# Planar Universal Graphs

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**Abstract**

A graph is $g$–universal if it satisfies two conditions. First it must contain a subdivision of every proper planar graph of degree at most three as a subgraph. Second, the function $g$ puts a restriction on the subdivision. In particular, for a planar graph $H$ of degree at most three, a fixed vertex $w_0$ of $H$, and an arbitrary vertex $w$ of $H$, the images of the vertices $w_0$ and $w$ in the universal graph are no more than $g(d(w_0, w))$ apart. We show that a large class of planar graphs are $O(n^3)$–universal.

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# 1 Introduction

The purpose of this paper is to exhibit a large family of planar graphs which have a certain universal property. A *proper embedding* of a graph in the plane is a planar embedding so the set of vertices has no accumulation point. By a *planar graph* we mean a graph that can be embedded properly in the plane. We allow loops and multiedges in graphs but require that each vertex has finite degree.

A *random walk* on a graph is a Markov chain where the state space is the set of graph vertices and the transition probability from vertex $v$ to vertex $w$ is 0 if $vw$ is not an edge, and $\frac{1}{\deg(v)}$ if $vw$ is an edge. Intuitively, a random walk is an infinite walk that proceeds from vertex to vertex along edges where each edge from a vertex has the same probability of being traversed. A random walk can be one of two types. It either has probability 1 of eventually returning to the starting vertex, or else the probability is less than 1. In the first case, the graph is said to be *recurrent*, while in the second case, the graph is called *transient*. Random walks on recurrent graphs visit the initial vertex an infinite number of times with probability 1, while random walks on a transient graph visit the initial vertex only a finite number of times with probability 1. Of course, finite graphs are recurrent while infinite graphs may be either recurrent or transient.

In [10] the following condition is given which implies that a graph is transient. Let $V$ be a set of vertices in the graph $\Gamma$. By $\partial V$ we mean the set of vertices in $V$ incident with a vertex outside of $V$. The graph $\Gamma$ is said to satisfy the $f$*-isoperimetric inequality* if $|\partial V| \geq f(|V|)$ for every finite set of vertices in $\Gamma$. If $\Gamma$ satisfies an $f$-isoperimetric inequality for some $f$ with $\sum_{n=1}^{\infty} f(n)^{-2}$ finite, then $\Gamma$ is transient. The special case where $f(n) = n^{\frac{1}{2}+\epsilon}$ is done in [11] where the condition $|\partial V| \geq f(|V|)$ is called an $\epsilon$-isoperimetric inequality. Also, in [11] it is shown that if $\Gamma$ satisfies an $\epsilon$-isoperimetric inequality with $\epsilon > 0$, then $\Gamma$ is transient.

The method of proof in [10] is based on a correspondence between transience and resistance of an electrical network, as presented in [5]. In fact, the proof of transience in [10] relies on the construction of an embedded subgraph $T$ of $\Gamma$ which is homeomorphic with a full infinite binary tree $B$. A flow is defined on the tree by labeling the edges. The two edges incident with the root are each labeled $\frac{1}{2}$. On any path from the root, each edge is given the same label as the previous edge if the common vertex has degree two. If the common vertex has degree 3 (a branch vertex), then the edge is labeled $\frac{1}{2}$ the previous edge. The energy of the flow is the sum of the squares of the labels. In [5] it is shown that if the energy of this flow is finite, then the graph is transient. In particular, let $T$ be homeomorphic with the full infinite binary tree $B$ and suppose that an edge from level $m$ to level $m+1$ in $B$ is mapped to a path of length at most $g(m)$ by the homeomorphism. The energy of the flow is bounded above by $\sum_{m=0}^{\infty}\left(\frac{1}{2^m}\right)^2 2^m g(m)$. Thus, to show a graph is transient, it is sufficient to find an embedded subtree $T$ homeomorphic with $B$ satisfying the condition that $\sum_{m=1}^{\infty}\frac{g(m)}{2^m}$ is finite.

**Definition 1** *Let $g$ be a function from the natural numbers to the reals. We say that a graph $\Gamma$ is $g$-universal if, for every connected planar graph $H$ with maximum degree at most three and every fixed vertex $w_0$ in $H$, there is a subgraph of $\Gamma$ that is homeomorphic with $H$. Furthermore, if $w$ is a vertex in $H$ and $v_0, v$ are the vertices in $\Gamma$ corresponding to $w_0, w$ by the homeomorphism, then $d_\Gamma(v_0, v) \leq g(d_H(w_0, w))$.*

The full infinite binary tree can be properly embedded in the plane, so if $\sum_{m=1}^{\infty}\frac{g(m)}{2^m}$ is finite, then any $g$-universal graph is transient. However, the full infinite binary tree is not universal since it doesn't contain a cycle. Consequently, transience is not enough to imply universal. Furthermore, the isoperimetric conditions of [10] are not sufficient to imply universal since the full infinite binary tree satisfies the $\epsilon$-isoperimetric condition for $\epsilon = \frac{1}{2}$.

The definition of universal used in this paper is similar to the definition of universal in [3, 4, 7]. Instead of requiring that every graph of a certain type be isomorphic with a subgraph or an induced subgraph, we only require that it be homeomorphic with a subgraph, but add the condition that distance from a fixed vertex not be changed more than a fixed function $g$.

The purpose of this paper is to show that a certain class of planar graphs consists of $g$-universal graphs where $g(m) = O(m^3)$. In order to define this class of graphs, we first define a few terms.

A graph $\Gamma$ is 1-*ended* if for any finite set of vertices $V$ of $\Gamma$, $\Gamma - V$ has only one infinite component. $\Gamma$ has *bounded degree* if there is a number $d$ such that every vertex has degree at most $d$. A planar graph has *bounded codegree* if there is a number $\ell$ such that each face is bounded by a polygon with at most $\ell$ sides.

We define a disk to be a union of closed faces of a planar graph that is homeomorphic with the usual disk in the plane. Recall that for a regular $k$-gon in the Euclidean plane an interior angle has size $(1 - \frac{2}{k})\pi$. Hence, for a simple graph, if there are $deg(v) \geq 3$ regular polygons incident to a vertex $v$ and these polygons have $n_i$ edges respectively, where $1 \leq i \leq deg(v)$, then the sum of the angles at $v$ is $\sum_{i=1}^{deg(v)} (1 - \frac{2}{n_i})\pi$. For convenience we omit the factor of $\pi$ and define the *excess* of a vertex in general as follows. (Loosely speaking the excess measures how far the sum of the angles incident to a vertex deviates from the normal Euclidean sum of $2\pi$.)

The *excess* of a vertex $v$ in a disk is given by

$$\mathrm{Ex}(v) = \left[ \sum_i (1 - \frac{2}{n_i}) \right] - 2 + b_v,$$

where $n_i$ is the number of edges bounding the $i^{\text{th}}$ face incident with $v$ and $b_v$ is one if $v$ is incident with the unbounded face and zero otherwise. The number $n_i$ is counted with multiplicity. Furthermore, the same face is counted with multiplicity in the sum.

It is not difficult to show that in a disk $D$, $\sum_{v \in D} \mathrm{Ex}(v) = -2$ [2]. This is simply Euler's formula written in terms of excess.

For an infinite planar graph $\Gamma$ the excess of a vertex is treated as if it were an interior vertex, that is, $b_v = 0$. If $D$ is a disk inside $\Gamma$, then we write $\mathrm{Ex}_D(v)$ to denote the excess at $v$ using the disk $D$. In this case, $\mathrm{Ex}_D(v) = \mathrm{Ex}(v)$ as long as $v$ is interior to $D$, but for vertices on the boundary of $D$ these two expressions are not the same.

In Section 3 we prove the following theorem.

**Theorem 2** *Let $\Gamma$ be an infinite 1-ended planar graph with bounded degree and codegree. If there is a number $\epsilon > 0$ such that every vertex of $\Gamma$ has excess at least $\epsilon$, then $\Gamma$ is $O(m^3)$-universal.*

By the comment above concerning transient graphs and $g$-universal graphs, Corollary 3 then follows.

**Corollary 3** *Let $\Gamma$ be an infinite 1-ended planar graph with bounded degree and codegree. If there is a number $\epsilon > 0$ such that every vertex of $\Gamma$ has excess at least $\epsilon$, then $\Gamma$ is transient.*

In Section 2 we prove that a certain graph is $O(m)$-universal. This result is used in Section 3 to prove Theorem 2. Section 4 contains an example that shows it is not sufficient to assume that the average excess is at least $\epsilon$ to imply that a graph is universal. Furthermore, in Section 4 we give an alternate proof of Corollary 3 using the isoperimetric condition of [10].

## 2 A universal graph

We first define a graph $\Lambda$. Figure 1 shows the graph and illustrates the definition. We start with the full infinite binary tree $B$ and label all the vertices at level $h$, 0 through $2^h - 1$, going from left to right. We identify a vertex in $B$ by specifying its level and label. We refer to the vertex to the right or left of a vertex $v$ as the vertex at the same level as $v$ whose label is one more or one less respectively than the label of $v$, where the label is read modulo $2^h$. We replace each edge of $B$ with a path of length 3 and refer to each path as a leg. An $h$–leg is a leg that replaces a vertex between vetices of $B$ at level $h$ and level $h+1$ The two internal vertices in an $h$–leg are referred to as the upper and the lower vertices, the upper vertex is incident with a vertex at level $h$ and the lower is incident with a vertex of level $h+1$ in the original tree. Left and right legs are defined in the obvious way. For each left

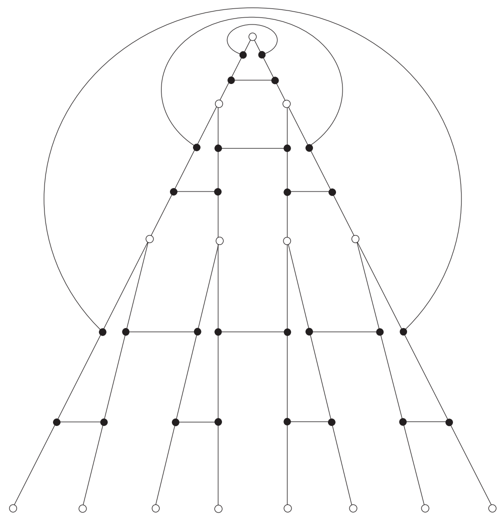

Figure 1: The graph $\Lambda$. Vertices corresponding to the infinite binary tree $B$ are represented with hollow dots, while the vertices corresponding to the subdivisions are represented with solid dots.

leg its lower vertex and the lower vertex of the leg to the right are connected with an edge and the upper vertex is connected with an edge to the upper vertex of the leg to the left. When referring to a vertex of $\Lambda$ of level $h$, we mean a vertex whose level in the original binary tree was at level $h$. Children in $\Lambda$ mean children in the original full binary tree. Horizontal and vertical edges in $\Lambda$ have the obvious meaning.

It is obvious that the graph $\Lambda$ can be properly embedded in the plane. Figure 1 gives an explicit embedding. It is also obvious that a subdivision of any binary tree $T$ can be embedded in $\Lambda$ in such a way that left and right children of a vertex $v$ in $T$ map to left and right children of the image of $v$ in $\Lambda$.

A *horizontal path* is a path in $\Lambda$ between vertices which are at the same level in $B$. Furthermore, the path follows a vertical edge down, a horizontal edge across, a vertical edge up, a vertical edge down, and so on as indicated by the two horizontal paths (highlighted in bold) in Figure 2. Note that any

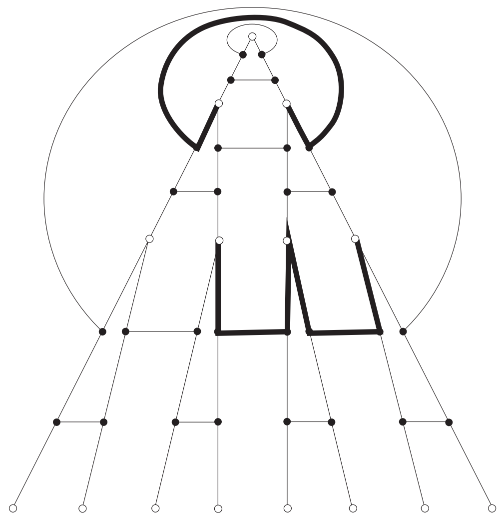

Figure 2: Two horizontal paths in $\Lambda$.

two distinct vertices at the same level have two horizontal paths between them.

We now show that $\Lambda$ is universal. Let $H$ be a properly embedded graph in the plane with maximum degree at most 3 and fix $w_0'$ a vertex in $H$. If $w_0'$ does not have degree 2, then add a vertex to an edge adjacent with $w_0'$ and call the new vertex $w_0$. Otherwise, let $w_0 = w_0'$. This modification changes distance between vertices by at most 1. Let $W_m$ denote those vertices whose edge distance from $w_0$ is exactly $m$. We call edges between two vertices in $W_m$ horizontal edges while edges from vertices in $W_m$ to $W_{m+1}$ are vertical edges. From $H$ remove all the horizontal edges and call the remaining subgraph $\bar{H}$. Note that for a vertex $w$ in $H$, $d_H(w_0, w) = d_{\bar{H}}(w_0, w)$.

Now remove selected vertical edges from $\bar{H}$ to form a tree $T$. It is not difficult to show by induction on $m$, that selected vertical edges can be removed so the remaining graph restricted to the vertex set $W_0 \cup W_1 \cup \cdots \cup W_m$ is a tree. This gives a subgraph $T$ of $\bar{H}$ such that $V(T) = V(H)$, distance in $T$ from $w_0$ is the same as distance in $H$, and $T$ is a tree rooted at $w_0$ with maximum degree at most 3. Furthermore, the fact that $H$ is embedded in the plane gives $T$ the structure of a binary tree. The vertices of $W_m$ are labeled from the set $\{0, 1, 2 \cdots, 2^m - 1\}$ by labeling each left child twice its parent and each right child one more than twice its parent. Since $H$ is embedded in the plane, for each vertex of degree three, the planar embedding determines if a child vertex in $T$ is a right child or a left child. In the case where a vertex in $H$ has degree 2, then its child is arbitrarily assigned to be a left or right child. An embedding of $T$ into the full infinite binary tree $B$ simply sends a vertex with label $s$ and level $h$ to a vertex with label $s$ and level $h$. Since a subdivision of the full binary tree sits in $\Lambda$, we have an embedding $\phi$ of a subdivision of $T$ in $\Lambda$. Furthermore if $w \in V(H)$, then $d(\phi(w_0'), \phi(w)) \leq 3d(w_0', w) + 3$.

In order to produce a subgraph of $\Lambda$ that is homeomorphic with $H$, it remains to map edges in $E(H) - E(T)$ to paths in $\Lambda$. For two edges $e, e' \in E(H) - E(T)$ we write $e < e'$ to mean the unique cycle in $T \cup e'$ encloses the interior of $e$. Note that it is impossible for $e < e'$ and $e' < e$.

**Lemma 4** *The edges of $E(H) - E(T)$ can be ordered $e_1, e_2, \cdots$ in such a way that $i < j$ implies that $e_i < e_j$ or else $e_i$ and $e_j$ are not comparable.*

Proof. Let $S \neq \emptyset$ be a subset of the edges of $E(H) - E(T)$. Pick $e \in S$ to be an edge incident with a vertex in $W_m$ for the smallest possible value of $m$. The graph $T \cup e$ has a unique cycle. Since this cycle bounds a compact face $D$ of the plane and the embedding is proper, there are only a finite number of vertices in the closed face $D$. Therefore, $e' < e$ for only a finite number of $e' \in S$. For any $e' < e$, if $e'' < e'$, then $e'' < e$. So, there is an element in $S$, say $g(S)$ with $g(S) < \bar{e}$ or else $g(S)$ and $\bar{e}$ are not comparable for all $\bar{e} \in S - \{g(S)\}$.

We define $S_0 = E(H) - E(T)$ and $e_1 = g(S_0)$. Then we define $e_n$ and $S_n$ recursively by $S_n = S_{n-1} - \{e_{n-1}\}$ and $e_n = g(S_{n-1})$. Note that the way $g(S)$ is defined by starting with an edge incident with $W_m$ for minimum $m$ implies that every edge appears in the sequence $e_1, e_2, \cdots$. Furthermore, the condition that $i < j$ implies $e_i < e_j$ or else $e_i$ and $e_j$ are not comparable is clear from the construction. ■

We shall assume that the edges of $E(H)-E(T)$ are ordered as in Lemma 4.

**Theorem 5** *There is an embedding $\phi$ of a subdivision of $H$ in $\Lambda$ with the property that for any vertex $w \in V(H)$, $d(\phi(w_0), \phi(w)) \leq 3d(w_0, w) + 3$.*

PROOF. We first embed a subdivision of the tree $T$ in $\Lambda$ as defined above. We then show by induction that the edges $e_1, e_2, \cdots, e_m$ can be mapped to paths in $\Lambda$ to complete the embedding to all of $H$. Note that regardless of how the edges in $E(H)-E(T)$ are mapped to $\Lambda$, the condition on the distance is not violated since adding more edges to $T$ can not increase distances.

We assume inductively that all the edges $e_1, \cdots, e_{m-1}$ have been mapped to paths in $\Lambda$ by $\phi$. We further assume that there is a homeomorphism $\theta$ of the plane to itself that extends $\phi$ and each edge $e_i$, for $i < m$, is mapped by $\phi$ to a path consisting of vertical edges followed by a horizontal path and then vertical edges. The base of the induction is obvious since $T$ is a binary tree. We need to show how to map the edge $e_m$ to a path. We do this by first considering the image $\theta(e_m)$. Of course $\theta(e_m)$ need not be a path in $\Lambda$. We know from the inductive assumption and the order of the edges in $E(H) - E(T)$ that $\theta(e_m)$ is not in a bounded component of $T \cup e_1 \cup \cdots \cup e_{m-1}$. Consequently, one can deform $\theta(e_m)$ into a path $\gamma$ that first follows vertical edges, then follows a horizontal path, and finally follows vertical edges. This path cannot intersect $\phi(T \cup e_1 \cdots \cup e_{m-1})$ since the vertical edges in $\theta(e_m)$ are following branches of the binary tree that are not in the image of $\phi(T \cup e_1 \cdots \cup e_{m-1})$. We then define $\phi(e_m)$ to be $\gamma$. Since we are deforming a path in the plane, the homeomorphism $\theta$ can be modified so that $\gamma$ is the image of $e_m$. ■

# 3 Proof of the main theorem

We now prove that any graph satisfying the conditions of Theorem 2 is universal. We assume that $\Gamma$ is an infinite 1-ended planar graph with degree bounded by $d$ and codegree bounded by $\ell$. Furthermore, we assume there is a number $\epsilon > 0$ such that every vertex of $\Gamma$ has excess at least $\epsilon$.

There are two distance measures for $\Gamma$ considered here. We let $d(v, u)$ denote the usual distance between vertices $v$ and $u$, while we let $d'(v, u)$ denote the face distance between $v$ and $u$. The face distance is the usual distance between vertices when each face is replaced by a complete graph on the incident vertices. That is, the face distance is the number of faces one must traverse in order to go between two vertices.

Fix a vertex $v_0$ of $\Gamma$ and let $V_m$ denote the set of vertices of $\Gamma$ whose face distance is exactly $m$ from $v_0$. As shown in [2], for each $m \geq 1$ there is a cycle $C_m$ whose vertices are in $V_m$ with the property that $v_0$ is in the finite component of $\Gamma - C_m$. Furthermore in [2], it is shown that $|V(C_m)|$ grows exponentially with $m$.

**Lemma 6** *For each vertex $v$ in $C_m$ with $m \geq 1$, there is a face incident with $v$ that is also incident with a vertex in $C_{m-1}$.*

PROOF. Since $v$ is in $V_m$, it is incident with a face containing a vertex $w$ in $V_{m-1}$. If $w$ is not in $C_{m-1}$, then there is a path from $v_0$ to $w$ that does not intersect $C_{m-1}$. Consequently, there is a path from $v_0$ to $v \in V(C_m)$. But then, $V(C_m)$ is in the component of $\Gamma - C_{m-1}$ containing $v_0$. This implies that $v_0$ is in the infinite component of $\Gamma - C_m$ which is a contradiction. ■

For each vertex of $v$ of $C_m$ we define $\mathrm{Ex}^-(v)$ to be the excess for the vertex $v$ on the disk bounded by $C_m$. That is, $\mathrm{Ex}^-(v) = \left[\sum_i (1 - \frac{2}{n_i})\right] - 1$, where the index $i$ labels faces incident to $v$ which are interior to the cycle. The quantity $\mathrm{Ex}^+(v) = \left[\sum_i (1 - \frac{2}{n_i})\right] - 1$, where in this sum the index $i$ labels faces incident to $v$ which are exterior to $C_m$. Note that $\mathrm{Ex}^-(v) + \mathrm{Ex}^+(v) = \mathrm{Ex}(v)$. We refer to $\mathrm{Ex}^-(v)$ as the *inner excess* and $\mathrm{Ex}^+(v)$ as the *outer excess*.

In [2] it is shown that for planar graphs where every region is a triangle, if the excess at each vertex is at least 0, then the graph is concentric. That is, all the vertices in $V_m$ are in $C_m$. A key part of the inductive argument is that $Ex^-(v) \leq 0$ for each vertex in $C_m$. In our setting, this is certainly not the case. However, we wish to establish an upper bound on the excess sum around consecutive vertices on the cycle $C_m$. The next few lemmas give the desired bound.

**Lemma 7** *On the cycle $C_m$, for any consecutive set of vertices $R$,*

$$\sum_{v \in R} \mathrm{Ex}^-(v) < \frac{2}{3} m\ell$$

*where $d$ is an upper bound on degree and $\ell$ an upper bound on codegree.*

Proof. In the cases where $R$ contains all vertices in $C_m$ or all except one vertex in $C_m$, the statement follows from Euler's formula involving excess since the excess of each internal vertex is positive. For the remaining cases, let $u$ and $w$ be the end vertices for $R$. That is, $u$ and $w$ are vertices not in $R$, but incident in $C_m$ with vertices in $R$. By Lemma 6 there are paths $p$ and $q$ of length at most $\frac{1}{2}\ell m$ starting at $u$ and $w$ respectively and ending at a common vertex $x$ on some $C_i$, $0 \leq i < m$. We furthermore can assume that $x$ is the only vertex common to the two paths. Let $D$ be the disk bounded by the cycle consisting of the paths $p$ and $q$ together with the part of $C_m$ induced by the vertices of $R \cup \{u, w\}$. Each vertex of the paths $p$ and $q$ have excess at least $-\frac{2}{3}$ and the sum of the disk excess for $D$ is $-2$. Let $E = \sum_{v \in R} \mathrm{Ex}^-(v)$ and $E' = \sum_{v \in \mathrm{int}(D)} \mathrm{Ex}(v) \geq 0$. Then $E + E' - \frac{2}{3}(\ell m + 1) \leq -2$. Therefore, $E < \frac{2}{3}\ell m$. ■

In the case that $|R|$ is large, Lemma 7 can be improved to insure that the excess sum is negative.

**Lemma 8** *If $R$ is a set of consecutive vertices on $C_m$ and $|R| \geq \frac{2d\ell^2 m}{3\epsilon}$, then $\sum_{v \in R} \mathrm{Ex}^-(v) < -1$.*

Proof. As in Lemma 7 construct paths from end vertices $u$ and $w$ of $R$ that meet at vertex $x$ and each having length at most $\frac{\ell m}{2}$. Also form the disk $D$ as in Lemma 7. Let $R'$ denote the vertices interior to $D$ which are on the cycle $C_{m-1}$.

Each vertex $v$ of $R$ is connected by a face to a vertex in $R'$. Therefore, $|R'| > \frac{|R|}{d\ell}$. We define $E$ and $E'$ as in Lemma 7 and note that $E' > \frac{|R|}{d\ell}\epsilon$. Since

$$E + \frac{|R|}{d\ell}\epsilon - \frac{2}{3}(\ell m + 1) \leq -2$$

we have

$$\begin{aligned} E &< \frac{2}{3}\ell m - \frac{|R|}{d\ell}\epsilon - 1 \\ &< \frac{2}{3}\ell m - \frac{2d\ell^2 m}{3\epsilon}\frac{\epsilon}{d\ell} - 1 \\ &= -1 \end{aligned}$$

■

Given two vertices $v$ and $w$ in $C_m$, we let the $B(v,w)$ denote the vertices encountered in a counterclockwise walk around $C_m$ from $v$ to $w$, but including neither $v$ nor $w$. We refer to the set $B(v,w)$ as the set of vertices between $v$ and $w$. Note that the set of vertices between $v$ and $w$ is not the same as the set of vertices between $w$ and $v$.

Let $v \in V(C_m)$. We say that the vertex $v$ *links outward* to $w$ if there is a face incident with both $v$ and $w$, and $w \in V(C_{m+1})$. A key part of the proof of Theorem 2 is to establish how the total excess grows from cycle $C_m$ to cycle $C_{m+1}$. Not every vertex of $C_m$ links outward. Lemmas 9 and 10 give bounds on the excess sum between consecutive vertices that link outward and the excess sum between the vertices in $C_{m+1}$ to which they link.

**Lemma 9** *Suppose that $p_1$ and $p_2$ are paths from $x \in V(C_m)$ to $x' \in V(C_{m+1})$ and $y \in V(C_m)$ to $y' \in V(C_{m+1})$ respectively, and there is a face $A$ whose bounding cycle contains $p_1$ and a face $B$ whose bounding cycle contains $p_2$. Furthermore, assume there are no vertices incident with either $A$ or $B$ that are in $B(y,x) \cup B(y',x')$. Then*

$$\sum_{v \in B(y',x')} \mathrm{Ex}^-(v) \leq \sum_{v \in B(y,x)} \mathrm{Ex}^-(v) - \epsilon |B(y,x)|.$$

PROOF. We let $D$ be the disk whose boundary consists of the cycle $C_m$ restricted to the vertices $\{x, y\} \cup B(y, x)$, the path $p_2$, the cycle $C_{m+1}$ restricted to $\{x', y'\} \cup B(y', x')$, and the path $p_1$. Then

$$\begin{aligned} -2 = \sum_{v \in V(p_1)} \mathrm{Ex}_D(v) \quad &+ \sum_{v \in V(p_2)} \mathrm{Ex}_D(v) + \sum_{v \in B(y,x)} \mathrm{Ex}^+(v) \\ &+ \sum_{v \in B(y',x')} \mathrm{Ex}^-(v) + \sum_{v \in \mathrm{int}(D)} \mathrm{Ex}(v). \end{aligned}$$

It is easy to verify that $\sum_{v \in V(p_1)} \mathrm{Ex}_D(v) + \sum_{v \in V(p_2)} \mathrm{Ex}_D(v) \geq -2$. Also, $\sum_{v \in \mathrm{int}(D)} \mathrm{Ex}(v) \geq 0$ since the excess at each vertex of $\Gamma$ is positive. Consequently, $\sum_{v \in B(y',x')} \mathrm{Ex}^-(v) \leq -\sum_{v \in B(y,x)} \mathrm{Ex}^+(v)$. Since $\mathrm{Ex}^+(v) + \mathrm{Ex}^-(v) = \mathrm{Ex}(v) \geq \epsilon$ for every vertex $v \in B(y, x)$, we have

$$\sum_{v \in B(y',x')} \mathrm{Ex}^-(v) \leq \sum_{v \in B(y,x)} \mathrm{Ex}^-(v) - \epsilon |B(y, x)|.$$

■

**Lemma 10** *If $x$ and $y$ are vertices in $C_m$ incident with faces containing vertices in $C_{m+1}$ and there is no vertex in $B(y, x)$ incident with a face containing a vertex of $C_{m+1}$, then*

$$\sum_{v \in B(y,x)} \mathrm{Ex}^+(v) \leq 0.$$

PROOF. We first note that by a minor modification of the construction of $C_m$ given in [2], it is possible to allow only edges in $C_m$ that bound faces incident with both a vertex in $V_m$ and a vertex in $V_{m-1}$. As a result, there are only two possible cases. Either the face incident with $x$ and incident with a vertex of $C_{m+1}$ is the same face as the face incident with $y$ and incident with a vertex in $C_{m+1}$ or else the two faces intersect in a vertex $t$ on $C_{m+1}$. See Figures 3a and 3b respectively.

We first consider the case shown in Figure 3a where the faces are the same. Let $A$ be the set of vertices on the boundary of the face $R$ in Figure 3a. Then $\sum_{v \in A} \mathrm{Ex}(v) + \sum_{v \in B(y,x)} \mathrm{Ex}^+(v) \leq -2$. But $\sum_{v \in A} \mathrm{Ex}(v) \geq -2$, as the sum of

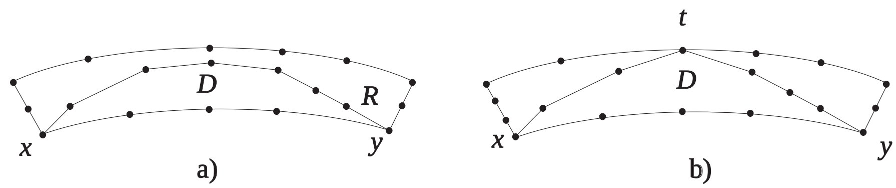


Figure 3: Faces having at most a point in common.

the excess over all the vertices bounding the face $R$ is $-2$. It follows that $\sum_{v \in B(y,x)} \mathrm{Ex}^+(v) \leq 0$.

For the case illustrated in Figure 3b, let $D$ be the disk whose boundary contains $x$, $y$ and $t$, then $\sum_{v \in D} \mathrm{Ex}_D(v) = -2$. The vertices $x$, $y$, and $t$ contribute at least $-2$ to the sum. Each vertex in the interior of $D$ contributes a positive amount to the sum, and each vertex on the interior of the paths from $x$ to $t$ and $t$ to $y$ on the boundary of $D$ contribute a positive amount to the sum. Therefore, $\sum_{v \in B(y,x)} \mathrm{Ex}^+(v) \leq 0$. ■

Corollary 11 gives a lower bound on how far around the cycle $C_m$ one must travel in order to come to a vertex that links outward.

**Corollary 11** *For any vertex $v \in V(C_m)$, there is a vertex $w$ on $C_m$ such that*

1. *$w$ links outward, and*
2. *the path from $v$ to $w$ on $C_m$ in a clockwise direction has length at most $\lceil \frac{2d\ell^2 m}{3\epsilon} \rceil$.*

*Furthermore, in condition 2) clockwise can be replaced with counterclockwise.*

Proof. From $v$ travel around $C_m$ in a clockwise direction until you find the first vertex that links outward. Call this vertex $x$. From $v$ travel counterclockwise to find the first vertex $y$ that links outward. Suppose there are at least $\lceil \frac{2d\ell^2 m}{3\epsilon} \rceil$ vertices in $B(y,x)$. Then $\sum_{v \in B(y,x)} \mathrm{Ex}^-(v) < 0$ by

Lemma 8. Since for any $v \in V(C_m)$, $\mathrm{Ex}(v) = \mathrm{Ex}^+(v) + \mathrm{Ex}^-(v)$, we have $\sum_{v\in B(y,x)} \mathrm{Ex}^+(v) > 0$. This contradicts Lemma 10. ■

We summarize Lemma 9 and Corollary 11 in Corollary 12.

**Corollary 12** *There is a $\beta > 0$ such that if $R$ is a set of consecutive vertices around $C_m$ with $|R| > \beta m$, then*

1. $\sum_{v\in R} \mathrm{Ex}^-(v) < -1$, *and*

2. *among the vertices of $R$ at least one vertex links outward.*

*Note that $\beta$ depends on $\epsilon$, $d$, and $\ell$, but does not depend on $m$.*

In order to place a copy of $\Lambda$ in $\Gamma$, we need to have some control of how the total excess changes from $C_m$ to $C_{m+1}$. We think of the excess sum between vertices on $C_m$ as a measure of the distance between the vertices. Consequently, we can form a branch vertex on $C_m$ when the absolute value of the excess sum between vertices is sufficiently large. Lemma 13 gives the desired bound.

**Lemma 13** *Let $x$ and $y$ be vertices in $C_m$ which link outward to $x'$ and $y'$ respectively. If $\sum_{v\in B(y,x)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\tau m$, where $m > \frac{2\delta}{\epsilon}$, $\delta = 1+\epsilon$, and $\tau$ is some positive number, then $\sum_{v\in B(y',x')} \mathrm{Ex}^-(v) < -\frac{2}{3}\tau(m+1)\delta$.*

Proof. By Lemma 9, $\sum_{v\in B(y',x')} \mathrm{Ex}^-(v) \leq \sum_{v\in B(y,x)} \mathrm{Ex}^-(v) - \epsilon|B(y,x)|$. Since $\sum_{v\in B(y,x)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\tau m$, $|B(y,x)| \geq \tau m$. Consequently,

$$\begin{aligned}
\sum_{v\in B(y',x')} \mathrm{Ex}^-(v) &\leq -\frac{2}{3}\tau m - \epsilon\tau m \\
&= -\tau m\left(\frac{2}{3}+\epsilon\right) \\
&= -\frac{2}{3}\tau\,(m+1)\,\delta + \frac{2}{3}\tau\,(m+1)\,\delta - \tau m\left(\frac{2}{3}+\epsilon\right) \\
&= -\frac{2}{3}\tau\,(m+1)\,\delta + \tau m\left(\frac{2}{3}\delta - \frac{2}{3} - \epsilon\right) + \frac{2}{3}\tau\delta
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{3}\tau\,(m+1)\,\delta + \tau m\left(-\frac{1}{3}\epsilon\right) + \frac{2}{3}\tau\delta \\
&= -\frac{2}{3}\tau\,(m+1)\,\delta + \tau\left(\frac{-\epsilon m}{3} + \frac{2}{3}\delta\right) \\
&< -\frac{2}{3}\tau\,(m+1)\,\delta + \tau\left(-\frac{\epsilon m}{3} + \frac{2}{3}\frac{\epsilon m}{2}\right) \\
&= -\frac{2}{3}\tau\,(m+1)\,\delta
\end{aligned}
$$

■

Lemma 14 is the main technical lemma needed in the proof of Theorem 2. The idea is to first embed a copy of a subdivision of the full infinite binary tree $B$ in $\Gamma$. In order to do this, the image of each leg of $B$ must be mapped to a path in $\Gamma$. Lemma 14 allows us to map part of two legs emanating from the same branch vertex into $\Gamma$ starting on $C_m$ and ending on $C_{m+1}$. Furthermore, condition 5 keeps the excess sum between vertices on the legs essentially the same on $C_{m+1}$ as on $C_m$. After the embedding of a subdivision of $B$ in $\Gamma$ has been established it is routine to extend it to an embedding of a subdivision of $\Lambda$ in $\Gamma$.

For Lemma 14 we assume that $\epsilon < 1$. Since we only assume that the excess at each vertex is at least $\epsilon$, we can always replace $\epsilon$ with a smaller positive value.

**Lemma 14** *Let $\beta' = \max\left(\frac{9}{\epsilon}\beta, \ell\right)$. Suppose that $x$ and $y$ are vertices in $C_m$ with $m > \frac{2\delta}{\epsilon}$, $\sum_{v \in B(y,x)} \mathrm{Ex}^{-}(v) \le -\frac{2}{3}\beta' m$, $x$ links outward to $x'$, and $y$ links outward to $y'$. Then there are vertices $u, w \in B(y', x')$ such that*

1. *both $u$ and $w$ link outward, and*
2. *on $C_{m+1}$ starting at $x'$ and walking clockwise the order in which vertices are traversed is $x', u, w, y'$, and*
3. *$-3\beta\,(m+1) < \sum_{v \in B(u,x')} \mathrm{Ex}^{-}(v) < -\beta\,(m+1)$, and*
4. *$-3\beta\,(m+1) < \sum_{v \in B(y',w)} \mathrm{Ex}^{-}(v) < -\beta\,(m+1)$, and*
5. *$\sum_{v \in B(w,u)} \mathrm{Ex}^{-}(v) \le -\frac{2}{3}\beta'(m+1)$.*

PROOF. Let $R$ be the set of vertices between $x'$ and $y'$. By Lemma 13 the inner excess sum for these vertices is at most $-\frac{2}{3}\beta'(m+1)(1+\epsilon)$. Each vertex has excess at least $-\frac{2}{3}$ so there must be at least $\beta'(m+1)(1+\epsilon)$ vertices in $R$. Since $\beta' \geq \frac{9\beta}{\epsilon}$ we have $\beta'(m+1)(1+\epsilon) > 9\beta(m+1)$, that is, there are at least $9\beta(m+1)$ vertices in $R$. By Corollary 12, if we start at $x'$ and move around $C_{m+1}$ in a clockwise direction, there must be a first vertex $u$ which links outward and satisfies condition 3. Similarly, by starting at $y'$ and moving around $C_{m+1}$ counterclockwise, there is a first vertex $w$ which links outwards and satisfies condition 4.

Suppose that the order of $u$ and $w$ indicated in condition 2 is reversed. Let

$$\begin{aligned} r' &= \sum_{v\in B(u,w)} \mathrm{Ex}^-(v) \\ r_1 &= \sum_{v\in B(u,x')} \mathrm{Ex}^-(v) \\ r_2 &= \sum_{v\in B(y',w)} \mathrm{Ex}^-(v) \\ r_3 &= \sum_{v\in B(y',x')} \mathrm{Ex}^-(v). \end{aligned}$$

Then the sum of the excess for the vertices in $B(u,w)$ is given by

$$\begin{aligned} r' &= r_1 + r_2 - r_3 \\ &> -6\beta(m+1) + \frac{2}{3}\beta'(m+1)(1+\epsilon) \\ &\geq -6\beta(m+1) + \frac{2}{3}\frac{9\beta}{\epsilon}(m+1)\epsilon + \frac{2}{3}\beta'(m+1) \\ &= \frac{2}{3}\beta'(m+1) \\ &\geq \frac{2}{3}\ell(m+1). \end{aligned}$$

Note that this contradicts Lemma 7. Therefore, the order indicated in condition 2 is correct.

It remains to show condition 5. Let $r_1$, $r_2$, and $r_3$ be defined as above, but let $r' = \sum_{v \in B(w,u)} \mathrm{Ex}^-(v)$. A calculation similar to the previous one gives:

$$\begin{aligned}
r' &= r_3 - r_1 - r_2 \\
&< -\frac{2}{3}\beta'(m+1)(1+\epsilon) + 6\beta(m+1) \\
&= -\frac{2}{3}\beta'(m+1) - \frac{2}{3}\beta'(m+1)\epsilon + 6\beta(m+1) \\
&\leq -\frac{2}{3}\beta'(m+1) - \frac{2}{3}\frac{9\beta}{\epsilon}(m+1)\epsilon + 6\beta(m+1) \\
&= -\frac{2}{3}\beta'(m+1)
\end{aligned}$$

■

**Lemma 15** *There is an embedding $\phi$ of a subdivision of $\Lambda$ in $\Gamma$ with the property that if $w_0$ is the root of $\Lambda$ and $w$ is a vertex in $\Lambda$ then $d(\phi(w_0), \phi(w)) = O(d(w_0, w)^3)$.*

Proof. As shown in [2], the number of vertices in $C_m$ grows exponentially. Consequently, there is an $m_0 > \frac{2\delta}{\epsilon}$ such that on $C_{m_0-1}$ there is a vertex $v'$ that links outward to a vertex $v_0$ and the number of vertices in $C_{m_0-1}$ is greater than $\beta\beta'(m_0-1)^2$. We can partition the vertices in $V(C_{m_0-1}) - \{v'\}$ into $\beta'(m_0-1)$ subsets of consecutive vertices around $C_{m_0-1}$ so that each of the subsets has size at least $\beta(m_0-1)$. By Corollary 12

$$\begin{aligned}
\sum_{v \in V(C_{m_0-1}) - \{v'\}} \mathrm{Ex}^-(v) &< -\beta'(m_0-1) \\
&< -\frac{2}{3}\beta'(m_0-1)
\end{aligned}$$

Let $\phi(w_0) = v_0$. We next wish to find the image of the legs of $\Lambda$ emanating from the root $w_0$. For the right leg follow $C_{m_0}$ clockwise to a vertex $u$ that links outward and satisfies $-3\beta m_0 < \sum_{v \in B(u,v_0)} \mathrm{Ex}^-(v) < -\beta m_0$ as in Lemma 14. Note that Corollary 12 implies that if $3\beta^2 m_0^2$ vertices are passed then the total excess of passed vertices is at most $-3\beta m_0$. Therefore, at most $3\beta^2 m_0^2$ vertices are passed before vertex $u$ is found. Follow the boundary of

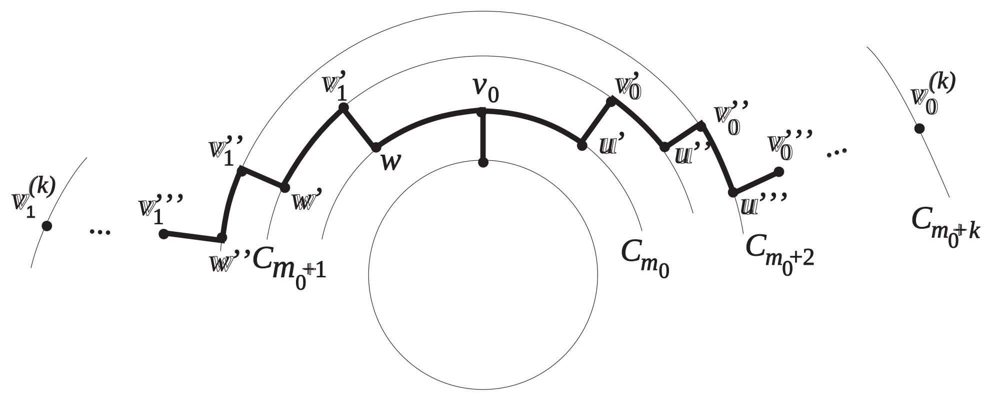


Figure 4: Embedding a subdivision of $B$ in $\Gamma$.

the face from $u$ to a vertex $v_0'$ in $C_{m_0+1}$. This path around the boundary has length at most $\frac{\ell}{2}$. From $v_0'$ follow $C_{m_0+1}$ clockwise as before until a vertex $u'$ is reached that links outward and satisfies $-3\beta(m_0+1) < \sum_{v\in B(u',v_0')} \mathrm{Ex}^-(v) < -\beta(m_0+1)$. As before, at most $3\beta^2(m_0+1)^2$ vertices are passed before arriving at $u'$. Continue this process until you reach $C_{m_0+k}$ where $\delta^k > \frac{\beta'}{3\beta}$ and $k \geq 2$. The path in $\Gamma$ which is the image of the left leg of $\Lambda$ is defined in the same way, except the direction is counterclockwise instead of clockwise. Let $v_0^{(k)}$ and $v_1^{(k)}$ be the end vertices of the right and left legs respectively as indicated in Figure 4.

Note that by construction $\sum_{v\in B(u,w)} \mathrm{Ex}^-(v) < -2\beta m_0$. By Lemma 13, $\sum_{v\in B(v_0',v_1')} \mathrm{Ex}^-(v) < -2\beta(m_0+1)\delta$, so $\sum_{v\in B(u',w')} \mathrm{Ex}^-(v) < -2\beta(m_0+1)\delta$. Continuing in this manner we see that for each $1 \leq j \leq k$

$$\begin{aligned}
\sum_{v\in B(v_0^{(j)},v_1^{(j)})} \mathrm{Ex}^-(v) &< -2\beta(m_0+j)\delta^j. \\
&< -2\beta(m_0+j)\frac{\beta'}{3\beta} \\
&= -\frac{2}{3}(m_0+j)\beta'.
\end{aligned}$$

Note that these inequality implies that the images of the left and right legs of a subdivision of the infinite binary tree $B$ intersect only at the vertex $v_0$.

We construct left and right legs from $v_1^{(k)}$ and $v_0^{(k)}$ using the same procedure as was used to construct left and right legs from $v_0$. Note that Lemma 14 implies that the images of nonintersecting legs in $B$ do not intersect in $\Gamma$.

By repeating this process, we construct a subgraph of $\Gamma$ homeomorphic with a full binary tree. The images of the horizontal edges of $\Lambda$ are mapped to paths along the cycles $C_r$ for appropriate values of $r$.

By summing up the lengths of the pieces of the paths defined above, the length of an $s$–leg (from level $s$ to level $s+1$) is at most

$$\sum_{j=0}^{k-1} \left( \frac{\ell}{2} + 3\beta^2 (m_0 + sk + j)^2 \right) = O(s^2).$$

Therefore,

$$\begin{aligned} d(\phi(w_0), \phi(w)) &= \sum_{s=0}^{d(w_0,w)} O(s^2) \\ &= O(d(w_0, w)^3). \end{aligned}$$

■

We note that Lemma 15 implies Theorem 2, namely that $\Gamma$ is $O(m^3)$-universal.

# 4 An example

We now construct a 1–ended planar graph $\Phi$ with bounded degree and codegree which has average excess positive at each level measured from a specified fixed vertex. The graph $\Phi$ does not satisfy the conditions of Theorem 2 because many of its vertices do not have positive excess. Furthermore, $\Phi$ is recurrent and not transient, consequently $\Phi$ cannot be universal. This shows that the condition of Theorem 2 stating that each vertex has positive excess cannot be replaced by a condition on the average excess.

The building blocks for graph $\Phi$ are given in Figure 5. The graph $\Phi$ starts with the graph on the left of Figure 5. The fixed vertex $v_0$ is labeled as $C_2$ in the figure. We first describe how to complete the inside of the triangle $A_2B_2C_2$. We place a copy of the right graph of Figure 5 which we call the bow tie graph inside $A_2B_2C_2$ identifying vertices $A_2$, $B_2$, and $C_2$ with vertices

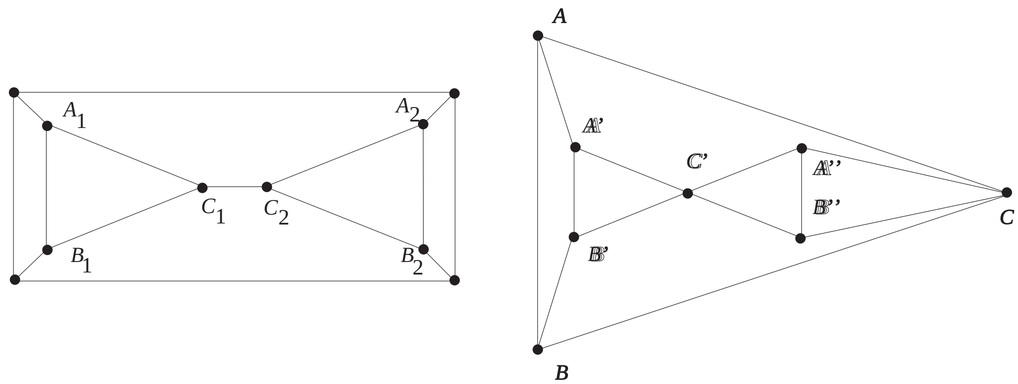


Figure 5: The building blocks of $\Phi$.

$A$, $B$, and $C$ respectively, and identifying edges between the same vertices. This bow tie creates two more internal triangles. We insert another bow tie inside $A'B'C'$.

In the left triangle $A_1B_1C_1$ we insert a bow tie, giving us two triangles. In each of these triangles we insert a bow tie, giving us four triangles. Into each of these triangles we insert a bow tie, giving us eight triangles. We think of these eight triangles as being at level 3. At level $m \geq 3$, we insert exactly $k_m$ bow ties and leave $s_m$ triangles empty. We always fill in the triangles consecutively starting from the left. Note that for triangles which are left empty the excess at each vertex of the triangle is negative, whereas if a triangle is filled with a bow tie the excess at each vertex is positive. The numbers $s_m$ and $k_m$ are required to satisfy

$$\frac{1}{8} \leq \frac{s_m}{k_m} \leq \frac{7}{8}.$$

It follows that

$$\frac{k_m - s_m}{k_m + s_m} \geq \frac{1}{15}$$

which implies that for some $\epsilon > 0$, the average excess of all vertices exactly $m$ away from $C_2$ (using face distance) is at least $\epsilon$ for every $m$.

Furthermore, it follows that

$$\frac{k_{m+1} + s_{m+1}}{k_m + s_m} \geq \frac{16}{15}$$

which implies the graph has exponential growth.

It is not hard to check that $\Phi$ is 1-ended. To see this, note that

$$\begin{aligned}\frac{k_{m+1}+s_{m+1}}{k_m+s_m} &\le \frac{16}{9}\\ &< 2.\end{aligned}$$

If at some level $m$, the triangle on the left has all the triangles inside it filled in at every level, then the growth of $k_m + s_m$ would have to be $O(2^n)$. But the equation above says the growth is $o(2^n)$ as $\frac{16}{9} < 2$. Consequently, at each level the only triangle containing an infinite number of filled in triangles is the one on the far left.

We note that the way we defined $\Phi$ the vertex set of $\Phi$ has a limit point in the plane. By mapping this limit point to infinity, we can properly embed $\Phi$ in the plane.

The graph $\Phi$ has average excess at least $\epsilon$ for some positive $\epsilon$ in the sense that if you average the excess of all the vertices at a fixed distance from $C_2$, you get a number larger than $\epsilon$. We note that $\Phi$ does not satisfy an $f$-isoperimetric condition for any function $f$ that approaches infinity, since there are vertex sets $V$ of arbitrarily high but finite cardinality whose boundary $\partial V$ consist of exactly three points. Furthermore, $\Phi$ is recurrent. This can be seen by using the method of shorting out edges as described in [5]. Since $\Phi$ is recurrent, it is not transient, and therefore, not universal.

We end by giving an alternate proof of Corollary 3 that uses the isoperimetric condition described in the Introduction, instead of Theorem 2.

Let $\Gamma$ be an infinite 1-ended planar graph with bounded degree and codegree. Suppose there is a number $\epsilon > 0$ such that every vertex of $\Gamma$ has excess at least $\epsilon$. Let $V$ be a set of vertices in $\Gamma$. Let $V'$ be all the vertices that are on a face incident with a vertex in $V$. Let $V''$ be all the vertices of $V'$ together with all the vertices in finite components of $\Gamma - V'$. The graph $H$ induced by $V''$ is finite. Furthermore, the union of the closed finite faces of $H$ form a disk $D$ since all the holes are filled in by including vertices in $V''$.

Let $A$ be the vertices on the boundary of the disk and $B$ the vertices inside the disk. Then $V'' = A \cup B$ and $|\partial V''| \geq \frac{|A|}{\ell}$. Now,

$$\begin{aligned} -2 &= \sum_{v \in A} \mathrm{Ex}_D(v) + \sum_{v \in B} \mathrm{Ex}_D(v) \\ &\geq -\frac{2}{3}|A| + \epsilon|B| \end{aligned}$$

It follows that $|\partial V''| \geq c_1|V''| + c_2$ for constants $c_1$ and $c_2$.

Note that $|\partial V'| \geq |\partial V''|$ and $|V| \leq |V'| \leq |V''|$. Also, for each face bounding cycle included in $V'$ but not in $V$, less than $d\ell$ new boundary vertices are created by including the cycle and $\partial V$ intersects the cycle. Also, each vertex on $\partial V'$ is on at most $\ell$ face bounding cycles. Therefore,

$$\begin{aligned} \frac{|\partial V|}{|V|} &\geq \frac{1}{d\ell^2}\frac{|\partial V'|}{|V'|} \\ &\geq \frac{1}{d\ell^2}\frac{|\partial V''|}{|V''|} \\ &\geq \frac{1}{d\ell^2}\left(c_1 + \frac{c_2}{|V''|}\right) \\ &\geq c_3 \end{aligned}$$

for some constant $c_3 > 0$. We let $f(n) = c_3 n$ and note that

$$\sum_n \left(\frac{1}{f(n)}\right)^2$$

converges. By the $f$-isoperimetric condition, the graph $\Gamma$ is transient.

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