On the Choquet-Dolecki Theorem

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Abstract

In this paper, we prove that if a multifunction \( \Phi : T \to X \) from a first countable space \( T \) into a space \( X \) with property (\( \ast \)) is usc at a point \( t_0 \in T \), then the active boundary of \( \Phi \) at \( t_0 \) is compact. Moreover, we also show that if \( X \) is an angelic space then the active boundary of \( \Phi \) at \( t_0 \) is the smallest kernel of \( \Phi \) at \( t_0 \).

1 Introduction

Let \( T \) and \( X \) be topological spaces and \( \Phi : T \to X \) a multifunction. We say that \( \Phi \) is upper semicontinuous, (abbreviated as \( \text{usc} \)), at \( t_0 \in T \) if for each open subset \( V \) containing \( \Phi(t_0) \) there exists a neighbourhood \( U \) of \( t_0 \) such that \( \Phi(U) \subseteq V \) and we shall call a subset \( K \subseteq \Phi(t_0) \) a kernel [2] of \( \Phi \) at \( t_0 \), if for every open set \( V \) containing \( K \) there is a neighbourhood \( U \) of \( t_0 \) such that \( \Phi(U) \setminus \Phi(t_0) \subseteq V \). Obviously, \( \Phi(t_0) \) is a kernel for \( \Phi \) at \( t_0 \) and so the interest here is in finding smaller kernels. Choquet stated in [2], without proof, the following result.

**Theorem 1.1.** Let \( \Phi : T \to X \) be a multifunction from a metric space \( T \) into a metric space \( X \). If \( \Phi \) is usc at \( t_0 \), then \( \Phi \) has a compact kernel at \( t_0 \).

AMS (1991) Subject Classification—Primary 54C60; Secondary 54D99, 54E99.

Key Words and Phrases: Multifunction, kernel, Dieudonné-complete, angelic.
Nearly 30 years later Dolecki [3] improved Theorem 1.1 and provided a natural candidate for a kernel of any multifunction. This candidate kernel is called the **active boundary** of $\Phi$ at $t_0 \in T$, denoted by $\text{Frac}\Phi(t_0)$, and defined by:

\[
\text{Frac}\Phi(t_0) \equiv \bigcap_{U \in \mathcal{U}(t_0)} \Phi(U) \setminus \Phi(t_0),
\]

where $\mathcal{U}(t_0)$ denotes the set of all neighbourhoods of $t_0$.

**Theorem 1.2. (Choquet-Dolecki Theorem)**

Let $\Phi$ be a multifunction from a topological space $T$ into a metric space $X$. If $t_0 \in T$ has a countable local base and $\Phi$ is usc at $t_0$ then $\text{Frac}\Phi(t_0)$ is compact and a kernel for $\Phi$ at $t_0$.

The proofs of both Theorem 1.1 and Theorem 1.2 rely on the following simple lemma [1].

**Lemma 1.3.** Let $\Phi$ be a multifunction from a topological space $T$ into a Hausdorff space $X$. If $\{U_n : n \in \mathbb{N}\}$ is a local base for $t_0 \in T$ and $\Phi$ is usc at $t_0$, then each sequence $(x_n : n \in \mathbb{N})$ in $X$ with $x_n \in \Phi(U_n) \setminus \Phi(t_0)$ has a cluster point in $\Phi(t_0)$. In particular, if $x = \lim_{n \to \infty} x_n$ and $x_n \in \Phi(U_n) \setminus \Phi(t_0)$ then $x \in \Phi(t_0)$.

Theorem 1.2 has subsequently been refined in terms of the following definitions. A space $X$ is **angelic** [8] if (i) each relatively countably compact subset (i.e. every sequence of distinct elements of the set has a cluster point) of $X$ is compact; (ii) each point in the closure of a relatively compact subset $A$ of $X$ is the limit of some sequence in $A$, and $X$ is called **Dieudonné-complete** [7] if it can be embedded as a closed subspace of the Cartesian product of a family of metrizable spaces. It was shown in [6] that if a multifunction $\Phi : T \to X$ from a first countable space $T$ into a Dieudonné-complete space $X$ is usc at $t_0$ and $\Phi(t_0)$ is closed then $\text{Frac}\Phi(t_0)$ is compact and a kernel for $\Phi$ at $t_0$. In addition, it was shown in [9] that $\text{Frac}\Phi(t_0)$ is compact and a kernel for $\Phi$ at $t_0$ when $X$ is angelic.

A common generalization of Dieudonné-completeness and angelicness is the property $(\ast)$. The main purpose of this note is to study kernels and active boundaries of multifunctions that map into spaces possessing property $(\ast)$. 

2
2 Main results

Let $X$ be a topological space. Consider the following property ($\ast$) of $X$:

($\ast$): The closure $\overline{A}$ of each relatively countably compact subset $A$ of $X$ is compact.

Property ($\ast$) is productive and hereditary, with respect to closed subsets, preserved by perfect maps and possessed by all metrizable topological spaces.

**Proposition 2.1.** Let $X$ be a regular Hausdorff space possessing property ($\ast$), and let $\{A_n : n \in \mathbb{N}\}$ be a decreasing sequence of non-empty subsets of $X$. If each sequence $(a_n : n \in \mathbb{N})$ with $a_n \in A_n$ has a cluster point in $X$, then $\bigcap_{n=1}^{\infty} \overline{A_n}$ is (non-empty) compact and for each open set $W$ containing $\bigcap_{n=1}^{\infty} \overline{A_n}$ there exists a $n_0 \in \mathbb{N}$ so that $A_{n_0} \subseteq W$.

**Proof.** Let $A = \{a \in X : a \in \bigcap_{m=1}^{\infty} \{a_k : k \geq m\} \text{ and } a_n \in A_n\}$. Clearly, from the regularity of $X$ we have that $\emptyset \neq A \subseteq \bigcap_{n=1}^{\infty} \overline{A_n} \subseteq \overline{A}$. So to show that $\bigcap_{n=1}^{\infty} \overline{A_n}$ is compact it suffices to show that $A$ is relatively countably compact. To this end, let $(a^n : n \in \mathbb{N})$ be a sequence in $A$. For each $n \in \mathbb{N}$, choose a sequence $(a^n_k : k \in \mathbb{N})$ with $a^n_k \in A_k$ so that $a^n$ is a cluster point of $(a^n_k : k \in \mathbb{N})$. Let

$$B = \{a^n_k : 1 \leq n \leq k, (n, k) \in \mathbb{N} \times \mathbb{N}\}.$$  

We claim that $B$ is relatively countably compact. Indeed, let $\{b_n : n \in \mathbb{N}\} \subseteq B$ be an arbitrary sequence of distinct elements of $B$. Since for each $n \in \mathbb{N}$, $\{b \in B : b \notin A_n\}$ is finite we see that the sequence $(b_n : n \in \mathbb{N})$ must eventually pass down through the sets $A_n$. In particular this means that we may extract a subsequence $(b_{n_k} : k \in \mathbb{N})$ of $(b_n : n \in \mathbb{N})$ so that $b_{n_k} \in A_{n_k}$. Hence by the hypothesis $(b_{n_k} : k \in \mathbb{N})$ has a cluster point, and so $B$ is relatively countably compact. This in turn implies that the sequence $(a^n : n \in \mathbb{N})$ has a cluster point, since $\{a^n : n \in \mathbb{N}\} \subseteq \overline{B}$. This shows that $A$ is relatively countably compact and hence $\bigcap_{n=1}^{\infty} \overline{A_n}$ is compact. The proof of the last claim of the proposition is trivial.  

If $X$ is a topological space, then the $G_{\delta}$-topology on $X$ is the topology generated by taking all the $G_{\delta}$-sets in $X$ as a base. The following lemma is contained in [9].
Lemma 2.2. Let $\Phi : T \to X$ be a multifunction from a topological space $T$ into a regular space $X$. If $t_0$ has a countable local base and $\Phi$ is usc at $t_0$, with $\Phi(t_0)$ closed in the $G_\delta$-topology then $\text{Frac}\Phi(t_0) \subseteq \Phi(t_0)$.

Theorem 2.3. Let $\Phi$ be a multifunction from a topological space $T$ into a regular Hausdorff space $X$ with property $(\ast)$. If $\Phi$ is usc at $t_0 \in T$, $t_0$ has a countable local base and $\Phi(t_0)$ is closed in the $G_\delta$-topology then $\text{Frac}\Phi(t_0)$ is compact and a kernel for $\Phi$ at $t_0$.

Proof. This follows directly from Lemma 2.2, Lemma 1.3 and Proposition 2.1 applied to the sets $A_n = \Phi(U_n) \setminus \Phi(t_0)$, where \{${U_n : n \in \mathbb{N}}$\} is a countable monotonic decreasing local base for $t_0$.

Next we will improve both Theorem 4.1 of [6] and Theorem 1 of [9].

Proposition 2.4. Let $X$ be a regular Hausdorff angelic space, and let \{${A_n : n \in \mathbb{N}}$\} be a decreasing sequence of non-empty subsets of $X$. If each sequence $(a_n : n \in \mathbb{N})$ with $a_n \in A_n$ has a cluster point in $X$, then \(\bigcap_{n=1}^{\infty} A_n\) is (non-empty) compact and \(\bigcap_{n=1}^{\infty} A_n = \{a : a = \lim_{n \to \infty} a_n, a_n \in A_n\}\).

Proof. \cite{From Proposition 2.1, it suffices to show that:} \(\bigcap_{n=1}^{\infty} A_n = \{a : a = \lim_{n \to \infty} a_n, a_n \in A_n\}\).

Let $A = \{a \in X : a \in \bigcap_{n=1}^{\infty} \{a_k : k \geq m\}$ and $a_n \in A_n\}$. As in Proposition 2.1, $\overline{A} = \bigcap_{n=1}^{\infty} A_n$. So consider $a \in \overline{A}$. If $a \in \bigcap_{n=1}^{\infty} A_n$, then the result is obvious, and so we will consider the case $a \notin \bigcap_{n=1}^{\infty} A_n$. In this situation there is a $n_0 \in \mathbb{N}$ so that $a \notin A_n$ for all $n \geq n_0$. Since $X$ is angelic there exists a sequence \{${a^n : n \in \mathbb{N}}$\} $\subseteq A$ such that $a = \lim_{n \to \infty} a^n$. For each $n \in \mathbb{N}$, choose a sequence $(a^n_k : k \in \mathbb{N})$ with $a^n_k \in A_k$ so that $a^n$ is a cluster point of $(a^n_k : k \in \mathbb{N})$. Let $b^n_k = a^n_{k+n_0}$ and

$$B = \{b^n_k : 1 \leq n \leq k, (n,k) \in \mathbb{N} \times \mathbb{N}\}.$$

Then $a \notin B$, but $a \notin \overline{B}$, since \{${a^n : n \in \mathbb{N}}$\} $\subseteq \overline{B}$. As shown in Proposition 2.1, $B$ is relatively countably compact. Hence, there exists a sequence of distinct elements $(a'_n : n \in \mathbb{N})$ of $B$ so that $a = \lim_{n \to \infty} a'_n$. After, possibly passing to a subsequence (and re-labeling), we may assume that $a'_n \in A_n$. This completes the proof.
Theorem 2.5. Let $\Phi : T \to X$ be a multifunction from a topological space $T$ into a regular Hausdorff angelic space $X$. If $\Phi$ is usc at $t_0$ and $t_0$ has a countable local base, then $\text{Frac} \Phi(t_0)$ is compact and is the smallest kernel of $\Phi(t_0)$ at $t_0$.

Proof. That $\text{Frac} \Phi(t_0)$ is compact and a kernel follows directly from Proposition 2.4 and Lemma 1.3. So we are left with showing that $\text{Frac} \Phi(t_0)$ is the smallest kernel for $\Phi$. To this end, let $K$ be an arbitrary kernel of $\Phi$ at $t_0$, and let $\{U_n : n \in \mathbb{N}\}$ be a countable monotonic decreasing local base for $t_0$. Assume that there is some point $a \in \text{Frac} \Phi(t_0) \setminus K$, then by Proposition 2.4, there is a sequence $(a_n : n \in \mathbb{N})$ with $a_n \in \Phi(U_n) \setminus \Phi(t_0)$ such that $\lim_{n \to \infty} a_n = a$. Set $B = \{a\} \cup \{a_n : n \in \mathbb{N}\}$. Then $B$ is closed and disjoint from $K$. Since $K$ is a kernel there is some $n_0 \in \mathbb{N}$ such that $\Phi(U_{n_0}) \setminus \Phi(t_0) \subseteq (X \setminus B)$. It follows then that $a_{n_0} \notin B$, which is impossible. Hence $\text{Frac} \Phi(t_0) \subseteq K$ for every kernel $K$ of $\Phi$ at $t_0$. This completes the proof. \qed

References