Growth of Infinite Planar Graphs

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Abstract

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The growth rate of infinite planar 1–ended graphs is considered. For those graphs which are also concentric a recurrence relation is given which determines the growth rate. In the more general case lower bounds on the growth rate are given. In both the concentric and the general cases, the formulae involve the local condition of excess at a vertex.

1 Introduction

Graphs considered in this paper are infinite, planar and of bounded degree unless otherwise indicated. The symbols $V(\Gamma)$ and $E(\Gamma)$ will denote respectively the vertex set and the edge set. We do not in general restrict to simple graphs, but allow multiedges and loops. A loop contributes two to the degree of the incident vertex while each multiedge contributes one to the degree of each incident vertex. An infinite graph is 1–ended if the removal of a finite set of vertices yields at most one infinite component.

Knowing the growth rate of the different classes of infinite planar graphs is a key concept in a possible solution to the conjecture [2] that a characterisation of infinite planar graphs can be obtained from the properties of their infinite paths. Bonnington, Imrich and Seifter [1] have obtained such a characterisation for infinite, locally finite, transitive, 1-ended graphs with polynomial growth. In a previous paper [3] Brand and Morton showed that a planar graph Γ having the properties assumed in [1] can be embedded nicely in either the Euclidean plane or the hyperbolic plane. In the former case Γ has quadratic growth and in the latter case it has exponential growth. In this paper the growth of a planar graph is studied based on local information from the embedding with no symmetry assumption.

Definition 1 *A disc, (Γ, D), is a finite connected graph embedded in the plane so that the union of its closed finite regions is a topological disc, D. An annulus (Γ, D) is a disc with the interior of a set of regions removed. A vertex is said to be exterior if it is incident with either the infinite region or* *a deleted region; otherwise it is said to be interior. We say that an annulus is missing k regions if it is obtained from a disc by deleting the interiors of $k-1$ regions.*

Recall that for a regular k-gon in the Euclidean plane an interior angle has size $(1-\frac{2}{k})\pi$. Hence, for a simple graph, if there are $deg(v) \geq 3$ regular polygons incident to a vertex v and these polygons have n_i edges respectively, where $1 \leq i \leq deg(v)$, then the sum of the angles at v is $\sum_{i=1}^{deg(v)}(1-\frac{2}{n_i})\pi$. For convenience we omit the factor of π and define the *excess* of a vertex in general as follows. (Loosely speaking the excess measures how far the sum of the angles incident to a vertex deviates from the normal Euclidian sum of 2π.)

The *excess* of a vertex v in an annulus is given by

$$Ex(v) = \left[\sum_i (1-\frac{2}{n_i})\right] - 2 + b_v,$$

where n_i is the number of edges bounding the i^{th} nondeleted region incident with v and b_v is the number of deleted regions incident with v. Intuitively, the reason for adding 1 for each deleted region incident with v is that the sum of the angles for the nondeleted regions should be less than 2π if some incident regions were deleted. Both n_i and b_v are counted with multiplicity. Furthermore, the same region is counted with multiplicity in the sum. In Figure **??** the shaded regions are not deleted while the white regions (including the infinite region) are deleted. Notice that the region labeled F counts twice for b_v and the edge labeled e counts twice to give $n_i = 6$ for this region. In the disc on the right the excess at three vertices is $-\frac{1}{3}$, the excess at the center vertex is $-\frac{4}{3}$, and the excess at the lower left vertex is $+\frac{1}{3}$. For an infinite graph Γ the excess of a vertex is treated as if it were an interior vertex, that is, $b_v = 0$.

The excess of an annulus (Γ, D) is defined to be $\sum_{v \in V(\Gamma)} Ex(v)$. The following result shows that every disc has excess -2.

Lemma 2 *If (Γ, D) is a disc then $\sum_{v \in V(\Gamma)} Ex(v) = -2$.*

Figure 1: Region F is counted twice in b_v and edge e is counted twice in n_i

Proof. From the definition of excess we have

$$\begin{aligned}
&\sum_{v\in V(\Gamma)} Ex(v)\\
&= \sum_{v\in V_{int}(\Gamma)} Ex(v) + \sum_{v\in V_{ext}(\Gamma)} Ex(v)\\
&= \sum_{v\in V_{int}(\Gamma)} \left(\left[\sum_{i=1}^{deg(v)} (1-\frac{2}{n_i})\right] - 2\right) + \sum_{v\in V_{ext}(\Gamma)} \left(\left[\sum_{i=1}^{deg(v)-1} (1-\frac{2}{n_i})\right] - 1\right)\\
&= \sum_{v\in V_{int}(\Gamma)} \sum_{i=1}^{deg(v)} (1-\frac{2}{n_i}) - 2|V_{int(\Gamma)}| + \sum_{v\in V_{ext}(\Gamma)} \sum_{i=1}^{deg(v)-1} (1-\frac{2}{n_i}) - |V_{ext(\Gamma)}|\\
&= \sum_{\text{all regions } R_j} (1-\frac{2}{n_{R_j}})n_{R_j} - 2|V_{int(\Gamma)}| - |V_{ext(\Gamma)}|\\
&= \sum_{\text{all regions } R_j} n_{R_j} - 2|V_{int(\Gamma)}| - |V_{ext(\Gamma)}| - 2f
\end{aligned}$$

where n_{R_j} is the number of edges of a region R_j and f is the number of regions in G.

Let v be the number of vertices in Γ, v_x be the number of external vertices, v_i be the number of internal vertices, and e be the number of edges. Then clearly $v = v_x + v_i$ and by Euler's formula we have $v_x + v_i - e + f = 1$. Since $e_x = v_x$ we have $\sum_{R_j} n_{R_j} = 2e - v_x$. Thus

$$\sum_{\text{all regions } R_j} n_{R_j} - 2|V_{int(\Gamma)}| - |V_{ext(\Gamma)}| - 2f$$

$$
\begin{aligned}
&= \sum_{\text{all regions } R_j} n_{R_j} - 2v_i - v_x - 2f \\
&= 2e - v_x - 2v_i - v_x - 2f \\
&= 2e - 2v - 2f \\
&= -2
\end{aligned}
$$

■

Corollary 3 *If* (Γ, D) *is an annulus missing* k *regions, then the excess of* (Γ, D) *is* $2(k-2)$.

Proof. The proof is by induction on the number of deleted regions. The case $k=1$ is simply Lemma **??**. So suppose that (Γ, D) is an annulus with $k>1$ missing regions. Let (Γ, D') be the annulus obtained by filling in one missing region R of D. To clarify which annulus we are using for excess we write $Ex(v,D)$ and $Ex(v,D')$ to denote the excess at the vertex v in the annulus (Γ, D) and (Γ, D') respectively. Let the vertices bounding R be $v_1, v_2, \ldots, v_r$ (listed without multiplicity). Furthermore we let the number of edges in the i^{th} nondeleted region (regions listed with multiplicity) incident with vertex v_j be denoted by $n_{i,j}$ for $1 \leq i \leq k_j$. For each vertex v_j, the i^{th} nondeleted region is incident with the region R at least once. Let t_j be the multiplicity of the incidence of v_j with R. We assume with no loss of generality that $n_{i,j} = n$ for $1 \leq i \leq t_j$ is the number of incident edges to R. Note that $t_1 + t_2 + \ldots + t_r = n$.

Then

$$
\begin{aligned}
&\sum_{v \in V(\Gamma)} Ex(v,D) - \sum_{v \in V(\Gamma)} Ex(v,D') \\
&= \sum_{j=1}^{r} \left[Ex(v_j,D) - Ex(v_j,D')\right] \\
&= \sum_{j=1}^{r} \left[\sum_{i=t_j+1}^{k_j} \left(1 - \frac{2}{n_{i,j}}\right) - 2 + b_{v_j} - \left(\sum_{i=1}^{k_j} \left(1 - \frac{2}{n_{i,j}}\right) - 2 + b'_{v_j}\right)\right] \\
&= \sum_{j=1}^{r} t_j \left(-\left(1 - \frac{2}{n}\right)\right) + \sum_{j=1}^{r} (b_{v_j} - b'_{v_j})
\end{aligned}
$$

$$
\begin{aligned}
&= -n+n\left(\frac{2}{n}\right)+n \\
&= 2
\end{aligned}
$$

So the formula follows by induction. ■

Recall that an infinite graph is 1–ended if the removal of a finite set of vertices yields at most one infinite component.

Definition 4 *An embedding of an infinite graph Γ in the plane is nice if the graph is 1–ended and connected and there is a number N such that each region is bounded by at most N edges (counting multiplicity).*

We will abuse notation and say Γ is nice to mean that we fix a nice embedding of Γ in the plane.

Definition 5 *Suppose that Γ is nice. Define a graph Γ' so that $V(\Gamma') = V(\Gamma)$, but for each pair of vertices v, w incident with a common region, vw is an edge in Γ'. The regional distance between two vertices v and w in Γ is the usual graph distance in the graph Γ' and is denoted d_R.*

Definition 6 *Let Γ be an infinite graph with a distance function d defined on its vertex set and $v_0 \in V(\Gamma)$. The growth of the graph is*

$$f(n) = |\{v \in \Gamma : d(v, v_0) < n\}|.$$

Note that if all the regions are triangles then regional distance corresponds to the usual definition of distance. In general, let d_R be the regional distance between two vertices and let d be the usual graph distance between the two vertices. Then since in a nice embedding, all regions have at most N edges, it follows that $d_R \leq d \leq \lfloor \frac{N}{2} \rfloor d_R$ where N is as in the definition of nice.

In particular, for a nicely embedded graph the growth using the usual graph distance and the growth using the regional distance have the same order. However, if the region size is allowed to be unbounded, then the

Figure 2: The growth depends on which distance function is used.

two growths may be quite different. The graph in Figure **??** shows a graph where the growth is exponential using regional distance while the usual graph distance gives quadratic growth. For the rest of this paper we will consider only nice graphs. Consequently, we use regional distance as either distance gives the same order of growth.

Theorem 7 *Let Γ be a nicely embedded simple graph where the excess at every vertex in Γ is zero, then Γ has quadratic growth.*

PROOF. The proof of this theorem follows from the techniques used in [3]. The idea is to replace each region with a regular polygon with edge length one. The condition that the excess at each point is zero ensures that the regions fit together to fill the plane. Since the area of a disc, πr^2, is a quadratic function of r it follows that Γ has quadratic growth. See [3] for details. ■

Common examples of such graphs are the tessellations of the plane.

Figure 3: A graph with linear growth, but average excess 0.

It is not sufficient to assume that the "average" excess of all vertices within m of v_0 is zero to have quadratic growth. Let Γ be the graph given in Figure **??** where all vertices with the same label are identified and parallel edges are replaced by a single edge. Note that for this planar 1–ended graph the average degree of all vertices at distance m from v_0 is six. Furthermore, all regions are triangles. However, since for $m \geq 3$ and odd, $|V_m| = 12$ and for $m \geq 4$ and even, $|V_m| = 13$, this graph has linear growth.

2 Concentric Graphs

For a nicely embedded graph Γ with specified vertex v_0, we let V_m be the set of vertices whose regional distance is exactly m from v_0. We use E_m to denote the set of edges between vertices in V_m and F_m to denote the set of edges having one incident vertex in V_m and the other in V_{m+1}. The set R_m is defined to be the set of regions with all incident vertices in $V_m \cup V_{m+1}$ and at least one incident vertex in V_m.

Definition 8 *Let Γ be a nicely embedded graph. Γ is said to be concentric if the subgraph of Γ induced by V_m is a cycle for each $m \geq 1$.*

In the case of concentric graphs, Proposition **??** gives a recurrence relation that determines the exact growth of a graph in terms of the average degree of vertices in V_m and the average number of sides in a region of R_m.

Proposition 9 *Let Γ be a concentric graph. Let the average degree of vertices in V_m be d_m and the average number of edges bounding regions in R_m be n_m. Then for $m \geq 1$*

$$\begin{aligned}(d_m - 2)|V_m| &= |R_m| + |R_{m-1}| \\ (n_m - 2)|R_m| &= |V_{m+1}| + |V_m|.\end{aligned}$$

Furthermore, $|V_0| = 1$ and $|R_0| = \deg(v_0)$.

Proof. Let C_m, $m \geq 1$, be the cycle containing the vertices in V_m and the edges in E_m. Note that $|E_m| = |V_m|$. Consider the annulus bounded by C_m and C_{m+1}. The number of regions in this annulus is equal to $|R_m|$, consequently the average number of edges bounding one of these regions is given by

$$n_m = \frac{|V_m| + |V_{m+1}| + 2|R_m|}{|R_m|},$$

and hence

$$(n_m - 2)|R_m| = (|V_m| + |V_{m+1}|)\,.$$

The average degree of a vertex in C_m is determined by dividing the total number of edges incident to vertices in C_m by the number of vertices in C_m.

$$d_m = \frac{|R_m| + |R_{m-1}| + 2|E_m|}{|V_m|}.$$

Since Γ is concentric, $|V_m| = |E_m|$. Consequently,

$$(d_m - 2)|V_m| = |R_m| + |R_{m-1}|.$$

■

Corollary 10 *Let* Γ *be a concentric graph. Then for* $m \geq 1$

$$0 = \frac{1}{n_m - 2}|V_{m+1}| + \left(\frac{1}{n_m - 2} + \frac{1}{n_{m-1} - 2} + 2 - d_m \right) |V_m| + \frac{1}{n_{m-1} - 2}|V_{m-1}|$$

where d_m *is the average degree for all vertices in* V_m *and* n_m *is the average number of edges bounding a region* R_m*.*

Proof. Substituting for $|R_m|$ in Proposition **??** gives the required recurrence relation. ■

Corollary 11 *Suppose that* Γ *is a concentric graph and each region has* n *incident edges. If* x_m *is the average excess of vertices in* V_m*, then*

$$0 = |V_{m+1}| - (2 + x_m n)|V_m| + |V_{m-1}|$$

for all $m \geq 1$*.*

Proof. This follows from the definition of average excess and Corollary **??**. ■

Fix a vertex v_0 in a nicely embedded graph Γ. For a cycle C a *dividing path* P is a path which intersects C only in the end points of P. Let $\Gamma[C]$ be the graph obtained from Γ by deleting all vertices in the same component as v_0 of the plane with C removed. A dividing path is *external* if it is contained in $\Gamma[C]$, otherwise it is *internal.*

Theorem 12 *Let* Γ *be a nicely embedded graph, where each region is a triangle, and the excess of every vertex is non-negative. For each integer* $m \geq 1$ *let* Q_m *be the conjunction of the following four statements.*

[$Q_m(i)$] *Each vertex in* V_m *is incident with one or two edges in* F_{m-1}*.*

[$Q_m(ii)$] *There is a cycle* C_m *in* Γ *with vertex set* V_m*.*

[$Q_m(iii)$] C_m *has no external path* P *in* $\Gamma - C_m$ *with* $|E(P)| = 1$*.*

[$Q_m(iv)$] *C_m has no external path P in $\Gamma - C_m$ with $|E(P)| = 2$, unless the two edges of P form a triangle with an edge of C_m.*

Then Q_m holds for every $m \geq 1$.

Proof. Induction is used. First we show that for any $m \geq 1$ $Q_m(i)$ and $Q_m(ii)$ together imply $Q_m(iii)$ and $Q_m(iv)$, and then that $Q_m(iii)$ and $Q_m(iv)$ together imply $Q_{m+1}(i)$ and $Q_{m+1}(ii)$.

If $Q_1(i)$ or $Q_1(ii)$ does not hold then there is a 1- or 2-edge circuit bounding a finite subgraph of Γ. But each internal vertex has non-negative excess and each external vertex has excess at least $-2/3$, contradicting Lemma **??**. Therefore, $Q_1(i)$ and $Q_1(ii)$ hold.

Suppose for some $m \geq 1$, $Q_m(i)$ and $Q_m(ii)$ hold but $Q_m(iii)$ and $Q_m(iv)$ do not both hold. By $Q_m(ii)$ there is a cycle C_m with vertex set V_m. Suppose that P is an external path of C_m which contradicts $Q_m(iii)$ or $Q_m(iv)$. Clearly P is in the graph $\Gamma[C_m]$. Let C_m, C'_m and C''_m be the three distinct cycles of $\Gamma|(E(C_m) \cup E(P))$. Since Γ is 1-ended, one of $\Gamma[C'_m]$ and $\Gamma[C''_m]$ is finite and hence forms with its finite regions a disc. Without loss of generality suppose that $\Gamma[C'_m]$ is finite. Since the excess of every vertex in Γ is non-negative every internal vertex of $\Gamma[C'_m]$ has non-negative excess. By $Q_m(i)$, every external vertex of $\Gamma[C'_m]$ has non-negative excess, except possibly the two or three vertices in $V(P)$ which have excess at least $-2/3$. Since the total excess for $\Gamma[C'_m]$ must be -2 it follows that $|V(C'_m)| = 3$, and that $\Gamma[C'_m]$ has no internal vertices and its only external vertices are on P. Thus for every $m \geq 1$, $Q_m(i)$ and $Q_m(ii)$ imply $Q_m(iii)$ and $Q_m(iv)$.

Now we show that Q_m implies $Q_{m+1}(i)$ and $Q_{m+1}(ii)$. Clearly $|V(C_m)| > 3$. If some vertex in $|V_{m+1}|$ is incident with at least three edges in F_m, then there is a 2-edge external path which contradicts $Q_m(iv)$, thus $Q_{m+1}(i)$ holds. Clearly H, the subgraph of Γ induced by V_m, is outerplanar as there is a path P' from v_0 to v for every $v \in V_m$ such that P' only contains vertices in $V_0 \cup V_1 \cup \ldots \cup V_m \cup \{v\}$. To show $Q_{m+1}(ii)$ it is sufficient to show that H is two connected. (If a graph is outerplanar and 2-connected then it has a

Hamiltonian cycle.) It is routine to check that if H is not two connected then, since Γ is a triangulation, there is a path contradicting $Q_m(iii)$ or $Q_m(iv)$. Thus $Q_m(ii)$ holds and the result follows. ■

Theorem 13 *Let Γ be a nicely embedded graph, where each region is a quadrilateral, and the excess of every vertex is non-negative. For each integer $m \geq 1$ let Q_m be the conjunction of the following four statements.*

[$Q_m(i)$] *Each vertex in V_m is incident with zero or one edges in F_{m-1}.*

[$Q_m(ii)$] *There is a cycle C_m with vertex set V_m.*

[$Q_m(iii)$] *C_m has no dividing path P in $\Gamma - C_m$ with $|E(P)| = 1$.*

[$Q_m(iv)$] *C_m has no external path P in $\Gamma - C_m$ with $|E(P)| = 2$ or 3, unless P forms three sides of a quadrilateral with an edge in E_m.*

Then Q_m holds for any $m \geq 1$

Proof. The same techniques are used in this proof as are used in Theorem **??**, so we omit the details. ■

Note that a similar proof for n-gons, $n \geq 5$ is hopeless since a critical part of the induction argument is that each vertex on the cycle has (exterior) excess of no more than 0. Even two pentagons incident with a vertex on a cycle would make the (exterior) excess positive at v.

Corollary 14 *Let Γ be nicely embedded planar graph with non–negative excess at each vertex. Furthermore assume that either each region is a triangle or each region is a quadrilateral. Then Γ is concentric.*

Proof. This follows imediately from Theorems **??** and **??**. ■

Corollary 15 *Let Γ be nicely embedded planar graph with all regions triangles or else all regions quadrilaterals. Furthermore, suppose the excess at each vertex is non-negative. If the average excess $x = x_m$ of all vertices in V_m is independent of m, then Γ has quadratic growth if $x = 0$ and exponential growth otherwise.*

PROOF. In the case where the average excess is 0, Theorem **??** states the growth is quadratic. By Corollary **??** we have the reccurence relation

$$0 = |V_{m+1}| - (2 + xn)|V_m| + |V_{m-1}|$$

where n, the number of edges bounding each region, is either 3 or 4. If $x > 0$, then the polynomial in t,

$$t^2 - (2 + xn)t + 1$$

has a real root larger than 1. Consequently $|V_m|$ grows exponentially. ■

3 Non–concentric Graphs

For a planar graph Γ, the radial graph Γ' is obtained by adding a vertex in each region, adding an edge from the vertex in a region to each vertex incident with the region, and deleting all the original edges. Note that each region in Γ' is a quadrilateral. Furthermore, regional distance between two vertices in Γ is double the usual graph distance in Γ' between the same two vertices. Consequently, we can conclude from Corollary **??** that a graph where each region has at least five sides and every vertex has degree at least four (or every region has at least four sides and every vertex has degree at least 5) grows exponentially. This is a special case of Corollary **??** below.

The example in Figure **??** shows that if we allow regions of triangles and hexagons, then we do not have a concentric graph even though the excess at each vertex is 0. Vertices in Figure **??** are labeled by their distance away from v_0 using regional distance. Theorems **??** and **??** give a lower bound on the growth of a graph when Theorems **??** and **??** are not satisfied.

The recursion formulae of the last section give neither an upper bound nor a lower bound on the growth rate of a graph which is not concentric. To see this consider the graphs in Figure **??**.

Figure **??** shows a graph Γ consisting of cycles C_m with the m^{th} cycle having length $c_m = 2^m$ for each $m > 0$. Between cycles m and $m + 1$

Figure 4: V_3 does not form a cycle.

Figure 5: Non-concentric graphs not satisfying recurrences.

there are $s_m = 2^m$ parallel edges and $q_m = 2^m$ edges with one vertex on the cycle C_m and the other vertex of degree 1 for each $m > 0$. Furthermore there are $l_m = 2^m$ loops based on cycle C_m between C_m and C_{m-1} for each $m > 0$. Note that the growth of Γ is on the order of 2^n. Also, $d_m - 2 = \frac{2c_m+2l_m+q_{m-1}+q_m+s_{m-1}+s_m}{c_m+q_{m-1}} - 2 = \frac{8}{3}$, and $n_m - 2 = \frac{2s_m+2q_m+l_m+l_{m+1}+c_m+c_{m+1}}{s_m+l_m} - 2 = 3$ for every m. If we solve the recurrence in Corollary **??** using $d_m - 2 = \frac{8}{3}$ and $n_m - 2 = 3$, we get a growth rate of $\left(3+2\sqrt{2}\right)^n$, which is much larger than the actual growth of Γ.

On the other hand, Figure **??** also shows a graph Γ' where the recurrence relation in Corollary **??** underpredicts the actual value of $|V_m|$. In Γ', there is exactly one vertex at each possible regional distance from v_0 except that there are 11 vertices at regional distance $m+1$ from v_0. Also at the vertex m from v_0 there are two loops instead of just one. In this graph we have $|V_m| = |R_{m-1}| = 1$, $d_m = 6$, $d_{m+1} = \frac{24}{11}$, $n_m = 8$, $n_{m+1} = 14$, and $d_k = n_k = 4$ for all other values of k. If we were to use the formulae of Propositon **??** to predict values of $|R_k|$ and $|V_k|$ for these parameters we would get $|R_m| = 3$, $|V_{m+1}| = 17$, $|R_{m+1}| = \frac{1}{11}$ and $|V_{m+2}| = -15\frac{10}{11}$. In this example we are underestimating the growth of the graph.

We now derive lower bounds on the growth rate of graphs which are not necessarily concentric.

Lemma 16 *Let Γ be a nice graph and let $m \geq 1$. There is a cycle C_m consisting of vertices from V_m and edges from E_m such that a connected component of the complement of C_m in the plane contains an infinite number of vertices of Γ, but none from $V_0 \cup V_1 \cup \ldots \cup V_m$.*

Proof. We partition the closed regions of Γ as follows. For each integer m, all regions R such that, either all the vertices incident with R are in V_m, or else some of the vertices are in V_m and some are in V_{m+1} are placed in P_m. We order the regions of Γ in such a way that if $R_i \in P_r$ and $R_j \in P_s$ with $r < s$, then $R_i < R_j$ and so that $D_j = \cup_{i=1}^{j} R_i$ is connected for each j. It is easy to see by induction that for each j, D_j is bounded by a finite union of cycles. Now take D_j to be the union of all the regions through P_m (but not containing any region from P_{m+1}). One of the bounding cycles of D_j separates the infinite component of $\Gamma - V_m$ from v_0. This is the desired cycle. ■

Definition 17 *Let Γ be an infinite planar graph and $v_0 \in \Gamma$. The average excess of Γ is the function $A(m)$ which gives the average excess among all the vertices in Γ whose regional distance from v_0 is less than m.*

Theorem 18 *Let Γ be a nice graph where the excess at every vertex in Γ is at least 0. Then $|V_m| \geq 3 + \frac{3}{2}A(m)\,(|V_0 \cup V_1 \cup \ldots \cup V_{m-1}|)$ for $m \geq 1$.*

PROOF. Let $V'_m = V_0 \cup V_1 \cup \ldots \cup V_m$ for each m. Let C be the cycle as in Lemma **??**. Consider the finite graph Γ' induced by all the vertices of Γ not in the infinite component of $\Gamma - V_m$. This graph, together with its finite regions, forms a disc. For each vertex v in the cycle C, $Ex(v) \geq -\frac{2}{3}$ since each vertex is incident with a region containing at least three sides. Therefore, by Lemma **??**,

$$\begin{aligned} \sum_{v\in V(C)} Ex(v) &= -2 - \sum_{v \in V(\Gamma')-V_m} Ex(v) \\ \frac{2}{3}|V(C)| &\geq 2 + \sum_{v\in V'_{m-1}} Ex(v) \\ |V_m| &\geq 3 + \frac{3}{2}A(m)\,(|V_0 \cup V_1 \cup \ldots \cup V_{m-1}|) \end{aligned}$$

■

An easy consequence of Theorem **??** is

Corollary 19 *Let Γ be a nice graph and $\epsilon > 0$. If the excess at each vertex in Γ is at least 0 and the average excess $A(m) > \epsilon$ for all m, then the growth of Γ is exponential.*

Theorem **??** requires the condition that excess is at least zero at each vertex. Theorem **??** drops this asumption at the price of assuming that the graph is simple and obtains a smaller lower bound.

Theorem 20 *Let Γ be a nice simple graph. Then*

$$|V_m| \geq 3 + \frac{3}{5}A(m)|V_0 \cup V_1 \cup \ldots \cup V_{m-1}|.$$

PROOF. Note that the graph Γ' induced by vertices in $V'_m = V_0 \cup V_1 \cup \ldots \cup V_m$ together with the regions incident only with vertices in V'_m is an annulus. Let k be the number of deleted regions in this annulus and let S_j be the set of vertices in V_m with $b_v = j$. (Recall that b_v is the number of deleted regions incident with v.) Then by Corollary **??** and the fact that for a vertex in S_j its excess is at least $-\frac{5}{3} + j$ we have

$$\begin{aligned}
\sum_{j=1}^{k} \left(-\frac{5}{3} + j\right) |S_j| &\leq 2(k-2) - A(m)|V'_{m-1}| \\
-\frac{5}{3}|V_m| + \sum_{j=1}^{k} j|S_j| &\leq 2(k-2) - A(m)|V'_{m-1}| \\
-\frac{5}{3}|V_m| + 3k &\leq 2k - 4 - A(m)|V'_{m-1}| \\
-\frac{5}{3}|V_m| &\leq -k - 4 - A(m)|V'_{m-1}| \\
|V_m| &\geq 3 + \frac{3}{5}A(m)|V_0 \cup V_1 \cup \ldots \cup V_{m-1}|
\end{aligned}$$

■

Corollary 21 *Let Γ be a nice simple graph and $\epsilon > 0$. If for each m $A(m) > \epsilon$, then Γ has exponential growth.*

PROOF. This follows easily from Theorem **??**. ■

References

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