On the Choquet-Dolecki Theorem

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Abstract

In this paper, we prove that if a multifunction $\Phi : T \to X$ from a first countable space T into a space X with property $(*)$ is usc at a point $t_0 \in T$, then the active boundary of Φ at t_0 is compact. Moreover, we also show that if X is an angelic space then the active boundary of Φ at t_0 is the smallest kernel of Φ at t_0.

1 Introduction

Let T and X be topological spaces and $\Phi : T \to X$ a multifunction. We say that Φ is *upper semicontinuous*, (abbreviated as a *usc*), at $t_0 \in T$ if for each open subset V containing $\Phi(t_0)$ there exists a neighbourhood U of t_0 such that $\Phi(U) \subseteq V$ and we shall call a subset $K \subseteq \Phi(t_0)$ a *kernel* [2] of Φ at t_0, if for every open set V containing K there is a neighbourhood U of t_0 such that $\Phi(U) \setminus \Phi(t_0) \subseteq V$. Obviously, $\Phi(t_0)$ is a kernel for Φ at t_0 and so the interest here is in finding smaller kernels. Choquet stated in [2], without proof, the following result.

Theorem 1.1. *Let $\Phi : T \to X$ be a multifunction from a metric space T into a metric space X. If Φ is usc at t_0, then Φ has a compact kernel at t_0.*

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Nearly 30 years later Dolecki [3] improved Theorem 1.1 and provided a natural candidate for a kernel of any multifunction. This candidate kernel is called the *active boundary* of Φ at $t_0 \in T$, denoted by $Frac\Phi(t_0)$, and defined by:

$$Frac\Phi(t_0) \equiv \bigcap_{U \in \mathcal{U}(t_0)} \overline{\Phi(U) \setminus \Phi(t_0)},$$

where $\mathcal{U}(t_0)$ denotes the set of all neighbourhoods of t_0.

Theorem 1.2. *(Choquet-Dolecki Theorem)*
Let Φ be a multifunction from a topological space T into a metric space X. If $t_0 \in T$ has a countable local base and Φ is usc at t_0 then $Frac\Phi(t_0)$ is compact and a kernel for Φ at t_0.

The proofs of both Theorem 1.1 and Theorem 1.2 rely on the following simple lemma [1].

Lemma 1.3. *Let Φ be a multifunction from a topological space T into a Hausdorff space X. If $\{U_n : n \in \mathbf{N}\}$ is a local base for $t_0 \in T$ and Φ is usc at t_0, then each sequence $(x_n : n \in \mathbf{N})$ in X with $x_n \in \Phi(U_n)\backslash\Phi(t_0)$ has a cluster point in $\Phi(t_0)$. In particular, if $x = \lim_{n\to\infty} x_n$ and $x_n \in \Phi(U_n)\backslash\Phi(t_0)$ then $x \in \Phi(t_0)$*

Theorem 1.2 has subsequently been refined in terms of the following definitions. A space X is *angelic* [8] if (i) each relatively countably compact subset (i.e. every sequence of distinct elements of the set has a cluster point) of X is compact; (ii) each point in the closure of a relatively compact subset A of X is the limit of some sequence in A, and X is called *Dieudonné-complete* [7] if it can be embeded as a closed subspace of the Cartesian product of a family of metrizable spaces. It was shown in [6] that if a multifunction $\Phi : T \to X$ from a first countable space T into a Dieudonné-complete space X is usc at t_0 and $\Phi(t_0)$ is closed then $Frac\Phi(t_0)$ is compact and a kernel for Φ at t_0. In addition, it was shown in [9] that $Frac\Phi(t_0)$ is compact and a kernel for Φ at t_0 when X is angelic.

A common generalization of Dieudonné-completeness and angelicness is the property $(*)$. The main purpose of this note is to study kernels and active boundaries of multifunctions that map into spaces possessing property $(*)$.

2 Main results

Let X be a topological space. Consider the following property $(*)$ of X:

$(*)$: The closure $\overline{A}$ of each relatively countably compact subset A of X is compact.

Property $(*)$ is productive and hereditary, with respect to closed subsets, preserved by perfect maps and possessed by all metrizable topological spaces.

Proposition 2.1. *Let X be a regular Hausdorff space possessing property $(*)$, and let $\{A_n : n \in \mathbf{N}\}$ be a decreasing sequence of non-empty subsets of X. If each sequence $(a_n : n \in \mathbf{N})$ with $a_n \in A_n$ has a cluster point in X, then $\bigcap_{n=1}^{\infty} \overline{A_n}$ is (non-empty) compact and for each open set W containing $\bigcap_{n=1}^{\infty} \overline{A_n}$ there exists a $n_0 \in \mathbf{N}$ so that $A_{n_0} \subseteq W$.*

Proof. Let $A = \{a \in X : a \in \bigcap_{m=1}^{\infty} \overline{\{a_k : k \geq m\}}$ and $a_n \in A_n\}$. Clearly, from the regularity of X we have that $\emptyset \neq A \subseteq \bigcap_{n=1}^{\infty} \overline{A_n} \subseteq \overline{A}$. So to show that $\bigcap_{n=1}^{\infty} \overline{A_n}$ is compact it suffices to show that A is relatively countably compact. To this end, let $(a^n : n \in \mathbf{N})$ be a sequence in A. For each $n \in \mathbf{N}$, choose a sequence $(a_k^n : k \in \mathbf{N})$ with $a_k^n \in A_k$ so that a^n is a cluster point of $(a_k^n : k \in \mathbf{N})$. Let

$$B = \{a_k^n : 1 \leq n \leq k, (n,k) \in \mathbf{N} \times \mathbf{N}\}.$$

We claim that B is relatively countably compact. Indeed, let $\{b_n : n \in \mathbf{N}\} \subseteq B$ be an arbitrary sequence of distinct elements of B. Since for each $n \in \mathbf{N}$, $\{b \in B : b \notin A_n\}$ is finite we see that the sequence $(b_n : n \in \mathbf{N})$ must eventually pass down through the sets A_n. In particular this means that we may extract a subsequence $(b_{n_k} : k \in \mathbf{N})$ of $(b_n : n \in \mathbf{N})$ so that $b_{n_k} \in A_k$. Hence by the hypothesis $(b_{n_k} : k \in \mathbf{N})$ has a cluster point, and so B is relatively countably compact. This in turn implies that the sequence $(a^n : n \in \mathbf{N})$ has a cluster point, since $\{a^n : n \in \mathbf{N}\} \subseteq \overline{B}$. This shows that A is relatively countably compact and hence $\bigcap_{n=1}^{\infty} \overline{A_n}$ is compact. The proof of the last claim of the proposition is trivial. □

If X is a topological space, then the *G_δ-topology* on X is the topology generated by taking all the G_δ-sets in X as a base. The following lemma is contained in [9].

Lemma 2.2. *Let $\Phi : T \to X$ be a multifunction from a topological space T into a regular space X. If t_0 has a countable local base and Φ is usc at t_0, with $\Phi(t_0)$ closed in the G_δ-topology then $Frac\Phi(t_0) \subseteq \Phi(t_0)$.*

Theorem 2.3. *Let Φ be a multifunction from a topological space T into a regular Hausdorff space X with property $(*)$. If Φ is usc at $t_0 \in T$, t_0 has a countable local base and $\Phi(t_0)$ is closed in the G_δ-topology then $Frac\Phi(t_0)$ is compact and a kernel for Φ at t_0.*

Proof. This follows directly from Lemma 2.2, Lemma 1.3 and Proposition 2.1 applied to the sets $A_n = \Phi(U_n) \setminus \Phi(t_0)$, where $\{U_n : n \in \mathbf{N}\}$ is a countable monotonic decreasing local base for t_0. □

Next we will improve both Theorem 4.1 of [6] and Theorem 1 of [9].

Proposition 2.4. *Let X be a regular Hausdorff angelic space, and let $\{A_n : n \in \mathbf{N}\}$ be a decreasing sequence of non-empty subsets of X. If each sequence $(a_n : n \in \mathbf{N})$ with $a_n \in A_n$ has a cluster point in X, then $\bigcap_{n=1}^{\infty} \overline{A_n}$ is (non-empty) compact and $\bigcap_{n=1}^{\infty} \overline{A_n} = \{a : a = \lim\limits_{n\to\infty} a_n, a_n \in A_n\}$.*

Proof. ¿From Proposition 2.1, it suffices to show that:

$$\bigcap_{n=1}^{\infty} \overline{A_n} = \{a : a = \lim_{n\to\infty} a_n, a_n \in A_n\}.$$

Let $A = \{a \in X : a \in \bigcap_{m=1}^{\infty} \overline{\{a_k : k \geq m\}}$ and $a_n \in A_n\}$. As in Proposition 2.1, $\overline{A} = \bigcap_{n=1}^{\infty} \overline{A_n}$. So consider $a \in \overline{A}$. If $a \in \bigcap_{n=1}^{\infty} A_n$, then the result is obvious, and so we will consider the case $a \notin \bigcap_{n=1}^{\infty} A_n$. In this situation there is a $n_0 \in \mathbf{N}$ so that $a \notin A_n$ for all $n \geq n_0$. Since X is angelic there exists a sequence $\{a^n : n \in \mathbf{N}\} \subseteq A$ such that $a = \lim\limits_{n\to\infty} a^n$. For each $n \in \mathbf{N}$, choose a sequence $(a_k^n : k \in \mathbf{N})$ with $a_k^n \in A_k$ so that a^n is a cluster point of $(a_k^n : k \in \mathbf{N})$. Let $b_k^n = a_{k+n_0}^n$ and

$$B = \{b_k^n : 1 \leq n \leq k, (n,k) \in \mathbf{N} \times \mathbf{N}\}.$$

Then $a \notin B$, but $a \notin \overline{B}$, since $\{a^n : n \in \mathbf{N}\} \subseteq \overline{B}$. As shown in Proposition 2.1, B is relatively countably compact. Hence, there exists a sequence of distinct elements $(a'_n : n \in \mathbf{N})$ of B so that $a = \lim\limits_{n\to\infty} a'_n$. After, possibly passing to a subsequence (and re-labeling), we may assume that $a'_n \in A_n$. This completes the proof. □

Theorem 2.5. *Let $\Phi : T \to X$ be a multifunction from a topological space T into a regular Hausdorff angelic space X. If Φ is usc at t_0 and t_0 has a countable local base, then $Frac\Phi(t_0)$ is compact and is the smallest kernel of $\Phi(t_0)$ at t_0.*

Proof. That $Frac\Phi(t_0)$ is compact and a kernel follows directly from Proposition 2.4 and Lemma 1.3. So we are left with showing that $Frac\Phi(t_0)$ is the smallest kernel for Φ. To this end, let K be an arbitrary kernel of Φ at t_0, and let $\{U_n : n \in \mathbf{N}\}$ be a countable monotonic decreasing local base for t_0. Assume that there is some point $a \in Frac\Phi(t_0) \setminus K$, then by Proposition 2.4, there is a sequence $(a_n : n \in \mathbf{N})$ with $a_n \in \Phi(U_n) \setminus \Phi(t_0)$ such that $\lim_{n\to\infty} a_n = a$. Set $B = \{a\} \bigcup \{a_n : n \in \mathbf{N}\}$. Then B is closed and disjoint from K. Since K is a kernel there is some $n_0 \in \mathbf{N}$ such that $\Phi(U_{n_0}) \setminus \Phi(t_0) \subseteq (X \setminus B)$. It follows then that $a_{n_0} \notin B$, which is impossible. Hence $Frac\Phi(t_0) \subseteq K$ for every kernel K of Φ at t_0. This completes the proof. □

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