{d}r$ is given by (21). Take $t = 0$ at the start A.

Then, the radius r is reached at time

$$t = \int_1^r \frac{\mathrm{d}x}{\mp v(x)\sqrt{1 - \frac{K^2 v(x)^2}{x^2}}} , \qquad (37)$$

with the sign of the square root chosen as for ϑ.

2.5.1 Brachistochrone With Apse

If an apsidal distance R exists, then it follows from (25) and (37) that

$$t = \int_1^r \frac{\mathrm{d}x}{\mp v(x)\sqrt{1 - \left(\frac{R}{v(R)}\right)^2 \frac{v(x)^2}{x^2}}} . \qquad (38)$$

Attractive Force Similarly to the treatment of ϑ as a function of r, we get that on the first half–arch of the brachistochrone the time t is given as a function of radius r by

$$t = T(r,R) \stackrel{\text{def}}{=} \int_r^1 \frac{\mathrm{d}x}{v(x)\sqrt{1 - \left(\frac{R}{v(R)}\right)^2 \frac{v(x)^2}{x^2}}} \qquad (R < 1) . \qquad (39)$$

Repulsive Force Similarly to the treatment of ϑ as a function of r, we get that on the first half–arch of the brachistochrone the time t is given as a function of radius r by

$$t = T(r,R) \stackrel{\text{def}}{=} \int_1^r \frac{\mathrm{d}x}{v(x)\sqrt{1 - \left(\frac{R}{v(R)}\right)^2 \frac{v(x)^2}{x^2}}} \qquad (R > 1) . \qquad (40)$$

For both attractive and repulsive force, the first half–arch of the brachistochrone ends at the apse with apsidal distance R, at time $t = T(R,R)$. On the second half–arch the time is $t \; = \; 2T(R,R) - T(r,R)$, and the complete brachistochrone is traversed at $t = 2T(R,R)$.

Or, if time τ is measured from the apse, then the complete brachistochrone is given by

$$\tau \;=\; \pm \int_R^r \frac{\mathrm{d}x}{v(x)\sqrt{1 - \left(\frac{R}{v(R)}\right)^2 \frac{v(x)^2}{x^2}}} \;, \tag{41}$$

for both attractive and repulsive force.

3 Inverse Square Force

Consider a particle constrained to a smooth curve, under potential

$$\varphi(r) \;=\; \frac{-\mu}{r} \;, \tag{42}$$

so that the force (per unit mass) is

$$F \;=\; \frac{\mu}{r^2} \;. \tag{43}$$

Scale the unit of time so that $\mu = 1$ for attractive force, or $\mu = -1$ for repulsive force — this makes the problem non–dimensional.

For a particle released at rest from radius 1, the speed at radius r is v, where:

$$v^2 \;=\; 2(\varphi(1) - \varphi(r)) \;=\; 2\mu\left(\frac{1}{r} - 1\right) \;=\; \frac{2\mu(1-r)}{r} \;=\; \frac{2\,|r-1|}{r} \;. \tag{44}$$

Thus, for repulsive force, as $r \nearrow \infty$, then $v \nearrow \sqrt{2}$.

Radial Motion For attractive inverse square force, if the endpoint B lies inside the interval OA then the line AB is the brachistochrone, with the particle moving freely towards the centre O, with speed (cf. (44))

$$\frac{\mathrm{d}r}{\mathrm{d}t} \;=\; -v \;=\; -\sqrt{\frac{2(1-r)}{r}} \;. \tag{45}$$

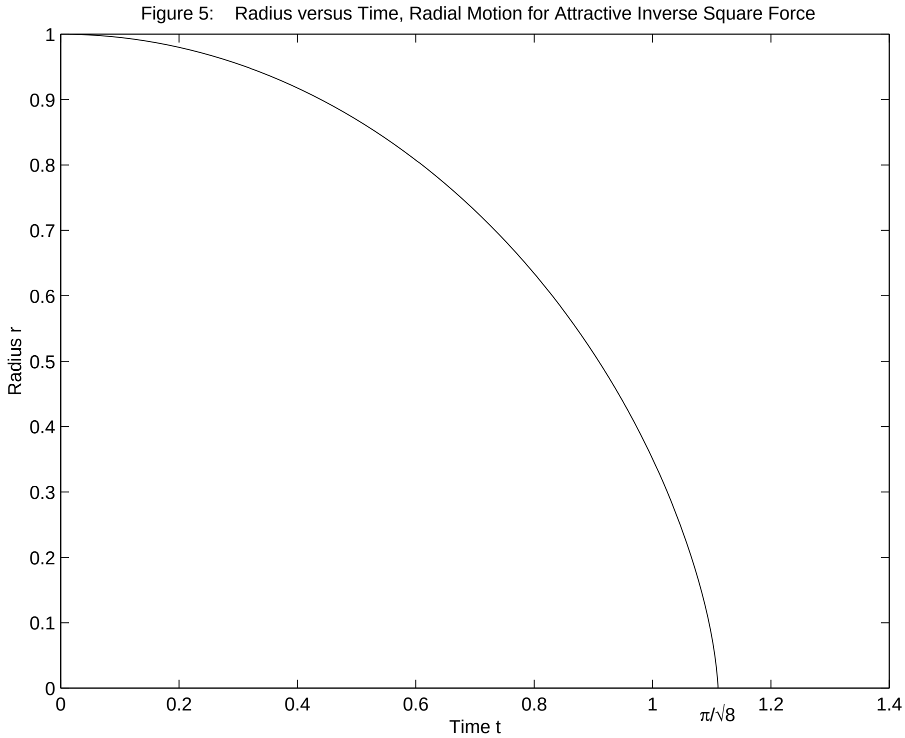
Hence the time to reach radius $r < 1$ is

$$t(r) \;=\; \frac{1}{\sqrt{2}} \int_r^1 \sqrt{\frac{x}{1-x}}\;\mathrm{d}x \;=\; \left[\tfrac{1}{2}\arccos(2r-1) + \sqrt{r(1-r)}\right]/\sqrt{2} \;, \tag{46}$$

so that as $r \searrow 0$, the time for falling to radius r approaches the limit:

$$t(0) \;=\; \frac{\pi}{\sqrt{8}} \;=\; 1{\cdot}1107207345385916 \;. \tag{47}$$

Figure 5 shews a graph of time versus radius, for a particle falling to the centre.



This limit can conveniently be abbreviated, by saying that the particle falls to the centre at time $t(0)$.

But, any mathematical model in which the particle passes through the singularity at the centre (e.g. by traversing a straight line passing through the singularity) introduces serious paradoxes, and cannot be regarded as a useful idealization of any physical motion. No physically meaningful path of a particle can pass through the singularity at the centre.

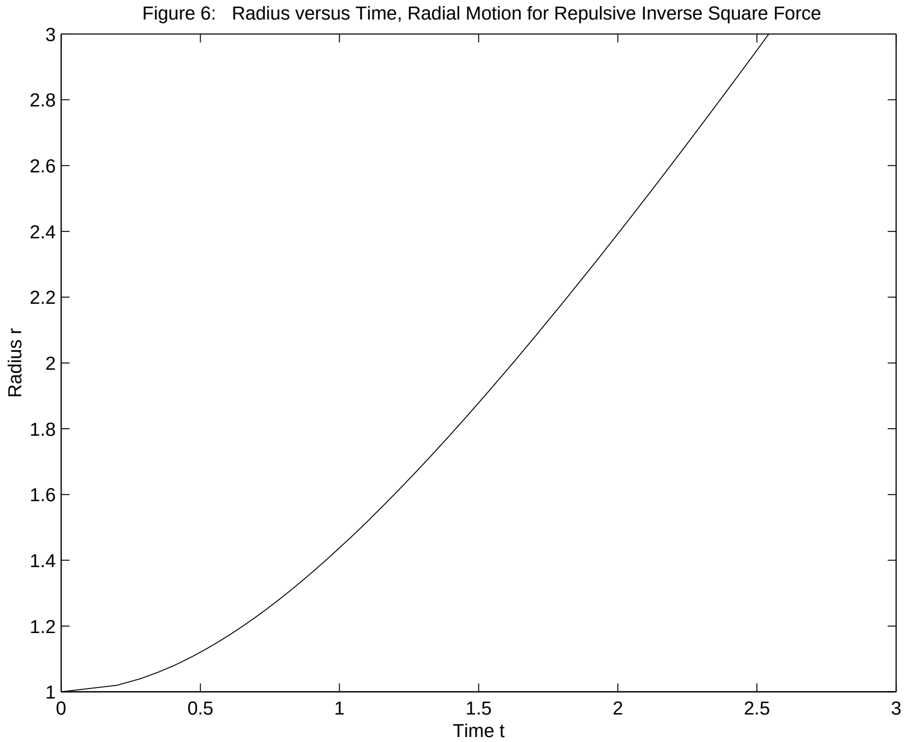
For repulsive inverse square force, if the starting point A is inside the interval OB then the line AB is the brachistochrone, with the particle moving freely outwards in the radial direction, with speed (cf. (44))

$$v \;=\; \frac{\mathrm{d}r}{\mathrm{d}t} \;=\; \sqrt{\frac{2(r-1)}{r}}\;. \tag{48}$$

Hence, the time to reach radius $r > 1$ is:

$$t(r) \;=\; \frac{1}{\sqrt{2}}\int_1^r \sqrt{\frac{x}{x-1}}\,\mathrm{d}x \;=\; \Big[\sqrt{r(r-1)} + \log(\sqrt{r} + \sqrt{r-1})\Big]/\sqrt{2}\;. \tag{49}$$

Figure 6 shews a graph of time versus radius, for a particle repelled from the centre.



In these two cases of radial motion, $K = 0$. Hereafter, we shall consider the general case where $K > 0$, and so AB is not a radial line.

3.1 Brachistochrones Under Inverse Square Force

For inverse square force, at radius r on the brachistochrone, the speed $v(r)$ of the particle is given by (44).

With this form of v, the general equations (23), (31) and (37) give angle, arclength and time as functions of radius with parameter K, the angular momentum of the associated free particle:

$$\begin{aligned} \vartheta &= \int_1^r \frac{\mathrm{d}x}{\pm x\sqrt{\frac{x^3}{2\mu K^2(1-x)} - 1}} \\ &= K\sqrt{2}\int_1^r \frac{1-x}{\pm x\sqrt{\mu(1-x)\,(x^3+2\mu K^2(x-1))}}\,\mathrm{d}x\ . \end{aligned} \tag{50}$$

$$s = \int_1^r \frac{\mathrm{d}x}{\pm\sqrt{1-2\mu K^2\frac{1-x}{x^3}}} = \int_1^r \frac{x^2}{\pm\sqrt{x(x^3+2\mu K^2(x-1))}}\,\mathrm{d}x\ . \tag{51}$$

$$\begin{aligned} t &= \int_1^r \frac{\mathrm{d}x}{\mp\sqrt{\frac{2\mu(1-x)}{x}\left(1-2\mu K^2\frac{1-x}{x^3}\right)}} \\ &= \int_1^r \frac{x^2}{\pm\sqrt{2\mu(1-x)(x^3+2\mu K^2(x-1))}}\,\mathrm{d}x\ . \end{aligned} \tag{52}$$

Thus, each of $\vartheta,\ s$ and t is given as an elliptic integral of r, with K as a parameter.

4 Attractive Inverse Square Force

With attractive force ($\mu = 1$), $r \leq 1$ and hence any apsidal distance (if it exists) is $R < 1$, so that $0 < R \leq r \leq 1$. It follows from (25) and (44) that

$$K^2 \;=\; \frac{R^2}{v(R)^2} \;=\; \frac{R^3}{2(1-R)}\ , \tag{53}$$

and hence as R increases from 0 to 1, K increases monotonically from 0 to infinity. Thus, for any positive K, R satisfies the cubic equation

$$2K^2(1-R) \;=\; R^3. \tag{54}$$

Now, the function R^3 increases with R, but the function $2K^2(1-R)$ decreases with R. At $R = 0$, $2K^2(1-R) = 2K^2 > 0 = R^3$, and at $R = 1$, $2K^2(1-R) = 0 < 1 = R^3$. Hence, for each $K > 0$, the cubic equation (25) has a single real root $0 < R < 1$; and so the brachistochrone does have an apse, with apsidal distance R. As $K \searrow 0$, $R \sim (2K^2)^{1/3}$, and as $K \nearrow \infty$, $1 - R \sim (2K^2)^{-1}$.

Therefore, with attractive inverse square force the family of brachistochrones through a fixed point A can be parametrized by the minimum radius R, as an alternative to K.

4.1 Equations For Brachistochrone

The equations (23), (31), (37) and (50), (51), (52) give angle, arclength and time on the first half-arch; each as an elliptic integral of radius with the minimum radius R as a parameter:

$$\begin{aligned}\Theta(r,R) \;&=\; \int_r^1 \frac{1-x}{x\sqrt{(1-x)\left(\left(\frac{1-R}{R^3}\right)x^3+x-1\right)}}\,\mathrm{d}x \\ &=\; \sqrt{\tfrac{R^3}{1-R}}\int_r^1 \frac{1-x}{x\sqrt{(1-x)(x-R)\left[x^2+Rx+\frac{R^2}{1-R}\right]}}\,\mathrm{d}x\ .\end{aligned} \tag{55}$$

$$\begin{aligned}S(r,R) \;&=\; \int_r^1 \frac{x^2}{\sqrt{x\left(x^3+\left(\frac{R^3}{1-R}\right)(x-1)\right)}}\,\mathrm{d}x \\ &=\; \int_r^1 \frac{x^2}{\sqrt{x(x-R)\left[x^2+Rx+\frac{R^2}{1-R}\right]}}\,\mathrm{d}x\ .\end{aligned} \tag{56}$$

$$T(r,R) \;=\; \int_r^1 \frac{x^2}{\sqrt{2(1-x)\left(x^3+\left(\frac{R^3}{1-R}\right)x-\frac{R^3}{1-R}\right)}}\,\mathrm{d}x$$

$$= \int_r^1 \frac{x^2}{\sqrt{2(1-x)(x-R)\left[x^2+Rx+\frac{R^2}{1-R}\right]}}\,\mathrm{d}x\ . \qquad (57)$$

Or, if angle ω (30), arclength σ (35) and time τ (41) are measured from the apse, then the complete brachistochrone is given in terms of r by the elliptic integrals

$$\begin{aligned}
\omega &= \pm\sqrt{\tfrac{R^3}{1-R}}\int_R^r \frac{1-x}{x\sqrt{(1-x)(x-R)\left[x^2+Rx+\frac{R^2}{1-R}\right]}}\,\mathrm{d}x\ ,\\
\sigma &= \pm\int_R^r \frac{x^2}{\sqrt{x(x-R)\left[x^2+Rx+\frac{R^2}{1-R}\right]}}\,\mathrm{d}x\ ,\\
\tau &= \pm\int_R^r \frac{x^2}{\sqrt{2(1-x)(x-R)\left[x^2+Rx+\frac{R^2}{1-R}\right]}}\,\mathrm{d}x\ . \qquad (58)
\end{aligned}$$

The discriminant of the polynomial $x^2+Rx+\frac{R^2}{1-R}$ is $R^2(R+3)/(R-1)$, and since $R<1$ that discriminant is negative. Therefore, that quadratic polynomial is positive for all real x.

4.1.1 Non-standardization of Elliptic Integrals

Every elliptic integral can be reduced to a standard form, as a linear combination of elementary functions and of Legendre's Elliptic Integrals of the First, Second and Third Kinds. But, the reduction of elliptic integrals to such a standard form is very complicated (Erdélyi [6], pages 294–312), with many hundreds of cases to be distinguished carefully (Byrd & Friedman [2], pages 42–161). Moreover, the only practical way to evaluate Elliptic Integrals of the Third Kind is to use some appropriate numerical quadrature. Hence, it is simpler to evaluate our various elliptic integrals directly, without reducing them to standard form.

4.1.2 Elimination of Singularities in Integrands

For minimum radius $R \ll 1$, the angle ϑ remains small until the particle has reached radius x which is slightly greater than R, after which ϑ increases (during very little time) to the apse angle $\Theta(R,R)$, during the small decrease of x to R. Thus, almost all of the apse angle is generated by values of the integrand near $x=R$, which are much larger than values of the integrand for smaller x, as a consequence of the factor x in the denominator in (55). That near–singularity of the integrand near $x=R\ll 1$ makes it difficult to evaluate the integral numerically, and it greatly complicates analysis of the asymptotic behaviour of the brachistochrones as $R \searrow 0$.

Accordingly, we shall apply the substitution

$$z = \frac{R}{x}\ , \qquad (59)$$

so that z increases from R to 1 as x decreases from 1 to R. Then, the equation (55) becomes

$$\Theta(r, R) \;=\; \int_{R}^{R/r} \frac{z - R}{\sqrt{(1-z)(z-R)\big(z^2 + (1-R)z + 1 - R\big)}}\, \mathrm{d}z \;. \tag{60}$$

This transformed integrand (60) still has an integrable singularity at $z = R$, and when $r = R$ there is an integrable singularity at $z = 1$, and similarly for (57). Accordingly, we shall apply substitution of variable in each integral to remove those integrable singularities, so that the transformed integrands are C^∞–smooth, and hence the transformed integrals can readily be evaluated to high accuracy by Romberg integration.

4.2 Angle as a Function of Radius

In the integral (60) for ϑ, substitute

$$z \;=\; z(\lambda) \;=\tfrac{1}{2}(1 + R + (1-R)\cos 2\lambda) \;= \cos^2\lambda + R\;\sin^2\lambda \;, \tag{61}$$

so that λ increases from 0 to $\frac{1}{2}\pi$ as r increases from R to 1 and z decreases from 1 to R; and

$$\lambda \;=\; \lambda(z) \;=\; \tfrac{1}{2}\arccos\left(\frac{2z - 1 - R}{1 - R}\right) . \tag{62}$$

Therefore

$$z - R \;=\; (1-R)\cos^2\lambda \;, \tag{63}$$

and

$$\frac{\mathrm{d}z}{\sqrt{(1-z)(z-R)}} \;=\; -2\,\mathrm{d}\lambda \;. \tag{64}$$

Thus, (60) becomes

$$\Theta(r, R) \;=\; \int_{\lambda(R/r)}^{\pi/2} \frac{2(z-R)}{\sqrt{z^2 + (1-R)z + 1 - R}}\,\mathrm{d}\lambda \;, \tag{65}$$

where the function $z = z(\lambda)$ is given by (61).

Now, $z(\lambda)$ is a C^∞–smooth function of λ, and the quadratic polynomial in the denominator is strictly positive, and hence the integrand in (65) is a C^∞–smooth function of λ. Thus, the integral (65) may readily be evaluated to high accuracy by Romberg integration with respect to λ.

In particular, the apse angle is

$$\Theta(R, R) \;=\; \int_{0}^{\pi/2} \frac{2(z-R)}{\sqrt{z^2 + (1-R)z + 1 - R}}\,\mathrm{d}\lambda \;, \tag{66}$$

Write the integrand as $f(R, \lambda)$, where z is given as a function of R and of λ by (61). Then,

$$\frac{\mathrm{d}}{\mathrm{d}R}\,\Theta(R, R) \;=\; \frac{\mathrm{d}}{\mathrm{d}R}\int_0^{\pi/2} f(R, \lambda)\,\mathrm{d}\lambda \;=\; \int_0^{\pi/2} \frac{\partial f}{\partial R}\,\mathrm{d}\lambda \;. \tag{67}$$

Holding λ fixed, we get

$$\frac{\partial z}{\partial R} \;=\; \sin^2\lambda \; . \tag{68}$$

Thus,

$$\frac{\partial f}{\partial R} \;=\; \frac{\partial}{\partial R}\left[2(1-R)\cos^2\lambda\,\left(z^2+(1-R)z+1-R\right)^{-1/2}\right]$$

$$\begin{aligned} &= -2\,\cos^2\lambda\,\left(z^2+(1-R)z+1-R\right)^{-1/2} - (1-R)\cos^2\lambda\,\times \\ &\quad \times\left(z^2+(1-R)z+1-R\right)^{-3/2}\left(2z\,\sin^2\lambda+(1-R)\,\sin^2\lambda-z-1\right) \\ &= -C\big[2(z^2+(1-R)(z+1)) \\ &\quad +(1-R)\left(z(2\,\sin^2\lambda-1)-1+(1-R)\,\sin^2\lambda\right)\big]\;, \end{aligned} \tag{69}$$

where

$$C \;=\; \cos^2\lambda\,\left(z^2+(1-R)z+1-R\right)^{-3/2} \;\geq\; 0\;, \tag{70}$$

with equality only at $\lambda=\frac{1}{2}\pi$. Hence, for $0\leq\lambda<\frac{1}{2}\pi$,

$$\begin{aligned} \frac{-1}{C}\frac{\partial f}{\partial R} &= 2(z^2+(1-R)(z+1)) \\ &\quad +(1-R)\left(z(2\,\sin^2\lambda-1)-1+(1-R)\,\sin^2\lambda\right) \\ &= 2z^2+(1-R)\left(2z+2+2z\,\sin^2\lambda-z+(1-R)\,\sin^2\lambda-1\right) \\ &= 2z^2+(1-R)\left(z+1+(2z+1-R)\,\sin^2\lambda\right) \;>\; 0\;; \end{aligned} \tag{71}$$

and so

$$\frac{\partial f}{\partial R} \;<\; 0 \tag{72}$$

for all $\lambda\in[0,\frac{1}{2}\pi)$.

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}R}\,\Theta(R,R) \;=\; \int_0^{\pi/2}\frac{\partial f}{\partial R}\,\mathrm{d}\lambda \;<\; 0\;, \tag{73}$$

and hence the apse angle $\Theta(R,R)$ is a strictly decreasing function of R, for all $R\in(0,1)$. As $R\nearrow 1$, then $\Theta(R,R)\searrow 0$.

4.2.1 Bound for Apse Angle

As $R\searrow 0$ then $1-R\nearrow 1$, and it follows from (61) that $z(\lambda)$ converges to $\cos^2\lambda$, uniformly for all $\lambda\in\left(0,\frac{1}{2}\pi\right)$. Hence, $z^2+(1-R)z+1-R$ converges to $1+\cos^2\lambda+\cos^4\lambda$, with relative error converging to 0, uniformly for all $\lambda\in\left(0,\frac{1}{2}\pi\right)$. Therefore, we get the limiting value of the apse angle:

$$\begin{aligned} \lim_{R\to 0}\Theta(R,R) &= \int_0^{\pi/2}\frac{2\,\cos^2\lambda}{\sqrt{1+\cos^2\lambda+\cos^4\lambda}}\,\mathrm{d}\lambda \\ &= \int_0^{\pi/2}\frac{2\,\cos^2\lambda\,\sin\lambda}{\sqrt{(1-\cos^2\lambda)(1+\cos^2\lambda+\cos^4\lambda)}}\,\mathrm{d}\lambda \\ &= \int_0^{\pi/2}\frac{2\,\cos^2\lambda\,\sin\lambda}{\sqrt{1-\cos^6\lambda}}\,\mathrm{d}\lambda \;=\; -\frac{2}{3}\int_0^{\pi/2}\frac{\mathrm{d}\left(\cos^3\lambda\right)}{\sqrt{1-\cos^6\lambda}} \\ &= \frac{2}{3}\int_0^1\frac{\mathrm{d}u}{\sqrt{1-u^2}} \;=\; \frac{\pi}{3}\;, \end{aligned} \tag{74}$$

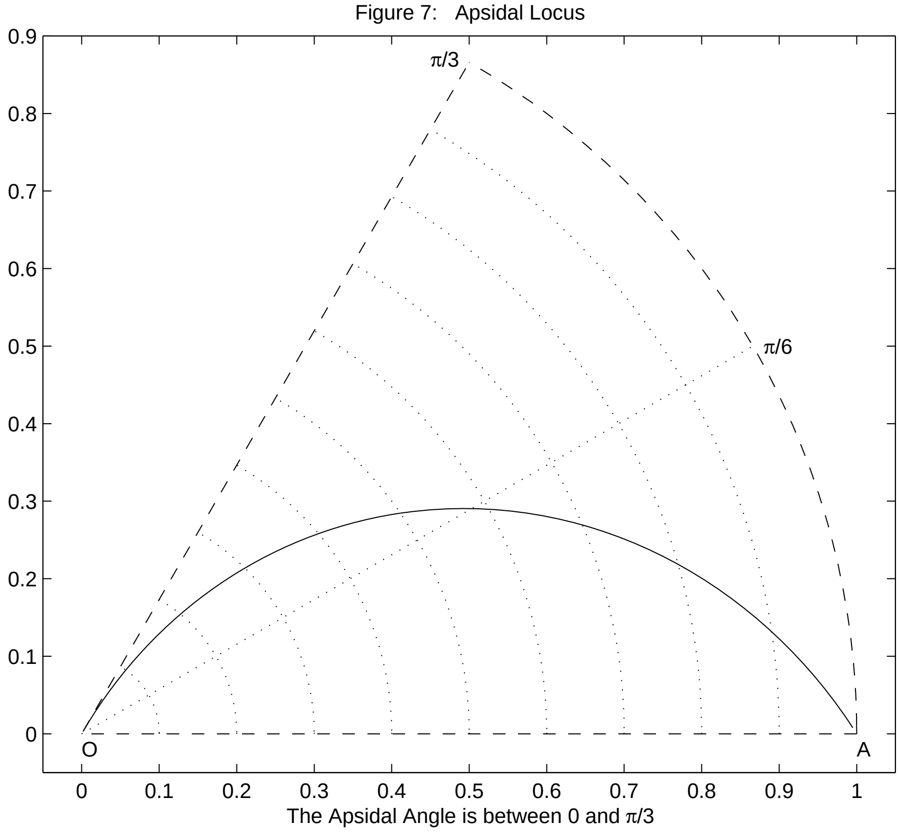
where $u = \cos^3 \lambda$.

Thus*, as $R \searrow 0$, then $\Theta(R, R) \nearrow \frac{1}{3}\pi$.

Apsidal Locus The curve $\vartheta = \Theta(R, R)$, which we shall call the *apsidal locus*, separates those points C for which the brachistochrone AC is part of the (open) first half–arch ((27) or (28)) of a complete brachistochrone AB, from those points C for which the brachistochrone AC includes part of the (open) second half–arch (29).

Figure 7 shews that the apsidal locus approximates quite closely to a circular arc with angle $\frac{2}{3}\pi$, spanning the chord OA.

Figure 7: Apsidal Locus



4.2.2 Sector of Inaccessibility

The apse angle has the least upper bound of $\frac{1}{3}\pi$, and hence *every* brachistochrone covers angle strictly less than $\frac{2}{3}\pi$. Therefore, for any point $C = (\rho, \psi)$ with $\frac{2}{3}\pi \leq \psi \leq \pi$, there does not exist any brachistochrone from A to C!

As the apse radius $R \searrow 0$, the first half-arch of the brachistochrone converges (non-uniformly) to the radius vector OA (with $\vartheta = 0$) and the second half-arch converges (non-uniformly) to the radius vector with $\vartheta = \frac{2}{3}\pi$. But (for $K > 0$)

0*Ron Keam has generalized this analysis, to shew that for inverse n-th power force, with real $n > 1$, the limit of the apse angle is $\pi/(n + 1)$.

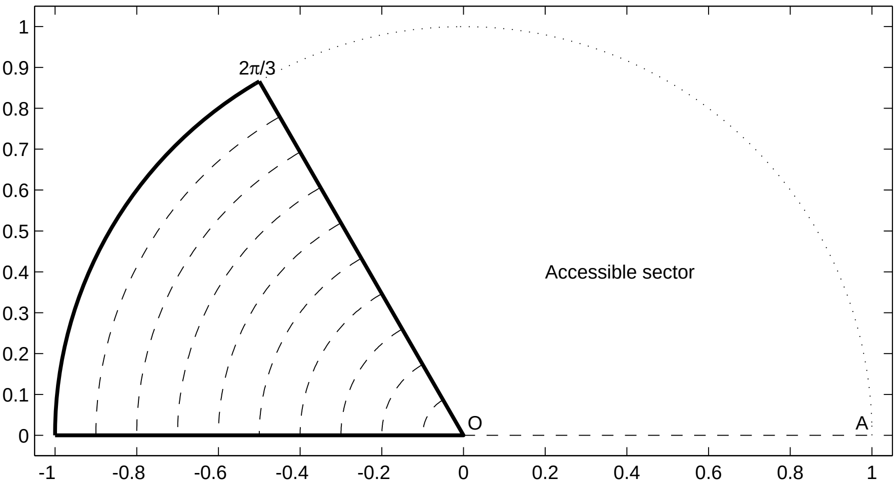
the starting point A is the only point on those bounding radii vectors through which passes any brachistochrone starting at A.

Rather, a particle sliding on a smooth curve which is close to the radius vector OA until it gets very close to the centre O, then swerving across to a path close to the radius vector OC, will reach C after a period slightly more than $\kappa(\rho)$, which is defined as the time for a particle to fall from A to O, plus the time for a particle (starting from rest at radius 1) to fall from C to O. Thus (cf. (44)),

$$\kappa(\rho) \;=\; 2t(0) - t(\rho) \;=\; \left[\pi - \tfrac{1}{2}\arccos(2\rho - 1) - \sqrt{\rho(1-\rho)}\right]/\sqrt{2}\,. \qquad (75)$$

But, there does not exist <u>any</u> smooth path which does get the particle from A to C in time $\kappa(\rho)$. In the sector of the unit circle with $\frac{2}{3}\pi \le \vartheta \le \pi$, for each point C on the arc of radius ρ, $\kappa(\rho)$ is the greatest lower bound for passage time from A to C. That sector is here called the *Sector of Inaccessibility* for brachistochrones from A — it is depicted in Figure 8,

Figure 8: Sector of Inaccessibility



On each arc of radius ρ, κ(ρ) is the greatest lower bound of Passage Time

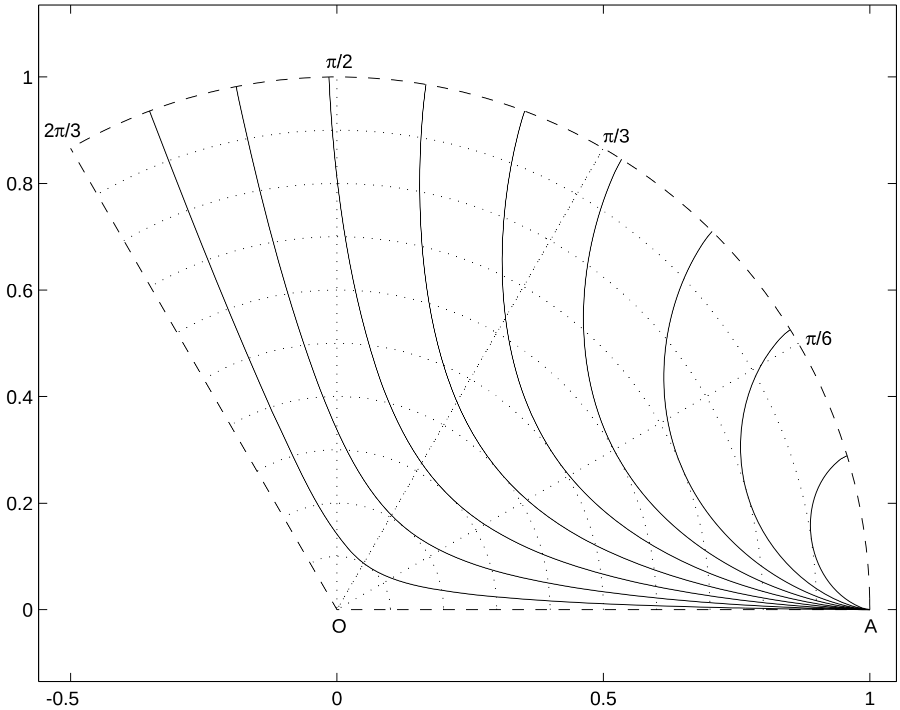
The sector with $0 \le \psi < \frac{2}{3}\pi$ is here called the *accessible sector.*

4.2.3 Pictures of Brachistochrones, for Attractive Inverse Square Force

Figure 9 shews the complete brachistochrones from A for various values of the minimum radius R, with ϑ computed for various values of r by Romberg integration of (65).

Note that the small complete brachistochrones do indeed resemble complete cycloid arches; and that the angle covered by the complete brachistochrone does increase towards $\frac{2}{3}\pi$ as $R \searrow 0$ and the first half-arch converges to the radius vector OA.

Figure 9: Brachistochrones, for minimum radius R = 0.1,0.2,...,0.9



One brachistochrone passes through each point in the accessible sector

4.3 Arclength as a Function of Radius

In the integral (56) for s, substitute

$$x = x(y) = y^2 + R, \tag{76}$$

so that $y = \sqrt{x - R}$ and $\mathrm{d}x = 2y\,\mathrm{d}y$, and hence

$$\begin{aligned} S(r,R) &= \int_{\sqrt{r-R}}^{\sqrt{1-R}} \frac{2yx^2}{y\sqrt{x\left[x^2 + Rx + \frac{R^2}{1-R}\right]}}\,\mathrm{d}y \\ &= \int_{\sqrt{r-R}}^{\sqrt{1-R}} \frac{2x}{\sqrt{x + R + \frac{R^2}{(1-R)x}}}\,\mathrm{d}y\ , \end{aligned} \tag{77}$$

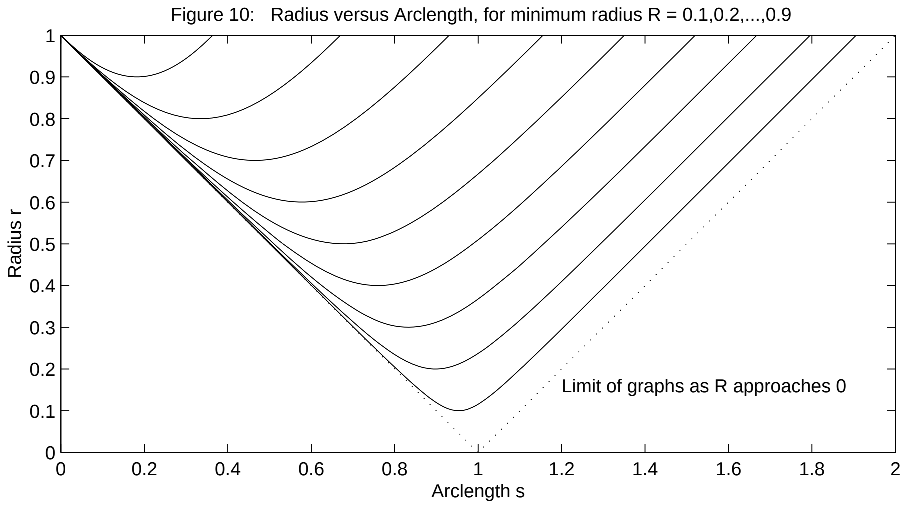
where $x = y^2 + R$. In particular, the arclength to the apse is

$$S(R,R) = \int_0^{\sqrt{1-R}} \frac{2x}{\sqrt{x + R + \frac{R^2}{(1-R)x}}}\,\mathrm{d}y\ . \tag{78}$$

And these transformed integrals are suitable for Romberg integration with respect to y.

4.3.1 Graphs of Radius versus Arclength

Figure 10 shews graphs of r versus s, for various values of the minimum radius R. Each graph is symmetric about the line $s = S(R, R)$.



The dotted line is the limit of the graphs as $R \searrow 0$, with the first half representing the radius vector at $\vartheta = 0$, and the second half representing the radius vector at $\vartheta = \frac{2}{3}\pi$.

4.4 Time as a Function of Radius

Substitute

$$x \;=\; x(\zeta) \;= \tfrac{1}{2}(1 + R + (1 - R)\cos\zeta), \tag{79}$$

so that as x decreases from 1 to R, ζ increases from 0 to π, and

$$\zeta \;= \zeta(x) \;=\; \arccos\left(\frac{2x - 1 - R}{1 - R}\right). \tag{80}$$

Then, (57) becomes

$$T(r, R) \;=\; \int_0^{\zeta(r)} \frac{x^2}{\sqrt{2\left[x^2 + Rx + \frac{R^2}{1-R}\right]}}\, \mathrm{d}\zeta\ , \tag{81}$$

where the function $x = x(\zeta)$ is given by (79). In particular, the time at the apse is

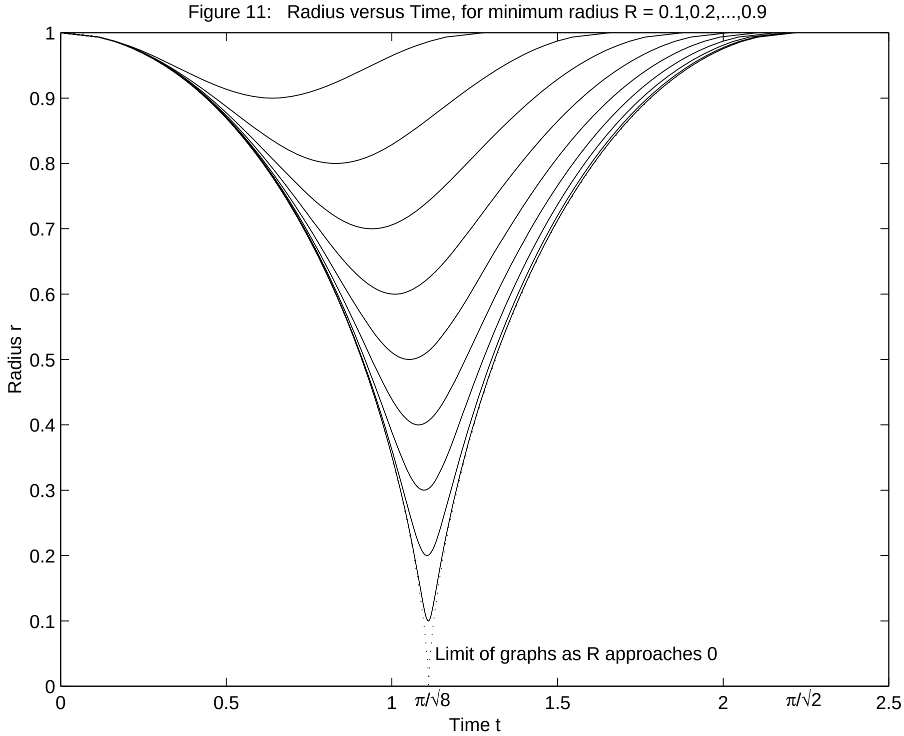
$$T(R, R) \;=\; \int_0^{\pi} \frac{x^2}{\sqrt{2\left[x^2 + Rx + \frac{R^2}{1-R}\right]}}\, \mathrm{d}\zeta\ . \tag{82}$$

And these transformed integrals are suitable for Romberg integration with respect to ζ.

4.4.1 Graphs of Radius versus Time

Figure 11 shews graphs of r versus t, for various values of the minimum radius R. Each graph is symmetric about the line $t = T(R, R)$.

Figure 11: Radius versus Time, for minimum radius R = 0.1,0.2,...,0.9



The dotted curve is the limit of the graphs as $R \searrow 0$, with the first half (cf. Figure 5) representing free fall down the radius vector at $\vartheta = 0$, and the second half representing free rise up the radius vector at $\vartheta = \frac{2}{3}\pi$.

4.5 Relation between Arclength and Angle

For each value of the minimum radius R, each of ϑ, s and t can be computed from (55), (56) and (57) as functions of r.

Figure 12 shews graphs of ϑ versus s for various values of R, with the points on each graph computed for the parameter r.

The dotted line is the limit of the graphs as $R \searrow 0$, with the first vertical line representing the radius vector at $\vartheta = 0$, and the second vertical line representing the radius vector at $\vartheta = \frac{2}{3}\pi$. The horizontal line represents the transition from the first radius vector to the second, with zero arclength involved.

Since angle, arclength (and likewise time), measured from the apse, are each even functions (58) of r, each of these graphs is symmetric about its midpoint, corresponding to the apse.

Figure 12: Arclength versus Angle, for minimum radius R = 0.1,0.2,...,0.9

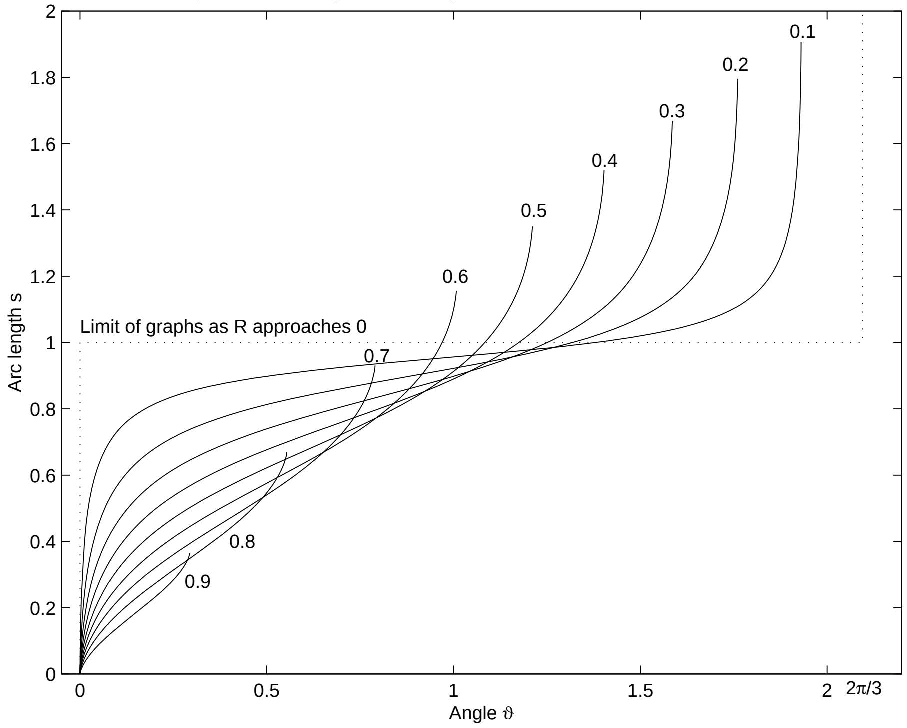
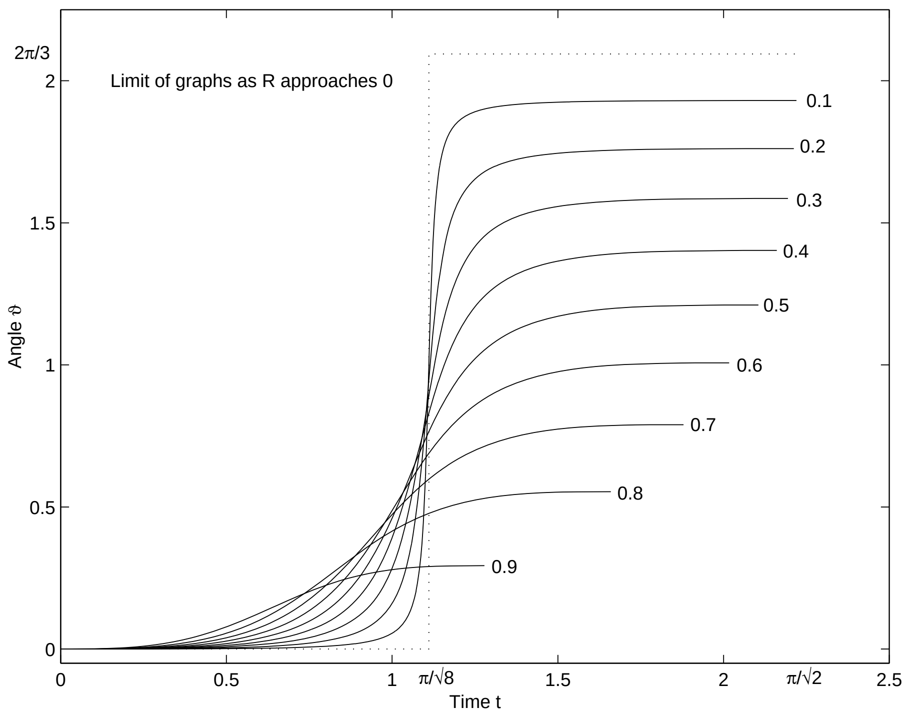


Figure 13: Angle versus Time, for minimum radius R = 0.1,0.2,...,0.9



4.6 Relation between Time and Angle

Figure 13 shews graphs of t versus ϑ for various values of R, with the points on each graph computed for the parameter r.

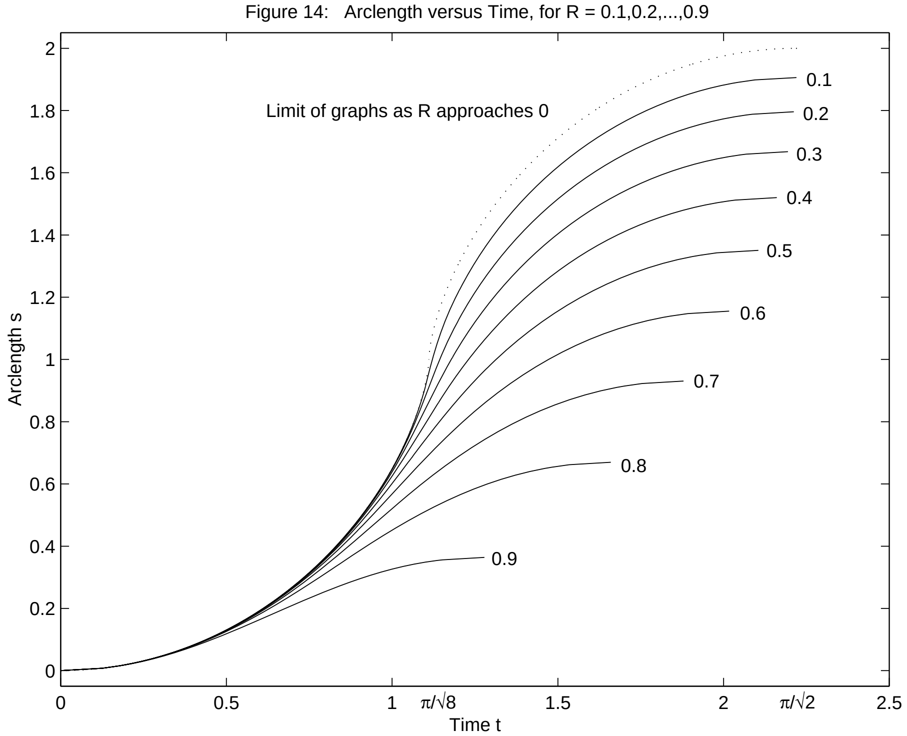
The dotted line is the limit of the graphs as $R \searrow 0$, with the first horizontal line representing the radius vector at $\vartheta = 0$, and the second horizontal line representing the radius vector at $\vartheta = \frac{2}{3}\pi$. The vertical line represents the transition from the first radius vector to the second, with zero time involved.

Since both angle and time, measured from the apse, are even functions (58) of r, each of these graphs is symmetric about its midpoint, corresponding to the apse.

4.7 Relation between Arclength and Time

Figure 14 shews graphs of s versus t for various values of R, with the points on each graph computed for the parameter r.

Figure 14: Arclength versus Time, for R = 0.1,0.2,...,0.9



The dotted curve is the limit of the graphs as $R \searrow 0$, with the first half (cf. Figure 5) representing free fall down the radius vector at $\vartheta = 0$, and the second half representing free rise up the radius vector at $\vartheta = \frac{2}{3}\pi$.

Since both arclength and time, measured from the apse, are even functions (58) of r, each of these graphs is symmetric about its midpoint, corresponding to the apse.

5 Brachistochrone Through Two Points

In view of (55) and (29), the polar equation of the brachistochrone with minimum radius R is

$$\vartheta \;=\; \int_r^1 \frac{1-x}{x\sqrt{(1-x)\left(\left(\frac{1-R}{R^3}\right)x^3+x-1\right)}}\,\mathrm{d}x \;=\; \Theta(r,R) \tag{83}$$

on the first half-arch; and $\vartheta = 2\Theta(R,R) - \Theta(r,R)$ on the second half-arch.

For $R \in [0,1)$, $R^3/(1-R)$ is an increasing function of R; and hence for each fixed x the integrand in (83) is a monotonically increasing function of R. Hence, for fixed $r \geq R$, $\Theta(r,R)$ increases monotonically with R. Therefore, for fixed $r \geq R$, ϑ is an increasing function of R on the first half-arch. But $2\Theta(R,R)$ is a decreasing function of R, and for fixed $r > R$, $-\Theta(r,R)$ is also a decreasing function of R; and hence $\vartheta = 2\Theta(R,R) - \Theta(r,R)$ is a decreasing function of R on the second half-arch.

Consider any point $B \neq A$ in the accessible sector with polar coordinates (ρ,ψ), where $0 \leq \rho \leq 1$ and $0 \leq \psi < \frac{2}{3}\pi$.

Could two brachistochrones, with distinct parameters $R_1 \neq R_2$, starting at the same point A, intersect at B? If so, then B could not be on the first half-arch of both brachistochrones, since ϑ increases monotonically with R on the first half-arch. Similarly, B could not be on the second half-arch of both brachistochrones, since ϑ decreases monotonically with R on the second half-arch. Therefore, B must be on the first half-arch of one brachistochrone and on the second half-arch of the other brachistochrone. But, for brachistochrones with startpoint A, the apsidal locus separates those points in the accessible sector which are on a first half-arch from those which are on a second half-arch. Therefore, two distinct brachistochrones with startpoint A cannot intersect at any point other than A,

Thus, through each point B in the accessible sector, there passes 1 and only 1 brachistochrone starting at A.

In order to construct the brachistochrone AB, we need to compute its minimum radius $\overline{R}$, which is the parameter for that particular brachistochrone starting from A and passing though B.

Note that $0 < \overline{R} \leq \rho \leq 1$.

5.1 Computation of the Parameter $\overline{R}$

If $0 < \psi \leq \Theta(\rho,\rho) < \frac{1}{3}\pi$, then B is on the first half-arch of the brachistochrone. Accordingly (cf. (27) & (65)), the angle at B satisfies the equation

$$\psi \;=\; \Theta(\rho,\overline{R}), \tag{84}$$

for some minimum radius $\overline{R} \leq \rho$.

Otherwise, B is on the second half-arch, and the angle at B satisfies the equation (29)

$$\psi \;=\; 2\Theta(\overline{R},\overline{R}) - \Theta(\rho,\overline{R}). \tag{85}$$

In particular, this is the case if $\frac{1}{3}\pi \leq \psi < \frac{2}{3}\pi$.

Thus, the construction of the brachistochrone AB has been reduced to computation of the root $R = \overline{R}$ of the equation

$$\Upsilon(R) - \psi \;=\; 0. \tag{86}$$

Here

$$L(R) \quad \stackrel{\text{def}}{=} \quad \tfrac{1}{2}\arccos\left(\frac{\frac{2R}{\rho} - 1 - R}{1-R}\right),$$

$$z = z(\lambda) \quad \stackrel{\text{def}}{=} \quad \cos^2\lambda + R\,\sin^2\lambda\ , \tag{87}$$

and for $\psi \le \Theta(R, R)$,

$$\Upsilon(R) \quad \stackrel{\text{def}}{=} \quad \int_{L(R)}^{\pi/2} \frac{2(z-R)}{\sqrt{z^2 + (1-R)z + 1 - R}}\,\mathrm{d}\lambda\ , \tag{88}$$

but otherwise

$$\Upsilon(R) \quad \stackrel{\text{def}}{=} \quad 2\Theta(R,R) - \int_{L(R)}^{\pi/2} \frac{2(z-R)}{\sqrt{z^2 + (1-R)z + 1 - R}}\,\mathrm{d}\lambda\ . \tag{89}$$

The equation (86) can be readily solved to high accuracy by the secant method, which requires evaluation of the function Υ and which requires two initial estimates of the root — e.g. $R_0 = 0{\cdot}9\rho$, $R_1 = 0{\cdot}7\rho$. When the pair of initial estimates are such that the secant method does converge to a root ($R_n \to \overline{R}$), then it converges (for a C^2 function) with order $\gamma = (\sqrt{5}+1)/2 = 1{\cdot}6180340$. Since the convergence is of order $\gamma > 1$ (which is faster than linear), then $(R_n - \overline{R})/(R_n - R_{n-1}) \to 0$ as $n \to \infty$, and so the limit $\overline{R}$ can reliably be estimated.

Once $\overline{R}$ has been computed, then the arclength AB can be computed from (77), and the minimum passage time from A to B can be computed from (81).

5.2 Examples of Construction of Brachistochrone Through Two Points

Many examples of brachistochrones through two points have been computed by a program written in Lightspeed PASCAL, using extended variables which have roundoff corresponding to 18 or 19 significant decimal figures. Some examples are presented in Table 1. In each case, the secant method ended with two successive estimates of R differing by less than 10^{-16}.

Here, ρ and ψ give the polar coordinates of the endpoint B,
R_0 and R_1 are the initial estimates given for apse radius R,
Steps is the number of steps performed of the secant method,
$\overline{R}$ is the computed value for R,
s is the arclength of the brachistochrone AB, and
t is the time for the particle to reach the endpoint B.

Table 1

ρ	ψ	R_0	R_1	**Steps**	**Brachistochrone** AB
0·2	1·2	0·2	0·1	10	$\overline{R}$ = 0·184781963328079 s = 0·967544327890300 t = 1·127755808164997
0·5	0·1	0·4	0·2	7	$\overline{R}$ = 0·252787093630160 s = 0·505594850833648 t = 0·914164272619962
0·7	1·8	0·6	0·5	7	$\overline{R}$ = 0·168913316595188 s = 1·531621194881120 t = 1·481218910277573
0·9	0·4	0·8	0·7	7	$\overline{R}$ = 0·838545794202334 s = 0·442680075643429 t = 1·058599268341289
1·0	2·0	0·2	0·1	6	$\overline{R}$ = 0·057854464393370 s = 1·947547388961997 t = 2·221060707125763

Note that, for $r = r(\vartheta)$, R is a stationary value of r at the apse. Accordingly, if the endpoint B is very close to the apse on the brachistochrone AB, then any small error in computing the apse radius $R = \overline{R}$ inevitably produces a much larger error $\vartheta - \psi$ in matching the angle at B.

6 Brachistochrones For Repulsive Inverse Square Force

The brachistochrones for repulsive inverse square force display some interesting complexities [18] associated with the radius $r = 3/2$ — and also a remarkable simplification.

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