On the subspaces of analytic and antianalytic functions in weighted L_2 space on the boundary of a multiply connected domain, analog of Helson-Sarason Theorem and Fredholm properties of the corresponding Toeplitz operators

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Abstract

The problem of description of those positive weights on the boundary Γ of a finitely connected domain Ω for which the angle in a weighted L_2 space on Γ between the linear space $\mathcal{R}(\Omega)$ of all rational functions on $\bar{\mathbf{C}}$ with poles outside of $Clos\Omega$ and the linear space $\mathcal{R}(\Omega)_- = \{\bar{f}|f \in \mathcal{R}(\Omega)\}$ of antianalytic rational functions, is a natural analog of the problem solved in a famous Helson-Szegö theorem. In this paper we solve more general problem and give a complete description (in terms of necessary and sufficient conditions) of those positive weights w on Γ for which the sum of the closures in $L_2(\Gamma, w)$ of the subspaces $\mathcal{R}(\Omega)$ and $\mathcal{R}(\Omega)_-$ is closed and their intersection is finite dimensional. The given description is similar to that one in the Helson-Sarason Theorem, i.e. the "modified" weight should satisfy the Muckenhoupt condition.

1. Introduction

Let us start with recalling a number of well-known facts from the harmonic analysis in the unit disk.

1. The famous Helson-Szegö [1, 2] and Hunt-Muckenhoupt-Wheeden [3] theorems give the descriptions of those finite positive measures $d\mu$ on the unit circle $\mathbf{T}$, for which the angle in $L_2(d\mu, \mathbf{T})$ between $\bigvee_{n<0}\{z^n\}$ and $\bigvee_{n\geq 0}\{z^n\}$ is nonzero.

2. Going further, Helson and Sarason had shown [4] that the angle in $L_2(d\mu, \mathbf{T})$ between $\bigvee_{n<0}\{z^n\}$ and $\bigvee_{n\geq N}\{z^n\}$ is nonzero iff $d\mu = wd\varphi$ and $w = |P(z)|^2 w_0$, where w_0 is a Helson-Szegö weight and $P(z)$ is a polynomial of degree at most N with all zeros lying on the unit circle $\mathbf{T}$.

3. It is easy to see that Helson-Sarason Theorem admits the following simple generalization. Let as usual denote by A the disk algebra, i.e. the uniform closure of analytic trigonometric polynomials on $\mathbf{T}$ and $A_- = \{\bar{f}|f \in A, f(0) = 0\}$. Let B be a finite Blaschke product of degree N, then the angle in $L_2(d\mu)$ between A_- and BA is nonzero iff $d\mu = wd\varphi$ and $w = |P(z)|^2 w_0$, where w_0 is a Helson-Szegö weight and $P(z)$ is a polynomial of degree at most N with all zeros lying on the unit circle $\mathbf{T}$.

4. The result which clarifies the nature of (2) and (3) (see for example [2]) asserts that the closures in $L_2(wd\varphi)$ of A and A_- have a finite dimensional intersection and the sum of these closures is closed if and only if w is a weight which satisfies the conditions of Helson-Sarason Theorem with the degree N of the polynomial equal to the dimension of the above cited intersection.

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In what follows we will be interested in analog of the last assertion for the weighted L_2 space on a boundary of multiply connected domain, but before switching ourselves to the discussion of the problem and the difficulties and differences arising from nontrivial connectivity of the domain let us mention that the assertion from item (5) is equivalent to following statement on Fredholm properties of the corresponding Toeplitz operator. Namely, let h be an outer function from H^2_+, $|h|^2_{\mathbf{T}} = w$, then the Toeplitz operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ is Fredholm with its index equal to N if and only if w is a Helson-Sarason weight. Note that in this case obviously the index of $\mathbf{T}_{\frac{\bar{h}}{h}}$ is equal to the dimension of its kernel.

Let Ω be a ($g+1$) connected planar domain, $g > 1$, with nondegenerate boundary components. Let us fix the point $a \in \Omega$ and denote by $d\eta_a$ the harmonic measure of $\Gamma = \partial\Omega$ with respect to the point a, i.e. the measure which gives rise to the continuous linear functional on $C(\Gamma)$: $u \longrightarrow u(a)$, where $u \in C(\Gamma)$ and $u(a)$ is the value of the harmonic extension of u inside Ω. Let us also denote by $\mathcal{R}(\Omega)$ the linear space of all rational functions on $\bar{\mathbf{C}}$ with poles outside of $Clos\Omega$. The closure of $\mathcal{R}(\Omega)$ in $L_2(\Gamma, d\eta_a) = L_2(\Gamma) = L_2$ gives us the usual Hardy space $H^2_+(\Omega)$ (see for example [6]). Let us denote by $(\mathcal{R}(\Omega))^\perp$ the orthogonal compliment to $\mathcal{R}(\Omega)$ in $L_2(\Gamma, d\eta_a)$ and by $\mathcal{R}(\Omega)_-$ the linear space $\{\bar{f} | f \in \mathcal{R}(\Omega), f(a) = 0\}$. Obviously $\mathcal{R}(\Omega)_- \subset (\mathcal{R}(\Omega))^\perp$. The natural problem similar to that in a case of a unit disk is to describe those finite positive measures $d\mu$ on Γ for which the closures in $L_2(\Gamma, d\mu)$ of $\mathcal{R}(\Omega)$ and $(\mathcal{R}(\Omega))^\perp \cap L_2(\Gamma, d\mu)$ have a finite dimensional intersection and the sum of these closures is closed.

The first and essential difference of the case under consideration from that in the unit disk is that co-dimension of the linear space antianalytic rational functions $\mathcal{R}(\Omega)_-$ in $(\mathcal{R}(\Omega))^\perp$ is equal to g. Namely, as opposed to the case of the unit disk, where $L_2(\mathbf{T}) = H^2_+ \oplus H^2_-$, $H^2_- = \{\bar{f} \mid f \in H^2_+, f(0) = 0\}$, (and in general the sum of the spaces of analytic and antianalytic trigonometric polynomials is dense in $L_2(d\mu, \mathbf{T})$ for any $d\mu$), in our case there exists a g-dimensional space of measures which are orthogonal to $\mathcal{R}(\Omega) + \mathcal{R}(\Omega)_-$ and consequently

$$L_2(\Gamma, d\eta_a) = H^2_+ \oplus H^2_- \oplus \mathcal{M},$$

where $H^2_- = \{\bar{f} \mid f \in H^2_+, f(a) = 0\}$ and $\mathcal{M}$ is g-dimensional so-called defect space. This means that the sum of the spaces $H^2_+ \cap L_2(d\mu)$ and $H^2_- \cap L_2(d\mu)$ is not necessarily dense in $L_2(d\mu)$ for arbitrary finite positive measure $d\mu$ on Γ. Another important difference arises immediately if we want as above in the case of the unit disk to reformulate our problem in terms of the Fredholm properties of the corresponding Toeplitz operators. It will not be difficult to observe that the measures under consideration should be absolutely continuous with respect to harmonic measure $d\eta_a$, i.e. $d\mu = w d\eta_a, w \in L_1(\Gamma)$, and that the weight w should have a summable logarithm. This enable us to represent the weight w as a square of the modulus of boundary values on Γ of some outer function h. But in general this outer function h turns out to be multivalued function in Ω. To be precise, h is a co-called character automorphic outer function from the character-automorphic Hardy space H^2_{+,κ_h}, corresponding to (parametrized by) the character – vector $\kappa_h = \{\kappa_1, \ldots, \kappa_g\}$ from the g-dimensional real torus $\mathbb{R}^g/\mathbb{Z}^g$ (see for example [8, 9, 10]).

Note that for each $\kappa \in \mathbb{R}^g\mathbb{Z}^g$ the space $H^2_{+,\kappa}(\Gamma)$ is a subspace of $L_2(\Gamma)$ and consists of the functions f, such that

1. f is locally analytic in Ω
2. $|f|$ is single-valued in Ω, and $|f|^2$ has a harmonic majorant in Ω.
3. any analytic element of f acquires the unimodular factor $e^{2\pi i \kappa_j}$ after analytic continuation along any closed curve in Ω homologous to Γ_j, $j = 1, \ldots, g$
4. f has non-tangential limit function $\tilde{f}$ a.e. $d\eta_a$ on Γ which belongs to $L_2(\Gamma)$.

Let us denote by P^κ_+ the "Riesz" orthogonal projection in $L^2(\Gamma)$ onto $H^2_{+,\kappa}$. Now if we transfer our initial problem in a usual (for the unit disk theory) way to the problem on Toeplitz operators, then the new object for study will be the so-called character-automorphic Toeplitz operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ acting from H^2_{+,κ_h} into $H^2_{+,-\kappa_h}$ in a following way

$$\mathbf{T}_{\frac{\bar{h}}{h}} f \stackrel{def}{=} P_+^{-\kappa_h} \frac{\bar{h}}{h} f, \ f \in H^2_{+,\kappa_h}.$$

That is we are in the position to study character automorphic Toeplitz operator acting from one character-automorphic Hardy space into another. But we actually have a whole g-dimensional torus of similar decompositions: $L_2(\Gamma) = H^2_{+,\kappa} \oplus H^2_{-,\kappa} \oplus \mathcal{M}_\kappa$, where $H^2_{-,\kappa} = \{\bar{f} | f \in H^2_{+,-\kappa}, f(a) = 0\}$ and $\mathcal{M}_\kappa$ is a g-dimensional κ-automorphic defect space. Now for any $\kappa, \nu \in \mathbb{R}^g/\mathbb{Z}^g$ and $f \in L^\infty$ we can consider the whole scale of Toeplitz operators $\mathbf{T}_f^{\kappa,\nu} \stackrel{def}{=} P_+^{\kappa+\nu} f | H^2_{+,\kappa}$ (see [8]). Thus we can consider the Fredholm properties of the whole scale in κ of Toeplitz operators $\mathbf{T}_{\frac{\bar{h}}{h}}^{\kappa,-2\kappa_h}$ and respectively in place of a single initial problem on the closures in $L_2(d\mu)$ of the linear spaces of analytic (from $H^2_{+,\mathbf{0}} \cap L_2(d\mu)$) and antianalytic (from $\{H^2_{-,\mathbf{0}} + \mathcal{M}\} \cap L_2(d\mu)$) single-valued functions we can consider the whole scale of problems posed for each character κ. This more general problem is for any fixed character $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$ to get a description of finite positive measures of the form $d\mu = w d\eta_a, w \in L_1(\Gamma)$ on γ for which the closures in $L_2(d\mu)$ of $H^2_{+,\kappa} \cap L_2(d\mu)$ and $\{H^2_{-,\kappa} + \mathcal{M}_\kappa\} \cap L_2(d\mu)$ have a finite dimensional intersection and the sum of these closures is closed.

Note that if we require that the intersection of these closures is zero then we have the problem on the nonzero angle in $L_2(d\mu)$ between $H^2_{+,\kappa} \cap L_2(d\mu)$ and $\{H^2_{-,\kappa} + \mathcal{M}_\kappa\} \cap L_2(d\mu)$. In terms of character-automorphic Hankel operators $\mathbf{H}_f^{\kappa,\nu} \stackrel{def}{=} \{I - P_+^{\kappa+\nu}\} f | H^2_{+,\kappa}$ the last problem is about the description of those character-automorphic outer functions $h \in H^2_{+,\kappa_h}$ for which $|\mathbf{H}_{\frac{\bar{h}}{h}}^{\kappa,-2\kappa_h}| < 1$. The essential difference of character-automorphic Hankel operators from the case of the unit disk is (see [8, Section 6]) that for any particular character κ we have only the inequality $|\mathbf{H}_f^{\kappa,\nu}| \leq dist_{L^\infty}\{f, H^\infty_{+,\nu}\}$ and the equality is attained only for the supremum (maximum) of the norms in the left-hand side over all characters, i.e.

$$\max_{\kappa \in \mathbb{R}^g/\mathbb{Z}^g} |\mathbf{H}_f^{\kappa,\nu}| = dist_{L^\infty}\{f, H^\infty_{+,\nu}\}.$$

This problem on nonzero angle has been solved completely by the author in [7] and the result asserts that the angle is nonzero for all characters simultaneously and the weight should satisfy the analog of Muckenhoupt condition. Another problem on the nonzero angle in $L_2(d\mu)$ between $H^2_{+,\kappa} \cap L_2(d\mu)$ and $H^2_{-,\kappa} \cap L_2(d\mu)$ was completely solved in ([16]) and the situation there turned out to be a particular case of Helson-Sarason type problem (see item (3) at the beginning of this section and the last section of this paper) and the answer was similar to that one in Helson-Sarason theorem, i.e. the weight could have "double zeroes" at finite number of points on Γ and the improved weight should satisfy the Muckenhoupt condition. But apart of this the final answer essentially depends on a character.

In view of absence of an apparent analog of Widom-Devinatz theorem in our situation, we will use the approach from the above mentioned paper [16], though the proofs will be simplified and modified in appropriate way.

2. Preliminaries

2.1.

Fix a point $P_0 \in \Gamma_0$ and let $\gamma_1, \ldots, \gamma_g$ be oriented crosscuts from P_0 to the boundary components $\Gamma_1, \ldots, \Gamma_g$ respectively, which except for their end points lie in Ω_+ and intersect each other only in the point P_0. We will consider the vector space $\mathcal{H}_\kappa,\ \kappa \in \mathbb{R}^g/\mathbb{Z}^g$ of functions f, locally analytic on Ω_+, with single-valued modulus and such that the analytic continuation of any functional element of f along a closed curve homologous to $\sum_{j=1}^g m_j b_j$ leads to multiplication of the initial value by the unimodular factor (character)

$$e^{2\pi i \sum_{j=1}^g m_j \kappa_j}.$$

From now on we fix the term character for the elements $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$, this means that in place of the usual multiplicative representation of the fundamental group of Ω_+ in $\mathbf{T}^g$ we use the (equivalent) additive one.

The functions from the space $\mathcal{H}_\kappa$ are called modulus automorphic or character automorphic corresponding to the character κ, or simply κ-automorphic. These functions can be considered as single-valued analytic functions f in the simply connected domain $\Omega'_+ = \Omega_+ \setminus \bigcup_{j=1}^g \gamma_j$ with $|f|$ continuous in Ω_+ and such that the limits $f(P\pm) = \lim_{z\to P\pm} f(z)$ exist on $\bigcup_{j=1}^g \gamma_j \cap \Omega_+$and satisfy $f(P+) = e^{2\pi i \lambda_j} f(P-),\ P \in \gamma_j,\ j = 1,\dots,g$. Here the limits as $z \to P+,\ z \to P-$ on γ_j are respectively from the left and from the right side of γ_j.

The spaces $H^p_{+,\kappa},\ 1 \le p \le \infty$, mentioned in the previous section are the spaces of functions f from $\mathcal{H}_\kappa$ which have non-tangential limits on Γ a.e. $d\eta_a$ belonging to $L_p(\Gamma)$.

By analogy with the theory in the unit disk we also consider the space

$$H^2_{-,\kappa} = \left\{\bar{f} : f \in H^2_{+,-\kappa}, f(a) = 0\right\}.$$

The spaces $H^2_{+,\kappa}$ and $H^2_{-,\kappa}$ are orthogonal since $< f, \bar{g} >= \int_\Gamma f g d\eta_a = f(a)g(a)$ for $f \in H^2_{+,\kappa},\ g \in H^2_{+,-\kappa}$ and the following decomposition takes place

$$L^2(\Gamma) = H^2_{-,\kappa} \oplus \mathcal{M}_\kappa \oplus H^2_{+,\kappa},$$

where $\mathcal{M}_\kappa$ is g-dimensional so-called κ-automorphic defect space.

2.2.

It is important to mention now that by considering the double $\hat{\Omega}$ of the domain $\Omega = \Omega_+$, which is a compact Riemann surface of genus g, obtained by gluing the the second copy Ω_- of of Ω to Ω_+ along the boundary Γ, we can treat $H^2_{-,\kappa}$ as a character-automorphic Hardy space on the second "sheet" Ω_- of the double $\hat{\Omega}$. By $J(z)$ we denote the natural antiholomorphic involution on $\hat{\Omega}$, which interchange the same points of Ω_+ and Ω_-. For the details of construction of the double of planar domain we refer the reader to expositions in [8, 12] In what follows we use freely the notations, definitions and basic results related to the function theory on the doubles quoted in [8, Preliminaries]. Here we only mention a number of important facts which will be of frequent use in what follows.

By $\omega_1,\dots,\omega_g$ we denote the normalized basis of holomorphic differentials on $\hat{\Omega}$, $\vec{\omega} \overset{def}{=} \{\omega_1,\dots,\omega_g\}$.

We will write down the divisors on $\hat{\Omega}$ in the additive form, $\mathcal{A} = \sum_{j=1}^{k} n_j P_j,\ n_j \in \mathbb{Z},\ P_j \in \hat{\Omega}$, using the standard notations: $\mathcal{A}$ is positive ($\mathcal{A} \ge 0$) if $n_j \ge 0$ for all j, $ord(\mathcal{A}) \overset{def}{=} \sum_{j=1}^{k} n_j$. The action of the antiholomorphic involution J is obviously extended on the divisors.

In what follows we make use of Abel-Jacoby mapping which is defined by

$$\varphi(P) = \int_{P0}^{P} \vec{\omega},\ P \in \hat{\Omega},$$

where $\vec{\omega} = (\omega_1,\dots,\omega_g)^t$. If we denote by$\tau$ the matrix of b-periods of differentials $\omega_1,\dots,\omega_g$, then $\mathbb{Z}^g + \tau\mathbb{Z}^g$ is a period lattice of the holomorphic differentials in $\mathbb{C}^g$ and hence the mapping

$\varphi : \hat{\Omega} \longrightarrow \mathbb{C}^g$, being multi-valued, is a correctly defined mapping from $\hat{\Omega}$ to the complex torus $Jac(\hat{\Omega}) = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$ which is called the Jacobian variety of the Riemann surface $\hat{\Omega}$.

The action of Abel-Jacoby mapping can obviously be extended on $D_0(\hat{\Omega})$ - the divisors of order 0 on $\hat{\Omega}$ (for details see [9, 10, 12, 13] or Preliminaries in [8]).

Note also that the harmonic measure $d\eta_P, P \in \Omega_+$, is a restriction on Γ of $d\eta_{P,J(P)}$, where for any two points $P, Q \in \hat{\Omega}, P \neq Q$, $d\eta_{P,Q}$ is the normalized meromorphic differential of the third kind on $\hat{\Omega}$ with the two simple poles at the points P and Q with the residues $1/2\pi i$ and $-1/2\pi i$ respectively. The divisor of the zeros of $d\eta_{P,J(P)}$ has order 2g and is symmetric with respect to J and has the form $Z^* + J(Z^*)$ where $Z^* \subset \Omega_+,\ Z^* = z_1^* + \ldots + z_g^*$, and $z_1^*, \ldots, z_g^*$ are the critical points of the Green function $G(z, P)$. This is why the divisor $Z^* = Z^*(P)$ is often referred to as the critical Green's divisor.

More generally by character automorphic meromorphic function on $\hat{\Omega}$ we mean a function of the form

$$f_{\mathcal{A},\vec{c}} = e^{2\pi i \int\limits_{P0}^{P} \omega(\mathcal{A},\vec{c})}, \quad \omega(\mathcal{A}, \vec{c}) = \sum_{j=1}^{k} d\eta_{Q_j,P_j} + \sum_{l=1}^{g} c_l\omega_l, \tag{2.1}$$

where $\mathcal{A}$ is an arbitrary divisor of order 0, $\mathcal{A} = \sum_{j=1}^{k} Q_j - \sum_{j=1}^{k} P_j$, such that $\kappa = \varphi(\mathcal{A}) \in\in \mathbb{R}^g/\mathbb{Z}^g$ modulo $\mathbb{Z}^g + \tau\mathbb{Z}^g$, and $\vec{c} = (c_1, \ldots, c_g) \in \mathbb{Z}^g$. Clearly such a function corresponds exactly to the character κ (see [8, 13]). The analog of Abel's theorem (see for example [13]) asserts that a divisor $\mathcal{A}$ of order zero, is a divisor of a character-automorphic function if and only if $\varphi(\mathcal{A}) = \kappa \in \mathbb{R}^g/\mathbb{Z}^g$, and κ is exactly the character of this function. The divisor of a character-automorphic function f will be denoted by (f) (clearly $ord(f) = 0$). We denote by $L_\kappa(\hat{\Omega})$ the vector space of κ-automorphic meromorphic functions on $\hat{\Omega}$ and by $L_\kappa(\mathcal{A})$ the vector space $\left\{f \in L_\kappa(\hat{\Omega}) : (f) \geq \mathcal{A}\right\}$, $r^\kappa[\mathcal{A}] = dimL_\kappa(\mathcal{A})$. We can also consider the character-automorphic meromorphic differentials on $\hat{\Omega}$. Actually all such differentials are obtained by multiplication of meromorphic differentials by character-automorphic functions on $\hat{\Omega}$. The vector space of κ-automorphic meromorphic differentials will be denoted by $\mathcal{N}_\kappa$ and $\mathcal{N}_\kappa(\mathcal{A}) = \{\omega \in \mathcal{N}_\kappa : (\omega) \geq \mathcal{A}\}, i^\kappa[\mathcal{A}] = dim\mathcal{N}_\kappa(\mathcal{A})$, where by (ω) we again denote the divisor of ω. The version of Riemann-Roch theorem for character-automorphic functions and differentials asserts that for any divisor $\mathcal{A}$ on $\hat{\Omega}$

$$r^\kappa[\mathcal{A}] = -ord(\mathcal{A}) - g + 1 + i^{-\kappa}[-\mathcal{A}].$$

In these notations for character-automorphic defect space we have

$$\mathcal{M}_\kappa = L_\kappa(J(a) - Z^* - J(Z^*)),$$

and consequently $r^\kappa[J(a) - J(Z^*)] = 0$, since any nonzero function from $L_\kappa(J(a) - J(Z^*))$ would be in both $\mathcal{M}_\kappa$ and $H^2_{+,\kappa}$ simultaneously.

2.3.

The character-automorphic meromorphic functions on $\hat{\Omega}$ of the form (2.1) with $P_j = J(Q_j),\ Q_j \in \overline{\Omega_+}, c_j = 0,\ j = 1, \ldots, g$ for the point P and the path of integration from P_0 to P contained in $\overline{\Omega_+}$ give important examples of the character-automorphic functions. Thus, the function

$$b_Q \overset{def}{=} f_{Q-J(Q),\vec{\mathbf{0}}},\ Q \in \Omega_+$$

is a character-automorphic function with $\kappa = \kappa(Q) = \Re\varphi(Q-J(Q))$, contractive in Ω_+, unimodular on Γ, with only one simple zero at the point Q (and only one simple pole at $J(Q)$). We will call this function an elementary Blaschke factor. By character automorphic Blaschke product we will mean the product (finite or infinite) of elementary Blaschke factors (for conditions on the zero set of Blaschke factors which are necessary and sufficient for the uniform convergence of Blaschke product see for example [11]).

In what follows we will denote by B_Z the Blaschke product with the divisor of zeros Z and the corresponding character by κ_Z and by $K^\kappa_Z = K^\kappa_{B_Z}$ the corresponding co-invariant subspace of $H^2_{+,\kappa}$, $K^\kappa_Z = H^2_{+,\kappa} \ominus B_Z H^2_{+,\kappa-\kappa_Z}$. Note that

$$K^\kappa_Z = L_\kappa(J(a) - J(Z) - J(Z^*)).$$

In particular B_{Z^*} is the finite Blaschke product corresponding to the critical Green's divisor Z^* (with the zeros at the critical points of the Green function $G(z,a)$) with character $\kappa^* = \kappa_{Z^*} = \Re\varphi(Z^* - J(Z^*)) = 2\Re\varphi(Z^*)$.

For the critical Green's divisor $Z^* = z_1^* + \ldots + z_g^*$ we consider the g-dimensional space $K^\kappa_{Z^*} = H^2_{+,\kappa} \ominus B_{Z^*} H^2_{+,\kappa-\kappa^*}$, $\kappa^* = \kappa(Z^*) = \Re\varphi(Z^* - J(Z^*))$, which is spanned by the reproducing kernels at the points z_j^*, $j = 1, \ldots, g$. Then by [8, Sec.3]

$$\mathcal{M}_\kappa = \frac{1}{B_{Z^*}} K^{\kappa+\kappa^*}_{Z^*} = B_{Z^*} \bar{K}^{\kappa-\kappa^*}_{Z^*},$$

where as above $\mathcal{M}_\kappa$ is κ-automorphic defect space.

3. The weight might be bad

3.1.

Let $\mathcal{B}^\kappa_+$ be the space of all functions from $\mathcal{H}_\kappa(\Omega_+)$ which are continuous up to the boundary Γ. It is known (see for example [7, Sec.3] that $\mathcal{B}^\kappa_+$ is a dense set in $H^2_{+,\kappa}$. By $\mathcal{B}^\kappa_-$ we denote the corresponding dense subset of $H^2_{-,\kappa}$, $\mathcal{B}^\kappa_- = \{\overline{f \circ J} : f \in \mathcal{B}^{-\kappa}_+, f(a) = 0\}$. Note that the sum $\mathcal{B}^\kappa_+ + \mathcal{B}^\kappa_- + \mathcal{M}_\kappa$ is obviously dense in $L_2(d\eta_a)$ and moreover (see [7, Sec.3]) is dense in $L_2(d\mu)$ for any finite positive measure $d\mu$ as well.

The question we are interested in is the following:

What are necessary and sufficient conditions on a finite positive measure $d\mu$ on Γ, which is absolutely continuous with respect to harmonic measure $d\eta_a$, so that the dimension of the intersection of the subspaces $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_- + \mathcal{M}_\kappa$ and $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+$ is finite and the sum of these subspaces is closed.

Proposition 3.1. *Let $d\mu = w d\eta_a$, $w \in L_1(\Gamma)$ be a finite positive measure on Γ. If the intersection in $L_2(d\mu)$ of $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_- + \mathcal{M}_\kappa$ and $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+$ is finite and the sum of these subspaces is closed, then* $\log w \in L_1(\Gamma)$.

Proof. The statement about w follows immediately from the character-automorphic analog of Szegö's theorem ([7, Theorem 3.1, Corollary 3.1]), which in particular asserts that if $\log w \notin L_1(d\eta_a)$ then $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+ = L_2(d\mu)$. But this is not true in our case since the closures of $\mathcal{B}^\kappa_+$ and $\mathcal{B}^\kappa_-$ in $L_2(d\mu)$ have only finite dimensional intersection. •

From now on we denote by h a character-automorphic outer function in H^2_{+,κ_0}, with a certain character κ_0, such that $|h|^2 \mid_\Gamma = w$, $d\mu = |h|^2 \, d\eta_a$.

Remark 3.2. It is easy to see that in place of the spaces $\mathcal{B}^\kappa_+, \mathcal{B}^\kappa_- + \mathcal{M}_\kappa$ we can use the spaces $H^\infty_{+,\kappa}, H^\infty_{-,\kappa} + \mathcal{M}_\kappa \subset L_2(d\mu)$. Indeed, we have $\mathcal{B}^\kappa_+ \subset H^\infty_{+,\kappa}$, $\mathcal{B}^\kappa_- \subset H^\infty_{-,\kappa}$ and

$$\mathcal{D}_+^\kappa = Clos_{L_2(d\mu)} H^\infty_{+,\kappa} = Clos_{L_2(d\mu)} \mathcal{B}_+^\kappa = \frac{1}{h} H^2_{+,\kappa+\kappa_0},$$

$$\mathcal{D}_-^\kappa = Clos_{L_2(d\mu)} \{H^\infty_{-,\kappa} + \mathcal{M}_\kappa\} = Clos_{L_2(d\mu)} \{\mathcal{B}_-^\kappa + \mathcal{M}_\kappa\} = \frac{1}{\bar{h}} \{H^2_{-,\kappa-\kappa_0} + \mathcal{M}_\kappa\}.$$

To see that, for example, the first of these chains of equalities is valid one should simply note that due to the properties of outer functions, $h\mathcal{B}^\kappa$ (as well as $hH^\infty_{+,\kappa}$) is dense in $H^2_{+,\kappa+\kappa_0}$ and that the convergence of a sequence of functions $\{f_n\}_{n=1}^\infty$ to the function f in $L_2(d\mu)$ is equivalent to the convergence of a sequence $\{hf_n\}_{n=1}^\infty$ to the function hf in L_2.

Remark 3.3. Note that if the dimension of the intersection of $\mathcal{D}_-^\kappa$ and $\mathcal{D}_+^\kappa$ is greater then zero then $1/h \notin L_2(\Gamma)$ and respectively $1/w \notin L_1(\Gamma)$. Indeed, if $1/h$ is square summable on Γ then it is an outer function from $H^2_{+,-\kappa_0}$ and for the nonzero function $f \in \mathcal{D}_- \cap \mathcal{D}_+$ $f = f_-/\bar{h} = f_+/h$, where $f_- \in H^2_{-,\kappa-\kappa_0} + \mathcal{M}_{\kappa-\kappa_0}$ and $f_+ \in H^2_{+,\kappa+\kappa_0}$. But then $f \in \{H^1_{-,\kappa} + \mathcal{M}_\kappa\} \cap H^1_{+,\kappa}$, which is impossible since the last intersection is zero by character-automorphic analog of Riesz brothers theorem (see [8]).

3.2.

In what follows we will reduced the problem on the weight $w = |h|^2|_\Gamma$ to the problem on the Fredholm properties of the corresponding Toeplitz operators. To do this we will use an approach similar to that in [2, 5, 14].

As above P_+^ν, $\nu \in \mathbb{R}^g/\mathbb{Z}^g$, is a "Riesz" orthogonal projection from $L_2(\Gamma)$ onto $H^2_{+,\nu}$ and let $P_-^\nu = I - P_+^\nu$ be the orthogonal projection from $L_2(\Gamma)$ onto $H^2_{-,\nu} + \mathcal{M}_\nu$. Recall that (see for example [8]) for any $f \in L_\infty(\Gamma)$ and $\mu, \nu \in \mathbb{R}^g/\mathbb{Z}^g$ the Hankel operator $\mathbf{H}_f^{\nu,\mu}$ with symbol f is an operator acting from $H^2_{+,\nu}$ into $H^2_{-,\nu+\mu} + \mathcal{M}_{\nu+\mu}$ according to the formula

$$\mathbf{H}_f^{\nu,\mu} g = P_-^{\nu+\mu} fg, \quad \forall g \in H^2_{+,\nu}.$$

The character-automorphic Toeplitz operator $\mathbf{T}_f^{\nu,\mu}$ with symbol f is acting from $H^2_{+,\nu}$ into $H^2_{+,\nu+\mu}$,

$$\mathbf{T}_f^{\nu,\mu} g = fg - \mathbf{H}_f^{\nu,\mu} g = P_+^{\nu+\mu} fg, \quad \forall g \in H^2_{+,\nu}.$$

Here the character ν describes the Hardy space, on which the operators act, and character μ prescribes the character of the symbol $f \in L_\infty(\Gamma)$ [1]

If f is a unimodular function on Γ function then clearly for any $g \in H^2_{+,\nu}$

$$\|\mathbf{H}_f^{\nu,\mu} g\|_2^2 + \|\mathbf{T}_f^{\nu,\mu} g\|_2^2 = \|g\|_2^2, \tag{3.1}$$

and the adjoint operator $\left(\mathbf{T}_f^{\nu,\mu}\right)^*$ is the Toeplitz operator $\mathbf{T}_{\bar{f}}^{\nu+\mu,-\mu}$.

Just as in the case of the unit disk theory the estimation of the angles between subspaces in weighted L_2 on Γ is equivalent to the estimation of the norms of corresponding Hankel operators, for example if B is a character-automorphic inner function corresponding to character κ_B then

$$\cos(\mathcal{D}_-^\kappa, B\mathcal{D}_+^{\kappa-\kappa_B}) = \sup_{f \in \mathcal{B}_+^{\kappa-\kappa_B}, g \in \mathcal{B}_-^\kappa + \mathcal{M}_\kappa} \frac{|< Bfh, gh >|}{\|fh\|_2 \cdot \|g\bar{h}\|_2} =$$

$$\sup_{\substack{f \in H^2_{+,\kappa+\kappa_0-\kappa_B} \\ g \in H^2_{-,\kappa-\kappa_0} + \mathcal{M}_{\kappa-\kappa_B}}} \frac{| < Bf\frac{\bar{h}}{h}, g > |}{\|f\|_2 \cdot \|g\|_2} = \sup_{\substack{f \in H^2_{+,\kappa+\kappa_0-\kappa_B} \\ g \in \left(H^2_{+,\kappa-\kappa_0}\right)^\perp}} \frac{| < B\frac{\bar{h}}{h} f, g > |}{\|f\|_2 \cdot \|g\|_2} =$$

$$= \left| \mathbf{H}^{\kappa+\kappa_0-\kappa_B, -2\kappa_0+\kappa_B}_{B\frac{\bar{h}}{h}} \right|.$$

[1] Note that (see [8]) $L^\infty(\Gamma)$ can be identified with $L^\infty_\mu(\Gamma)$ for all characters μ.

In what follows, to simplify the formulae, we will omit the upper indices for Hankel and Toeplitz operators and the reader should keep in mind that the characters of the symbols, which would be the products or quotients of character-automorphic functions, could be restored from these expressions, and the characters of the Hardy spaces could be restored from the context. For example, as above, the character of the symbol $\bar{h}/h$ is the difference of the characters of $\bar{h}$ and h, i.e. is equal to $-2\kappa_0$ and the character of the symbol $B_{Z^*}\bar{h}/h$ in the same manner is equal to $\kappa^* - 2\kappa_0$.

Later on we will reformulate our problem in terms of Fredholm properties of the corresponding Toeplitz operators. Recall that the operator A on a Hilbert space H is called Fredholm iff the dimensions of $KerA$ and $KerA^*$ are finite and the image (range) of A is closed, i.e. $A(H) = ClosA(H)$. Index of Fredholm operator is defined as $indA = dimKerA - dimKerA^*$. If A and B are two Fredholm operators $A : H \longrightarrow H'$, $B : H' \longrightarrow H''$, then BA is Fredholm, $indBA = indA + indB$.

Now it is very easy to reformulate our initial problem in terms of Toeplitz operators.

Proposition 3.4. *Let $d\mu = |h|^2 d\eta_a$, where h is an outer function from H^2_{+,κ_0}. The following conditions are equivalent*

1. *The sum in $L_2(\Gamma, d\mu)$ of subspaces $\mathcal{D}^\kappa_+ = \frac{1}{h} H^2_{+,\kappa+\kappa_0}$ and $\mathcal{D}^\kappa_- = \frac{1}{h}\{H^2_{-,\kappa-\kappa_0} + \mathcal{M}_\kappa\}$ is closed and dimension of their intersection is equal to n.*
2. *The Toeplitz operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ acting on the space $H^2_{+,\kappa+\kappa_0}$ is Fredholm with the index equal to $dimKer\mathbf{T}_{\frac{\bar{h}}{h}} = n$.*

Proof. By remark 3.2 the conditions that $\mathcal{D}^\kappa_- + \mathcal{D}^\kappa_+$ is closed, $\mathcal{D}^\kappa_- \cap \mathcal{D}^\kappa_+$ is finite dimensional are equivalent to the conditions that $\{H^2_{-,\kappa-\kappa_0} + \mathcal{M}_{-,\kappa-\kappa_0}\} + \frac{\bar{h}}{h} H^2_{+,\kappa+\kappa_0}$ is closed in $L_2(\Gamma)$ and $\{H^2_{-,\kappa-\kappa_0} + \mathcal{M}_{-,\kappa-\kappa_0}\} \cap \frac{\bar{h}}{h} H^2_{+,\kappa+\kappa_0}$ is finite dimensional. Note that the last intersection is exactly equal to $Ker\mathbf{T}_{\frac{\bar{h}}{h}}$ – the kernel of the operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ acting on the space $H^2_{+,\kappa+\kappa_0}$. The kernel of the adjoint Toeplitz operator $\mathbf{T}_{\frac{h}{\bar{h}}}$ acting on the space $H^2_{-,\kappa-\kappa_0}$ is obviously zero, since otherwise there exist $f_- \in H^2_{-,\kappa+\kappa_0} + \mathcal{M}_{\kappa+\kappa_0}$, $f_+ \in H^2_{+,\kappa-\kappa_0}$, such that $f_- = \frac{h}{\bar{h}} f_+$ and hence $\bar{h} f_- = h f_+$, which is impossible by analog of Riesz brothers theorem, as we have $\bar{h} f_- \in H^1_{-,\kappa} + \mathcal{M}_\kappa$ and $h f_+ \in H^1_{+,\kappa}$.

Now since $H^2_{-,\kappa-\kappa_0} + \frac{\bar{h}}{h} H^2_{+,\kappa+\kappa_0}$ is closed and obviously equal to the whole space $L_2(\Gamma)$, we see that $P^{\kappa-\kappa_0}_+ \frac{\bar{h}}{h} H^2_{+,\kappa+\kappa_0} = \mathbf{T}_{\frac{\bar{h}}{h}}(H^2_{+,\kappa+\kappa_0})$ is closed and equal to $H^2_{+,\kappa-\kappa_0}$. This means that the operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ acting on the space $(H^2_{+,\kappa+\kappa_0})$ is Fredholm with the index equal to $dimKer\mathbf{T}_{\frac{\bar{h}}{h}} - dimKer\mathbf{T}_{\frac{h}{\bar{h}}} = dimKer\mathbf{T}_{\frac{\bar{h}}{h}}$. The converse statement is also true, since if the operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ is Fredholm then the closedness of $\mathbf{T}_{\frac{\bar{h}}{h}}(H^2_{+,\kappa+\kappa_0})$ implies that $H^2_{-,\kappa-\kappa_0} + \frac{\bar{h}}{h} H^2_{+,\kappa+\kappa_0}$ is closed, which, as we already have checked, is equivalent to the closedness of $\mathcal{D}^\kappa_- + \mathcal{D}^\kappa_+$ in $L_2(\Gamma, d\mu)$. •

If for $A \in L_\infty(\Gamma)$ the Toeplitz operator T_A acting from H^2_κ to $H^2_{\kappa+\nu}$ is Fredholm for one certain value of character κ, at the moment it is not clear if it is also Fredholm for other values of κ. The following rather simple theorem clarifies this question and plays an important role in what follows.

Theorem 3.5. *Let the Toeplitz operator T_A acting from $H^2_{+,\kappa}$ to $H^2_{+,\kappa+\nu}$ is Fredholm for one certain value of character κ. Then it is Fredholm for all other values $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$ and the value of $indT_A$ does not depend on κ.*

Proof. Note first that for any finite Blaschke product B with the divisor of zeros Z_B, corresponding to character κ_B the Toeplitz operators with symbols B and $\bar{B}$ are Fredholm for all characters κ and $KerT_{\bar{B}}$ on $H^2_{+,\kappa}$ is equal to $K^{\kappa}_B = H^2_{+,\kappa} \ominus BH^2_{+,\kappa-\kappa_B}$, $T_{\bar{B}}(H^2_{+,\kappa}) = H^2_{+,\kappa-\kappa_B}$ and $KerT_B = 0, T_B(H^2_{+,\kappa}) = BH^2_{+,\kappa}$. So $indT_{\bar{B}} = -indT_B = ordZ_B$ for all values of κ .

Now let κ be that value of character for which the operator $T_A = T^{\kappa,\nu}_A$ is Fredholm. Let us consider the operators T_B acting from the space $H^2_{+,\kappa-\kappa_B}$ to $H^2_{+,\kappa}$, and $T_{\bar{B}}$ acting from the space $H^2_{+,\kappa+\nu}$ to $H^2_{+,\kappa+\nu-\kappa_B}$. Then by the above mentioned properties of Fredholm operators the operator

$$T_{\bar{B}}T_aT_B = P^{\kappa+\nu-\kappa_B}_{+}\bar{B}P^{\kappa+\nu}_{+}AP^{\kappa}_{+}BP^{\kappa-\kappa_B}_{+} = P^{\kappa+\nu-\kappa_B}_{+}AP^{\kappa-\kappa_B}_{+} = T^{\kappa-\kappa_B,\nu}_a$$

is Fredholm with the index equal to the index of T_A on $H^2_{+,\kappa}$. So T_A is also Fredholm on $H^2_{+,\kappa-\kappa_B}$ with the same value of index, where κ_B is a character of a finite Blaschke product B. But since for any character in $\mathbb{R}^g/\mathbb{Z}^g$ there exists a finite Blaschke product with the number of zeros less or equal to $g(g+1)$ with exactly this character (see for example [11, Lemma 13]), we actually proved that T_A is Fredholm for all value of κ with the same value of index.

•

3.3.

Lemma 3.6. *If the Toeplitz operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ on the space $H^2_{\kappa+\kappa_0}$ is Fredholm with the index equal to n, then there exists a finite single-valued Blaschke product B with simple zeros such that $KerT_{B\frac{\bar{h}}{h}} = 0$ that is the angle in $L_2(\Gamma, d\mu)$ between $\mathcal{D}^{\kappa}_{-}$ and $B \cdot \mathcal{D}^{\kappa}_{+}$ is nonzero.*

Proof. Note that since for any finite character-automorphic Blaschke product B the Toeplitz operator on $H^2_{+,\kappa}$ with the symbol B is obviously Fredholm for any character κ, the zero kernel of $T_{B\frac{\bar{h}}{h}}$ means that this operator is an isomorphism on its image, which by (3.1) is equivalent to the inequality $|\mathbf{H}_{B\frac{\bar{h}}{h}}| = \cos(\mathcal{D}^{\kappa}_{-}, B\mathcal{D}^{\kappa}_{+}) < 1$. Thus the nonzero angle condition is definitely equivalent to the zero kernel condition.

Now let θ be any single valued finite Blaschke product (see for example [11]) with m zeros $(m \geq g+1)$ and construct the function

$$B = \prod_{j=1}^{n} b_{\alpha_j}(\theta), \quad \alpha_k \neq \alpha_j, \quad k \neq j, \quad |\alpha_j| < 1 \ , 1 \leq j \leq n,$$

where $b_{\alpha_j}(\theta) = \frac{\theta-\alpha_j}{1-\bar{\alpha}_j\theta}$, B is obviously a finite Blaschke product with the set of zeros equal to $\bigcup_{j=1}^{n}\{\theta^{-1}(\alpha_j)\}$. Such a function may have multiple zeros if and only if for certain $1 \leq j \leq n$ the set $\theta^{-1}(\alpha_j)$, which is a θ-preimage of α_j, contains strictly less then m distinct points, or equivalently when α_j is equal to the value of θ at one of its critical points (zeros of θ'). Note that θ is m-valent in Ω_+ and all its critical points are the zeros of the meromorphic on $\hat{\Omega}$ differential $d\theta/\theta$, that is there is only finite number of them. Therefore by choosing α_j properly, we get the Blaschke product with simple zeros. This function B suits our purpose. Indeed, suppose that $f \neq 0, f \in H^2_{+,\kappa+\kappa_0}$ and $f \in KerT_{B\frac{\bar{h}}{h}}$. Then by the definition of the Toeplitz operator $P^{\kappa-\kappa_0}_{+}\frac{\bar{h}}{h}\prod_{j=1}^{n} b_{\alpha_j}(\theta)f = 0$, but then $P^{\kappa-\kappa_0}_{+}\frac{\bar{h}}{h}f = 0$ and $P^{\kappa-\kappa_0}_{+}\frac{\bar{h}}{h}\prod_{j=1}^{k} b_{\alpha_j}(\theta)f = 0$, for $k = 1, 2, \ldots n$. This means that all the functions $f, b_{\alpha_1}(\theta)f, \ldots, \prod_{j=1}^{n} b_{\alpha_j}(\theta)f \in KerT_{\frac{\bar{h}}{h}}$ in $H^2_{+,\kappa+\kappa_0}$ and since the system $1, b_{\alpha_1}(\theta), \prod_{j=1}^{2} b_{\alpha_j}(\theta), \ldots, \prod_{j=1}^{n} b_{\alpha_j}(\theta)$ is linearly independent, it follows that we managed to find $n+1$ nonzero linearly independent elements in n dimensional space. This contradiction proves the lemma. •

The next proposition is similar to that one in [16] but describes a little more general situation and will be of use throughout this paper. The idea of the proof, which is similar to that one in [16], may be also used to get alternative proof of Helson-Sarason theorem in the unit disk theory.

Proposition 3.7. *Let $d\mu = wd\eta_a = |h|^2 d\eta_a$ for character-automorphic outer function $h \in H^2_{+,\kappa_0}$ and let B be a κ_B-automorphic finite Blaschke product of degree m with the divisor of zeros Z_B, $ordZ_B = m$. If the angle in $L_2(\Gamma, d\mu)$ between $\mathcal{D}^\kappa_-$ and $B\mathcal{D}^{\kappa-\kappa_B}$ is nonzero, but $\mathcal{D}^\kappa_- \bigcap \mathcal{D}^\kappa \neq \{0\}$, then*

1. *there exists a positive divisor $U_h = u_1 + \ldots + u_k$ of order $k \leq m$ contained totally in Γ, such that $1/h$ is locally square summable (from L_2^{loc}) at any point from $\Gamma \setminus U_h$, is not square summable at any of the points u_j, $1 \leq j \leq k$, but at each of these points the function $(z-u_j)^{\nu_j}/h$ is locally square summable, where $\nu_j = ordU_h|_{u_j}$, the order of divisor U_h at the point u_j (note that some of the points u_j may coincide). Moreover U_h is a minimal divisor possessing these properties in the sense that if U is any other divisor with the same properties, then $U_h \leq U$.*

2. $$r^\kappa[(Z_B) + J(a) - U_h - J(Z^*)] = 0.$$

Proof. First of all let us clarify the assumption on nonzero intersection of $\mathcal{D}^\kappa_-$ and $\mathcal{D}^\kappa_+$. It is natural to assume that the angle between these two subspaces is zero, since otherwise the answer is known (see [7]). But since we know that the angle between $\mathcal{D}^\kappa_-$ and $B\mathcal{D}^{\kappa-\kappa_B}_+$ is nonzero, i.e. $|\mathbf{H}_{B\frac{\bar h}{h}}| < 1$ on $H^2_{\kappa-\kappa_B+\kappa_0}$, we see from (3.1) that the operator $T_{B\frac{\bar h}{h}}$ on $H^2_{\kappa-\kappa_B+\kappa_0}$ is an isomorphism on its image, i.e. has a zero kernel and closed range. Moreover $dimKerT_{\bar B\frac{h}{\bar h}} \leq dimKerT_{\bar B} = ordZ_B$ on $H^2_{+,\kappa-\kappa_0}$ so that $T_{B\frac{\bar h}{h}}$ is Fredholm with the index $indT_{B\frac{\bar h}{h}} = -dimKerT_{\bar B\frac{h}{\bar h}} \geq -ordZ_B = -m$. Therefore since B is a finite Blaschke product $T_{\frac{\bar h}{h}}$ is also Fredholm and $dimKerT_{\frac{\bar h}{h}} = indT_{\frac{\bar h}{h}} \leq m$. But now if $KerT_{\frac{\bar h}{h}} = 0$ or equivalently $\mathcal{D}^\kappa_- \bigcap \mathcal{D}^\kappa_+ = 0$ then $T_{\frac{\bar h}{h}}$ is invertible on $H^2_{+,\kappa-\kappa_0}$ and consequently the angle between $\mathcal{D}^\kappa_-$ and $\mathcal{D}^\kappa_+$ is nonzero.

The assumption of nonzero angle between $\mathcal{D}^\kappa_-$ and $B\mathcal{D}^{\kappa-\kappa_B}_+$ is equivalent to the assumption that the Riesz projection P^κ_+ can be extended from $\mathcal{B}^\kappa_- + \mathcal{M}_\kappa + B\mathcal{B}^\kappa_-$ to its closure in $L_2(\Gamma, d\mu)$ as a bounded operator, i.e. as a skew projection on $B\mathcal{D}^{\kappa-\kappa_B}_+$ parallel to $\mathcal{D}^\kappa_-$.

Note that since $\log w \in L_1$, we know (see [7, Sec.3]) that for any character κ and for any fixed $\lambda \in \Omega_+$ the distance in $L_2(wd\eta_a)$ between the reproducing kernel k^κ_λ in $H^2_{+,\kappa}$ at the point λ and $b_\lambda \mathcal{B}^{\kappa-\kappa(\lambda)}_+$ is nonzero and, hence the orthogonal projection $P^+_{k^\kappa_\lambda}$ in $H^2_{+,\kappa}$ onto the one-dimensional space spanned by k^κ_λ can be extended from $\mathcal{B}^\kappa_+$ as a bounded operator to $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+$:

$$|P^+_{k^\kappa_\lambda}|_{L_2(d\mu)} = \frac{\|k^\kappa_\lambda\|_{L_2(d\mu)}}{dist_{L_2(d\mu)}(k^\kappa_\lambda, b_\lambda \mathcal{B}^{\kappa-\kappa(\lambda)}_+)} < \infty.$$

Since in $L_2(\Gamma)$ the orthogonal projection on one-dimensional subspace spanned by the element $Bk^{\kappa-\kappa_B}_\lambda$ can be expressed in the way

$$P_{Bk^{\kappa-\kappa_B}_\lambda} = BP_{k^{\kappa-\kappa_B}_\lambda}\bar B,$$

we see that the projection $P^+_{Bk^{\kappa-\kappa_B}_\lambda} = P_{Bk^{\kappa-\kappa_B}_\lambda}|BH^2_{+,\kappa-\kappa_B}$ also can be extended from $B\mathcal{B}^{\kappa-\kappa_B}_+$ as a bounded operator to $Clos_{L_2(d\mu)}B\mathcal{B}^{\kappa-\kappa_B}_+$. Now we see that $P_{Bk^{\kappa-\kappa_B}_\lambda}$ as well can be extended from $\mathcal{B}^\kappa_- + \mathcal{M}_\kappa + B\mathcal{B}^{\kappa-\kappa_B}_+$ as a bounded operator to $Clos_{L_2(d\mu)}\{\mathcal{B}^\kappa_- + \mathcal{M}_\kappa + B\mathcal{B}^{\kappa-\kappa_B}_+\}$. Indeed, this fact immediately follows from the expression $P_{Bk^{\kappa-\kappa_B}_\lambda} = P^+_{Bk^{\kappa-\kappa_B}_\lambda} P^\kappa_{BH^2_{\kappa-\kappa_B}}$. On $Clos_{L_2(d\mu)}\{\mathcal{B}^\kappa_- + \mathcal{M}_\kappa + B\mathcal{B}^{\kappa-\kappa_B}_+\}$ this operator is a skew projection on one dimensional space spanned by $\{Bk^{\kappa-\kappa_B}_\lambda\}$ parallel to $Clos_{L_2(d\mu)}\{\mathcal{B}^\kappa_- + \mathcal{M}_\kappa + b_\lambda B\mathcal{B}^{\kappa-\kappa_B-\kappa_\lambda}_+\}$. Therefore

$$|P_{\{Bk^{\kappa-\kappa_B}_\lambda\}}|Clos_{L_2(d\mu)}\{\mathcal{B}^\kappa_- + \mathcal{M}_\kappa + B\mathcal{B}^{\kappa-\kappa_B}_+\}|_{L_2(d\mu)} = \frac{\|k^{\kappa-\kappa_B}_\lambda\|_{L_2(d\mu)}}{dist_{L_2(d\mu)}(Bk^{\kappa-\kappa_B}_\lambda, \mathcal{B}^\kappa_- + \mathcal{M}_\kappa + b_\lambda B\mathcal{B}^{\kappa-\kappa_B-\kappa(\lambda)}_+)} < \infty. \tag{3.2}$$

Let us put for any positive ε, $w_\varepsilon = w + \varepsilon$ and $d\mu_\varepsilon = w_\varepsilon d\eta_a$. Then for any $f \in \mathcal{B}_-^\kappa + \mathcal{M}_\kappa + \mathcal{B}_+^\kappa$

$$\lim_{\varepsilon \to 0^+} \int_\Gamma |f|^2 d\mu_\varepsilon = \int_\Gamma |f|^2 d\mu$$

by Lebesgue's theorem on monotone convergence Now using the measures $d\mu_\varepsilon$ and taking into account that in $L_2(\Gamma)$

$$\left(\sqrt{w_\varepsilon}\{\mathcal{B}_-^\kappa + \mathcal{M}_\kappa + b_\lambda B \mathcal{B}_+^{\kappa-\kappa_B-\kappa_\lambda}\}\right)^\perp = \frac{1}{\sqrt{w_\varepsilon}}(\{\mathbb{C}\, Bk_\lambda^{\kappa-\kappa_B}\} + K_B^\kappa),$$

where $K_B^\kappa = H_{+,\kappa}^2 \ominus BH_{+,\kappa-\kappa_B}^2$, we get

$$dist^2_{L_2(d\mu_\varepsilon)}(Bk_\lambda^{\kappa-\kappa_B}, \mathcal{B}_-^\kappa + \mathcal{M}_\kappa + b_\lambda B\mathcal{B}_+^{\kappa-\kappa_B-\kappa(\lambda)}) =$$
$$dist^2_{L_2}(\sqrt{w_\varepsilon}Bk_\lambda^{\kappa-\kappa_B}, \sqrt{w_\varepsilon}(\mathcal{B}_-^\kappa + \mathcal{M}_\kappa + b_\lambda B\mathcal{B}_+^{\kappa-\kappa_B-\kappa(\lambda)})) =$$
$$\sup_{\substack{\varphi \in \left(\sqrt{w_\varepsilon}\{\mathcal{B}_-^\kappa + \mathcal{M}_\kappa + b_\lambda B\mathcal{B}_+^{\kappa-\kappa_B-\kappa_\lambda}\}\right)^\perp \\ \|\varphi\|_2 = 1}} |< \sqrt{w_\varepsilon}Bk_\lambda^{\kappa-\kappa_B}, \varphi >|^2 =$$
$$\sup_{\varphi \in \{\mathbb{C}\, Bk_\lambda^{\kappa-\kappa_B}\} + K_B^\kappa} \frac{|< Bk_\lambda^{\kappa-\kappa_B}, \varphi >|^2}{\int_\Gamma |\varphi|^2 \frac{1}{w_\varepsilon} d\eta_a} = (k_\lambda^{\kappa-\kappa_B}(\lambda))^2 \left(\inf_{\varphi \in K_B^\kappa} \int_\Gamma |Bk_\lambda^{\kappa-\kappa_B} + \varphi|^2 \tfrac{1}{w_\varepsilon} d\eta_a\right)^{-1},$$

which, by compactness arguments in the finite dimensional space K_B^κ, means that there exists $\varphi_\varepsilon \in K_B^\kappa$, such that

$$dist^2_{L_2(d\mu_\varepsilon)}(Bk_\lambda^{\kappa-\kappa_B}, \mathcal{B}_-^\kappa + \mathcal{M}_\kappa + b_\lambda B\mathcal{B}_+^{\kappa-\kappa_B-\kappa(\lambda)}) =$$
$$(k_\lambda^{\kappa-\kappa_B}(\lambda))^2 \left(\int_\Gamma |Bk_\lambda^{\kappa-\kappa_B} + \varphi_\varepsilon|^2 \frac{1}{w_\varepsilon} d\eta_a\right)^{-1},$$

Since

$$\lim_{\varepsilon \to 0^+} dist_{L_2(d\mu_\varepsilon)}(Bk_\lambda^{\kappa-\kappa_B}, \mathcal{B}_-^\kappa + \mathcal{M}_\kappa + b_\lambda B\mathcal{B}_+^{\kappa-\kappa_B-\kappa(\lambda)}) =$$
$$dist^2_{L_2(d\mu)}(Bk_\lambda^{\kappa-\kappa_B}, \mathcal{B}_-^\kappa + \mathcal{M}_\kappa + b_\lambda B\mathcal{B}_+^{\kappa-\kappa_B-\kappa(\lambda)}),$$

taking into account (3.2) in the same way as it has been done in [16] we get that there exists $\varphi_\lambda \in K_B^\kappa$,such that

$$\int_\Gamma |Bk_\lambda^{\kappa-\kappa_B} + \varphi_\lambda|^2 \frac{1}{w} d\eta_a = \lim_{\varepsilon \to 0^+} \int_\Gamma |Bk_\lambda^{\kappa-\kappa_B} + \varphi_\varepsilon|^2 \frac{1}{w_\varepsilon} d\eta_a < \infty. \tag{3.3}$$

Now let us look at the expression (3.3). On the one hand we know that $1/w \notin L_1(\Gamma)$ but on the another hand (3.3) tells us that for any $\lambda \in \Omega_+$ it is possible to find $\varphi_\lambda \in K_B^\kappa$, such that $|Bk_\lambda^{\kappa-\kappa_B} + \varphi_\lambda|^2/w \in L_1(\Gamma)$. Since $Bk_\lambda^{\kappa-\kappa_B} + \varphi_\lambda$ is locally analytic on Γ, it means that $1/w$ may be not locally summable only at the zeros of $Bk_\lambda^{\kappa-\kappa_B} + \varphi_\lambda$ on Γ. Since $Bk_\lambda^{\kappa-\kappa_B} + \varphi_\lambda \in L_\kappa(J(a) - J(\lambda) - J(Z_B) - J(Z^*))$ (see for example [8, Proposition 3.2]) it may have at most $m + g$ zeros on Γ. Thus the number of points on Γ at which $1/w$ is not locally summable is also finite. So $1/h$ is not locally square summable on Γ only at a finite number of points.

Let us denote by $U = U_h$ the positive divisor of these points on Γ. Note that due to (3.3) the order of U_h at a zero of $Bk_\lambda^{\kappa-\kappa_B} + \varphi_\lambda$ is less or equal to the order of this zero. Therefore if $\nu = ordU_h|_u$, $u \in \Gamma$ then $(z-u)^\nu/h$ is locally square summable at u.

To finish the proof of Proposition 3.7 we must show that $ordU_h \leq m = ordZ_B$ and prove the statement (2) of the proposition. Note that since (3.3) is valid for all $\lambda \in \Omega_+$, $Bk_\lambda^{\kappa-\kappa_B} + \varphi_\lambda \in$

$L^\kappa(U_h+J(a)-J(\lambda)-J(Z_B)-J(Z^*))$ for all $\lambda\in\Omega_+$. Hence $r^\kappa[U_h+J(a)-J(\lambda)-J(Z_B)-J(Z^*)]\geq 1$ for all $\lambda\in\Omega_+$. The application of the Riemann-Roch theorem gives us

$$1\leq r^\kappa[U_h+J(a)-J(\lambda)-J(Z_B)-J(Z^*)]=$$
$$ordZ_B+1-ord(U_h)+i^{-\kappa}[J(\lambda)+J(Z_B)+J(Z^*)-U_h-J(a)],$$

$$r^\kappa[U_h+J(a)-J(Z_B)-J(Z^*)]= \tag{3.4}$$
$$ordZ_B-ord(U_h)+i^{-\kappa}[J(Z_B)+J(Z^*)-U_h-J(a)].$$

Clearly $r^\kappa[U_h+J(a)-J(\lambda)-J(Z_B)-J(Z^*)]\geq r^\kappa[U_h+J(a)-J(Z_B)-J(Z^*)]$ and $i^{-\kappa}[J(Z_B)+J(Z^*)-U_h-J(a)]\geq i^{-\kappa}[J(\lambda)+J(Z_B)+J(Z^*)-U_h-J(a)]$. We will show that

$$i^{-\kappa}[J(Z_B)+J(Z^*)-U_h-J(a)]=0. \tag{3.5}$$

This will imply that $i^{-\kappa}[J(\lambda)+J(Z_B)+J(Z^*)-U_h-J(a)]=0$ and, by (3.4), that $ordU_h\leq m$. Suppose for the moment that $i^{-\kappa}[J(Z_B)+J(Z^*)-U_h-J(a)]>0$, then the equality $i^{-\kappa}[J(Z_B)+J(Z^*)-U_h-J(a)]=i^{-\kappa}[J(\lambda)+J(Z_B)+J(Z^*)-U_h-J(a)]$ can be valid only for a finite number of points $\lambda\in\Omega_+$. Indeed, this equality means that all differentials from $\mathcal{N}_\kappa(J(Z_B)+J(Z^*)-U_h-J(a))$ have a zero at the point λ, which is not true if λ does not belong to the set of zeros of one certain differential from this space. Now for such λ $i^{-\kappa}[J(Z_B)+J(Z^*)-U_h-J(a)]=i^{-\kappa}[J(\lambda)+J(Z_B)+J(Z^*)-U_h-J(a)]+1$ and, by (3.4)

$$r^\kappa[U_h+J(a)-J(Z_B)-J(Z^*)]=$$
$$ordZ_B+1-ord(U_h)+i^{-\kappa}[J(\lambda)+J(Z_B)+J(Z^*)-U_h-J(a)]=$$
$$r^\kappa[U_h+J(a)-J(\lambda)-J(Z_B)-J(Z^*)].$$

But due to (3.3) $Bk_\lambda^{\kappa-\kappa_B}+\varphi_\lambda\in L_\kappa(U_h+J(a)-J(\lambda)-J(Z_B)-J(Z^*))$ and $Bk_\lambda^{\kappa-\kappa_B}+\varphi_\lambda\notin L_\kappa(U_h+J(a)-J(Z_B)-J(Z^*))$. This contradiction proves (3.5) and to finish the proof of the Proposition one should simply note that statement (2) of the proposition follows from (3.5) by dividing the corresponding space of $(-\kappa)$- automorphic differentials by single-valued (0-automorphic) differential $d\eta_a$ with the divisor $Z^*+J(Z^*)-a-J(a)$ and taking the complex conjugation over the resulting space of $(-\kappa)$-automorphic functions. •

Corollary 3.8. *For any $\lambda\in\Omega_+$ and function $Bk_\lambda^{\kappa-\kappa_B}+\varphi_\lambda$ from (3.3)*

$$\frac{Bk_\lambda^{\kappa-\kappa_B}+\varphi_\lambda}{h}\in H^2_{+,\kappa-\kappa_0}.$$

This fact immediately follows from the observations that this function is square summable on Γ and analytic on Ω_+ and that h is outer and $Bk_\lambda^{\kappa-\kappa_B}+\varphi_\lambda$ is locally analytic in the closure of Ω_+.

Corollary 3.9.

1.

$$r^\kappa[U_h+J(a)-J(Z_B)-J(Z^*)]=n-ordU_h,$$

and for any function $\varphi_B\in L_\kappa(U_h+J(a)-J(Z_B)-J(Z^*))$

$$\frac{\varphi_B}{h}\in H^2_{+,\kappa-\kappa_0},\ \ \frac{\varphi_B}{\bar h}\in B(\mathcal{M}_{\kappa+\kappa_0}+H^2_{-,\kappa+\kappa_0}).$$

2. Any function $\varphi_B\in K_B^\kappa$ *for which*

$$\frac{\varphi_B}{h}\in H^2_{+,\kappa-\kappa_0},$$

belongs to $L_\kappa(U_h+J(a)-J(Z_B)-J(Z^*))$

The proof immediately follows from (3.4), (3.5) and the same observation as for Corollary 3.8.

Remark 3.10. It follows from the proof of Proposition 3.7, that for any character $\nu \in \mathbb{R}^g/\mathbb{Z}^g$ and any $\lambda \in \Omega_+$

$$r^{\nu}[U_h + J(a) - J(\lambda) - J(Z_B) - J(Z^*)] \geq 1,$$

and for any $\psi \in L_\nu(U_h + J(a) - J(\lambda) - J(Z_B) - J(Z^*))$ the function ψ/h is square summable on Γ.

The converse statement is also true: If $\psi/h \in L^2(\Gamma)$, $\psi \in Bk_\lambda^{\nu-\kappa_B} + K_B^\nu$, then $\psi \in L_\nu(U_h + J(a) - J(\lambda) - J(Z_B) - J(Z^*))$.

4. The weight could be improved. Necessary and sufficient conditions

4.1.

Let us now apply the Proposition 3.7 to our problem. By proposition 3.4 the Toeplitz operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ is Fredholm with the index equal to $dimKer\mathbf{T}_{\frac{\bar{h}}{h}} = n$ and by lemma 3.6 there exists a finite single-valued Blaschke product B, such that the angle in $L_2(d\mu)$ between $\mathcal{D}_-^\kappa$ and $B\mathcal{D}_+^\kappa$ is nonzero. Therefore the Proposition 3.7 is applicable to this situation and implies that for the divisor U_h of "bad behavior" of h on Γ $ordU_h \leq ordZ_B$, where as above Z_B is a divisor of zeros of B, which by construction of B is greater or equal then n. Now we are going to show that $ordU_h = n$.

Proposition 4.1.

$$\mathcal{M} = \mathbf{T}_{B\frac{\bar{h}}{h}}\left(H^2_{+,\kappa+\kappa_0}\right) \bigcap K_B^{\kappa-\kappa_0} \neq \{0\}, \quad dim\mathcal{M} = ordU_h = n$$

$$dimKer\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}} = r^\kappa[U_h + J(a) - J(Z_B) - J(Z^*)] = ordZ_B - ordU_h,$$

$$Ker\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}} = \frac{1}{h} L_\kappa(U_h + J(a) - J(Z_B) - J(Z^*)).$$

Proof. As we already mentioned in the proof of Lemma 3.6 the Fredholm property for the Toeplitz operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ on $H^2_{+,\kappa-\kappa_0}$ implies that the operator $\mathbf{T}_{B\frac{\bar{h}}{h}}$ is an isomorphism of $H^2_{+,\kappa+\kappa_0}$ onto its image $\mathbf{T}_{B\frac{\bar{h}}{h}}\left(H^2_{+,\kappa+\kappa_0}\right)$. Since $\mathbf{T}_{\frac{\bar{h}}{h}} = \mathbf{T}_{\bar{B}}\mathbf{T}_{B\frac{\bar{h}}{h}}$ and $Ker\mathbf{T}_{\bar{B}}$ on $H^2_{+,\kappa-\kappa_0}$ is equal to $K_B^{\kappa-\kappa_0}$, we see that $Ker\mathbf{T}_{\frac{\bar{h}}{h}} = \{f : \mathbf{T}_{B\frac{\bar{h}}{h}} f \in K_B^{\kappa-\kappa_0}\}$. Therefore

$$\mathcal{M} = \mathbf{T}_{B\frac{\bar{h}}{h}}\left(H^2_{+,\kappa+\kappa_0}\right) \bigcap K_B^{\kappa-\kappa_0} = \mathbf{T}_{B\frac{\bar{h}}{h}}\left(Ker\mathbf{T}_{\frac{\bar{h}}{h}}\right) \neq \{0\}$$

and $dim\mathcal{M} = n$. To show that $ordU_n = n$ let us calculate the dimension of space $\mathcal{M}$ in another way.

To do this note that

$$\mathbf{T}_{B\frac{\bar{h}}{h}}\left(H^2_{+,\kappa+\kappa_0}\right) \oplus Ker\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}} = H^2_{+,\kappa-\kappa_0}.$$

Let us first find the dimension of $Ker\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}}$ in $H^2_{+,\kappa-\kappa_0}$. If $q \in Ker\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}}$ then, $\bar{B}\frac{h}{\bar{h}}q = q_-$ for some $q_- \in H^2_{-,\kappa+\kappa_0} + \mathcal{M}_{\kappa+\kappa_0}$. This means that

$$B\overline{h}q_- = hq \tag{4.1}$$

Note that the both sides of this equality are summable on Γ and, moreover $hq \in H^1_{+,\kappa}, B\bar{h}q_- \in B(H^1_{-,\kappa} + \mathcal{M}_\kappa)$. Therefore the function hq is "orthogonal" to (annihilated by) all functions from $BH^\infty_{+,\kappa} + H^\infty_{-,\kappa} + \mathcal{M}_\kappa$ and by the analog of the Riesz brothers theorem (see for example [8]), it belongs to K^κ_B. If we denote this function by φ_B, then, by (4.1),

$$\frac{\varphi_B}{h} \in H^2_{+,\kappa-\kappa_0}, \; \frac{\varphi_B}{\bar{h}} \in B(\mathcal{M}_{\kappa+\kappa_0} + H^2_{-,\kappa+\kappa_0}),$$

and, by part (2) of Corollary 3.9 $\varphi_B \in L_\kappa(U_h + J(a) - J(Z_B) - J(Z^*))$.

Conversely, if $\varphi_B \in L_\kappa(U_h + J(a) - J(Z_B) - J(Z^*))$ then, by part (1) of Corollary 3.9 $q = \frac{\varphi_B}{h} \in H^2_{+,\kappa-\kappa_0}$ and

$$\bar{B}\frac{h}{\bar{h}}q = \frac{\varphi_{\mathcal{M}_\kappa}}{B\bar{h}} \in \mathcal{M}_{\kappa+\kappa_0} + H^2_{-,\kappa+\kappa_0}.$$

Hence $q \in Ker\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}}$. Thus we have proved that

$$dimKer\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}} = r^\kappa[U_h + J(a) - J(Z_B) - J(Z^*)] = ordZ_B - ordU_h,$$
$$Ker\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}} = \frac{1}{h}L_\kappa(U_h + J(a) - J(Z_B) - J(Z^*)). \tag{4.2}$$

Now for any $q \in Ker\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}}$ write $q = k_q + Bf_q$, where $k_q = P_{K_B^{\kappa-\kappa_0}}q$, $f_q \in H^2_{+,\kappa-\kappa_0}$, then $\mathcal{M} = \{g \in K_B^{\kappa-\kappa_0} : 0 =< g, q >=< k_q, g >, \forall q \in Ker\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}}\}$. Note, that for any such q, $k_q \neq 0$. This last fact follows from (4.2), since otherwise the expression

$$q = \frac{\varphi_B}{h} = Bf_q, \quad \varphi_B \in L_\kappa(U_h + J(a) - J(Z_B) - J(Z^*)) \subset K^\kappa_B$$

would imply that $\varphi_B = hf_q \in BH^1_{+,\kappa}$.

This means that

$$dimP_{K_B^{\kappa-\kappa_0}}Ker\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}} = dim \bigvee_{q \in Ker\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}}} \{k_q\} = ordZ_B - ordU_h.$$

Since $\mathcal{M} = K_B^{\kappa-\kappa_0} \ominus Ker\mathbf{T}_{\bar{B}\frac{h}{\bar{h}}}$, we have proved that $dim\mathcal{M} = ord(U_h) = n$. •

Now let us consider the set $\mathcal{Z}$ of all positive divisors Z of order $ordU_h = n$,such that $Z \leq Z_B$. For each such divisor Z let us consider the finite Blaschke product B_Z, for which Z is a divisor of zeros. Note that each such B_Z divides B.

Proposition 4.2. *There exists at least one divisor $Z = Z_h \in \mathcal{Z}$, for which the Toeplitz operator with symbol $B_Z\frac{\bar{h}}{h}$ is an isomorphism of $H^2_{+,\kappa+\kappa_0}$ onto its image.*

Proof. Suppose on the contrary that for all such divisors Z, $Ker\mathbf{T}_{B_Z\frac{\bar{h}}{h}} \neq 0$. Then the same reasoning as above in the proof of Proposition 4.1 gives us that for the complementary divisor $Z_B - Z$ of order $ordZ_B - ordU_h$ (for which $B = B_Z B_{Z_B - Z}$), $\mathbf{T}_{B_Z\frac{\bar{h}}{h}} = \mathbf{T}_{\bar{B}_{Z_B-Z}}\mathbf{T}_{B\frac{\bar{h}}{h}}$ and, hence

$$\mathbf{T}_{B\frac{\bar{h}}{h}}\left(H^2_{+,\kappa+\kappa_0}\right) \bigcap K^{\kappa-\kappa_0}_{Z_B-Z} \neq \{0\},$$

where $K^{\kappa-\kappa_0}_{Z_B-Z} =\subset K^{\kappa-\kappa_0}_B$, $dimK^{\kappa-\kappa_0}_{Z_B-Z} = ordZ_B - ordU_h$, is spanned by the reproducing kernels at points of the divisor $Z_B - Z$. But we already know that

$$\mathbf{T}_{B\frac{\bar{h}}{h}}\left(H^2_{+,\kappa+\kappa_0}\right) \bigcap K^{\kappa-\kappa_0}_{Z^*} = \mathcal{M}, \; dim\mathcal{M} = ordU_h.$$

Therefore our assumption implies that for all divisors Z, $ordZ = ordU_h, 0 < Z < Z_B$, $K^{\kappa-\kappa_0}_{Z_B-Z} \cap \mathcal{M} \neq \{0\}$.

But this is impossible due to the following simple Lemma.

Lemma 4.3. *Let $\mathcal{H}$ be an n-dimensional vector space with basis $V = \{v_1, \dots, v_n\}$ and let $\mathbf{K_k}$, $k < n$, be the family of all k-dimensional subspaces of $\mathcal{H}$, each of which is spanned by some k vectors from V. Then for any $(n-k)$-dimensional subspace $\mathcal{M} \subset \mathcal{H}$, there exists at least one subspace $\mathcal{K} \in \mathbf{K_k}$, such that $\mathcal{M} \cap \mathcal{K} = \{0\}$.*

Proof. Without loss of generality we can assume that $\mathcal{H} = \mathbb{R}^n$ and V is the standard basis of $\mathbb{R}^n$. Let $A = \{a_{jl}\}_{j=1,l=1}^{n,n-k}$ be $n \times (n-k)$ matrix, whose columns form the basis in $\mathcal{M}$. Clearly $rankA = n-k$. Let $1 \le j_1 < j_2 < \dots < j_{n-k} \le n$ be the indices of $n-k$ linear independent rows of A. Then the k-dimensional space $\mathcal{K}$ from $\mathbf{K_k}$ which is defined by the condition that all its vectors have zeros at the $j_1, j_2, \dots, j_{n-k}$ coordinate positions is the desired one. Indeed, if $\mathcal{M} \cap \mathcal{K} \neq \{0\}$ then there exists a vector $\vec{c} \in \mathbb{R}^{n-k}$ such that for nondegenerate $(n-k) \times (n-k)$ matrix $A_{n-k} = \{a_{j_s,l}\}_{s,l=1}^{n-k}$, $A_{n-k}\vec{c} = \mathbf{0}$. •

All we need now is to apply this Lemma to the case $\mathcal{H} = K_B^{\kappa-\kappa_0}, n = ordZ_B,\ k = ordZ_B - ordU_h$ and V is the basis of reproducing kernels at the points of Z_B (note that by construction B has only simple zeros). •

Corollary 4.4. *For the divisor Z_h of Proposition 4.2 the Toeplitz operator $\mathbf{T}_{B_{Z_h}\frac{\bar h}{h}}$ acting on the space $H^2_{+,\kappa+\kappa_0}$ is invertible (onto and zero kernel) as well as its adjoint operator $\mathbf{T}_{\overline{B_{Z_h}}\frac{h}{\bar h}}$, acting on the space $H^2_{+,\kappa-\kappa_0+\kappa_h}$ i. Here $\kappa_h = \Re\varphi(Z_h - (ordU_h)b_0)$ is a character of finite Blaschke product B_{Z_h}.*

Proof. The operator $\mathbf{T}_{B_{Z_h}\frac{\bar h}{h}}$ acting on the space $H^2_{+,\kappa+\kappa_0}$ is obviously Fredholm (together with its adjoint) since $\mathbf{T}_{B_{Z_h}\frac{\bar h}{h}} = \mathbf{T}_{\frac{\bar h}{h}}\mathbf{T}_{B_{Z_h}}$. Now, taking into account Proposition 4.2 we have

$$-dimKer\mathbf{T}_{\overline{B_{Z_h}}\frac{h}{\bar h}} = ind\mathbf{T}_{B_{Z_h}\frac{\bar h}{h}} = ind\mathbf{T}_{\frac{\bar h}{h}} + ind\mathbf{T}_{B_{Z_h}} = n - n = 0.$$

Therefore from

$$\mathbf{T}_{B_{Z_h}\frac{\bar h}{h}}(H^2_{+,\kappa+\kappa_0}) \oplus Ker\mathbf{T}_{\bar B_{Z_h}\frac{h}{\bar h}} = H^2_{+,\kappa-\kappa_0+\kappa_h}. \tag{4.3}$$

we conclude that $\mathbf{T}_{B_{Z_h}\frac{\bar h}{h}}$ is onto.

•

4.2.

Proposition 4.5. *Let U_h be the divisor from Proposition 3.7, then for $k = ordU_h|_u$, $u \in \Gamma$, the expression $h/(z-u)^k$ is locally square summable at u and for any character-automorphic meromorphic on $\hat\Omega$ function ϕ with the divisor on Γ exactly equal to U_h: $(\phi)|_\Gamma = U_h$, the weight $|h/\phi|^2$ satisfies the Muckenhoupt condition, i.e.*

$$\sup_{\lambda \in \Omega_+} \int_\Gamma \left|\frac{h}{\phi}\right|^2 d\eta_\lambda \cdot \int_\Gamma \left|\frac{\phi}{h}\right|^2 d\eta_\lambda < \infty.$$

Proof. Since the character automorphic Toeplitz operator $\mathbf{T}_{\overline{B_{Z_h}}\frac{h}{\bar h}}$ acting on a space $H^2_{+,\kappa-\kappa_0+\kappa_h}$ is invertible and its symbol is unimodular on Γ, we can apply the result from [17, Theorem 5.7] on the invertibility of character-automorphic Toeplitz operators with unimodular symbols, by which there exists an outer function $\chi \in H^2_{+,\nu}$, for certain character ν such that the weight $|\chi|^2$ is the Muckenhoupt one, and there exists a finite character-automorphic Blashke product B_Q with the divisor of zeros $Q, ordQ = g$, such that

$$\overline{B_{Z_h}}\frac{h}{\bar{h}} = \frac{\bar{\chi}}{\chi}\frac{B_Z^*}{B_Q}$$

The last equality implies that

$$\chi h B_Q = \overline{\chi h} B_{Z^*} B_{Z_h}$$

The same reasoning as we used above gives us that $\chi h \in K^{\nu+\kappa_0}_{B_{Z_h} b_a} = L^{\nu+\kappa_0}(-J(Z_h) - J(Z^*))$. But then $\chi = \phi/h$ for some function $\phi \in L^{\nu+\kappa_0}(-J(Z_h) - J(Z^*))$ and since $\chi \in H^2_{+,\nu}$, by the definition of divisor U_h, we see that $\phi \in L^{\nu+\kappa_0}(U_h - J(Z_h) - J(Z^*))$. It means that $(\phi)|_\Gamma \geq U_h$. Note now that since $1/\chi = h/\phi \in L_2(\Gamma)$ the simple application of Hölder inequality

$$\int_X \left|\frac{1}{\phi}\right| d\eta_a = \int_X \left|\frac{h}{\phi}\frac{1}{h}\right| d\eta_a \leq \left\{\int_X \left|\frac{h}{\phi}\right|^2 d\eta_a\right\}^{1/2} \cdot \left\{\int_X \left|\frac{1}{h}\right|^2 d\eta_a\right\}^{1/2}$$

for arbitrary measurable set $X \subset \Gamma$, gives us the converse inequality $(\phi)|_\Gamma \leq U_h$. That is $(\phi)|_\Gamma = U_h$ and for this certain character-automorphic meromorphic on $\hat{\Omega}$ function ϕ with $(\phi)|_\Gamma = U_h$, the weight $|\phi/h|^2 = |\chi|^2$ satisfies the Muckenhoupt condition. Thus we find one particular character-aumomorphic meromorphic on $\hat{\Omega}$ function ϕ for which the assertion of the proposition holds. Now the statement of the proposition that the same is true for any character-automorphic meromorphic on $\hat{\Omega}$ function with the divisor on Γ equal to U_h follows from the fact that the ratio of the absolute values of two such functions is a function continuous on Γ and separated from zero and infinity. •

Let us reverse the statement of the last proposition and show that this condition on function h is also sufficient for the Fredfolm property of Toeplitz operator with the symbol $\bar{h}/h$.

Proposition 4.6. *If for the outer function $h \in H^2_{+,\kappa_0}$ there exist a nonnegative divisor $U = U_h$ on Γ, such that for any character-automorphic meromorphic on $\hat{\Omega}$ function ϕ with the divisor on Γ exactly equal to U_h: $(\phi)|_\Gamma = U_h$, the weight $|h/\phi|^2$ satisfies the Muckenhoupt condition, i.e.*

$$\sup_{\lambda\in\Omega_+} \int_\Gamma \left|\frac{h}{\phi}\right|^2 d\eta_\lambda \cdot \int_\Gamma \left|\frac{\phi}{h}\right|^2 d\eta_\lambda < \infty,$$

then the operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ on a space $H^2_{+,\kappa+\kappa_0}$ is Fredholm for any character κ and

$$ind\mathbf{T}_{\frac{\bar{h}}{h}} = dimKer\mathbf{T}_{\frac{\bar{h}}{h}} = ordU_h.$$

Proof. We start with the following usefull remark.

Remark 4.7. Note that [16, Lemma 3.2] asserts that for any positive divisor Z on $\hat{\Omega}$ of order g with $\varphi(Z) = \mu_Z + \tau\nu_Z, \mu_Z, \nu_Z \in \mathbb{R}^g/\mathbb{Z}^g$, there exists a positive divisor T of order g which is contained totaly in side Ω_+ and for which $\varphi(Z) = \mu_Z + \tau\nu_Z$. That is $\Im\varphi(T - Z) = 0$ and by Abel's theorem for character-automorphic functions, there exists a $(\mu_T - \mu_Z)$-automorphic function with the divisor $T - Z$. Here we will need a little bit more strong statement that the same is true in a case $ordZ = ordT \geq g$. Indeed, if $ordZ > g$ let us take arbitrary positive divisor P of order $ordZ - g$ which is contained totaly inside Ω_+. Now taking into account the above cited lemma from [16] and the fact that Abel-Jacobi transform maps the set of positive divisors of order g onto Jacobian variety, we see that there exixts the positive divisor Q of order g contained totaly inside Ω_+, such that $\Im\varphi(Q - (Z - P)) = 0$, that is there exists a character-automorphic function on $\hat{\Omega}$ with the divisor $P + Q - Z$.

As for the positive divisors Z of order $ordZ < g$, it is easy to see that one can find two positive divisors P and Q, $ordP = g - ordZ, orqQ = g$, both contained totaly inside Ω_+,such that there exists a charcter-automorphic function on $\hat{\Omega}$ with the divisor $Q - Z - P$.

Now let us first consider the case when $ordU_h = n \geq g$, then by Remark 4.7 we can find the finite character-automorphic Blaschke product B_Q with the divisor of zeros Q of degree $ordQ = n$, such that there exists a charcter-automorphic function ϕ with the divisor $(\phi) = U_h - J(Q)$. Note that $(\phi)|_\Gamma = U_h$ and therefore by the assumption of the proposition the weight $|h/\phi|^2$ satisfies the Muckenhoupt condition. Note also that $\chi = h/\phi$ is an outer function and since $\chi/\bar{\chi} = B_Q$

$$\frac{\bar{h}}{h} B_Q = \frac{\bar{\chi}}{\chi}. \tag{4.4}$$

Since by [7] the norms of Hankel operators with the symbols $\bar{\chi}/\chi$ and $\chi/\bar{\chi}$ are strictly less then 1 on $H^2_{+,\kappa}$ for all characters κ, we see that the Toeplitz operator $\mathbf{T}_{\frac{\bar{\chi}}{\chi}}$ is invertible (onto and with zero kernel) on $H^2_{+,\kappa}$ for any κ. Therefore by 4.4 the Toeplitz operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ is Fredholm on $H^2_{\kappa+\kappa_0}$ for all κ and

$$dimKer\mathbf{T}_{\frac{\bar{h}}{h}} = ind\mathbf{T}_{\frac{\bar{h}}{h}} = ind\mathbf{T}_{\overline{B_Q}} + ind\mathbf{T}_{\frac{\bar{\chi}}{\chi}} = ordQ = n.$$

If $ordU_h < g$ then by Remark 4.7 we can find two Blaschke products B_P and B_Q with the divisors of zeros P and Q respectively, $ordP = g - ordU_h, ordQ = g$, such that there exists the character-automorphic function ϕ with the divisor $(\phi) = U_h + J(P) - J(Q)$. So in this case $\phi/\bar{\phi} = B_Q/B_P$, the weight $|h/\phi|^2$ is the Muckenhoupt one, the function $\chi = h/\phi$ is outer and moreover

$$\frac{\bar{h}}{h} = \frac{\bar{\chi}}{\chi} \frac{B_P}{B_Q}. \tag{4.5}$$

Therefore $\mathbf{T}_{\frac{\bar{h}}{h}} = \mathbf{T}_{\overline{B_Q}} \mathbf{T}_{\frac{\bar{\chi}}{\chi}} \mathbf{T}_{B_P}$ and the assertion on the Fredholm property and index of $\mathbf{T}_{\frac{\bar{h}}{h}}$ on $H^2_{\kappa+\kappa_0}$ for any κ can be obtained just in the same way as above

$$ind\mathbf{T}_{\frac{\bar{h}}{h}} = ind\mathbf{T}_{\overline{B_Q}} + ind\mathbf{T}_{\frac{\bar{\chi}}{\chi}} + ind\mathbf{T}_{B_P} = g - (g - ordU_h) = orgU_h = n.$$

•

Now combining the Propositions 4.5 and 4.6 we get the complete answer to our initial problem.

Theorem 4.8. *Let $h \in H^2_{+,\kappa_0}$ be an outer function, then the following assertions are equivalent*

1. *The Toeplitz operator $\mathbf{T}_{\frac{\bar{h}}{h}}$ on $H^2_{+,\kappa+\kappa_0}$ is Fredholm for some character κ with $ind\mathbf{T}_{\frac{\bar{h}}{h}} = dimKer\mathbf{T}_{\frac{\bar{h}}{h}} = n$.*

2. *The same is true for $\mathbf{T}_{\frac{\bar{h}}{h}}$ on $H^2_{+,\kappa+\kappa_0}$ for all $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$.*

3. *There exist the positive divisor $U = U_h$ of order n on Γ, such that for for any character-automorphic meromorphic on $\hat{\Omega}$ function ϕ with divisor on Γ exactly equal to U_h, $(\phi)|_\Gamma = U_h$, the weight $|h/\phi|^2$ satisfies the Muckenhoupt condition, i.e.*

$$\sup_{\lambda \in \Omega_+} \int_\Gamma \left|\frac{h}{\phi}\right|^2 d\eta_\lambda \cdot \int_\Gamma \left|\frac{\phi}{h}\right|^2 d\eta_\lambda < \infty.$$

Theorem 4.9. *Let $d\mu = wd\eta_a$ be a finite positive measure on Γ absolutely continuous with respect to the harmonic measure $d\eta_a$, then the sum of the subspaces $\mathcal{D}^\kappa_- = Clos_{L_2(d\mu)}\{B^\kappa_- + \mathcal{M}_\kappa\}$ and $\mathcal{D}^\kappa_+ = Clos_{L_2(d\mu)} B^\kappa_+$ is closed and the dimension of their intersection is finite and equal to n if and only if*

1. $\log w \in L_1(\Gamma)$

2. *There exists a character-automorphic meromorphic on* $\hat{\Omega}$ *function* ϕ *with nonnegative divisor on* Γ, $(\phi)|_\Gamma = U_w$, $ordU_w = n$, *such that the "improved" weight* $w/|\phi|^2$ *satisfies the Muckenhoupt condition*

5. On a Helson-Sarason type problem and Fredholm properties of general character-automorphic Toeplitz operators

5.1.

Note that above we have started with the problem posed for certain fixed character $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$ but finally we have got a necessary and sufficient condition on the weight which does not depend on κ, i.e. holds for all characters simultaneously. Thus as well as the problem from [7] on the nonzero angle in $L_2(d\mu, \Gamma)$ between $\mathcal{B}_-^\kappa + \mathcal{M}_\kappa$ and $\mathcal{B}_+^\kappa$ (which can be treated as a particular case of our problem when the intersection of $\mathcal{D}_-^\kappa$ and $\mathcal{D}_+^\kappa$ is zero) our problem has exactly the same answer as the corresponding problem in the unit disk theory. However our way of solving it has been much more complicated as we actually solved it for all characters κ.

The Helson-Sarason type problem is that one on description of those positive measures on Γ for which the angle in $L_2(d\mu, \Gamma)$ between $\mathcal{D}_-^\kappa = Clos_{L_2(d\mu)}\{\mathcal{B}_-^\kappa + \mathcal{M}_\kappa\}$ and $B\mathcal{D}_+^{\kappa-\kappa_B} = B\{Clos_{L_2(d\mu)}\mathcal{B}_+^{\kappa-\kappa_B}\}$ is nonzero for fixed κ and finite character-automorphic Blaschke product B, corresponding to the character κ_B.

Note that we actually have a partial answer on such a problem given by Proposition 3.7 and by applying Lemma 3.6 we have reduced our initial problem to the situation discussed in Proposition 3.7. But note that together with statement (1) on existence of the divisor U_h of "bad behavior" of h on Γ this proposition contains also the statement (2) which is actually the necessary condition on B, U_h and the character κ.

The problem on nonzero angle in $L_2(d\mu, \Gamma)$ between $Clos_{L_2(d\mu)}\mathcal{B}_-^\kappa$ and $Clos_{L_2(d\mu)}\mathcal{B}_+^\kappa$ solved in [16] is actually of Helson-Sarason type, since it can be reduced to this type by multiplication of both subspaces by finite Blaschke product $B = B_Z^*$. The condition on a character similar to condition from statement (2) of Proposition 3.7 turned out to be essential in the last problem. Namely (see [16]) this condition together with the Muckenhoupt condition for the "improved" (at the divisor U_h) weight forms the necessary and sufficient condition for the nonzero angle at that problem.

Therefore one should expect that condition (2) of Proposition 3.7 would play an essential role in the solution of general Helson-Sarason type problem as well.

Now we are going to give a solution of Helson-Sarason type problem by reducing it to the initial problem of the present paper.

Theorem 5.1.
Let $d\mu$ *be the finite positive measure on* Γ *and* B *be the finite* κ_B*-automorphic Blaschke product with the divisors of zeros* Z_B, *then the angle in* $L_2(d\mu, \Gamma)$ *between subspaces* $\mathcal{D}_-^\kappa$ *and* $B\mathcal{D}^{\kappa-\kappa_B}$ *is nonzero if and only if*

1. *The measure* $d\mu$ *is absolutely continuous with respect to* $d\eta_a$, $d\mu = w d\eta_a$, *with* $\log w \in L_1(\Gamma)$,

2. *There exists character-automorphic meromorphic on* $\hat{\Omega}$ *function* ϕ *with nonnegative divisor* $(\phi)|_\Gamma = U = U_w$ *on* Γ, $ordU_w \leq ordZ_B$ *such that the weight* $w/|\phi|^2$ *satisfies the Muckenhoupt*

condition

$$\sup_{\lambda\in\Omega_+}\int_\Gamma \frac{w}{|\phi|^2}d\eta_\lambda\cdot\int_\Gamma \frac{|\phi|^2}{w}d\eta_\lambda<\infty \tag{5.1}$$

and

$$r^\kappa[Z_B+J(a)-U_w-J(Z^*)]=0 \tag{5.2}$$

Proof. First of all it can be shown in exactly the same way as it was done in [16, Proposition 4.1] that if the angle in $L_2(d\mu,\Gamma)$ between $\mathcal{D}^\kappa_-$ and $B\mathcal{D}^{\kappa-\kappa_B}_+$ is nonzero then the measure $d\mu$ is absolutely continuous with respect to measure $d\eta_a$, $d\mu=wd\eta_a$, and $\log w\in L_1(\Gamma)$. So in the same way as at the beginning of the present paper we get, that there exist an outer function $h\in H^2_{+,\kappa_0}$ such that $w=|h|^2|_\Gamma$ and

$$\mathcal{D}^\kappa_+=\frac{H^2_{+,\kappa+\kappa_0}}{h},\ \mathcal{D}^\kappa_-=\frac{H^2_{-,\kappa-\kappa_0}+\mathcal{M}_{\kappa-\kappa_0}}{\bar h}.$$

Now following the reasoning at the very beginning of the proof of Proposition 3.7 we see that if the angle under consideration is nonzero then the Toeplitz operator $\mathbf{T}_{\frac{\bar h}{h}}$ on the space $H^2_{+,\kappa+\kappa_0}$ is Fredholm with $dimKer\mathbf{T}_{\frac{\bar h}{h}}=ind\mathbf{T}_{\frac{\bar h}{h}}=ind\mathbf{T}_{\overline{B}}+ind\mathbf{T}_{B\frac{\bar h}{h}}=ordZ_B-dimKer\mathbf{T}_{\bar B\frac{h}{\bar h}}\le ordZ_B$. So this operator satisfy the conditions of Propositions 3.7 and 4.5 and for the divisor $U_w=U_h$ of "bad behavior" of h on Γ, $ordU_h=dimKer\mathbf{T}_{\frac{\bar h}{h}}\le ordZ_B$ by statement (2) of Proposition 3.7 $r^\kappa[(Z_B)+J(a)-U_h-J(Z^*)]=0$.

Now let the outer function $h\in H^2_{+,\kappa_0}, w=|h|^2|_\Gamma$, satisfy the conditions of the Theorem 5.1 and let us show that the desired angle is nonzero. By Theorem 4.8 the Toeplitz operator $\mathbf{T}_{\frac{\bar h}{h}}$ is Fredholm on $H^2_{+,\kappa+\kappa_0}$ and $dimKer\mathbf{T}_{\frac{\bar h}{h}}=ind\mathbf{T}_{\frac{\bar h}{h}}=ordU_h$. So operator $\mathbf{T}_{B\frac{\bar h}{h}}$ is also Fredholm and

$$ind\mathbf{T}_{B\frac{\bar h}{h}}=ordU_h-ordZ_B. \tag{5.3}$$

All we need is to show that $\mathbf{T}_{B\frac{\bar h}{h}}$ has a zero kernel on $H^2_{+,\kappa+\kappa_0-\kappa_B}$. Indeed, this will imply that this operator is an isomorphism on the image and hence $\cos(\mathcal{D}^\kappa_-,B\mathcal{D}^{\kappa-\kappa_B})=|H_{B\frac{\bar h}{h}}|<1$.

By (5.3) this is equivalent to the fact that for an adjoint operator $\mathbf{T}_{\bar B\frac{h}{\bar h}}$ on $H^2_{+,\kappa-\kappa_0}$ $dimKer\mathbf{T}_{\bar B\frac{h}{\bar h}}=ordZ_B-ordU_h$. But in the same way as we have done it in the proof of Proposition 4.1 we get that

$$Ker\mathbf{T}_{\bar B\frac{h}{\bar h}}=\frac{1}{h}L_\kappa(U_h+J(a)-J(Z_B)-J(Z^*)).$$

So $dimKer\mathbf{T}_{\bar B\frac{h}{\bar h}}=r^\kappa[U_h+J(a)-J(Z_B)-J(Z^*)]$ But by Riemann-Roch theorem

$$r^\kappa[U_h+J(a)-J(Z_B)-J(Z^*)]=ordZ_B-ordU_h+i^{-\kappa}[J(Z_B)+J(Z^*)-U_h-J(a)],$$

and we already know (see the end of the proof of Proposition 3.7) that $i^{-\kappa}[J(Z_B)+J(Z^*)-U_h-J(a)]=r^\kappa[Z_B+J(a)-U_h-J(Z^*)]$, which is equal to zero by conditions of the Theorem 5.1. •

Remark 5.2. We see that as opposed to the Theorems 4.8 and 4.9 which deal with the Fredholm property of the corresponding Toeplitz operators, in the above solution of Helson-Sarason type problem the condition (5.2) plays the essential role. This situation differs also from that one in the unit disk theory. This difference can be explained somehow by the fact that in the unit disk theory Coburn alternative holds, i.e. for $a\in L_\infty,|a|=1$ a.e. on the unit circle either $KerT_a=0$ or $Ker\mathbf{T}_{\bar a}=0$, which in particular means that if for an outer function h the operator $T_{\frac{\bar h}{h}}$ is Fredholm with $ind\mathbf{T}_{\frac{\bar h}{h}}=dimKer\mathbf{T}_{\frac{\bar h}{h}}=m$, then for $\mathbf{T}_{B\frac{\bar h}{h}}$, where B is a finite Blaschke product of degree m, index is equal to 0 and hence $Ker\mathbf{T}_{B\frac{\bar h}{h}}=0$. Therefore in the unit disk theory the Toeplitz operator with unimodular symbol and zero index is invertible.

In our situation Coburn alternative is no longer true (see also [15]), as well as the invertibility of zero index Toeplitz operators. By Theorem 5.1 we see that even if condition (5.1) is fulfilled (and provides the Fredholm property for the operator $\mathbf{T}_{\frac{\bar{h}}{h}}$) the condition (5.2) is still necessary for the nonzero angle. Note that in trivial situation when $U_h = 0$ condition (5.2) is obviously true for all characters, but it is not necessary in this situation , since this is exactly the problem on invertibility of $\mathbf{T}_{\frac{\bar{h}}{h}}$, which was actually solved in [7]. It is easy to see that if $ordB_Z > ordU_h + g - 1$ then the condition (5.2) is automatically fulfilled for all characters κ, since by the definition of the space $L_\kappa(Q)$ its dimension $r^\kappa[Q]$ is equal to zero for any divisor $Q, ordQ > 0$. But for the fixed outer function $h \in H^2_{+,\kappa_0}$, such that for the weight $w = |h|^2|_\Gamma$ condition (5.1) holds, it is easy to produce the situation (character κ and function B) for which condition (5.2) is no longer true and , hence, the corresponding angle is zero. It can be done even if we require the condition $ordZ_B = ordU_h + g - 1$. Indeed, since $ordU_h + g - 1 \geq g$, by Remark 4.7 we can find the divisor Z contained totally in Ω_+ with $ordZ = ordU_h + g - 1$ such that there exists a character-automorphic function ϕ (with some character κ) with the divisor $(\phi) = Z + J(a) - U_h - J(Z^*)$. This character κ and Blaschke product with divisor Z_B of zeros equal to Z provide the necessary example since by construction

$$\phi \in L_\kappa(Z_B + J(a) - U_h - J(Z^*)).$$

5.2.

Now let us briefly discuss the necessary and sufficient conditions for the Toeplitz operator $T_a^{\kappa,\nu}$, $a \in L_\infty(\Gamma)$ to be Fredholm.

First of all note that in Lemma 3.6, Propositions 4.1 and 4.2 and Corollary 4.4 we have actually proved the following more general statements

Lemma 5.3. *If the Toeplitz operator $\mathbf{T}_a$ acting from the space H^2_κ to the space $H^2_{+,\kappa+\nu}$ is Fredholm with the index equal to n, then there exists a finite single-valued Blaschke product B with simple zeros such that $KerT_{Ba} = 0$.*

Proposition 5.4. *Let the Toeplitz operator $\mathbf{T}_a, a \in L_\infty(\Gamma)$ acting from $H^2_{+,\kappa}$ to $H^2_{+,\kappa+\nu}$ be Fredholm and $dimKer(\mathbf{T}_a)^* = 0$, so that $ind\mathbf{T}_a = dimKer\mathbf{T}_a = n$, then for the single-valued finite Blaschke product B from Lemma 5.3*

$$\mathcal{M} = \mathbf{T}_{aB}(H^2_{+,\kappa}) \bigcap K^\kappa_B \neq \{0\}, dim\mathcal{M} = ord\mathbf{T}_a = n.$$

Proposition 5.5. *Let the Toeplitz operator $\mathbf{T}_a$ satisfy the conditions of Proposition 5.4, then there exists at least one positive divisor $Z = Z_a \leq Z_B$ of $ordZ_a = ind\mathbf{T}_a = n$, such that for the finite Blaschke product B_Z with divisor of zeros equal to $Z = Z_a$ the Toeplitz operator $\mathbf{T}_{aB_Z} = 0$ acting from the space $H^2_{+,\kappa}$ is invertible*

Now applying Lemma 5.3 to the adjoint operator $\mathbf{T}_{\bar{a}}$ acting from the space $H^2_{+,\kappa+\nu}$ to $H^2_{+,\kappa}$, we can find the finite single-valued Blaschke product B such that for $A = \overline{B}a$ the operator $(\mathbf{T}_A)^* = \mathbf{T}_{\bar{a}B}$ has a zero kernel.

Now let $dimKer\mathbf{T}_A = ind\mathbf{T}_A$ be equal to n. Applying to operator $\mathbf{T}_A$ the Propositions 5.4 and 5.5, we find a finite Blaschke product B_Z of degree n, such that operator $\mathbf{T}_{AB_Z}$ on $H^2_{+,\kappa}$ is invertible. Therefore we reduced the problem to a description of invertible character-automorphic Toeplitz operators.

It is easy (but boring) to show that in analogy with the unit disk theory if the operator $\mathbf{T}_f, f \in L_\infty$ acting from $H^2_{+,\kappa}$ to $H^2_{+,\kappa+\nu}$ is invertible then $ess\inf|f|$ on Γ is greater then zero. Now if $f_0 \in H^\infty_{+,\kappa_0}$ is an outer function such that $|f_o| = |f|$ then the Toeplitz operator $\mathbf{T}_{f_0}$ is invertible from $H^2_{+,\kappa}$ to $H^2_{+,\kappa+\kappa_0}$ for all κ.

Therefore , since $\mathbf{T}_f = \mathbf{T}_{\frac{f}{f_0}} \mathbf{T}_{f_0}$, the invertibility of operator $\mathbf{T}_f$ is equivalent to invertibility of the Toeplitz operator with unimodular symbol f/f_0. Using the description of such operators, given in [17, Theorem 5.7] and reversing the above chain of arguments we get the following result

Theorem 5.6. *The character-automorphic Toeplitz operator* $\mathbf{T}_a, a \in L_\infty(\Gamma)$ *acting from* $H^2_{+,\kappa}$ *to* $H^2_{+,\kappa+\nu}$ *is Fredholm if and only if*

1. $$ess \inf_{\Gamma} |a| > 0.$$

   and consequently there exists an outer function $a_0 \in H^\infty_{+,\kappa_a}$ *such that* $|a_0|_\Gamma = |a|$.

2. *There exists an outer function* $h \in H^{+,\kappa_h}$, *such that the weight* $|h|^2$ *satisfies the Muckenhoupt condition*
   $$\sup_{\lambda \in \Omega_+} \int_\Gamma |h|^2 d\eta_\lambda \cdot \int_\Gamma \left|\frac{1}{h}\right|^2 d\eta_\lambda < \infty,$$
   and such that
   $$\frac{a}{a_0} = \frac{\bar{h}}{h} \frac{B_P}{B_Q},$$
   where B_P *and* B_Q *are finite Blaschke products with the divisors of zeros* P *and* Q *and corresponding to the characters* κ_P *and* κ_Q *respectively.*

   In this situation $\nu = -2\kappa_h + \kappa_P - \kappa_Q + \kappa_a$ *and* $ind\mathbf{T}_a = ordQ - ordP$.

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