Towards Axiomatisation of Social Epistemic Logic

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Abstract

This thesis is a theoretical development of epistemic logic to problems concerning the relationship between perception and knowledge. We closely follow the approach of Seligman, Liu and Girard's "Logic in the Community" [55] which proposes a two-dimensional multi-agent epistemic logic, in which the model operator K (knows) is supplemented with a 'social' operators which allow reasoning about relations between agents. The logic also uses operators from hybrid logic, such as nominals n, which name agents, the perspective shifting operator $@_n$, which moves to agent n's perspective, and the downarrow operator $\downarrow_x$, which names the current agent a rigid name x. We review axiomatic and tableaux systems for this logic and propose a new axiomatisation and completeness proof, using the step-by-step method, first for the basic logic and then for the case of downarrow, which is more involved. While the framework is very general, we are specifically interested in a perceptual agent-oriented operator S (sees). Axioms for the interaction of seeing and knowing are explored. We then consider dynamic extensions of the basic logic with public announcement and "observational" announcement, in which information is given only to agents who can see the announcer. Various subtleties are discussed and connections are made to dynamic epistemic logic.

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Contents

Chapter 1

Introduction

This thesis is a contribution to recent work in epistemic logic that aims to incorporate reasoning about relations between agents. In the last ten years, logicians have developed formal systems for reasoning about many topics on the interface between logic and sociology: social influence [55, 41, 56, 43, 6, 18, 19, 22, 4, 32], pluralistic ignorance [31, 29, 17, 8, 49, 52] and information cascades [5, 51].

I start with the paper "Logic in the Community" by Seligman, Liu and Girard [55]. This proposes a general framework for social epistemic logic based on two main ideas. First, it uses a two-dimensional model logic, in which formulas are evaluated at a pair. The pair consists of a 'world' to denote an epistemic possibility, and an agent. In other words, formulas of this logic are 'agent-indexical' propositions, whose truth value depends on the perspective of a particular agent. Second, this logic borrows the logical machinery of hybrid logic [10] to increase its expressive power, which is useful in applications.

The main contribution of this thesis is an axiomatisation of social epistemic logic and the proof of its completeness.

I propose an axiomatisation for not only the minimal logic, but also some extensions for various classes of logics (Chapter 3) that are useful in applications. I also consider the distinction between rigid and non-rigid names. Chapter 4 covers an axiomatisation of the extension that includes the down arrow operator of hybrid logic. Dynamic extensions of the logic are discussed more briefly (Chapter 5), by incorporating the ideas from Baltag, Moss and Solecki [7].

Although general in scope, the thesis focuses on a specific social relation: the seeing relation. To demonstrate the applicability of the logical tools I have developed here, I will show how they can be used to discuss the interactions between seeing and knowing.

A Chinese proverb says: "What you hear about may be false; what you see is true." The following scenarios show how seeing affects agents epistemic reasoning.

1.1 Seeing and Knowing

A classical puzzle of modern epistemic logic is the "Muddy Children" story, in which a strict father scolds his children for having mud on their faces after playing outside [61]. We adapt it to give the story of the Muddy Monks.

SCENARIO 0

Three monks Andrew, Bob and Charlie may have mud on the top of their heads without being aware of this. Of course, they cannot see their own mud if they have any. However, they can see whether the other monks are muddy. They are perfect logicians and not allowed to communicate with each other. And all this is commonly known. Let us assume Andrew is the only clean monk. The Abbot now says: "At least one of you is muddy. If you know you are, say it out loud." If nobody reacts, then the Abbot repeats his request. Then the result is that Bob and Charlie shout out "muddy". Only after that will Andrew know that he's clean. From the perspective of the muddy monks, the reasoning is simple. In the first round, it is obvious that no one knows. The fact that no one knows becomes common knowledge when the Abbot's first question is met with silence. Both muddy monks know that they can see a clean monk. And both of them also know that if he is clean, the other one will see two clean monks and say "muddy" on the first round. But that didn't happen. So he knows that he is also muddy! After two muddy monks shout out "muddy", the clean monk will know he's clean since he knows that the only case in which the other monks know that they are muddy is one in which he is clean. So every monk gets to know if he is muddy or not. We show this in Figure 1.1. For Andrew (a), Bob (b) and Charlie (c), gray means muddy. The dotted arrows represent their line of vision. The number means how many rounds (how many times the Abbot has to repeat the question) are needed for each monk to know whether he is muddy.

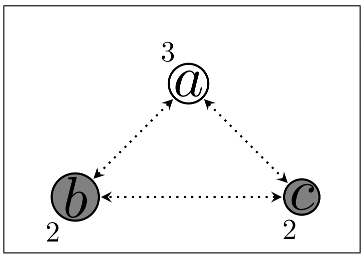


Figure 1.1: Muddy Monks

There are two issues to discuss. First, there is the question of whether the facts about who sees who is common knowledge. If they are, we will say that there is <u>public vision</u>, abbreviated as PV. Figure 1.1 is a very special case in which "everyone can see everyone else". And in this case we can assume there is public vision. Another slightly less public setting we want to look at is the one in which each agent knows that she is seen by another agent if and only if she sees that agent, and knows whether other agents that can see each other or not if and only if she can see both of them. In this case, we'll say that vision is semi-private, denoted as SV. The second issue is whether any announcements made are public. If so, we call it public announcement, denoted as PA. In this story, the Abbot and the monks communicate with their voice. To make the question and announcements public, it should be assumed that none of them has hearing problems.

Now let's consider announcements that depend on seeing rather than hearing. For example, we can suppose that the monks stay silent and only raise their hands when they know they are muddy. We call this an observable announcement (OA) because the message is received by all and only the agents who can see the announcer.

We have four possible combinations of the two issues above: public announcement with public vision (PA+PV); observable announcement with public vision (OA+PV); public announcement with semi-private vision (PA+SV) and observable announcement with semi-private vision (OA+SV). In the Muddy Monks story, these four combinations make no difference because all the monks can see each other. We now consider some variants.

SCENARIO 1

The three monks Andrew, Bob and Charlie, stand in a line in such a way that the only mud a monk can see is the mud on the heads of the monks in front of him. Only one of the monks is clean. But this is known only to the Abbot. The Abbot also tells them that at least one of them is muddy and asks who now knows he is muddy. If any of the muddy monks fails to know it, the Abbot repeats his question. Now we consider the four possible combinations we mentioned above. We assume that a monk's reply by shouting "muddy" is PA and by silently raising hands is OA. For the facts of who sees whom, if it is common knowledge, then it is PV; if it is only known by those who can see the people involved, it is SV. Figure 1.2 shows the four variants and the result for each initial state. "N" means that the agent will never find out if she is muddy. If there is a number instead, it is the number of times the Abbot asks his question.

We only briefly explain Case Two of PV+OA. Andrew and Bob cannot see Charlie, so they cannot know whether Charlie made his observable announcement (raise hands) or not. Without receiving Charlie's reply, Bob just sees that Andrew is clean and concludes nothing about himself. So does Andrew.

The weaker vision, the harder reasoning.

In Scenario 1, we investigated many possible cases based on the same seeing settings. The distinct difference between Scenario 1 and the original puzzle is that the monk's vision is restricted. And this difference makes the monks performance worse. There are twelve variants in Scenario One, but none of them allows the monks to all know whether they are muddy or clean. The best performing case is Case Two of PA+PV. It is not surprising that their performance is better when their announcements are public. But it is better under the condition of public vision. For example, they can't achieve anything in PA+SV. So the agent's vision may play the dominant role for the monks' reasoning here.

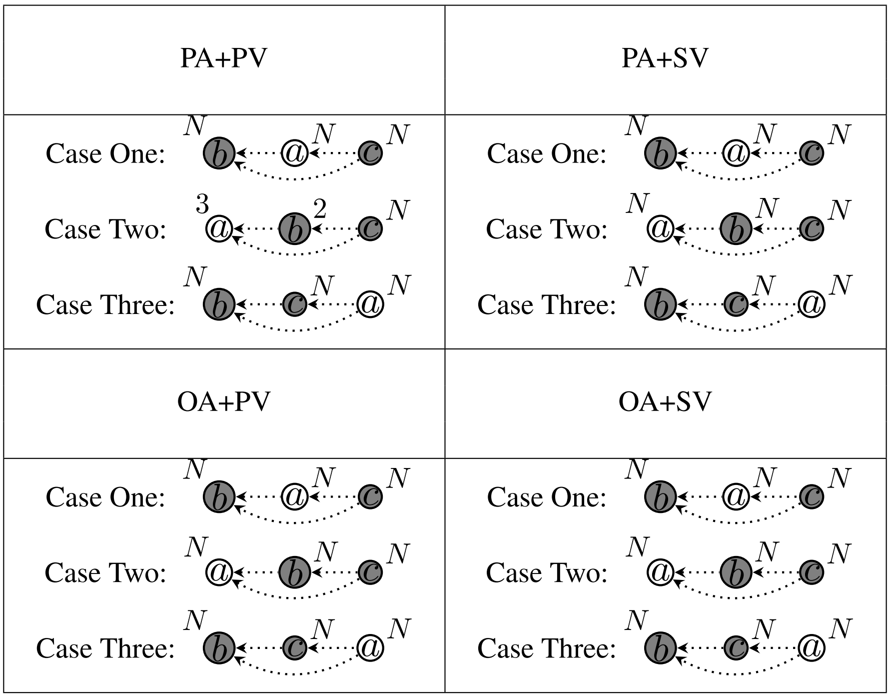


Figure 1.2: SCENARIO 1

SCENARIO 2

We introduce another variant of Muddy Monks. Bob's eyesight has some problem such that he can only see Andrew who stands nearer.

Figure 1.3 shows that Bob will shout out "clean" in round two of the PA+PV case because if he is clean then Charlie will see two clean monks and shout out "muddy" on round one. Then after Bob's announcement, Andrew will know that he is clean based on the same reasoning as Bob's from round one.

How about the PA+SV case? By definition, the only uncertainty is whether Bob knows Charlie can see him or not. But this is enough to stop Bob knowing that he is muddy because no matter whether he is muddy or not Charlie will keep silent.

In the last two cases, we assume it is Andrew (not Bob) who has a problem with his eyes such that he can only see Bob.[1] Figure 1.3 shows the change of setting. Andrew knows he is muddy finally in both cases. It seems that SV in the last case doesn't affect Andrew's reasoning. Why?

[1] Since we fix that Andrew is clean and the other two are muddy, we change who has eyesight problem to continue our discussion.

Your new knowledge may bring a bonus of knowing who can see you.

After Bob raises his hand in round 2, Andrew sees it. Then, Andrew will think how Bob knows that he is muddy: Bob didnt raise his hand on round 1, so either Charlie is, or I am muddy. If Charlie cannot see me, then Bob knows it. Then Charlies silence in round 1 doesnt give any significant information to Bob. And Bob cannot raise his hand on round 2. So Charlie sees me. And if both of them cannot see a clean monk, no one raises hands in round 2. Since Bob is muddy, I am clean. So Bob's announcement provides the significant knowledge for Andrew to know that Charlie can see him. In this case, SV is really not more restrictive than PV.

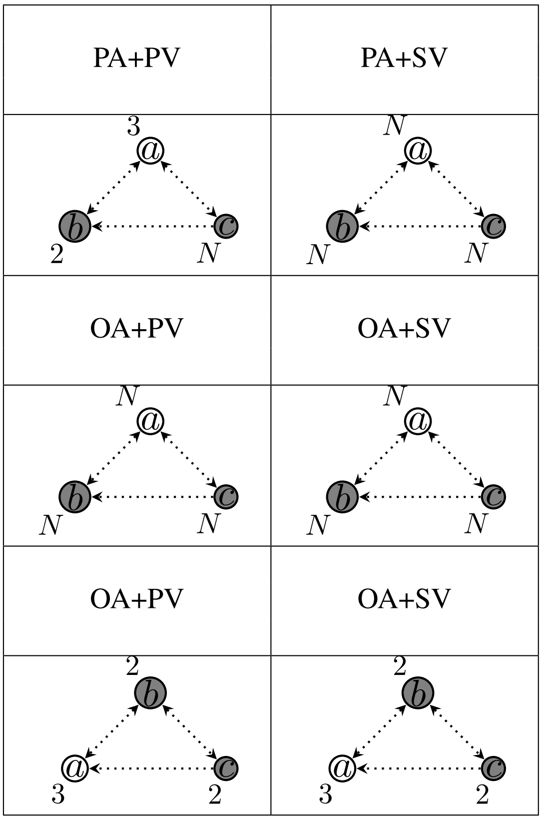


Figure 1.3: SCENARIO 2

Seeing the right person is helpful but not significant.

Perhaps we restricted the monks' vision too much in Scenario 1. So here only one monk's vision is restricted. But for the first four cases, it seems that no benefit arises from the improvement of vision. The last two cases have weaker announcement and weaker vision than the original puzzle. However, the monks achieve the same knowledge.

Why? If we compare the last two cases with the first four cases, we will realise the only difference is whether the muddy monks commonly know that both of them can see one muddy monk and one clean monk. If yes, then both of them will exclude the case where there is only one muddy monk. This leads to all three finally knowing whether they are muddy or not finally. Does this mean that vision is not so important if you can see the right monks? No. Take the last case (SV+OA) as an example. If Bob cannot see Charlie, then no monk will find out if he is muddy. In this sense, vision still plays the most significant role.

SCENARIO 3

Andrew and Charlie see each other. They also see Bob. This is common knowledge. Bob's eye sight problem is so severe that he is almost blind. So he has no idea who can see him. Two cases are shown in Figure 1.4.

Case One: Bob's eye sight problem is common knowledge.

On the first two rounds of PV+PA, no monk makes an announcement. Andrew sees two muddy monks and Charlie sees one clean and one muddy. This is not enough for them to know whether they are muddy or not. But on round 3, blind Bob will say that he is muddy. Why is that? How can the blind monk know anything? Well, from the perspective of Bob, if he is clean, then either Andrew or Charlie is muddy. If there is only one muddy monk, then that monk will see two clean monks and know that he is the only muddy one in the first round and so he would have made an announcement. If both are muddy, then the story is similar to the original scenario. The muddy ones will announce they are muddy on the second round. But we have two silent rounds. So Bob knows he is muddy.

Seeing more, knowing more? Not really.

The most surprising result of PV+PA is that poor blind Bob knows he is muddy but normal Andrew and Charlie fail to know. Ironic? This phenomenon has indeed already occurred: Case Two of PV+PA in Scenario 1 and PV+PA of Scenario 2. In these cases, the monk with the worst visual access to the others knows whether he is muddy or not, but at least one of the others doesn't. Why? Because monks with better vision will get more information and make valuable announcements, however, they cannot get help from the monk with the worst vision, but their announcements help him reason. But there is a precondition: their announcements mustn't be observable ones. Otherwise, the announcements won't be so helpful. It is demonstrated by the first OA+PV.

Case Two: Bob's eye sight problem is a secret.

We discuss the OA+BV where BV means believed vision, e.g. who you believe you can see and whom you believe can see you. How does Charlie reason now? Charlie thinks Bob sees both him and Andrew. In other words, Charlie thinks the visual situation is the same as in the original scenario: the Muddy Monk. When Bob is silent on round 1, Charlie will announce that he is muddy on round 2. Andrew has the same reasoning as Charlie's. Moreover, since Andrew and Charlie can see each other and Bob, they commonly know that the other is reasoning with the same way as himself. After Andrew and Charlie raise their hands, Bob doesn't. So Andrew and Charlie realise that their conclusions are wrong.

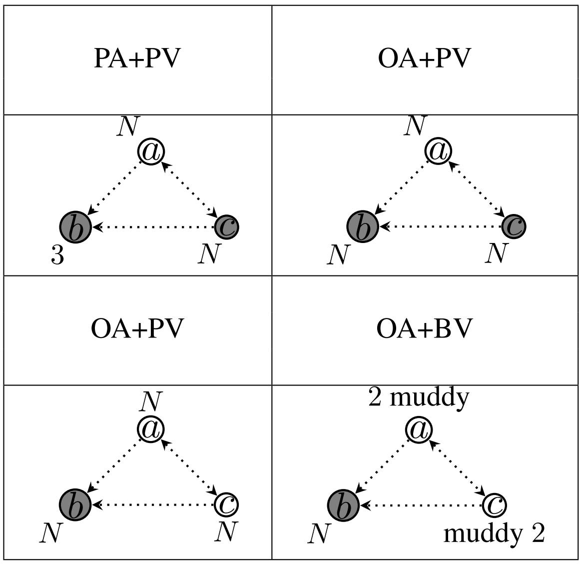


Figure 1.4: SENARIO 3

Seeing and believing

The BV case in this scenario is tricky. Bob keeps the fact that he is blind a secret. Then it turns out that Andrew and Charlie mistakenly think they are in the original Muddy Monks scenario. The outcome depends crucially on the fact that Charlie and Andrew have false beliefs. But they finally will realise that their beliefs about mud and vision are wrong. After belief revision, they will realise that they are both clean is the only case that makes this dramatic result happen.

1.2 Outline of the Thesis

Our plan is to propose a series of logical systems to discuss the reasoning in a range of different scenarios. The main technical idea we adopt to achieve this goal is the 'step-by-step' method for constructing models from consistent sets of formulas. This method was famously used for tense logic by Burgess [14] and by Xu [64]. In Chapter 2, we develop a generalised form of the technique and apply it to ordinary (one-dimensional) modal logic.

Chapter 3 addresses minimal social epistemic logic, **BSEL**. This is the system introduced in the 2010 paper 'Logic in the Community' [55]. At that time, only semantic definitions were given, without a formal system. Since then a tableau system was provided by Christoff, Hansen and Proietti[19] and a tree sequent calculus and a Hilbert axiomatisation were provided by Sano [54]. Instead we give a standard Hilbert axiomatisation and prove the completeness directly, using the step-by-step method. This was reported briefly as [40]. A discussion of the relationship between our system and those of Sano and Christoff et al. is given. Chapter 3 also discusses the rigidity of names, and some important extensions of the minimal logic such as $\mathbf{S4}$ and $\mathbf{S5}$.

Chapter 4 concerns the full logic $\mathbf{SEL}$ obtained by adding the downarrow operator. The axiomatisation and completeness proof is much harder, but also uses step-by-step.

Chapter 5 explores the dynamics of $\mathbf{BSEL}$. We first review dynamic epistemic logic, following the seminal work of A. Baltag, L. Moss and S Solecki [7], and show how it can be adapted to formalise the action of "observable announcements". We provide a semantic treatment of dynamic operators to axiomatise $\mathbf{DSEL}$ with the methods of dynamic epistemic logic. We propose an indexical action structures in which the epistemic accessibility relations between actions are defined indirectly and use them to analyse the scenarios presented in this chapter.

Finally, Chapter 6 summarises what we take to be the conclusions of this thesis, and points at some perspectives for future research.

Chapter 2

Logical Preliminaries

2.1 Basic Epistemic Logic

There is a long tradition of logicians analysing information and the change of information. What do we mean by *information*? We agree with the explanation given by [61] on Page 1:

> We regard information as something that is relative to a subject who has a certain perspective on the world, called an *agent*, and the kind of information we have in mind is meaningful as a whole, not just loose bits and pieces. This makes us call it *knowledge* and, to a lesser extent, *belief*.

The logicians propose formal languages to analyse not only the notions: *knowledge*, *belief* and so on, but also reasoning about valid arguments. Let us now briefly give an overview of the history of epistemic logic, the branch of philosophical logic that aims to formalise the logic of discourse about knowledge and belief.

Most logical systems can be specified either syntactically or semantically. The idea of epistemic logic goes back at least to Von Wright [63], which is one of the most important syntactic driven papers. Based on modal logic, he is one of the first to propose a formal system of reasoning about knowledge. Following this, there were many syntactic results about deduction, axiomatisation and so on. But typically the most interesting problems and solutions are found when logicians concentrate on semantics. There is a longer story of the semantically driven results. Early *Possible world semantics* can be traced to Carnap [15]. Then it was developed by Hintikka [33] which proposed the concept of "accessibility relation between worlds". Another book by Hintikka, *Knowledge and Belief: An Introduction to the Logic of the Two Notions* [34], is the first work to give a clear semantic account of *knowledge* and *belief*. Then Kripke developed the semantical analysis [37].

Thanks to the work of these people, possible world semantics was established by the 1960s. What then is possible world semantics?

One day two years ago, I took a flight from China to New Zealand. During my flight the All Blacks had an important match against one of the best National rugby teams in the world. I couldn't access any information about the match that night. So I didn't know the result when I was on the flight. But I could still consider two possible situations, one in which the All Blacks won, and one in which they are not. We can let p denote "The All Blacks won" and assume that it is true in the first situation, then "It is not the case that p" could be denoted as $\neg p$ and it is true in the second situation. The situations are called "possible worlds"[1]. The more issues you consider at the same time, the more possible worlds you will have to consider. For example, I also had no idea whether it was raining during the match (call this q). Then there are four possible worlds: p, q both true, p true q false, p false q true, and both false. In fact, there are 2^n possible worlds if you consider n issues at one time.

The method that represents agents' knowledge or lack of knowledge by possible worlds can be fitted perfectly into the framework of Kripke-style possible world semantics. Assume that we have a countably infinite set Prop of propositional atoms, which correspond to the issues under consideration. We define a truth assignment function V from Prop to the subsets of all possible worlds such that: for any $p \in \mathsf{Prop}$ and any possible world in the subset assigned to p, p is true in this possible world. Formally, a Kripke model is a structure of the form:

Definition 2.1.1 *A Kripke model is a structure M of the form $\langle W, R, V \rangle$, where*

- *W is a non-empty set of possible worlds;*
- *$R \subseteq W \times W$ is the knowledge accessibility relation;*
- *$V : \mathsf{Prop} \longrightarrow pow(W)$ is a function from atoms to sets of possible worlds.*

We give a brief explanation of the *knowledge accessibility relation*. For any possible worlds w and v, w accessibly relates to v, Rwv, if and only if v is indistinguishable from w. Figure 2.1 represents the above "win or lose" example.

In this figure, w and v denote two possible worlds: w is the one in which p is true and v is the other one that p is false. The underline of w means the actual situation is that All Blacks won. We have $w \in V(p)$.

We hope that a Kripke model M can tell us that something is true or false in a given possible world (maybe for every possible world). For example, what is the condition to be able to claim $K\varphi$ where K is a model operator denoting "The agent

[1]Readers must notice that possible worlds are different from the rows of a truth table, at least in multi-agent settings.

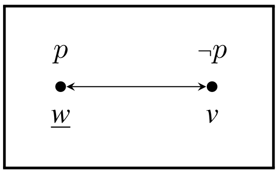


Figure 2.1: Win or lose?

knows a formula φ"? Then what is the condition to say that a possible world satisfies (denoted as $\models$) $K\varphi$ in a Kripke model? This is defined semantically as follows: for any $M = \langle W, R, V \rangle$ and $w \in W$,

$$M, w \models K\varphi \quad \text{iff} \quad \text{for all } v \in W \text{ with } Rwv,\ M, v \models \varphi$$

This definition states that the agent *knows* the formula φ at a possible world w if and only if φ holds in all possible worlds deemed by the agent at w. Then in Figure 2.1, the bi-directional arrow between w and v makes both $M, w \not\models Kp$ and $M, w \not\models K\neg p$ hold.

So far we haven't said clearly which symbols are formulas of the basic epistemic logic. Given a set Prop of propositional atoms, its BNF is:

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K\varphi$$

For the semantic definitions of atoms $p \in$ Prop and the usual propositional connectives: $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implication) and $\leftrightarrow$ (bi-implication), we define them the usual way as:

$$\begin{array}{lll}
M, w \models p & \text{iff} & w \in V(p) \\
M, w \models \neg\varphi & \text{iff} & M, w \not\models \varphi \\
M, w \models \varphi \wedge \psi & \text{iff} & M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \varphi \vee \psi & \text{iff} & M, w \models \varphi \text{ or } M, w \models \psi \\
M, w \models \varphi \rightarrow \psi & \text{iff} & M, w \models \varphi \text{ implies } M, w \models \psi \\
M, w \models \varphi \leftrightarrow \psi & \text{iff} & M, w \models \varphi \text{ if and only if } M, w \models \psi
\end{array}$$

Some formulas, for example $K\varphi \vee \neg K\varphi$, are satisfied by any Kripke model and any possible world in the model. Formally, we say that φ is *valid*, denoted as $\models \varphi$, if $M, w \models \varphi$ for all M and $w \in W$. Then the valid formulas with respect to the Kripke models can be axiomatised by the following Hilbert system **K** depicted by the Figure 2.2 :

$\vdash \varphi$ if φ is a propositional tautology	Taut
$\vdash K(\varphi \to \psi) \to (K\varphi \to K\psi)$	$\mathsf{K_K}$
from φ infer $K\varphi$	$\mathsf{Nec_K}$
from φ and $\varphi \to \psi$, infer ψ	MP

Figure 2.2: An axiomatisation of $\mathbf{K}$

The most fundamental properties of $\mathbf{K}$ are *soundness* and *completeness* with respect to the class of all Kripke models. The soundness theorem states that all $\mathbf{K}$-derivable formulas are valid. The completeness theorem states the converse. The readers are referred to [36] for details into how to prove it is sound and complete.

The above Hilbert system $\mathbf{K}$ is a minimal logic in the sense that R has no property. Is the concept of knowledge expressed well by $\mathbf{K}$? Unfortunately, no. The reason is that R is too weak. For instance, one of the properties of knowledge is that knowledge has to be true and this could be stated as the schema: $K\varphi \to \varphi$. However, this formula is not valid in all Kripke models. This suggests that we should constrain the underlying Kripke models by putting conditions on the accessibility relation R.

Proposition 1 *Any Kripke model where R is reflexive validates $K\varphi \to \varphi$.*

By adding $K\varphi \to \varphi$ to $\mathbf{K}$ as an axiom called T, we obtain the system $\mathbf{T}$. It can be shown to be sound and complete with respect to the class of all Kripke models in which R is reflexive; see Chapter 2 of [36] for details.

Is there any other property required for R to describe knowledge? Unlike $\mathbf{T}$, some other properties are conversial to add. For example, in many circumstances, people should know that they know what they know. This is expressed by the schema $K\varphi \to KK\varphi$. This however is not valid in either $\mathbf{K}$ or $\mathbf{T}$. We need another extra property for R to make it valid: transitivity. The accessibility relation R of $M = \langle W, R, V \rangle$ is transitive if it satisfies:

$$Rwv \text{ and } Rvu \text{ implies } Rwu \text{ for any } u, v, w \in W$$

Proposition 2 *Any Kripke model where R is transitive validates $K\varphi \to KK\varphi$.*

The readers may anticipate that the combination of $\mathbf{T}$ and $K\varphi \to KK\varphi$ is sound and complete with respect to the class of Kripke models where R is reflexive and transitive. It is, and this combination is well known as $\mathbf{S4}$.

We can go further: R is an *equivalence* relation it it satisfies the following three properties:

- Reflexivity: Rww for all $w \in W$
- Symmetry: Rwv implies Rvw for all $w, v \in W$
- Transitivity: Rwv and Rvu implies Rwu for all $w, u, v \in W$

Then we have the system known as $\mathbf{S5}$. Although it is controversial to claim it is perfect to formalise knowledge (actually, we haven't got a well accepted one), $\mathbf{S5}$ is still regarded as a more or less standard logic tool to analyse knowledge and is introduced by textbooks of epistemic logic, e.g. [61]. Figure 2.3 shows an axiomatisation of $\mathbf{S5}$, which includes $\mathbf{K}$ and the following:

$\vdash K\varphi \rightarrow \varphi$	T
$\vdash K\varphi \rightarrow KK\varphi$	4
$\vdash \neg K\varphi \rightarrow K\neg K\varphi$	5

Figure 2.3: Additional axioms for $\mathbf{S5}$

One of the reasons that epistemic logic is of such interest to philosophers and AI researchers is that it deals with not only knowledge about propositional facts, but also knowledge about knowledge, i.e., higher-order knowledge. Two of the most controversial principles of higher-order knowledge are *positive introspection* and *negative introspection*. Positive introspection says that you know what you know (axiom 4) and negative introspection says that you know what you don't know (axiom 5). These axioms correspond to requiring the knowledge accessibility relation to be transitive and Euclidean, respectively. For human knowledge, these principles, especially negative introspection, are too strong. To be introspective, you need to be aware of the issue first. You may not know the name of someone who passes you on the street, but until I ask you, it is unlikely that you know that you don't. Despite this objection, using $\mathbf{S5}$ as the basis for epistemic logic is adequate when awareness (or the lack of it) is not an issue. Although some progress has been made in applying dynamic epistemic logic to the problem of introspection (e.g. [59, 60, 23, 27, 50]) the matter will not be consider further in this thesis.

Epistemic logic for a single knower extended multi-agent epistemic logic. Syntactically it augments the language of propositional logic with knowledge operators K_a, for each agent a. Semantically we need an accessibility relations R_a for each agent a. As a result, different possible worlds may be accessible to different agents and so agents may have different knowledge about not only themselves but also others. For example, K_aK_bp says that a knows that b knows p. This implies K_bp that b knows p, but not that K_bK_ap, b knows a knows p, even if K_ap that a does know p. A variation of Kripke model may be used to describe the knowledge in the multi-agent case.

Definition 2.1.2 *A multi-agent Kripke model is a structure* $M = \langle W, \{R_a | a \in A\}, V\rangle$ *where* W *and* V *are defined as before,* A *is a finite set (of agents) and*

$R_a \subseteq W \times W$ *is an equivalence relation for each* $a \in A$.

The following is a model of Muddy Monks (after the Abbot's first request) we introduced in Chapter 1. The underlined possible world is the actual one. The number 0 means the monk is clean. And 1 means the monk is dirty. Each possible world is denoted by three numbers which denotes a, b and c's being muddy or not respectively.

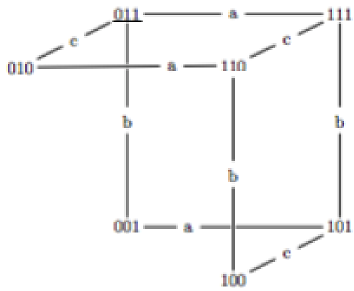


Figure 2.4: A model of the Muddy Monks

Then "group knowledge", "distributed knowledge", "common knowledge" can be defined [28].

Group knowledge is the knowledge that every agent in a group has. For any finite group B of agents, $E_B\varphi$, which means "everybody in B knows φ", is defined as:

$$E_B\varphi = \bigwedge_{b \in B} K_b\varphi$$

Semantically, the modal operator E_B is associated with $R_{E_B} = \bigcup_{a \in B} R_a$ and defined as:

$$M, w \models E_B\varphi \quad \text{iff} \quad M, v \models \varphi \text{ for every } v \text{ with } R_{E_B}wv$$

This definition means that a proposition is known to the group when it is satisfied at all the possible worlds that every group member can access. Any group knowledge is true in all of the alternatives of every person in the group.

As a normal modal operator, E_B satisfies properties such as $\mathsf{K}_{\mathsf{E_B}}$: $E_B(\varphi \to \psi) \to (E_B\varphi \to E_B\psi)$ and Nec_E: from φ infer $E_B\varphi$. Moreover, it satisfies $E_B\varphi \to \varphi$. Also note that it doesn't satisfy 4 or 5, and this is interesting, e.g. K_ap, K_bp so $E_{a,b}p$ but $\neg K_aE_{a,b}p$ because $\neg K_aK_bp$.

In the model of Figure 2.4 , $M, 010 \models E_{\{a,b,c\}}Kp_b$ where p_b denotes "b is muddy".

Another variant of knowledge is distributive knowledge ($D_B\varphi$), which means that if everyone *shares* their knowledge with everyone in the group, then it becomes *general knowledge*. We have $D_B\varphi$ if φ is distributed over the members of B, e.g. a knows ψ implies φ and b knows ψ for some $a, b \in B$. In another word, $D_B\varphi$ requires φ is true at the intersection of b's accessible worlds for $b \in B$: $R_{D_B} = \bigcap\limits_{a\in B} R_a$.

$$M, w \models D_B\varphi \quad \text{iff} \quad M, v \models \varphi \text{ for every } v \text{ with } R_{D_B}wv$$

So $M, 010 \models D_{\{a,c\}}K(\neg p_a \wedge \neg p_c)$ in the above model. This means monk a and c knows they are clean after sharing their knowledge at the possible world 010.

Common knowledge is a very strong concept that can be defined via group knowledge. φ is common knowledge of a finite group B ($C_B\varphi$) if "everybody in B knows φ and everybody in B knows that everybody in B knows that φ and so on. Let $E_B^n\varphi$ denote the iteration of $E_B\varphi$ n times, then $C_B\varphi$ is defined as

$$C_B\varphi = \bigwedge_{n\in\mathbb{N}} E_B^n\varphi$$

(Of course, this uses infinite conjunction and so cannot be expressed in basic epistemic logic.) From the perspective of semantics, $C_B\varphi$ requires φ is to be satisfied not only at the possible worlds of R_{E_B}, but also at all those possible worlds that are in any degree accessible via R_{E_B}. This can be realised by the *reflexive transitive closure* of R_{E_B}.

The reflexive transitive closure of R_{E_B} is the smallest of $R^*_{E_B}$ where

1. $R_{E_B} \subseteq R^*_{E_B}$
2. $R^*_{E_B}(w, w)$ for all $w \in W$
3. for all $w, v, u \in W$, if $R^*_{E_B}(w, v)$ and $R^*_{E_B}(v, u)$, then $R^*_{E_B}(w, u)$.

Then in the above model, we have $M \models C_{\{a,b,c\}}(p_a \vee p_b \vee p_c)$, which means "It is commonly known that at least a monk of a, b and c is muddy in every possible case."

Now we are ready for the axiomatisation of multi-agent epistemic logic, which is based on $\mathbf{S5}$ and with the following additional axioms ([25], p.199), which is shown by Figure 2.5.

$\vdash C_B(\varphi \to \psi) \to (C_B\varphi \to C_B\psi)$	$\mathsf{K_C}$
$\vdash D_B(\varphi \to \psi) \to (D_B\varphi \to D_B\psi)$	$\mathsf{K_D}$
$\vdash E_B(\varphi \to \psi) \to (E_B\varphi \to E_B\psi)$	$\mathsf{K_E}$
$\vdash C_B\varphi \to \varphi$	$\mathsf{T_C}$
$\vdash D_B\varphi \to \varphi$	$\mathsf{T_D}$
$\vdash D_B\varphi \to D_BD_B\varphi$	$\mathsf{4_D}$
$\vdash \neg D_B\varphi \to D_B\neg D_B\varphi$	$\mathsf{5_D}$
$\vdash E_B\varphi \leftrightarrow (K_a\varphi \wedge \ldots \wedge K_m\varphi)$	KE
$\vdash C_B\varphi \to E_BC_B\varphi$	EC
$\vdash C_B(\varphi \to E_B\varphi) \to (\varphi \to C_B\varphi)$	Cind
from φ infer $K_a\varphi$	$\mathsf{Nec_{K_a}}$
from φ infer $C_B\varphi$	$\mathsf{Nec_C}$
from φ infer $D_B\varphi$	$\mathsf{Nec_D}$

Figure 2.5: Additional axioms for multi-agent epistemic logic

The system can again be proven sound and complete, although the common knowledge operator brings much complexivity. For further details, please refer to [44].

Another goal of modern epistemic logic is to discuss "belief" formally. It is well known that the main difference between knowledge and belief is that belief can be false. It is acceptable if you say "I believe that they are friends" when they are strangers, but it does not make sense to say "I know that they are friends." Although we do not require belief to be true, it should be "internally consistent". From a technical point of view, it seems that $\mathbf{KD45}$, which is the same as $\mathbf{S5}$ except replacing T with D is a good choice for logics to discuss belief. The reason to delete T is obvious. Belief is not always true. The operator B usually denotes belief. Then D is the schema: $\neg B(\varphi \wedge \neg\varphi)$, which means belief has to be consistent. In other words, any agent shouldn't believe a contradiction. Figure 2.6 shows an axiomatisation of $\mathbf{KD45}$. Note that B is not an agent set here. It is a model operator switched from K meaning "believes that" instead of "knows that".

Similarly, one can define general/distributed/common belief, which are usually interpreted with respect to the transitive closure of the union of each agent's plausible relations.

At the individual level, multi-agent epistemic logic can talk about each agent's knowl-

$\vdash \varphi$ if φ is a propositional tautology	Taut
$\vdash B(\varphi \to \psi) \to (B\varphi \to B\psi)$	$\mathsf{K_B}$
$\vdash B\varphi \to BB\varphi$	$\mathsf{4_B}$
$\vdash \neg B\varphi \to B\neg B\varphi$	$\mathsf{5_B}$
$\vdash \neg B(\varphi \wedge \neg\varphi)$	D
from φ infer $B\varphi$	$\mathsf{Nec_B}$
from φ and $\varphi \to \psi$, infer ψ	MP

Figure 2.6: An axiomatisation of $\mathbf{KD45}$

edge (or belief). At the community level, it can discuss group epistemic properties like distributed knowledge, group belief and so on. But there are still at least two directions for it to be extended. One direction is dynamic epistemic logic. With such extensions, knowledge upgrade and belief update on either level can be modeled. The other direction is to add social relations between the agents. Most research on logic considers agents as elements of simple agent sets without any structure. But humans are social beings. In real life, we all live in social networks which constrain and influence our thinking and behaviour. So we need to enrich the logic's perspective and put a social structure in the agent set to strengthen our research.

2.2 Semantics of Basic Social Epistemic Logic

We will start by introducing the semantics of the logic proposed in [55] and give the axiomatisation and completeness proof in the next chapter. This logic has been used to reason about peer pressure in [41], knowledge and communication in social networks in [56], [24] and epistemic influence in [43], [17], [5] and[18]. It is a two-dimensional modal language: One dimension ranges over epistemic possible worlds, and the other over agents. Each dimension has an associated modal operator, one representing some sort of epistemic attitude (knowledge, belief, etc.) and the other quantifying over a social relation (friendship, influence, communication, etc.). The language also includes names for agents and the hybrid-logic-like operators $@_n$. Formulas are evaluated at a world-agent pair.

Definition 2.2.1 *The basic social epistemic logic language* L *is defined using a countably infinite set of propositional atoms* Prop *whose elements are usually denoted* p, q, r, *and so on, a set* Nom *of agent names whose elements are denoted* n, m, *and so on, and two unary modal operators:* K *and* S. *The well-formed formulas* φ *of this language are given by the rule*

$$\varphi \quad ::= \quad p \mid n \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K\varphi \mid S\varphi \mid @_n\varphi$$

We use standard abbreviations for the other Boolean operators: $\vee$*,* $\rightarrow$ *and* $\leftrightarrow$*. The dual modal operators are also defined:* $\langle K\rangle = \neg K\neg$ *and* $\langle S\rangle = \neg S\neg$*.*

The epistemic logic we introduced at the previous section is a one-dimensional modal language. The formulas are evaluated only at a possible world of a model. However, the basic social epistemic logic, $\mathbf{BSEL}$, is agent-indexical: formulas must be evaluated at a world-agent pair, which means they are also evaluated from the perspective of an agent. World-agent pairs are sometimes known as "centred worlds". They were introduced by David Lewis [39] to characterise self-idexical or de se thought, and have been used subsequently, in a modified form by David Chalmers in Chapter 2, section 4 of [16] and others to do more work in the philosophy of mind and language. Recall the above formula, $K_a p_b$ denotes "agent a knows that b is muddy." To express the same meaning in $\mathbf{BSEL}$, we firstly give agents a and b each a name, say $\underline{a}$ and $\underline{b}$ respectively. Then the name itself is a formula according to the definition of $\mathbf{BSEL}$. The formula $\underline{a}$ would be interpreted by "I am $\underline{a}$". Here "I am" is what the agent-indexical means; the objective of each atomic proposition is the current agent, say "I". Then let p denote the muddy property and $(\underline{a} \wedge p)$ says "I am $\underline{a}$ and I am muddy". Then $(\underline{a} \wedge Kp)$ says that "I am $\underline{a}$ and I know I am muddy". But it is far from the same expression as $K_a p_b$. Now we introduce the operator @, the operator to shift perspective. Then $@_{\underline{b}}p$ means "$\underline{b}$ is muddy". So $(\underline{a} \wedge K@_{\underline{b}}p)$ says "I am $\underline{a}$ and I know that $\underline{b}$ is muddy". But this is still a little different from $K_a p_b$ since it assumes that the current agent is a. We only need one more step: replace $\underline{a}$ with $@_{\underline{a}}$ for the last formula to get $@_{\underline{a}}K@_{\underline{b}}p$, which exactly says "agent a (named $\underline{a}$) knows that agent b (named $\underline{b}$) is muddy." The @ operator has a long history in modal logic dating back to the work of Arthur Prior, who used it in tense logic [48], and has been subsequently developed in hybrid logic [2, 53, 12, 9, 57, 35].

Now we introduce the operator S. $S\varphi$ says that I stand in some relationship to everyone else, from whose perspective φ is true. Examples of such relationships are: seeing, hearing, following on a social network or being followed by, being a friend of, and so on. In this thesis, we interpret S as a "seeing" relation. So, for example Sn says that I can only see the agent named n if I see someone. And $\langle S\rangle n$ says that I can see n. A more complex example is $\langle S\rangle K@_{\underline{a}}K@_{\underline{b}}p$, which says that I can see someone who has the knowledge that $\underline{a}$ knows $\underline{b}$ is muddy.

We are introducing the semantics now. The semantics of $\mathbf{BSEL}$ is based on that of epistemic logic but is more complex.

Definition 2.2.2 *A basic social epistemic model* M *is a tuple* $\langle W, A, k, s, g, V\rangle$ *where* W *is a set (of* <u>possible worlds</u>*),* A *is a non-empty set (of* <u>agents</u>*),* $k = \{k_a | a \in A\}$ *is a binary relation on* W*,* $s = \{s_w | w \in W\}$ *is a binary relation on* A*,* $g_w\colon \mathsf{Nom} \rightarrow A$ *is a function assigning an agent to each name in each world, and*

$V\colon \mathsf{Prop} \to pow(W \times A)$ *is a valuation function assigning a subset of* $W \times A$ *(i.e., an agent-indexical proposition) to each propositional variable.*

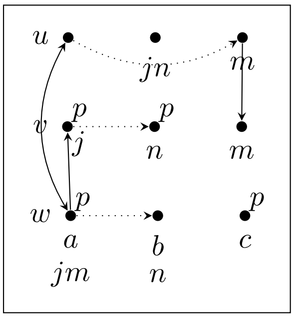


Figure 2.7: A simple model of social epistemic logic

Fig. 3.5 illustrates a simple social epistemic model with worlds $W = \{w, v, u\}$ and agents $A = \{a, b, c\}$. Vertical arrows represent epistemic relations, e.g. $k_a(w, v)$ and the horizontal dotted arrows represent social relations, e.g. $s_v(a, b)$. The distribution of 'p' represents $V(p)$, e.g. $\langle v, b\rangle \in V(p)$ and $g_v(n) = b$ is indicated by placing 'n' at the intersection of the v row and b column. Note that n is a rigid name in this model but j is not. We say a name n is rigid in M if $g_w(n) = g_v(n)$ for any $w, v \in W$; "Rigid" will be discussed at the end of the next chapter and we assume that all names are non-rigid elsewhere. Moreover, not all agents need have names: in world w agent c is nameless.

Saul Kripke [38] first introduced the distinction between rigid designators and non-rigid designators: a rigid designator designates the same thing in all possible worlds, and a non-rigid one designates different things in some possible worlds. Much of the philosophical debate about rigid designation concerns natural kind terms like "H_2O" and "water", but here we will be more concerned with personal names. When we use names in ordinary life, we often assume that they are rigid. Take the muddy monk as an example. Suppose monk a is called "Andrew". Although he thinks he is muddy in another possible world, he wouldn't think that he is not Andrew in that world. But some names we use are non-rigid. Take the name Satoshi Nakamoto for example. It is a non-rigid name. Why? Satoshi Nakamoto refers to the person who invented BitCoin in the actual world. But it could refer some other person, say Andrew, for Andrew might have invented BitCoin. That is, there are possible worlds in which it refers someone other than Satoshi Nakamoto.

Definition 2.2.3 *Given a basic social epistemic model* $M = \langle W, A, k, s, g, V\rangle$*, the relation* $\models$ *(satisfies) is defined recursively as follows:*

$M, w, a \models p$	*iff*	$\langle w, a\rangle \in V(p)$
$M, w, a \models n$	*iff*	$g_w(n) = a$
$M, w, a \models \neg\varphi$	*iff*	$M, w, a \not\models \varphi$
$M, w, a \models (\varphi \wedge \psi)$	*iff*	$M, w, a \models \varphi$ *and* $M, w, a \models \psi$
$M, w, a \models K\varphi$	*iff*	$M, v, a \models \varphi$ *for all* v *if* $k_a(w, v)$
$M, w, a \models S\varphi$	*iff*	$M, w, b \models \varphi$ *for all* b *if* $s_w(a, b)$
$M, w, a \models @_n\varphi$	*iff*	$M, w, g_w(n) \models \varphi$

As usual, $M, w, a \models \Sigma$ *if* $M, w, a \models \varphi$ *for all* φ *of* Σ*;* φ *is valid in* M *if* $M, w, a \models \varphi$ *for all* w *and* a*; it is valid if it is valid in every social epistemic model.*

Then we say that φ *is a semantic consequence of* Σ *(notation:* $\Sigma \models \varphi$*) if for all* M*, all worlds* w *and agents* a *in* M*, if for all* $\psi \in \Sigma$ $M, w, a \models \psi$*, then* $M, w, a \models \varphi$*.*

Figure 2.8 is a $\mathbf{BSEL}$ model representing Figure 2.4. Since k is an equivalence relation, we denote the bidirectional relations with lines (no arrows) and avoid drawing the reflexive relations. Moreover, we assume it is a rigid name model and $\underline{a}$, $\underline{b}$ and $\underline{c}$ are a's, b's and c's names respectively. We show how formulas are evaluated via Definition 2.2.3.

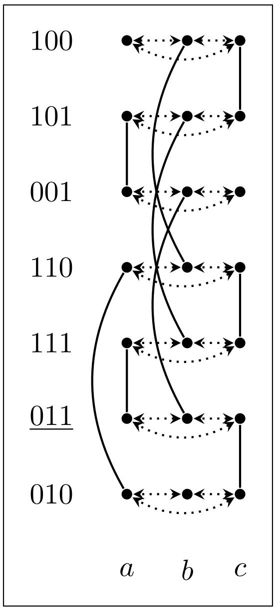


Figure 2.8: A $\mathbf{BSEL}$ model of Muddy Monks

From the story, it is true that $\underline{a}$ knows $\underline{b}$ is muddy in the case in which only $\underline{b}$ is muddy, that is, the possible world 010. How do we show that $M, 010, a \models K@_{\underline{b}}p$?

By the definition, $M, 010, a \models K@_{\underline{b}}p$ iff for all possible worlds v if $k_a(010, v)$ then $M, v, a \models @_{\underline{b}}p$. By M, there are two such v: 010 and 110. Obviously, $M, 010, b \models p$ and $M, 110, b \models p$. By M is rigid and $\underline{b}$ is b's name, $g_v(\underline{b}) = b$ for all $v \in W$. By the last clause of the semantic definition, we have $M, 010, a \models @_{\underline{b}}p$ and $M, 110, a \models @_{\underline{b}}p$. Hence, $M, 010, a \models K@_{\underline{b}}p$.

As a more complicated example, we show $M, 011, a \models \langle S\rangle K@_{\underline{a}}K@_{\underline{b}}p$. By the semantic definition of K and S, the dual $\langle K\rangle$ and $\langle S\rangle$ are semantically defined as:

$$
\begin{array}{lll}
M, w, a \models \langle K\rangle\varphi & \text{iff} & M, v, a \models \varphi \text{ for some } v \text{ such that } k_a(w, v) \\
M, w, a \models \langle S\rangle\varphi & \text{iff} & M, w, b \models \varphi \text{ for some } b \text{ such that } s_w(a, b)
\end{array}
$$

Then we have $M, 011, a \models \langle S\rangle K@_{\underline{a}}K@_{\underline{b}}p$ iff $M, 011, j \models K@_{\underline{a}}K@_{\underline{b}}p$ for some j such that $s_{011}(a, j)$. Both b and c could be a candidate for j. We let $c = j$. To prove $M, 011, c \models K@_{\underline{a}}K@_{\underline{b}}p$, we need to show for all v, $M, v, c \models @_{\underline{a}}K@_{\underline{b}}p$, if $k_c(011, v)$. Checking the model, we know v could be 011 and 010. So, we have to show:

(1) $M, 011, c \models @_{\underline{a}}K@_{\underline{b}}p$ and

(2) $M, 010, c \models @_{\underline{a}}K@_{\underline{b}}p$

From the last result, (2) is obvious. So we only prove (1), which is needed to show $M, 011, a \models K@_{\underline{b}}p$. Then 011 and 111 are the only possible worlds to check whether $@_{\underline{b}}p$ are satisfied. And the answer is obviously yes. So we are done.

Chapter 3

Axiomatisation of Basic Social Epistemic Logic

At the beginning of this chapter, we will review two recent pieces of research: [19] and [54]. Both are very close to the topic that we are discussing in this chapter.

Then we introduce a "step-by-step" method. It is a technique from the completeness proof of modal logic. Then as the main component of this chapter, we will show the completeness of the basic social epistemic logic ($\mathbf{BSEL}$), which has no restriction on the modal operators in either dimension and whose names are non-rigid. An earlier version of this result was published as [40].

We then extend this basic logic by: (1) restricting the modal operators in either dimension to represent some sort of epistemic attitude, social structure and the interactions between them; (2) restricting the non-rigid names to rigid ones. Then we update the completeness proof to show that all the extensions are complete too. We also compare my system with Sano's Hilbert system.

Like the standard hybrid logic, our language has a mixture of devices from modal propositional logic (modalities) and predicate logic (names and predication). The difference is that names of $\mathbf{BSEL}$ do not refer to points of evaluation (worlds, in the case of hybrid logic) but to agents. This imbalance upsets the canonical model method of proving completeness in a way that is hard to restore using the technique of "witnesses" familiar from predicate logic. In the last section, we will explain why we abandon the canonical model method.

3.1 A Tableau System for BSEL

In [19], the authors proposed a formal framework for representing reasoning about an individual's private opinion and public behaviour under social influence in the social network. Their research is also based on the "Facebook Logic" of Seligman et al. To model knowledge and friendship, the epistemic relation is an equivalence relation and the social relation is irreflexive and symmetric. Inspired by [13] from Bolander and Blackburn and [3], the work of Balbiani et al., they introduced a sound and complete tableaux system for some fragment of their logic.

In this section, I am going to adapt the method of their tableaux system to $\mathbf{BSEL}$. Since $\mathbf{BSEL}$ is minimal, we will ignore any assumption of the epistemic relation and social relation. Our tableaux system is represented by Fitting's method in [21].

Let W and A be sets of natural numbers. We regard them as prefix sets such that W is called the world prefix set and A is the individual prefix set.

We will normally denote elements of W by $\mathsf{w}, \mathsf{v}, \ldots$ and elements of A by $\mathsf{a}, \mathsf{b}, \ldots$. If φ is a $\mathbf{BSEL}$ formula, then we call $\mathsf{w}, \mathsf{a} : \varphi$ a prefixed formula.

Note that if $\mathsf{w}, \mathsf{a} : n$ and $\mathsf{v}, \mathsf{b} : n$ are both in a set of prefixed formulas Γ, then for Γ to be realised in a model, the individual-indices a and b must be identified. This is just like the situation in predicate logic with the identity symbol. The construction of a model works by first defining an equivalence relation between individual-indices and then constructing the agents of the model out of the equivalence classes. Specifically, we define individual-equivalence in Γ to be the smallest equivalence relation between individual-indices such that if for some agent name n, there is a world-index w such that both $\mathsf{w}, \mathsf{a} : n$ and $\mathsf{w}, \mathsf{b} : n$ are in Γ, then a is individual-equivalent to b in Γ. We abbreviate this to $\mathsf{a} \approx \mathsf{b}$ in Γ. Then we define $[\mathsf{a}]^\Gamma = \{\mathsf{b} | \mathsf{a} = \mathsf{b} \text{ or } \mathsf{a} \approx \mathsf{b} \text{ in } \Gamma\}$. Also, we define $\mathsf{W}^\Gamma = \{\mathsf{w} | \mathsf{w} \text{ is a world-index occurring in } \Gamma\}$ and $\mathsf{A}^\Gamma = \{\mathsf{a} | \mathsf{a} \text{ is an individual-index occurring in } \Gamma\}$. We will denote $[\mathsf{a}]^\Gamma$ as $[\mathsf{a}]$ if no ambiguity arises.

For a tableau, we need another two kinds of formulas besides the prefixed formulas: (1) network accessibility formulas of the form $\mathfrak{s}_\mathsf{w}(\mathsf{a}, \mathsf{b})$; and (2) epistemic accessibility formulas of the form $\mathfrak{k}_\mathsf{a}(\mathsf{w}, \mathsf{v})$. In the following parts of this section, when we say a set of prefixed formulas, we mean a set of prefixed formulas and network/epistemic accessibility formulas, if no ambiguity arises.

A realisation of a set of prefixed formulas Γ is given by a model $M = \langle W, A, k, s, g, V \rangle$ and functions r_W and r_A from world indices W^Γ and agent indices A^Γ to W and A, respectively, such that

1. $M, r_\mathsf{W}(\mathsf{w}), r_\mathsf{A}(\mathsf{a}) \models \varphi$ for every $\mathsf{w}, \mathsf{a} : \varphi$ in Γ,
2. $s_{r_\mathsf{W}(\mathsf{w})}(r_\mathsf{A}(\mathsf{a}), r_\mathsf{A}(\mathsf{b}))$ for every $\mathfrak{s}_\mathsf{w}(\mathsf{a}, \mathsf{b})$ in Γ, and
3. $k_{r_\mathsf{A}(\mathsf{a})}(r_\mathsf{W}(\mathsf{w}), r_\mathsf{W}(\mathsf{v}))$ for every $\mathfrak{k}_\mathsf{a}(\mathsf{w}, \mathsf{v})$ in Γ

Figure 3.1 shows the tableau rules for $\mathbf{BSEL}$. A tableau is a downward branching tree. To draw a tableau for a finite set of formulas Γ, we first list all $\varphi \in \Gamma$ as $0, 0 : \varphi$ and put them at the root. So every formula of Γ has the same prefix at the root. Now we discuss how to apply rules in Figure 3.1. Suppose we have $\mathsf{w}, \mathsf{a} : \varphi$ on a branch

α	α_1	α_2
$\mathsf{w},\mathsf{a} : (\varphi \wedge \psi)$	$\mathsf{w},\mathsf{a} : \varphi$	$\mathsf{w},\mathsf{a} : \psi$
$\mathsf{w},\mathsf{a} : \neg\neg\varphi$	$\mathsf{w},\mathsf{a} : \varphi$	

β	β_1	β_2
$\mathsf{w},\mathsf{a} : \neg(\varphi \wedge \psi)$	$\mathsf{w},\mathsf{a} : \neg\varphi$	$\mathsf{w},\mathsf{a} : \neg\psi$

π	π_1	π_2
$\mathsf{w},\mathsf{a} : @_n\varphi$	$\mathsf{w},\mathsf{b} : n$	$\mathsf{w},\mathsf{b} : \varphi$
$\mathsf{w},\mathsf{a} : \neg @_n\varphi$	$\mathsf{w},\mathsf{b} : n$	$\mathsf{w},\mathsf{b} : \neg\varphi$
$\mathsf{w},\mathsf{a} : \neg S\varphi$	$\mathsf{w},\mathsf{b} : \neg\varphi$	$\mathfrak{s}_{\mathsf{w}}(\mathsf{a},\mathsf{b})$
$\mathsf{w},\mathsf{a} : \neg K\varphi$	$\mathsf{v},\mathsf{a} : \neg\varphi$	$\mathfrak{k}_{\mathsf{a}}(\mathsf{w},\mathsf{v})$

γ	γ_1
$\mathsf{w},\mathsf{a} : S\varphi$	$\mathsf{w},\mathsf{b} : \varphi$
$\mathsf{w},\mathsf{a} : K\varphi$	$\mathsf{v},\mathsf{a} : \varphi$

Closure: a branch is closed if the set of indexed formulas Γ on the branch contains $\mathsf{w},\mathsf{a},\mathsf{b}$ and φ such that $\mathsf{w},\mathsf{a} : \varphi$ and $\mathsf{w},\mathsf{b} : \neg\varphi$ are in Γ and $\mathsf{a} \approx \mathsf{b}$ in Γ.

$\approx$ is the smallest equivalence class of individual indices such that
if $\mathsf{w},\mathsf{a} : n$ and $\mathsf{w},\mathsf{b} : n$ occur on the same branch, then $\mathsf{a} \approx \mathsf{b}$ on this branch.

Figure 3.1: Tableau rules for BSEL

for some world-index w and individual-index a. If $\mathsf{w},\mathsf{a} : \varphi$ is an α formula, then α_1 and α_2 can be added to the end of the branch. If $\mathsf{w},\mathsf{a} : \varphi$ is a β formula, then the branch can be split into two branches, one with β_1 and the other with β_2. If $\mathsf{w},\mathsf{a} : \varphi$ is a π formula, then π_1 and π_2 can be added to the end of the branch. However, both π_1 and π_2 have to be associated with a "new" individual-index or world-index such that neither of them occurs on the branch where $\mathsf{w},\mathsf{a} : \varphi$ is. Specifically, if φ is $\neg K\psi$ for some ψ, then π_1 is $\mathsf{v},\mathsf{a} : \neg\psi$ and π_2 is $\mathfrak{k}_{\mathsf{a}}(\mathsf{w},\mathsf{v})$ where v is the new world-index. If φ is one of the other three forms of the π formulas, say $@_n\psi$, then π_1 is $\mathsf{w},\mathsf{b} : n$ and π_2 is $\mathsf{w},\mathsf{b} : \psi$ where b is the new individual-index. The new world-index (or individual-index) is chosen to be numerically greater than any world-index (or individual-index) occurring on the same branch so far. For example, if you get $\mathsf{w},\mathsf{b} : n$ and $\mathsf{w},\mathsf{b} : p$ by applying the π rule to $\mathsf{w},\mathsf{a} : @_np$, then b is the new individual-index and b is currently the numerically greatest individual-index on the branch. Each time we apply an α rule, a β rule or a π rule to $\mathsf{w},\mathsf{a} : \varphi$, we check $\mathsf{w},\mathsf{a} : \varphi$ and mark it with the notation ✓. No rule can be applied to a checked formula any more. Finally, if $\mathsf{w},\mathsf{a} : \varphi$ is a γ formula, then γ_1 can be added to the end of the branch. If φ starts with the S operator, then γ_1 has to be associated with individual indices b and c such that $\mathfrak{s}_{\mathsf{w}}(\mathsf{c},\mathsf{b})$ has already occurred on the same branch where $\mathsf{a} \approx \mathsf{c}$. If γ starts with the K operator, then γ_1 has to be associated with an individual-index b and a world-index v such that $\mathfrak{k}_{\mathsf{b}}(\mathsf{w},\mathsf{v})$ has already occurred on the same branch where $\mathsf{a} \approx \mathsf{b}$. Any time we apply a γ rule, we have to tag the formula with \, not ✓. So after we apply the γ rule to $\mathsf{w},\mathsf{a} : S\psi$ and have $\mathsf{w},\mathsf{b} : \psi$ as a result, we have to mark $\mathsf{w},\mathsf{a} : S\psi$ with $\backslash\mathsf{b}$. Similar for the K case.

A <u>closed branch</u> of a tableau is a branch that satisfies the condition of closure given in Figure 3.1. And a <u>closed tableau</u> is a tableau where each branch is closed. We say a formula φ is provable, $\vdash \varphi$, if there is a closed tableau where only $\neg\varphi$ is at the root. The following is a tableau calculation as an example:

(1)	$w, a : (\neg K \neg(Sp \wedge \neg S \neg q) \wedge K \neg S \neg n)$	
(2)	$w, a : \neg K \neg(Sp \wedge \neg S \neg q)$ ✓	$\wedge$ from 1
(3)	$w, a : K \neg S \neg n \setminus v$	$\wedge$ from 1
(4)	$k_a(w, v)$	$\neg K$ from 2
(5)	$v, a : \neg\neg(Sp \wedge \neg S \neg q)$ ✓	$\neg K$ from 2
(6)	$v, a : (Sp \wedge \neg S \neg q)$ ✓	$\neg\neg$ from 5
(7)	$v, a : Sp \setminus b \setminus c$	$\wedge$ from 6
(8)	$v, a : \neg S \neg q$ ✓	$\wedge$ from 6
(9)	$s_v(a, b)$	$\neg S$ from 8
(10)	$v, b : \neg\neg q$ ✓	$\neg S$ from 8
(11)	$v, b : q$	$\neg\neg$ from 10
(12)	$v, b : p$	S from 9, 7
(13)	$v, a : \neg S \neg n$ ✓	K from 4, 3
(14)	$s_v(a, c)$	$\neg S$ from 13
(15)	$v, c : \neg q$ ✓	$\neg S$ from 13
(16)	$v, c : p$	S from 13, 6

3.1.1 Soundness of the tableau system

Lemma 3.1.1 *Let Γ be a set of prefixed formulas.*

1. *If $\Gamma \cup \{\alpha\}$ has a realisation, so has $\Gamma \cup \{\alpha, \alpha_1, \alpha_2\}$;*
2. *If $\Gamma \cup \{\beta\}$ has a realisation, so has $\Gamma \cup \{\beta, \beta_1\}$ or $\Gamma \cup \{\beta, \beta_2\}$;*
3. *If $\Gamma \cup \{\pi\}$ has a realisation, so has $\Gamma \cup \{\pi, \pi_1, \pi_2\}$;*
4. *If $\Gamma \cup \{\gamma\}$ has a realisation, so has $\Gamma \cup \{\gamma, \gamma_1\}$;*

Proof: Suppose $\Gamma \cup \{X\}$ has a realisation where $X \in \{\alpha, \beta, \gamma, \pi\}$. Then there is an $M = \langle W, A, k, s, g, V\rangle$, r_{W} and r_{A} such that $M, r_{\mathsf{W}}(\mathsf{w}), r_{\mathsf{A}}(\mathsf{a}) \models \varphi$ for every $\mathsf{w}, \mathsf{a} : \varphi$ in $\Gamma \cup \{X\}$, $s_{r_{\mathsf{W}}(\mathsf{w})}(r_{\mathsf{A}}(\mathsf{a}), r_{\mathsf{A}}(\mathsf{b}))$ for every $\mathfrak{s}_{\mathsf{w}}(\mathsf{a}, \mathsf{b})$ in $\Gamma \cup \{X\}$, and $k_{r_{\mathsf{A}}(\mathsf{a})}(r_{\mathsf{W}}(\mathsf{w}), r_{\mathsf{W}}(\mathsf{v}))$ for every $\mathfrak{k}_{\mathsf{a}}(\mathsf{w}, \mathsf{v})$ in $\Gamma \cup \{X\}$.

Case One: $X = \alpha = \mathsf{w}, \mathsf{a} : (\varphi \wedge \psi)$

Then $M, r_{\mathsf{W}}(\mathsf{w}), r_{\mathsf{A}}(\mathsf{a}) \models \varphi \wedge \psi$. So $M, r_{\mathsf{W}}(\mathsf{w}), r_{\mathsf{A}}(\mathsf{a}) \models \varphi$ and $M, r_{\mathsf{W}}(\mathsf{w}), r_{\mathsf{A}}(\mathsf{a}) \models \psi$. So M, r_{W} and r_{A} is a realisation of $\Gamma \cup \{\alpha, \alpha_1, \alpha_2\}$.

The case in which $X = \alpha = \mathsf{w}, \mathsf{a} : \neg\neg\varphi$ is straightforward to prove.

Case Two: $X = \beta = \mathsf{w}, \mathsf{a} : \neg(\varphi \wedge \psi)$

Straightforward to prove.

Case Three: $X = \pi = \mathsf{w}, \mathsf{a} : @_n\varphi$

Then $M, r_{\mathsf{W}}(\mathsf{w}), r_{\mathsf{A}}(\mathsf{a}) \models @_n\varphi$. And $\pi_1 = \mathsf{w}, \mathsf{b} : n$ and $\pi_2 = \mathsf{w}, \mathsf{b} : \varphi$ for some new individual-index b. Let $r'_{\mathsf{W}} = r_{\mathsf{W}}$ and

$$r'_{\mathsf{A}}(\mathsf{c}) = \begin{cases} g_{r_{\mathsf{W}}(\mathsf{w})}(n) & \text{if } \mathsf{c} = \mathsf{b} \\ r_{\mathsf{A}}(\mathsf{c}) & \text{Otherwise} \end{cases}$$

where the domain is $\mathsf{A}^{\Gamma \cup \{\pi, \pi_1, \pi_2\}}$. Then we have $M, r'_{\mathsf{W}}(\mathsf{w}), r'_{\mathsf{A}}(\mathsf{b}) \models n$. But b is a new individual-index to $\Gamma \cup \{\pi\}$. So b doesn't occur in $\Gamma \cup \{\pi\}$. Then $r_{\mathsf{A}} \subseteq r'_{\mathsf{A}}$. So M, r'_{A} and r'_{W} agree with M, r_{W} and r_{A} for all formulas in $\Gamma \cup \{\pi\}$. So $M, r'_{\mathsf{W}}(\mathsf{w}), r'_{\mathsf{A}}(\mathsf{a}) \models @_n\varphi$. Then $M, r'_{\mathsf{W}}(\mathsf{w}), r'_{\mathsf{A}}(\mathsf{b}) \models \varphi$. So M, r'_{A} and r'_{W} is a realisation of $\Gamma \cup \{\pi, \pi_1, \pi_2\}$.

Case Four: $X = \pi = \mathsf{w}, \mathsf{a} : \neg S\varphi$

Then $\pi_1 = \mathsf{w}, \mathsf{b} : \neg\varphi$ and $\pi_2 = \mathfrak{s}_{\mathsf{w}}(\mathsf{a}, \mathsf{b})$ for some new individual-index b. Also, $M, r_{\mathsf{W}}(\mathsf{w}), r_{\mathsf{A}}(\mathsf{a}) \models \neg S\varphi$. So there is some $b \in A$ such that $s_{r_{\mathsf{W}}(\mathsf{w})}(r_{\mathsf{A}}(\mathsf{a}), b)$ and $M, r_{\mathsf{W}}(\mathsf{w}), b \models \neg\varphi$. Let $r'_{\mathsf{W}} = r_{\mathsf{W}}$ and

$$r'_{\mathsf{A}}(\mathsf{c}) = \begin{cases} b & \text{if } \mathsf{c} = \mathsf{b} \\ r_{\mathsf{A}}(\mathsf{c}) & \text{Otherwise} \end{cases}$$

where the domain is $\mathsf{A}^{\Gamma \cup \{\pi, \pi_1, \pi_2\}}$. By a similar proof to the Case Three, we have $M, r'_{\mathsf{W}}(\mathsf{w}), r'_{\mathsf{A}}(\mathsf{b}) \models \neg\varphi$. Also, we have $s_{r'_{\mathsf{W}}(\mathsf{w})}(r'_{\mathsf{A}}(\mathsf{a}), r'_{\mathsf{A}}(\mathsf{b}))$ by $s_{r_{\mathsf{W}}(\mathsf{w})}(r_{\mathsf{A}}(\mathsf{a}), b)$. So M, r'_{A} and r'_{W} is a realisation of $\Gamma \cup \{\pi, \pi_1, \pi_2\}$.

The case in which $X = \pi = \mathsf{w}, \mathsf{a} : \neg K\varphi$ is similar to prove.

Case Five: $X = \gamma = \mathsf{w}, \mathsf{a} : S\varphi$

Then $\gamma_1 = \mathsf{w}, \mathsf{b} : \varphi$ where $\mathfrak{s}_{\mathsf{w}}(\mathsf{a}, \mathsf{b})$ is in $\Gamma \cup \{\gamma\}$. So $s_{r_{\mathsf{W}}(\mathsf{w})}(r_{\mathsf{A}}(\mathsf{a}), r_{\mathsf{A}}(\mathsf{b}))$ is in $\Gamma \cup \{\gamma\}$. Also, $M, r_{\mathsf{W}}(\mathsf{w}), r_{\mathsf{A}}(\mathsf{a}) \models S\varphi$. Then $M, r_{\mathsf{W}}(\mathsf{w}), r_{\mathsf{A}}(\mathsf{b}) \models \varphi$. So M, r_{A} and r_{W} is a realisation of $\Gamma \cup \{\gamma, \gamma_1\}$.

The case in which $X = \gamma = \mathsf{w}, \mathsf{a} : K\varphi$ is similar to prove.

♣

A branch B of a tableau can be regarded as a set of prefixed formulas such that any formula occurs on B if and only if it is in the set of prefixed formulas. If no ambiguity arises, we will borrow the notation B to denote the set of the prefixed formulas on the branch B. So we define a tableau to be <u>realisable</u> if it has some branch B that has a realisation.

Lemma 3.1.2 *No closed branch has a realisation.*

Proof: Suppose there is a closed branch B. For contradiction, suppose the branch has a realisation M. Since B is closed, there exists $\mathsf{w}, \mathsf{a}, \mathsf{b}$ and φ such that $\mathsf{w}, \mathsf{a} : \varphi$ and $\mathsf{w}, \mathsf{b} : \neg\varphi$ are on the branch B where $\mathsf{a} \approx \mathsf{b}$ in B. Then there is $\mathsf{v} \in \mathsf{W}$ such that $\mathsf{v}, \mathsf{a} : n$ and $\mathsf{v}, \mathsf{b} : n$ in B for some n. Since M is a realisation of B, there are r_{W} and r_{A} such that $M, r_{\mathsf{W}}(\mathsf{v}), r_{\mathsf{A}}(\mathsf{a}) \models n$ and $M, r_{\mathsf{W}}(\mathsf{v}), r_{\mathsf{A}}(\mathsf{b}) \models n$. Then $r_{\mathsf{A}}(\mathsf{a}) = r_{\mathsf{A}}(\mathsf{b})$. Also, $M, r_{\mathsf{W}}(\mathsf{w}), r_{\mathsf{A}}(\mathsf{a}) \models \varphi$ and $M, r_{\mathsf{W}}(\mathsf{w}), r_{\mathsf{A}}(\mathsf{b}) \models \neg\varphi$. Then M doesn't exist.

♣

Lemma 3.1.3 *Every tableau is realisable if the formulas at its root have a realisation.*

Proof:

Let T be a tableau where the root has a realisation. Let $t \in \mathbb{N}$ be the number of rules that have been applied to get T. We prove T is realisable by induction on t.

> Base Case: $t = 0$
>
> Since the root has a realisation, T is realisable.
>
> Inductive Case: $t = i + 1$ where $i \in \mathbb{N}$
>
> By Inductive hypothesis, T is realisable after i rules have been applied. Then there is some branch B of T that has a realisation. When we apply the $(i+1)^{\text{th}}$ rule to B, we will get a branch $B' \supset B$ as a result.[1] By Lemma 3.1.1, B' still has a realisation no matter what kind of rule the $(i + 1)^{\text{th}}$ is. So T is still realisable.

♣

Theorem 3.1.1 (Soundness) *If φ is provable, then φ is valid.*

Proof:

Suppose φ is not valid, we prove that φ is not provable. Let T be a tableau where $\neg\varphi$ is at the root. We need to show that T is not closed. Since φ is not valid, there is a model $M = \langle W, A, k, s, g, V\rangle$, w and a such that $M, w, a \not\models \varphi$. Then $M, w, a \models \neg\varphi$. Define $r_{\mathsf{W}}(0) = w$ and $r_{\mathsf{A}}(0) = a$. Then M is a realisation of the root since $M, r_{\mathsf{W}}(0), r_{\mathsf{A}}(0) \models \neg\varphi$. By Lemma 3.1.3, T is realisable. So there is some branch B of T such that B has a realisation. By Lemma 3.1.2, B is not closed. So T is not closed.

♣

[1]We will get two branches if it is the β rule. But that will not affect the following proof.

3.1.2 Completeness of the tableau system

To prove the completeness, we use contraposition. For any unprovable φ, a tableau starting from $0, 0 : \neg\varphi$ is not closed. So there exists at least one branch not closed. However, it would make no sense to say a branch is not closed if some rule in Figure 3.1 is still available to apply on this branch. So we have to decide at which stage the applications of rules bring no more progress for a tableau. If so, we say that the tableau is complete. To define a completable tableau, we need the following definitions.

A closed branch is complete. To say an open branch is complete, we need the following conditions: every prefixed formula in the branch is (1) basic (an atomic formula or its negation); or (2) checked (notation: ✓) or (3) tagged with $\backslash\mathsf{a}$ or $\backslash\mathsf{w}$ for every index a or w required by the γ rule respectively. A tableau is complete if all its branches are complete. A descendant of the branch B is a branch that results from the branch B after a finite sequence of rule applications. A completable branch B of a tableau is a branch where there is a finite sequence of rule application after which all of the descendants of B are complete. Now a completable tableau T is a tableau where every branch of T is completable.

Every tableau is completable. A standard way to prove this is to appeal the complexity of formulas by the assumption that the more rules we apply on a branch, the less complexity that the branch will be. Unfortunately, it is not always so in our tableau system. When we apply γ rule, the total complexity of the unchecked formulas on the branch increases. So we have to assign a natural number $c(B)$ to each branch B carefully, measuring its complexity in such a way that rule applications are guaranteed to reduce the complexity.

Definition 3.1.1 *Given a formula φ, we define the complexity $c_N(\varphi)$ of formula φ bounded by $N \in \mathbb{N}$ by induction as follows:*

$$c_N(p) = c_N(\neg p) = c_N(n) = c_N(\neg n) = 0$$

$$c_N(\varphi \wedge \psi) = c_N(\varphi) + c_N(\psi) + 1$$

$$c_N(\neg(\varphi \wedge \psi)) = c_N(\neg\varphi) + c_N(\neg\psi) + 1$$

$$c_N(\neg\neg\varphi) = c_N(@_n\varphi) = c_N(\varphi) + 1$$

$$c_N(\neg @_n\varphi) = c_N(\neg S\varphi) = c_N(\neg K\varphi) = c_N(\neg\varphi) + 1$$

$$c_N(S\varphi) = c_N(K\varphi) = N \times c_N(\varphi)$$

For prefixed formulas, $c_N(\mathsf{w}, \mathsf{a} : \varphi) = c_N(\varphi)$, also for relation formulas: $c_N(\mathfrak{s}_{\mathsf{w}}(\mathsf{a}, \mathsf{b})) = c_N(\mathfrak{k}_{\mathsf{a}}(\mathsf{w}, \mathsf{v})) = 0$. Then we define the complexity $c_N(B)$ of a branch B bounded by N to be the sum of $c_N(\theta)$ for each unchecked and untagged prefix formula θ on the branch B added to the sum of $c_N(\gamma) - n \times c_N(\gamma_1)$ for each formula γ on B having n tags. Then if the maximum number of tags on any prefixed formula is less than N, $c_N(B)$ will be positive.

Lemma 3.1.4 *Given a branch B and descendant branch B', there is a function $\mathtt{t}$ mapping each prefixed formula on B' to its unique "trace", a sub-formula occurring in a prefixed formula on B.*[2] *We denote "sub-formula occurrences" with the underlining, e.g. $\neg(\underline{Sp} \wedge SSp)$.*

Proof:

First we want to make it clear that we don't add a formula which already exists on the branch if we apply a rule, i.e., that if θ is already on the branch and a rule requires us to add θ then we don't add another copy. Similarly to $\mathsf{w}, \mathsf{a} : (p \wedge p)$ we only add one copy of $\mathsf{w}, \mathsf{a} : p$.

Now we give an inductive definition of $\mathtt{t}$. If θ is on B, then $\mathtt{t}(\theta)$ is the result of underlining the whole formula of θ. If θ was obtained by applying a rule to a prefixed formula $\mathsf{w}, \mathsf{a} : \varphi$, then θ has the form $\mathsf{v}, \mathsf{b} : \psi$, where ψ is either a sub-formula of φ or the negation of a sub-formula of φ. Then we define $\mathtt{t}(\theta)$ to be the result of underlining that sub-formula in $\mathtt{t}(\mathsf{v}, \mathsf{b} : \psi)$. Here is an example. If B has $\mathsf{w}, \mathsf{b} : \neg Sp$ then $\mathtt{t}(\mathsf{w}, \mathsf{a} : \neg Sp) = \mathsf{w}, \mathsf{a} : \underline{\neg Sp}$. The π rule can be applied to it such that $\mathsf{w}, \mathsf{b} : \neg p$ and $\mathfrak{s}_{\mathsf{w}}(\mathsf{a}, \mathsf{b})$ are added to get B, then $\mathtt{t}(\mathsf{w}, \mathsf{b} : \neg p) = \mathsf{w}, \mathsf{a} : \neg S\underline{p}$.

Suppose $\mathtt{t}(\mathsf{v}, \mathsf{b} : \psi_1) = \mathtt{t}(\mathsf{v}, \mathsf{b} : \psi_2)$ which is an underlining of θ. We show $\psi_1 = \psi_2$. For the cases that θ is an α, β, or π formula, then it will be checked off when the rule is applied. So $\mathsf{v}, \mathsf{b} : \psi_1$ and $\mathsf{v}, \mathsf{b} : \psi_2$ can only come from the same rule application. Since we underlined the same sub-formula of θ, $\psi_1 = \psi_2$. For the case that θ is a γ formula, then it won't be checked off, but the rule can only be applied to get $\mathsf{v}, \mathsf{b} : \psi_1$ or $\mathsf{v}, \mathsf{b} : \psi_2$ if θ is not tagged with $\backslash b$. So $\mathsf{v}, \mathsf{b} : \psi_1$ and $\mathsf{v}, \mathsf{b} : \psi_2$ must be from the same rule, and the underlining argument can be applied again.

♣

Lemma 3.1.5 *For any branch B, there is a number N such that $c_N(B')$ is positive for every descendant B' of B.*

Proof: Let N be the total number of sub-formulas of prefix formulas occurring on B added to the number of relation formulas on B.[3] Now let B' be any descendant branch of B and γ be a prefix formula on B'. Suppose γ is of the form $\mathsf{w}, \mathsf{a} : S\varphi$ and is tagged with $\backslash \mathsf{b}_1, \ldots, \backslash \mathsf{b}_n$. Then $\mathfrak{s}_{\mathsf{w}}(\mathsf{a}, \mathsf{b}_i)$ is on B for each $1 \leq i \leq n$. Each of them

[2]By unique we mean that $\mathtt{t}$ is injective for the formulas with the same prefix, i.e., if $\mathtt{t}(w, a : \varphi) = \mathtt{t}(w, a : \psi)$ then $\varphi = \psi$.

[3]If a given formula is a sub-formula that occurs twice in a prefix formula or in two prefix formulas with different prefixes, then we count it twice.

is either already in B or was introduced by applying the rule for some $\mathsf{w}, \mathsf{a} : \neg S\psi$, which is checked off and cannot be used again. But there are no more than N of these since every prefixed formula on B' with a given prefix can be traced back to a unique sub-formula of a prefixed formula on B by Lemma 3.1.4. So $n \leq N$. The argument for the K case is similar.

♣

We define c_B to be c_N.

Lemma 3.1.6 *If the branch B' is a descendant of the branch B and B'' is obtained from B' by the application of a tableau rule, then we have:*

$$0 < c_B(B'') < c_B(B')$$

Proof: First, by Lemma 3.1.5, $c_B(B'')$ is positive. For the cases that B'' results from B' by an application of the α, β or π rule, we have:

α) $c_B(B'') = c_B(B') - c_B(\alpha) + c_B(\alpha_1) + c_B(\alpha_2)$
but $c_B(\alpha) > c_B(\alpha_1) + c_B(\alpha_2)$ and so $c_B(B'') < c_B(B')$.

β) $c_B(B'') = c_B(B') - c_B(\beta) + c_B(\alpha_i)$ (where $i = 1$ or 2)
but $c_B(\beta) > c_B(\beta_i)$ and so $c_B(B'') < c_B(B')$.

π) $c_B(B'') = c_B(B') - c_B(\pi) + c_B(\pi_1) + c_B(\pi_2)$
but $c_B(\pi) > c_B(\pi_i)$ and so $c_B(B'') < c_B(B')$.

For the γ rule, suppose it is the S case in which the rule is applied to $\mathsf{w}, \mathsf{a} : S\varphi$ on B' with $\mathfrak{s}_{\mathsf{w}}(\mathsf{a}, \mathsf{b})$ also on B'. If so, it cannot be tagged with $\backslash\mathsf{b}$ in B' but with $\backslash\mathsf{b}$ in B''. Suppose $\mathsf{w}, \mathsf{a} : S\varphi$ has one more tag in B'' than in B' and so the contribution of $\mathsf{w}, \mathsf{a} : S\varphi$ to $c_B(B'')$ is $c_B(\mathsf{w}, \mathsf{a} : S\varphi)$, which is less than it is to $c_B(B')$. But $\mathsf{w}, \mathsf{b} : \varphi$ is added to B''. So we have:

γ) $c_B(B'') = c_B(B') - c_B(\mathsf{w}, \mathsf{a} : S\varphi) + c_B(\mathsf{w}, \mathsf{b} : \varphi)$
but $c_B(\mathsf{w}, \mathsf{a} : S\varphi) > c_B(\mathsf{w}, \mathsf{b} : \varphi)$, so $c_B(B'') < c_B(B')$.

And similarly for K.

♣

Lemma 3.1.7 *Every tableau is completable.*

Proof:

For contradiction, we suppose that a tableau T is not completable. Then it has a branch B_1 that is not completable. But then there must be a rule that can be applied to B_1 resulting the branch B_2 that is also not completable. By Lemma

3.1.6, $c_B(B_2) < c_B(B_1)$. Iterating this process, we get an infinite sequence of branches $B_1, B_2, B_3, \ldots$ with $c_B(B_{i+1}) < c_B(B_i)$. But this is impossible since each $c_B(B_i) > 0$ is a whole number.

♣

Theorem 3.1.2 *Given a complete tableau, any open branch of the tableau has a realisation.*

Proof: Let T be a complete tableau. And suppose B is an open branch of it. First, we define a structure. Then we prove that it is a $\mathbf{BSEL}$ model. Then we show that B has a realisation.

We define $M = \langle W, A, k, s, g, V\rangle$ as follows:

$$W = \{\mathsf{w} | \mathsf{w}, \mathsf{a} : \varphi \in B\}$$

$$A = \{[\mathsf{a}] | \mathsf{w}, \mathsf{b} : \varphi \in B \text{ and } \mathsf{a} \approx \mathsf{b} \text{ in } B\} \cup \{e_\mathsf{w} | \mathsf{w} \in W\}$$

$$k_{[\mathsf{a}]}(\mathsf{w}, \mathsf{v}) \text{ if } \mathfrak{k}_\mathsf{b}(\mathsf{w}, \mathsf{v}) \text{ is on the branch } B \text{ for some } \mathsf{b} \text{ where } \mathsf{a} \approx \mathsf{b} \text{ in } B$$

$$s_\mathsf{w}([\mathsf{a}], [\mathsf{b}]) \text{ if } \mathfrak{s}_\mathsf{w}(\mathsf{c}, \mathsf{d}) \text{ is on the branch } B \text{ where } \mathsf{c} \approx \mathsf{a} \text{ and } \mathsf{d} \approx \mathsf{b} \text{ in } B$$

$$g_\mathsf{w}(n) = \begin{cases} [\mathsf{a}] & \text{if } \mathsf{w}, \mathsf{b} : n \text{ is on } B \text{ for some } \mathsf{b} \approx \mathsf{a} \\ e_\mathsf{w} & \text{Otherwise} \end{cases}$$

$$V(p) = \{(\mathsf{w}, [\mathsf{a}]) | \mathsf{w}, \mathsf{a} : p \in B\}$$

where $\{e_\mathsf{w} | \mathsf{w} \in W\}$ is a disjoint set from A. Why do we introduce this set? We have to handle the cases in which some name n is not on B. If so, we assign it something not in A, say e_w, for each world-index w. For each $\mathsf{w} \in W$, e_w is different from all individual indices and each name, which doesnt occur on the branch but has w as its world-index.

To prove M is a model of $\mathbf{BSEL}$, we have to show k, s, g and V are well defined. Obviously, g and V are well defined since A is a set of equivalence classes. So we only prove the first two.

To show k is well defined, we suppose $\mathsf{a}' \in [\mathsf{a}]$ and prove $k_{[\mathsf{a}']}(\mathsf{w}, \mathsf{v})$ iff $k_{[\mathsf{a}]}(\mathsf{w}, \mathsf{v})$. We only prove the left to right direction since the reverse can be proven similarly. Suppose $k_{[\mathsf{a}']}(\mathsf{w}, \mathsf{v})$. Then $\mathfrak{k}_{\mathsf{b}'}(\mathsf{w}, \mathsf{v})$ is on the branch B for some b' and $\mathsf{a}' \approx \mathsf{b}'$ in B. By $\mathsf{a}' \in [\mathsf{a}]$, $\mathsf{a} \approx \mathsf{a}'$. Then $\mathsf{a} \approx \mathsf{b}'$. So $k_{[\mathsf{a}]}(\mathsf{w}, \mathsf{v})$.

Now we show s is well defined by proving

(1) $\mathsf{a}' \in [\mathsf{a}]$ implies $s_\mathsf{w}([\mathsf{a}'], [\mathsf{b}])$ iff $s_\mathsf{w}([\mathsf{a}], [\mathsf{b}])$;

(2) $\mathsf{b}' \in [\mathsf{b}]$ implies $s_\mathsf{w}([\mathsf{a}], [\mathsf{b}'])$ iff $s_\mathsf{w}([\mathsf{a}], [\mathsf{b}])$;

We only prove the left to right direction of (1) since the others are similar to prove. Suppose $s_{\mathsf{w}}([\mathsf{a}'], [\mathsf{b}])$. Then $\mathfrak{s}_{\mathsf{w}}(\mathsf{c}, \mathsf{d})$ is on the branch B where $\mathsf{c} \approx \mathsf{a}'$ and $\mathsf{d} \approx \mathsf{b}$ in B. By $\mathsf{a}' \in [\mathsf{a}]$, $\mathsf{a} \approx \mathsf{a}'$ in B. So $\mathsf{a} \approx \mathsf{c}$ in B. So $s_{\mathsf{w}}([\mathsf{a}], [\mathsf{b}])$.

Let r_{W} be the identity function. Define $r_{\mathsf{A}}(\mathsf{a}) = [\mathsf{a}]$ for each $\mathsf{a} \in \mathsf{A}$. Now we prove M is a realisation of B by inductively showing that $\mathsf{w}, \mathsf{a} : \varphi$ is on B implies $M, \mathsf{w}, [\mathsf{a}] \models \varphi$.

Base Case: $\varphi = p$, $\varphi = \neg p$, $\varphi = n$ or $\varphi = \neg n$

If $\varphi = p$, then by definition of V, $\mathsf{w}, \mathsf{a} : p$ is on B implies $(\mathsf{w}, [\mathsf{a}]) \in V(p)$.
So $M, \mathsf{w}, [\mathsf{a}] \models p$.

If $\varphi = \neg p$, then $\mathsf{w}, \mathsf{a} : p$ is not on B (Otherwise B is closed). So $(\mathsf{w}, [\mathsf{a}]) \notin V(p)$. So $M, \mathsf{w}, [\mathsf{a}] \not\models p$, which means $M, \mathsf{w}, [\mathsf{a}] \models \neg p$.

If $\varphi = n$, then $\mathsf{w}, \mathsf{a} : n$ is on B implies $g_{\mathsf{w}}(n) = [\mathsf{a}]$. So $M, \mathsf{w}, [\mathsf{a}] \models n$

If $\varphi = \neg n$, for contradiction we suppose $M, \mathsf{w}, [\mathsf{a}] \not\models \neg n$. Then $M, \mathsf{w}, [\mathsf{a}] \models n$. So $g_{\mathsf{w}}(n) = [\mathsf{a}]$. Since B is not closed, for any b, if $\mathsf{a} \approx \mathsf{b}$ in B, then it is not the case that both $\mathsf{w}, \mathsf{a} : \neg n$ and $\mathsf{w}, \mathsf{b} : n$ are on B. Since $\mathsf{w}, \mathsf{a} : \neg n$ is on B, $\mathsf{w}, \mathsf{b} : n$ is not on B for any $\mathsf{a} \approx \mathsf{b}$. So $g_{\mathsf{w}}(n) \neq [\mathsf{a}]$. Contradiction. So $M, \mathsf{w}, [\mathsf{a}] \models \neg n$.

Inductive Case:

$\varphi = (\psi \wedge \theta)$. Suppose $\mathsf{w}, \mathsf{a} : (\psi \wedge \theta)$ is on B. Because B is complete, the α rule has been applied and so both $\mathsf{w}, \mathsf{a} : \psi$ and $\mathsf{w}, \mathsf{a} : \theta$ are on B. By the inductive hypothesis, $M, w, [a] \models \psi$ and $M, w, [a] \models \theta$. So $M, w, [a] \models (\psi \wedge \theta)$.
The case in which $\varphi = \neg\neg\psi$ is similar to prove.

$\varphi = \neg(\psi \wedge \theta)$. Suppose $\mathsf{w}, \mathsf{a} : \neg(\psi \wedge \theta)$ is on B. Because B is complete, the β rule has been applied and so either $\mathsf{w}, \mathsf{a} : \neg\psi$ or $\mathsf{w}, \mathsf{a} : \neg\theta$ is on B. By the inductive hypothesis, $M, w, [a] \models \neg\psi$ or $M, w, [a] \models \neg\theta$. So $M, w, [a] \models \neg(\psi \wedge \theta)$.

$\varphi = @_n\psi$. Suppose $\mathsf{w}, \mathsf{a} : @_n\psi$ is on B. Because B is complete, the π rule has been applied and so both $\mathsf{w}, \mathsf{b} : n$ and $\mathsf{w}, \mathsf{b} : \psi$ are on B for some b. By inductive hypothesis, $M, \mathsf{w}, [\mathsf{b}] \models \psi$. Also, $M, \mathsf{w}, [\mathsf{b}] \models n$, which means $g_{\mathsf{w}}(n) = [\mathsf{b}]$. So $M, \mathsf{w}, [\mathsf{a}] \models @_n\psi$.

$\varphi = \neg @_n\psi$. Similar to the above case.

$\varphi = \neg S\psi$. Suppose $\mathsf{w}, \mathsf{a} : \neg S\psi$ is on B. Because B is complete, the π rule has been applied and so both $\mathsf{w}, \mathsf{b} : \neg\psi$ and $\mathfrak{s}_{\mathsf{w}}(\mathsf{a}, \mathsf{b})$ are on B for some b. By the inductive hypothesis, $M, \mathsf{w}, [\mathsf{b}] \models \neg\psi$. Since $\mathfrak{s}_{\mathsf{w}}(\mathsf{a}, \mathsf{b})$ is on B, $s_{\mathsf{w}}([\mathsf{a}], [\mathsf{b}])$. So $M, \mathsf{w}, [\mathsf{a}] \not\models S\psi$. So $M, \mathsf{w}, [\mathsf{a}] \models \neg S\psi$

$\varphi = \neg K\psi$. Suppose $\mathsf{w}, \mathsf{a} : \neg K\psi$ is on B. Because B is complete, the π rule has been applied and so both $\mathsf{v}, \mathsf{a} : \neg\psi$ and $\mathfrak{k}_{\mathsf{a}}(\mathsf{w}, \mathsf{v})$ are on B for some v. Then we have $M, \mathsf{v}, [\mathsf{a}] \models \neg\psi$ by the inductive hypothesis. Also, $k_{[\mathsf{a}]}(\mathsf{w}, \mathsf{v})$. So $M, \mathsf{w}, [\mathsf{a}] \not\models K\psi$. Then $M, \mathsf{w}, [\mathsf{a}] \models \neg K\psi$.

$\varphi = S\psi$. Suppose $\mathsf{w}, \mathsf{a} : S\psi$ is on B. To prove $M, \mathsf{w}, [\mathsf{a}] \models S\psi$, we show that for any $[\mathsf{b}]$ such that $s_{\mathsf{w}}([\mathsf{a}], [\mathsf{b}])$, we have $M, \mathsf{w}, [\mathsf{b}] \models \psi$. Suppose $s_{\mathsf{w}}([\mathsf{a}], [\mathsf{b}])$. Then $\mathfrak{s}_{\mathsf{w}}(\mathsf{c}, \mathsf{d})$ is on B where $\mathsf{c} \approx \mathsf{a}$ and $\mathsf{d} \approx \mathsf{b}$ in B. Since $\mathsf{w}, \mathsf{a} : S\psi$ is on B and B is complete, the γ rule has been applied and so $\mathsf{w}, \mathsf{d} : \psi$ is on B. By the inductive hypothesis, $M, \mathsf{w}, [\mathsf{d}] \models \psi$. Since $\mathsf{d} \approx \mathsf{b}$ in B, $[\mathsf{d}] = [\mathsf{b}]$. So $M, \mathsf{w}, [\mathsf{b}] \models \psi$.

$\varphi = K\psi$. Suppose $\mathsf{w}, \mathsf{a} : K\psi$ is on B. To prove $M, \mathsf{w}, [\mathsf{a}] \models K\psi$, we show that for any v such that $k_{[\mathsf{a}]}(\mathsf{w}, \mathsf{v})$, we have $M, \mathsf{v}, [\mathsf{a}] \models \psi$. Suppose $k_{[\mathsf{a}]}(\mathsf{w}, \mathsf{v})$. Then $\mathfrak{k}_{\mathsf{b}}(\mathsf{w}, \mathsf{v})$ is on B for some $\mathsf{b} \approx \mathsf{a}$ in B. Since $\mathsf{w}, \mathsf{a} : K\psi$ is on B and B is complete, the γ rule has been applied and so $\mathsf{v}, \mathsf{a} : \psi$ is on B. By the inductive hypothesis, $M, \mathsf{v}, [\mathsf{a}] \models \psi$.

The last two clauses of the definition of the realisation are satisfied by the construction of M. So B has a realisation.

♣

Theorem 3.1.3 (Completeness) *If* $\models \varphi$*, then* $\vdash \varphi$.

Proof:

Suppose φ is valid but $\not\vdash \varphi$. By φ is not provable, for any tableau T with $\neg\varphi$ at the root, T is not closed. By Lemma 3.1.7, T is completable. Let T' be an extended complete tableau, which is not closed. So there is an open branch in T'. By Theorem 3.1.2, any formula on the open branch of T' is satisfiable. Since T' is extended by T and $\neg\varphi$ is at the root of T, $\neg\varphi$ is also at the root of T'. So $\neg\varphi$ is satisfiable. And this contradicts the validity of φ, as required.

♣

The system in [19] has several rules for characterising the equivalence relation for k and irreflexive and symmetric relation for s. $\mathbf{BSEL}$ doesnt assume any property for either relation. So the tableau system in this section is much simpler than the one in [19]. Nonetheless, the essential insight as to how to build the system is the same as that of [19], and what we have here is entirely derived from that work.

3.2 Sano's Hilbert System

In [54], Sano provides a Hilbert system for Epistemic Logic of Friendship. This minimal logic is almost the same as our social epistemic logic except that it regards

$\vdash \varphi$ if φ is a propositional tautology	Taut
$\vdash S(p \to q) \to (Sp \to Sq)$	$\mathsf{K_S}$
$\vdash K(p \to q) \to (Kp \to Kq)$	$\mathsf{K_K}$
$\vdash @_n(p \to q) \to (@_n p \to @_n q)$	$\mathsf{K_@}$
$\vdash @_n p \leftrightarrow \neg @_n \neg p$	Selfdual@
$\vdash @_n n$	Ref@
$\vdash @_n @_m p \to @_m p$	Agree
$\vdash n \to (p \leftrightarrow @_n p)$	Intro
$\vdash @_n p \to S @_n p$	Back
$\vdash @_n m \to K @_n m$	Rigid$=$
$\vdash \neg @_n m \to K \neg @_n m$	Rigid$\neq$
$\vdash @_n K @_n p \leftrightarrow @_n K p$	DCom@K
from $\vdash \varphi$ infer $\vdash \varphi[p/\psi]$ or $\vdash \varphi[n/m]$	Sub
from $\vdash \varphi$ and $\vdash \varphi \to \psi$, infer $\vdash \psi$	MP
from $\vdash \varphi$ infer $\vdash S\varphi$	$\mathsf{Nec_S}$
from $\vdash \varphi$ infer $\vdash @_n\varphi$	$\mathsf{Nec_@}$
from $\vdash \varphi$ infer $\vdash K\varphi$	$\mathsf{Nec_K}$
from $\vdash n \to \varphi$, infer $\vdash \varphi$, where n is not in φ	Name
from $\vdash L(@_n \langle S \rangle m \to @_m \varphi)$,	
infer $\vdash L(@_n S\varphi)$, where m is not in $L(@_n S\varphi)$	L(BG)

Figure 3.2: Sano's Hilbert system

the social relation as "friendship" not vision. However, Sano discovers his Hilbert system indirectly: He first presents a tree sequent calculus for the logic and proves this calculus is sound and complete. Then he translates a tree sequent into an ordinary formula to specify a Hilbert system. Finally he proves that any formula is provable in the Hilbert system if and only if a corresponding tree sequent is provable in the tree sequent calculus. Based on the result that the tree sequent calculus is sound and complete, the soundness and completeness of the Hilbert system are proven.

Figure 3.2 shows Sano's Hilbert system where $\varphi[p/\psi]$, $\varphi[n/m]$ are uniform substitutions. The system has a special rule called L(BG). We are now introducing the notion of $L(\varphi)$ in the L(BG) rule. It is called the "necessity form", and originally proposed in [26] by the Sofia School of modal logic as a method to investigate different modal logics that have names.

Definition 3.2.1 *Let $\#$ be a symbol not occurring in the language. We define the set of necessity forms as follows:*

1. *$\#$ is a necessity form;*
2. *if L is a necessity form, then $\varphi \to L$ is also a necessity form;*
3. *if L is a necessity form and n is a nominal, then $@_n KL$ is also a necessity form; and*
4. *nothing else is a necessity form.*

Note that every necessity forms L has exactly one occurrence of the symbol $\#$. So $@_n K(\varphi \to (\psi \to \#))$ is a necessity form. We use $L(\varphi)$ to denote the formula obtained from L by replacing the unique occurrence of $\#$ by a formula φ.

The reason that Sano introduces the necessity form is to track the unique path from a label in a tree of a tree sequent to the root label of the tree.

We will review Sano's Hilbert system in more detail after we present ours.

3.3 Step-by-Step Method

The classical way of proving completeness of systems of modal logic is to construct the canonical model. For some logics, the canonical model is the unique one where all possible maximal consistent sets of the language are regarded as the worlds of the model. For others, the canonical model can be derived from a given consistent set in the sense that each maximal consistent set is chosen because it is somehow relevant to the given set. And the valuation of the canonical model is usually defined in terms of whether the atomic proposition is in the actual world (maximal consistent set) or not. Then there is one more thing to do: the relations between the worlds of the canonical model have to be defined in such a way that the semantic requirement of the modal operators can be mirrored. We call this kind of relation the "canonical relation". Although there are many modal logic techniques to allow us to construct modally equivalent varieties of the canonical model, we are still unable to ensure that the mirror can always be realised.

In this section, we will take the basic modal logic $\mathbf{K}$ as an example. And we show how to classically prove it is complete via a canonical model and step-by-step method respectively.

The syntax of our basic modal language $\mathcal{L}$ is as follows:

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

where $p \in \mathsf{Prop}$, a countably infinite set of propositional atoms.

A model of $\mathcal{L}$ is a triple $M = \langle W, R, V\rangle$ consisting of the following ingredients:

1. a non-empty set W,
2. R is a binary relation of the elements of W,
3. V is a valuation with domain Prop and range $\mathrm{p}ow(W)$

For the semantics, the clauses for the atomic and Boolean cases are the same as for the epistemic logic in Chapter Two. As for the modal case, we define

$$M, w \models \Box\varphi \quad \text{iff} \quad \text{for all } v \in W \text{ if } Rwv \text{ then } M, v \models \varphi.$$

Define the theorems of the basic modal logic $\mathbf{K}$, written $\vdash \varphi$ by the following axiom and rule schemas:

Taut	$\vdash \varphi$ if φ is an instantiation of propositional tautologies
K	$\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
MP	if $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$
Nec	if $\vdash \varphi$ then $\vdash \Box\varphi$

A set of formulas Γ is inconsistent if there is a conjunction φ of formulas in Γ such that $\vdash \neg\varphi$; otherwise, Γ is consistent. A set of formulas Γ is maximally consistent if it is consistent and has no strictly larger consistent superset, i.e., if $\Gamma \subset \Delta$, then Δ is inconsistent. In the latter parts of this thesis, we will see more logics. We assume these definitions remain unchanged (of course, $\vdash$ has a different meaning for different logic, but we will keep using this notation when no ambiguity arises) for these logics.

To prove completeness with the canonical model, we first have to build it. The canonical model $M^c = \langle W^c, R^c, V^c\rangle$ is defined as follows:

W^c is the set of all maximal consistent sets,

$R^c wv$ iff for all formulas φ, $\Box\varphi \in w$ implies that $\varphi \in v$

$V^c(p) = \{w \mid p \in w\}$

Naturally, we have some desired properties of the maximal consistent set Γ:

1. Γ is closed under modus ponens,

2. Γ contains all provable formulas of Λ,

3. $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for any φ,

4. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$

And Lindenbaum's Lemma guarantees that any consistent set can be extended to a maximal consistent set.

For the canonical relation, we have to prove the Existence Lemma: If $\neg\Box\varphi \in w$ then there exists $v \in W$ such that $R^c wv$ and $\neg\varphi \in v$. To prove this lemma, we need to find v. In fact, any maximal consistent set containing $\{\neg\varphi\} \cup \{\psi | \Box\psi \in w\}$ could be v. The reader can check that such v satisfies $R^c wv$ and $\neg\varphi \in v$. Does such a maximal consistent set exist? Yes, by the following lemma and Lindenbaum's Lemma.

Lemma 3.3.1 *Given a maximal consistent set Σ of $\mathcal{L}$, if $\neg\Box\varphi \in \Sigma$, then $\{\neg\varphi\} \cup \{\psi | \Box\psi \in \Sigma\}$ is consistent.*

Proof: For contradiction, suppose $\{\neg\varphi\} \cup \{\psi | \Box\psi \in \Sigma\}$ is not consistent. Then there exists a conjunction θ of $\{\psi | \Box\psi \in \Sigma\}$ such that $\vdash \neg(\theta \wedge \neg\varphi)$. So $\vdash \theta \to \neg\neg\varphi$. Then by Nec, $\vdash \Box(\theta \to \neg\neg\varphi)$. So we have $\vdash \Box\theta \to \Box\neg\neg\varphi$ by K and MP. Keep on applying theorem: $\vdash \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$[4], we will have $\Box\theta \in \Sigma$. Since Σ is a maximal consistent set, $\Box\neg\neg\varphi \in \Sigma$. So $\Box\varphi \in \Sigma$, which contradicts $\neg\Box\varphi \in \Sigma$.

♣

Then by induction on $\varphi \in \mathcal{L}$ we finally have the Truth Lemma:

$$M^c, w \models \varphi \quad \text{iff} \quad \varphi \in w$$

Now the proof method is clear. Given any consistent set Γ of $\mathcal{L}$, we extend Γ to a maximal consistent set Γ'. Let M^c be the canonical model. Then $\Gamma' \in W^c$. By the Truth Lemma, φ is satisfied at Γ' for each $\varphi \in \Gamma$. So Γ is satisfied. So the completeness is proven.

We'll now start to develop some definitions and concepts that will be useful in describing the step-by-step method for constructing a model. Some parts of the construction are familiar. The basic building materials are maximal consistent sets of formulas, and these are used in a way that is similar to the usual canonical model methods. But unlike the canonical model, which is built in one go, the step-by-step method builds a model in stages, like building a wall from brick. You start with one brick, then add another and another until you get the wall.

This is a useful guiding metaphor, but there are several differences between wall-building and model-building. First, the step-by-step construction of a model is less

[4]The readers can find the derivation of this theorem in any modal logic textbook.

uniform. It's like building a wall with lots of holes, which need filling later. At each stage, we look at the defects of the construction, pick one of them and repair it. We only repair one defect at a time, but we're careful not to undo any repairs we've already made, until all defects have been removed.

The second difference is that, unlike a wall, the structures we build are infinite, and so there can be no end to the step-by-step process. Instead, we take a mathematical limit of the process and we must be careful that this does not re-introduce defects.

A final point is that whereas partially built walls are still walls, the stages of our model construction are not actually models. That's because they have defects. The name we use for these intermediates structures is "network". A network which has no defects is said to be "perfect". It's only from perfect networks that we can define a model, and perfection occurs only at the limit.

Normally the key aspect of the step-by-step method is to figure out the specific properties we need and find a way to construct a network that satisfies these properties.

Definition 3.3.1 *A network of $\mathcal{L}$ is a tuple $\mathcal{N} = \langle N, \mathsf{r}, \delta \rangle$ where N is a non-empty countable set (of nodes),* r *is a relation on N, and δ maps nodes to maximal consistent sets.*

From a network $\mathcal{N}$, we define a model induced by $\mathcal{N}$: $M_{\mathcal{N}} = \langle N, \mathsf{r}, V \rangle$ by

$$w \in V(p) \quad \text{iff} \quad p \in \delta(w) \text{ for any } w \in N \text{ and } p \in \mathsf{Prop}.$$

We are aiming at a network $\mathcal{N}$ in which $\varphi \in \delta(w)$ iff $M_{\mathcal{N}}, w \models \varphi$. This is now ensured if φ is just an atomic formula by the definition of $V(p)$, but in general not. We list structural properties of the network designed to ensure coordination between the formulas in the sets $\delta(w)$ and those satisfied by the model at w.

But before that, we need to define a set of relations on sets of formulas for r by mimicing the canonical relations R^c:

$$\mathsf{r}_{wv} \quad \text{iff} \quad \text{for all formulas } \varphi,\ \Box\varphi \in \delta(w) \text{ implies that } \varphi \in \delta(v).$$

Now comes the structural properties of the network.

Definition 3.3.2 *A network $\mathcal{N}$ is coherent if for all $\varphi \in \mathcal{L}$, $\Box\varphi \in \delta(w)$ implies $\varphi \in \delta(v)$ for any r_{wv}.*

Definition 3.3.3 *A network $\mathcal{N}$ is saturated if for all formulas $\varphi \in \mathcal{L}$, $\neg\Box\varphi \in \delta(w)$ implies that there exists $v \in N$ such that $\neg\varphi \in \delta(v)$ and r_{wv}.*

Definition 3.3.4 *A network $\mathcal{N}$ is perfect if it is coherent and saturated.*

Assuming the network is perfect, we have the Truth Lemma:

Lemma 3.3.2 *Given any perfect network $\mathcal{N}$, $w \in N$ and $\varphi \in \mathcal{L}$, we have:*

$$M_{\mathcal{N}}, w \models \varphi \quad \textit{iff} \quad \varphi \in \delta(w)$$

Proof: We perform induction on φ.

Case One: $\varphi = p$

Straightforward by the definition of V.

Case Two: φ is a Boolean formula.

By the properties of maximal consistent sets

Case Three: $\varphi = \Box\psi$

The left to right direction is guaranteed by $\mathcal{N}$ is saturated; The other direction is because $\mathcal{N}$ is coherent.

♣

The Truth Lemma makes the completeness proof straightforward. But don't forget the condition of this lemma: $\mathcal{N}$ has to be perfect.

So far we have assumed a perfect $\mathcal{N}$ exists. But how can we guarantee that we can build a perfect network for any consistent set of formulas so that we can use the Extended Truth Lemma?

Here is our method. To make $\mathcal{N}$ saturated, we list all the cases when saturation fails. These cases are called "defects". Remove ONE defect in each step and keep our repair for all the latter steps. Finally we will get a saturated network without any defects. Moreover, for any coherent network, if we can keep the coherency when we remove defects, then the final network should be also coherent. The method requires the following to be established:

(1) Remove only one defect each time and any coherent network will still be coherent after that.

(2) Whenever we remove a defect, all defects that have been removed should not occur again.

(3) Based on the above results, we should finally be able to remove all defects of a network and make it perfect.

Definition 3.3.5 $[w, \varphi]$ *is a defect if* $\varphi = \neg\Box\psi$ *and* $\varphi \in \delta(w)$*, but no* $v \in N$ *exists such that* $\neg\psi \in \delta(v)$ *and* r_{wv}.

Fact: A network $\mathcal{N}$ is not saturated iff it has some defect.

Since N of $\mathcal{N}$ is a countable node set, all potential defects $[w, \varphi]$ is also enumerable. So Rule (1) is easily followed.

To realise the "remove" in Rule (2), we define a structure called "repair".

Definition 3.3.6 *Let $\mathcal{N}$ and $\mathcal{N}'$ be networks. We say that $\mathcal{N}' = \langle N', \mathsf{r}', \delta' \rangle$ is an extension of $\mathcal{N} = \langle N, \mathsf{r}, \delta \rangle$ and write $\mathcal{N} \trianglelefteq \mathcal{N}'$ if all the obvious inclusions hold: $N \subseteq N'$, $\mathsf{r} \subseteq \mathsf{r}'$ and $\delta(w) \subseteq \delta'(w)$ for each $w \in N$.*

Fact: If $\mathcal{N} \trianglelefteq \mathcal{N}'$, then $\delta(n) = \delta'(n)$ for each $n \in N$.

The reason is obvious. If we do not change (for example, extend) the language when we extend networks, then any maximal consistent set will be kept unchanged during the extension.

Definition 3.3.7 *If D is a defect of a coherent $\mathcal{N}$ then $\mathcal{N}'$ is a repair of D of $\mathcal{N}$ iff*

(1) *$\mathcal{N}'$ is a coherent extension of $\mathcal{N}$*
(2) *D is not a defect of $\mathcal{N}'$*
(3) *$\mathcal{N}'$ is finite if $\mathcal{N}$ is.*

Now we propose a Repair Lemma for the basic modal logic to illustrate how to construct a repair.

Lemma 3.3.3 (Repair Lemma) *For any defect $[w, \varphi]$ of a finite, coherent network $\mathcal{N}$ of the basic modal language, there is an $\mathcal{N}'$ such that $\mathcal{N}'$ is a repair of $[w, \varphi]$ of $\mathcal{N}$.*

Proof: Let $\mathcal{N} = \langle N, \mathsf{r}, \delta \rangle$ be a finite coherent network. The defect $[w, \varphi]$ means that $\neg\Box\psi \in \delta(w)$ and no $v \in N$ such that r_{wv} and $\neg\psi \in \delta(v)$. Define $\mathcal{N}' = \langle N', \mathsf{r}', \delta' \rangle$ such that

(1) $N' = N \cup \{w^\dagger\}$ where $w^\dagger \notin N$ ($w^\dagger$ is a new node)

(2)$\mathsf{r}' = \mathsf{r} \cup \{\langle n, n^\dagger \rangle\}$

(3)$\delta'(v) = \begin{cases} \delta(v) & \text{if } v \neq w^\dagger \\ \Gamma & \text{Otherwise} \end{cases}$

where Γ is a maximal consistent set extended by $\{\theta | \Box\theta \in \delta(w)\} \cup \{\neg\psi\}$, which is consistent by Lemma 3.3.1.

Now we check that $\mathcal{N}'$ is a repair of $[n, \varphi]$ of $\mathcal{N}$.

Firstly, $\mathcal{N}'$ exists since Γ exists by Lindenbaum Lemma. Secondly, it is an extension of $\mathcal{N}$. Thirdly, $[n, \varphi]$ is not a defect of $\mathcal{N}'$ any longer since $\mathsf{r}'(n, n^\dagger)$ and $\neg\psi \in \delta'(n^\dagger)$.

Fourthly, $\mathcal{N}'$ is obviously finite. Is $\mathcal{N}'$ coherent? Since $\mathcal{N}$ is coherent and we didn't change anything in $\mathcal{N}$ when we constructed $\mathcal{N}'$, we only need to check $\mathsf{r}'(n, n^\dagger)$, the thing we added for extension. But it is straightforward since any $\Box\theta \in \delta'(n)$ iff $\Box\theta \in \delta(n)$, and $\{\theta | \Box\theta \in \delta(n)\} \subseteq \delta'(n^\dagger)$.

♣

We would like to make some comments now. We regard the Repair Lemma as the line of demarcation of the whole procedure of the step-by-step method.

To the left of the line are all the parts that are decided by the logic being discussed. Given a specific logic, we have to devise the suitable coherence conditions according to the semantics to ensure that the model induced by the network is a model of this logic. Saturated conditions also have to be designed according to what we need for the model existence lemma. Of course, this is also decided by the logic. The repair method is also decided by how we define the coherent and saturated network. The readers will see that the proof for the basic modal logic is not strong enough for our logics since their semantics are more complicated. In the next chapter, we must figure out the boundary of the language. For example, if we repair some defects by introducing a new element to the language, and this means we are extending the signature, then obviously the original maximal consistent sets are no longer maximal in the extended language. So we have to rebuild all maximal consistent sets for the recovery. Under this circumstance, a much more complicated case we have to face is that the language has both free and bound variables.

To the right of the line, which we are going to discuss now, are the properties of extensions of the networks to realise Rule (3) and (4). We think these results are less reliant on the logic, so could be treated as a more or less general part of the step-by-step method and applied similarly in Chapter 4 and 5. So we will avoid repeating this part as much as we can in those chapters.

Lemma 3.3.4 *Let $\mathcal{N}'$ be an extension of $\mathcal{N}$. Then $\mathcal{N}'$ does not introduce the defects which have already been removed, e.g. if $[w, \varphi]$ is a defect of $\mathcal{N}'$ where w is a node of $\mathcal{N}$, then $[w, \varphi]$ is already a defect of $\mathcal{N}$.*

Proof: For contraposition, suppose $[w, \varphi]$ is not a defect of $\mathcal{N}$. Then by Definition 3.3.5, we have $\varphi = \neg\Box\psi \in \delta(w)$ implies there exists $v \in N$ such that $\neg\psi \in \delta(v)$ and r_{wv}. Suppose $\varphi \in \delta(w)$. As $\mathcal{N}'$ is an extension of $\mathcal{N}$, $v \in N'$. Also $\neg\psi \in \delta'(v)$ and r'_{wv}. So $[w, \varphi]$ is not a defect of $\mathcal{N}'$.

♣

Even though we have the Repair Lemma for each defect, it doesn't mean that we know how to remove all defects and obtain a perfect network, which is requested by Rule (4). Intuitively, we have to find some way to save the repair we gain from each step.

Lemma 3.3.5 (Limit Coherence) *If $\mathcal{N}_0 \trianglelefteq \mathcal{N}_1 \trianglelefteq \ldots$ is a chain of coherent networks then the limit of the chain is also coherent.*

Proof:

Define the limit network $\mathcal{N}$ by taking unions of all the components of the networks. To show that $\mathcal{N}$ is coherent, suppose $\mathsf{r}_{\mathcal{N}}(w, v)$ and $\Box\varphi \in \delta_{\mathcal{N}}(w)$. There must be an i such that $\mathsf{r}_{\mathcal{N}_i}(w, v)$ and a j such that $\Box\varphi \in \delta_{\mathcal{N}_j}(w)$. Let $k = max(i, j)$. Then $\mathsf{r}_{\mathcal{N}_k}(w, v)$ and $\Box\varphi \in \delta_{\mathcal{N}_k}(w)$. So by the coherence of $\mathcal{N}_k$, $\varphi \in \delta_{\mathcal{N}_k}(v)$ and so $\varphi \in \delta_{\mathcal{N}}(v)$.

♣

Lemma 3.3.6 (Perfect Extension) *If $\mathcal{N}$ is coherent then there is a perfect $\mathcal{N}'$ such that $\mathcal{N} \trianglelefteq \mathcal{N}'$.*

Proof: Enumerate the set $\mathcal{L} \times \mathbb{N}$ of *potential defects*. So every defect of every network is in this set. Now let the sequence of networks $\mathcal{N}_0 \trianglelefteq \mathcal{N}_1 \trianglelefteq \ldots$ be defined by $\mathcal{N}_0 = \mathcal{N}$ and $\mathcal{N}_{i+1}$ is a repair of the first defect of $\mathcal{N}_i$, i.e., the one that occurs first in our enumeration of potential defects by the Repair Lemma. If $\mathcal{N}_i$ has no defects, then $\mathcal{N}_{i+1} = \mathcal{N}_i$. Let $\mathcal{N}' = \bigvee_{i \in \mathbb{N}} \mathcal{N}_i$ (which exists by Lemma 3.3.5). By Repair Lemma, every network in the sequence is coherent. And so $\mathcal{N}'$ is coherent by Lemma 3.3.5. Now suppose for contradiction that $\mathcal{N}'$ is not perfect. Then it has a defect $[w, \varphi]$. Let D be the set of defects that occur before this one in the enumeration of potential defects. By Lemma 3.3.4, $[w, \varphi]$ is also a defect of $\mathcal{N}_i$ for some i. For each $j \in \mathbb{N}$, there is a defect of $\mathcal{N}_{i+j}$ that is repaired in $\mathcal{N}_{i+j+1}$ (by definition, since none of these networks is perfect) and so it is not a defect of $\mathcal{N}_{i+k}$ for any $k > j$ by Lemma 3.3.4. Since it is the first defect of $\mathcal{N}_{i+j}$ in the enumeration of potential defects, it must occur before $[w, \varphi]$. So it is in the set D. Then there is an infinite sequence of distinct defects, in D which is a finite set. Contradiction.

♣

Since all the four rules have been kept, we are ready for completeness:

Lemma 3.3.7 *Every consistent set of formulas is satisfiable.*

Proof: Suppose Γ is consistent. By the Lindenbaum Lemma, we extend Γ to a maximal consistent set Γ'. Let $\mathcal{N}_0 = \langle \{0\}, \emptyset, \delta_0 \rangle$ where $\delta_0(0) = \Gamma'$. Then $\mathcal{N}_0$ is coherent. Extend $\mathcal{N}_0$ to a perfect network $\mathcal{N}'$ by Lemma 3.3.6. Then $\Gamma \subseteq \delta_{\mathcal{N}'}(0)$. And by the Truth Lemma, $M_{\mathcal{N}'}, 0 \models \varphi$ for every $\varphi \in \Gamma$.

♣

3.4 Axiomatisation of Basic Social Epistemic Logic

In this section, we are going to propose an axiomatisation of the basic social epistemic logic introduced in Section 2.2 and prove its completeness. Since we have already proposed syntax and semantics in that section, we directly introduce the axiomatisation here.

3.4.1 Axiomatisation

$\vdash \varphi$ if φ is a tautology	Taut
$\vdash S(p \to q) \to (Sp \to Sq)$	$\mathsf{K_S}$
$\vdash K(p \to q) \to (Kp \to Kq)$	$\mathsf{K_K}$
$\vdash @_n(p \to q) \to (@_n p \to @_n q)$	$\mathsf{K_@}$
$\vdash @_n p \leftrightarrow \neg @_n \neg p$	Selfdual@
$\vdash @_n n$	Ref@
$\vdash @_n @_m p \leftrightarrow @_m p$	Agree
$\vdash n \to (p \leftrightarrow @_n p)$	Intro
$\vdash @_n p \to S @_n p$	Back
from $\vdash \varphi$ infer $\vdash \varphi[p/\psi]$ or $\vdash \varphi[n/m]$[5]	Sub
from $\vdash \varphi$ and $\vdash \varphi \to \psi$, infer $\vdash \psi$	MP
from $\vdash \varphi$ infer $\vdash S\varphi$	$\mathsf{Nec_S}$
from $\vdash \varphi$ infer $\vdash @_n\varphi$	$\mathsf{Nec_@}$
from $\vdash \varphi$ infer $\vdash K\varphi$	$\mathsf{Nec_K}$
from $\vdash @_n\varphi \to (n_1 \vee n_2 \vee \ldots \vee n_k)$ infer $\vdash \neg @_n \varphi$	@Name

Figure 3.3: The system of $\mathbf{BSEL}$.

The basic social epistemic logic, denoted by $\mathbf{BSEL}$, consists of the axioms and rules listed in Figure 3.3.

$\mathbf{BSEL}$ is closely based on the usual axiomatisation for hybrid logic; see, for example [10], p. 833. There are only two small differences. First, the Back Axiom is restricted to the social modality; this is because our names are names of agents and not world-agent pairs. Second, we have an additional rule: the @Name rule. To understand this rule, let's look at a special case. If $@_n\varphi \to m$ were valid, then $@_n\varphi$ must be

[5] $\varphi[p/\psi]$ is the result of replacing p by ψ in φ. $\varphi[n/m]$ is the result of replacing n by m in φ.

contradictory. Why? Because if $@_n\varphi$ is satisfied by any agent it is satisfied by every agent, and so cannot imply that agent is named m, or any finite disjunction of name and so cannot imply that agent is named m, or any finite disjunction of names. We want to emphasise that Lemma 3.4.2 in the next section, is the only case to apply the @Name rule. We don't know if the @Name rule is necessary for the proof. If it is admissible, then this rule is not needed in the Hilbert system of $\mathbf{BSEL}$.

By the soundness of hybrid logic, we only need to show the @Name rule is sound. To show this rule is sound, we need the following lemma:

Proposition 3 *The* @Name *rule is sound in* $\mathbf{BSEL}$.

Proof: For contradiction, suppose $\not\models \neg @_n\varphi$. Then there exists an M, w and a such that $M, w, a \models @_n\varphi$. Let $M' = \langle W, A \cup \{a^\dagger\}, k, s, g, V\rangle$ where $a^\dagger \notin A$. Then by the well-known agreement result, $M', w, a \models @_n\varphi$. So $M', w, a^\dagger \models @_n\varphi$. So $M', w, a^\dagger \models (n_1 \vee n_2 \vee \ldots \vee n_k)$. But g doesn't map any name to agent $a^\dagger$. Contradiction.

♣

The other axioms and rules are standard and can be checked for soundness, giving:

Theorem 3.4.1 ***Soundness of*** $\mathbf{BSEL}$.

$\vdash \varphi$ *implies that* φ *is valid.*

3.4.2 Theorems and Consistency

Theorem 3.4.2 *The following formulas of* $\mathbf{BSEL}$ *are the theorems we need for the completeness proof.*

(1) $\vdash K\varphi \wedge K\psi \leftrightarrow K(\varphi \wedge \psi)$
(2) $\vdash S\varphi \wedge S\psi \leftrightarrow S(\varphi \wedge \psi)$
(3) $\vdash @_n\varphi \wedge @_n\psi \leftrightarrow @_n(\varphi \wedge \psi)$
(4) $\vdash \langle S\rangle(\varphi \wedge \psi) \rightarrow (\langle S\rangle\varphi \wedge \langle S\rangle\psi)$
(5) $\vdash \langle S\rangle(n \wedge \varphi) \rightarrow @_n\varphi$
(6) $\vdash S\varphi \rightarrow (\langle S\rangle n \rightarrow @_n\varphi)$
(7) $\vdash @_m n \rightarrow @_n m$

Proof: The proof of (1) needs $\mathsf{K_K}$, MP, Sub and Taut. (2) and (3) can be proven similarly with $\mathsf{K_S}$ and $\mathsf{K_@}$. To prove (4), we need to prove "from $\varphi \rightarrow \psi$ infer $\langle S\rangle\varphi \rightarrow \langle S\rangle\psi$ first. This is easily proven by Taut and "from $\varphi \rightarrow \psi$ infer $S\varphi \rightarrow S\psi$, which is straightforward from $\mathsf{K_S}$. We prove (5) by (4), Intro and Back. Then (6) is got by (5) and Taut. Finally, (7) needs Intro Axiom, Ref@, $\mathsf{Nec_@}$, MP, Agree, Sub and Taut.

♣

Lemma 3.4.1 *Let* $\varphi = @_{n_1}\psi_1 \wedge \ldots \wedge @_{n_j}\psi_j$*, then for any* $n \in \mathsf{Nom}$ *we have* (1) $\vdash @_n\varphi \leftrightarrow \varphi$ *and* (2) $\vdash @_n\neg\varphi \leftrightarrow \neg\varphi$.

Proof: For (1), by applying Agree repeatedly, we have $\vdash @_{n_1}\psi_1 \wedge \ldots \wedge @_{n_j}\psi_j \leftrightarrow @_n@_{n_1}\psi_1 \wedge \ldots \wedge @_n@_{n_j}\psi_j$. Then applying Theorem 3.4.2.(3) repeatedly, $\vdash @_n@_{n_1}\psi_1 \wedge \ldots \wedge @_n@_{n_j}\psi_j \leftrightarrow @_n(@_{n_1}\psi_1 \wedge \ldots \wedge @_{n_j}\psi_j)$.

For (2), we have $\vdash \neg@_n\varphi \leftrightarrow \neg\varphi$ by 1. $\vdash @_n\neg\varphi \leftrightarrow \neg@_n\varphi$ by Selfdual.

♣

Definition 3.4.1 *Let* e *be an enumeration of all formulas of the language. We now define a* <u>*maximal consistent function*</u> $e(\Gamma)$ *from any consistent set* Γ *to the maximal consistent set extending* Γ *by* e.

Here are some abbreivations:

1. $\Gamma^e = e(\Gamma)$
2. $@\Sigma = \{@_n\varphi | @_n\varphi \in \Sigma\}$
3. $\mathsf{Nom}(\Sigma) = \{n | n \in \Sigma\} \cup \{\neg n | \neg n \in \Sigma\}$
4. $\neg\mathsf{Nom} = \{\neg n | n \in \mathsf{Nom}\}$

Lemma 3.4.2 *Given any maximal consistent set* Σ*,* $@\Sigma \cup \neg\mathsf{Nom}$ *is consistent.*

Proof: For contradiction, suppose $@\Sigma \cup \neg\mathsf{Nom}$ is not consistent. Then there exists $@_{n_1}\varphi_1 \ldots @_{n_k}\varphi_k \in @\Sigma$ and $\neg m_1 \ldots \neg m_j \in \neg\mathsf{Nom}$ for some $k, j \in \mathbb{N}$ such that

$$\vdash \neg(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k \wedge \neg m_1 \ldots \wedge \neg m_j)$$

.

Then $\vdash (@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \rightarrow (m_1 \vee \ldots \vee m_j)$. Then by Lemma 3.4.1,

$$\vdash @_n(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \rightarrow (m_1 \vee \ldots \vee m_j)$$

for some $n \in \mathsf{Nom}$. Then by the @Name rule, we have $\vdash \neg@_n(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k)$.

By Lemma 3.4.1 again, $\vdash \neg(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k)$. So $\neg(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \in \Sigma$. But by $@_{n_1}\varphi_1 \ldots @_{n_k}\varphi_k \in @\Sigma$, $@_{n_1}\varphi_1 \ldots @_{n_k}\varphi_k \in \Sigma$.

So $(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \in \Sigma$. This contradicts that Σ is maximal consistent.

♣

Lemma 3.4.3 *For any maximal consistent sets Σ and Γ, if $\{\psi | S\psi \in \Sigma\} \subseteq \Gamma$, then $@\Sigma = @\Gamma$.*

Proof:

($\Rightarrow$) Suppose $@_n\varphi \in \Sigma$. Then $S@_n\varphi \in \Sigma$ by Back. So we have $@_n\varphi \in \Gamma$.

($\Leftarrow$) Suppose $@_n\varphi \in \Gamma$. Then $@_n\neg\varphi \notin \Gamma$. So $S@_n\neg\varphi \notin \Sigma$. By Back, $@_n\neg\varphi \notin \Sigma$. So we have $@_n\varphi \in \Sigma$.

♣

3.4.3 Completeness

Based on the discussion for applying step-by-step method to the basic modal logic, we now only need to focus on the following in this section:

1. The network of $\mathbf{BSEL}$
2. The coherence and saturation
3. The induced model and Extended Truth Lemma
4. The kinds of potential defects
5. Repair Lemma

For all the rest, we will give a brief discussion afterward.

Definition 3.4.2 *A network of $\mathbf{BSEL}$ $\mathcal{N} = \langle W, A, \mathsf{k}, \mathsf{s}, \delta\rangle$ consists of non-empty sets $W, A \subseteq \mathbb{N}$, a binary relation $\mathsf{k}_a \subseteq W \times W$ for each $a \in A$, a binary relation $\mathsf{s}_w \subseteq A \times A$ for each $w \in W$, and a maximal consistent set: $\delta(w, a)$ for each $\langle w, a\rangle$ (the node of $\mathcal{N}$).*[6]

When we induced a model from a network for the basic modal language, we only needed to define the valuation function V. However, we now have to consider g as well.

Definition 3.4.3 *From a network $\mathcal{N} = \langle W, A, \mathsf{k}, \mathsf{s}, \delta\rangle$, we define a model $M_{\mathcal{N}} = \langle W, A, \mathsf{k}, \mathsf{s}, g, V\rangle$ by*

$$V_p(w, a) \quad \textit{iff} \quad p \in \delta(w, a)$$

$$g_w(n) = \begin{cases} \textit{the first element of } A \textit{ such that } a \in \delta(w, a) & \text{if } n \in \delta(w, b) \textit{ for some } b \\ \textit{an arbitrary element} & \text{Otherwise} \end{cases}$$

[6] Although W and A are both sets of numbers and so could overlap, this will have no significance in the proof. They are just indices.

The purpose of the second disjunct in the definition of g_w is merely to ensure that n is assigned *some* element of A in the case that it isn't in any of the $\delta(w, b)$. The "first element" refers to the first formula in our enumeration of formulas.

Coherence and saturation are structural properties of the network designed to ensure coordination between the formulas in the sets $\delta(w, a)$ and those satisfied by the model at (w, a).

Definition 3.4.4 *A network $\mathcal{N}$ is coherent iff for every $w, v \in W, a, b \in A$*

(k)	*if $\mathsf{k}_a(w, v)$ and $K\varphi \in \delta(w, a)$, then $\varphi \in \delta(v, a)$ for any φ*
(n)	$\delta(w, a) \cap \delta(w, b) \cap \mathsf{Nom} = \emptyset$ *if $a \neq b$*
(s)	*if $\mathsf{s}_w(a, b)$ and $S\varphi \in \delta(w, a)$, then $\varphi \in \delta(w, b)$ for any φ*
$(@)$	$@\delta(w, a) = @\delta(w, b)$

$\mathcal{N}$ is saturated iff

(S)	*if $\langle S\rangle\varphi \in \delta(w, a)$ then there is a $b \in A$ such that $\mathsf{s}_w(a, b)$ and $\varphi \in \delta(w, b)$*
$(@_n)$	*if $@_n\varphi \in \delta(w, a)$ then there is a $b \in A$ such that $n, \varphi \in \delta(w, b)$*
(K)	*if $\langle K\rangle\varphi \in \delta(w, a)$ then there is a $v \in W$ such that $\mathsf{k}_a(w, v)$ and $\varphi \in \delta(v, a)$*
(N)	*if $n \notin \delta(w, a)$ then there is a $b \in A$ such that $n \in \delta(w, b)$*

$\mathcal{N}$ is perfect iff it is both coherent and saturated.

Then we have the following lemma:

Lemma 3.4.4 (Hintikka) *If $\mathcal{N}$ is perfect then $M_{\mathcal{N}}$ satisfies the following conditions, which mirror the semantics conditions for $\models$, for all formulas in L:*

$n \in \delta(w, a)$	*iff*	$g_w(n) = a$
$p \in \delta(w, a)$	*iff*	$V_p(w, a)$
$\neg\varphi \in \delta(w, a)$	*iff*	$\varphi \notin \delta(w, a)$
$(\varphi \wedge \psi) \in \delta(w, a)$	*iff*	$\varphi \in \delta(w, a)$ *and* $\psi \in \delta(w, a)$
$S\varphi \in \delta(w, a)$	*iff*	$\varphi \in \delta(w, b)$ *for every* $b \in A$ *such that* $s_w(a, b)$
$K\varphi \in \delta(w, a)$	*iff*	$\varphi \in \delta(v, a)$ *for every* $v \in W$ *such that* $k_a(w, v)$
$@_n\varphi \in \delta(w, a)$	*iff*	$\varphi \in \delta(w, g_w(n))$

for any φ, ψ, p and n.

Proof: We perform induction on φ.

1. Case $\varphi = n$.

($\Rightarrow$) Suppose $n \in \delta(w,a)$. We prove that a is the only element such that $n \in \delta(w,a)$. Then by Definition 3.4.3, $g_w(n) = a$. For contradiction suppose not. By the clause (n), there is $b \in A$ such that $b \neq a$ and $n \in \delta(w,b)$. Then by $\mathcal{N}$ is coherent, $n \notin \delta(w,a)$. Contradiction.

($\Leftarrow$) Suppose $n \notin \delta(w,a)$. To prove $g_w(n) \neq a$, we need to show that the second disjunct in Definition 3.4.3 cannot be satisfied. By $n \notin \delta(w,a)$, $\neg n \in \delta(w,a)$. By $\mathcal{N}$ is saturated and the clause (N), $n \in \delta(w,b)$. And this is the negation of the second disjunct.

2. Case $\varphi = p$. By definition of $M_{\mathcal{N}}$.

3. Boolean cases. By properties of maximal consistent set.

4. Modal cases: S and K. The left to right direction is straightforward by $\mathcal{N}$ is coherent. And the other direction is proved as $\mathcal{N}$ is saturated.

5. Case $\varphi = @_n\psi$

   ($\Rightarrow$) Suppose $@_n\psi \in \delta(w,a)$. By the first induction, $n \in \delta(w, g_w(n))$. By $\mathcal{N}$ is coherent, $@_n\psi \in \delta(w, g_w(n))$. Then by Intro, $\psi \in \delta(w, g_w(n))$.

   ($\Leftarrow$) Similar.

♣

Lemma 3.4.5 (Extended Truth Lemma) *Given any perfect network $\mathcal{N}$, for any $w \in W, a \in A$ and any formula φ,*

$$M_{\mathcal{N}}, w, a \models \varphi \quad \textit{iff} \quad \varphi \in \delta(w,a)$$

Proof: Induction on φ. For the case that $\varphi = p$ or $\varphi = n$, it is straightforwardly proven by the definition of $M_{\mathcal{N}}$. For the other cases, it is then a simple inductive consequence of Lemma 3.4.4.

♣

To gain a perfect network, we are now discussing the defects and repair method. Similarly to the case of basic modal logic, defects correspond to the cases in which the saturation fails.

Definition 3.4.5 *Given $\mathcal{N}$, the following are potential defects of $\mathcal{N}$:*

$[w, a, \varphi]$ *is a* K*-defect*	*iff*	$\mathcal{N}$ *fails* (K) *clause of saturation for* φ *at* w, a.
$[w, a, \varphi]$ *is an* S*-defect*	*iff*	$\mathcal{N}$ *fails* (S) *clause of saturation for* φ *at* w, a.
$[w, a, \varphi]$ *is an* $@$*-defect*	*iff*	$\mathcal{N}$ *fails* $(@_n)$ *clause of saturation for* φ *at* w, a.
$[w, a, \varphi]$ *is an* n*-defect*	*iff*	$\mathcal{N}$ *fails* (N) *clause of saturation for* φ *at* w, a.

By this definition, $[w, a, \langle S\rangle\varphi]$ will be an S-defect if $\langle S\rangle\varphi \in \delta(w, a)$ but no $b \in A$ such that $\mathsf{s}_w(a, b)$ and $\varphi \in \delta(w, b)$. In fact, if a network satisfies Clause $(@_n)$, it will satisfy Clause (N). Suppose $n \notin \delta(w, a)$, as the condition of (N). Then by $\vdash @_n n$, $@_n n \in \delta(w, a)$. Then by $(@_n)$, there exists some b such that $n \in \delta(w, b)$.

So we only have three kinds of potential defects.

We are ready to define a repair. The repair is defined via the extension of the network in the case of the basic modal logic. We use the same strategy here.

Definition 3.4.6 *We say that* $\mathcal{N}' = \langle W', A', \mathsf{k}', \mathsf{s}', \delta'\rangle$ *is an* <u>*extension*</u> *of network* $\mathcal{N} = \langle W, A, \mathsf{k}, \mathsf{s}, \delta\rangle$ *and write* $\mathcal{N} \trianglelefteq \mathcal{N}'$ *if all the obvious inclusions hold:* $W \subseteq W'$*,* $A \subseteq A'$*,* $\mathsf{k}_a \subseteq \mathsf{k}'_a$*,* $\mathsf{s}_w \subseteq \mathsf{s}'_w$*, and* $\delta(w, a) \subseteq \delta'(w, a)$ *for each* $w \in W$ *and* $a \in A$*.*

Definition 3.4.7 *If* D *is a defect of* $\mathcal{N}$ *then* $\mathcal{N}'$ *is a* <u>*repair*</u> *of* D *in* $\mathcal{N}$ *iff*

(1) $\mathcal{N}'$ *is a coherent extension of* $\mathcal{N}$

(2) D *is not a defect of* $\mathcal{N}'$

(3) $\mathcal{N}'$ *is finite if* $\mathcal{N}$ *is.*

The readers may notice that Definition 3.4.7 is the same as Definition 3.3.7 for the basic modal logic. Now we are going to discuss how to repair the potential defects.

Lemma 3.4.6 *Any* S *defect of a finite coherent network* $\mathcal{N}$ *has a repair.*

Proof:

Suppose we have a defect $[w, a, \langle S\rangle\varphi]$ of a finite coherent $\mathcal{N}$. We discuss how to repair it by cases.

Case One: $\langle S\rangle(m \wedge \varphi) \in \delta(w, a)$ and $m \in \delta(w, b)$ for some $m \in \mathsf{Nom}$ and $b \in A$.

As $\mathcal{N}$ is coherent, b is unique for m.

To repair it, we only need to add an s link from a to b at w to get an extension $\mathcal{N}'$ of $\mathcal{N}$. So $\mathsf{s}_w(a, b)$.

Claim I: $[w, a, \langle S\rangle\varphi]$ is no longer a defect in $\mathcal{N}'$.

Proof of Claim I:

We have $@_m\varphi \in \delta(w,a)$ by $\vdash \langle S\rangle(m\wedge\varphi) \to @_m\varphi$ from Theorem 3.4.2. So $@_m\varphi \in \delta(w,b)$ since $\mathcal{N}$ is coherent and $\mathsf{s}_w(a,b)$. Then $\varphi \in \delta(w,b)$ since $m \in \delta(w,b)$ by the Intro Axiom. So $[w,a,\langle S\rangle\varphi]$ is not a defect of $\mathcal{N}'$.

Here is a formal definition of $\mathcal{N}' = \langle W, A, \mathsf{k}, \mathsf{s}', \delta\rangle$:

$$\mathsf{s}'_v = \begin{cases} \mathsf{s}_v & \text{if } v \neq w \\ \mathsf{s}_v \cup \{\langle a,b\rangle\} & \text{Otherwise} \end{cases}$$

Claim II: $\mathcal{N}'$ is a coherent extension of $\mathcal{N}$.

Proof of Claim II:

It is easy to see that $\mathcal{N}'$ is an extension of $\mathcal{N}$. Notice that $\mathcal{N}'$ is the same as $\mathcal{N}$ except the addition of $\mathsf{s}'_w(a,b)$ As $\mathcal{N}$ is coherent, this also means that we only need to check clause (s) of the coherent definition holds for $\mathsf{s}'_w(a,b)$.

Suppose $S\theta \in \delta(w,a)$. We prove $\theta \in \delta(w,b)$. As $\langle S\rangle(m\wedge\varphi) \in \delta(w,a)$ and $\vdash \langle S\rangle(m\wedge\varphi) \to \langle S\rangle m \wedge \langle S\rangle\varphi$ (from Theorem 3.4.2), we have $\langle S\rangle m \in \delta(w,a)$. Again by $\vdash S\theta \to (\langle S\rangle m \to @_m\theta)$ from Theorem 3.4.2, $@_m\theta \in \delta(w,a)$. Then $@_m\theta \in \delta(w,b)$ as the $@$ clause holds in coherent $\mathcal{N}$. Since $m \in \delta(w,b)$ by Intro, $\theta \in \delta(w,b)$.

$\mathcal{N}'$ is also finite since $\mathcal{N}$ is finite.

Based on the above claims, $\mathcal{N}'$ is a repair of $[w,a,\langle S\rangle\varphi]$ of a finite coherent $\mathcal{N}$ by Definition 3.4.7.

Case Two: Otherwise.

This is equivalent to: for any $m \in \mathsf{Nom}$, if $\langle S\rangle(m\wedge\varphi) \in \delta(w,a)$, then for all $b \in A$, $m \notin \delta(w,b)$. Here is our repair method:

We denote by $a^\dagger$ the least number not in A, which exists because A is finite. As depicted by Figure 3.4, this will add nodes $(v,a^\dagger)$ to $\mathcal{N}$ for all $v \in W$, which makes a new column. Then we assign $[\{\varphi\}\cup\{\psi|S\psi \in \delta(w,a)\}]^e$ to $(w,a^\dagger)$ and $[@\delta(v,a)\cup\neg\mathsf{Nom}]^e$ to $(v,a^\dagger)$ for every $v \in W$ and $v \neq w$ in the new column.

Both maximal consistent sets exist: By Lemma 3.3.1, $\{\varphi\} \cup \{\psi|S\psi \in \delta(w,a)\}$ is consistent. And by Lemma 3.4.2, $@\delta(v,a) \cup \neg\mathsf{Nom}$ is consistent.

And the new structure $\mathcal{N}'$ is an extension of $\mathcal{N}$ since we didn't change anything in the original. By $\varphi \in \delta(w,a^\dagger)$, $[w,a,\langle S\rangle\varphi]$ is not a defect in the extension.

Claim I: $[w,a,\langle S\rangle\varphi]$ is no longer a defect in $\mathcal{N}'$.

By construction, the node $(w,a^\dagger)$ is in $\mathcal{N}'$ such that $\mathsf{s}'_w(a,a^\dagger)$ and $\varphi \in \delta'(w,a^\dagger)$.

Claim II: $\mathcal{N}'$ is a coherent extension of $\mathcal{N}$.

Proof of Claim II:

Obviously, $\mathcal{N}'$ is an extension of $\mathcal{N}$. To prove it is still coherent, we need to check that the coherency still holds at the new nodes (all the nodes in the new $a^\dagger$ column).

Case One: the new node is $(v, a^\dagger)$ where $v \neq w$. Then it has no s relation with any other nodes. So we only need to check the $@$ and n clauses. Since no name is contained by $\delta(v, a^\dagger)$, n is does not need to be checked. To prove $@$ is kept, we show $@\delta(v, b) = @\delta(v, a^\dagger)$ for any $b \in A$. By $\mathcal{N}$ is coherent, $@\delta(v, b) = @\delta(v, a)$. By $@\delta(v, a) \subseteq \delta(v, a^\dagger)$ and $\delta(v, a^\dagger)$ is maximal, $@\delta(v, a) = @\delta(v, a^\dagger)$.

Case Two: the new node is $(w, a^\dagger)$. Then $(@)$ is kept by Lemma 3.4.3. And (s) is satisfied by the definition of $\delta(w, a^\dagger)$. And (k) is no need to consider since $\mathsf{k} = \mathsf{k}'$. To prove it does not break (n), suppose $n \in \delta(w, a^\dagger) \cap \delta(w, b)$ for any $b \in A$ for contradiction. Then $n \in \delta(w, b)$. But by the condition of Case Two, we have $\langle S\rangle(n \wedge \varphi) \notin \delta(w, a)$. Then $S\neg(n \wedge \varphi) \in \delta(w, a)$. So $\neg(n \wedge \varphi) \in \delta(w, a^\dagger)$ by our construction. By $\varphi \in \delta(w, a^\dagger)$, we have $\neg n \in \delta(w, a^\dagger)$. So $n \notin \delta(w, a^\dagger)$ and then $n \notin \delta(w, a^\dagger) \cap \delta(w, b)$. Contradiction.

We now give a formal definition of this extension: $\mathcal{N}' = \langle W, A', \mathsf{k}, \mathsf{s}', \delta'\rangle$

$$A' = A \cup \{a^\dagger\}$$

$$\mathsf{s}'_v = \begin{cases} \mathsf{s}_v & \text{if } v \neq w \\ \mathsf{s}_v \cup \{\langle a, a^\dagger\rangle\} & \text{Otherwise} \end{cases}$$

$$\delta'(v, b) = \begin{cases} [\{\varphi\} \cup \{\psi | S\psi \in \delta(w, a)\}]^e & \text{if } v = w, b = a^\dagger \\ [@\delta(v, a) \cup \neg\mathsf{Nom}]^e & \text{if } v \neq w, b = a^\dagger \\ \delta(v, b) & \text{Otherwise} \end{cases}$$

We give a brief explanation of δ'. The first case is to define the maximal consistent set assigned to $\langle w, a^\dagger\rangle$. Since our logic is two dimensional, we also have to consider which maximal consistent sets to assign to the other nodes of the new $a^\dagger$ column. This is why we have Case Two. Our method is using $@\delta(v, a)$ to copy all $@$ formulas from agent a's node of that row[7], say v, then make it not contain any names to satisfy (n). Then Clause Three is to keep the old maximal consistent sets unchanged during the repair.

When we say no new defect arises for a repair, we mean there is no defect such that it is not a defect before the repair, however, it is a defect after the repair, and particularly, it occurs at the node that exists before the repair. To prove that no new defect arises in this case, for contradiction, we suppose $[v, b, \psi]$ is such a defect. It could be one of the three kinds of defects. Suppose it is an S defect. Then $\psi = \langle S\rangle\theta$ for some θ. So $\langle S\rangle\theta \in \delta'(v, b)$ and $\langle v, b\rangle$ is a node of $\mathcal{N}$. By $\delta(v, b) = \delta'(v, b)$, $\langle S\rangle\theta \in \delta(v, b)$. By $[v, b, \psi]$ is not a defect of $\mathcal{N}$, there is a c such that $\mathsf{s}_w(b, c)$ and $\theta \in \delta(v, c)$. By $\mathsf{s}_w \subseteq \mathsf{s}'_w$ and $\delta(v, c) \subseteq \delta'(v, c)$, we have $\mathsf{s}'_w(b, c)$ and $\theta \in \delta'(v, c)$.

[7]It is not necessary to fix a here. In fact, any agent can be chosen since $\mathcal{N}$ satisfies the $(@)$ clause of the coherency definition.

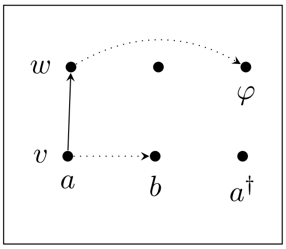


Figure 3.4: The addition of $a^\dagger$.

Then $[v, b, \psi]$ is not a defect of $\mathcal{N}'$. Contradiction. Then suppose it is a K defect. Let $\psi = K\theta$ now. So $\langle K\rangle\theta \in \delta'(v, b)$ and $\langle v, b\rangle$ is a node of $\mathcal{N}$. Again, $\langle K\rangle\theta \in \delta(v, b)$ and there is a node $\langle u, b\rangle$ of $\mathcal{N}$ such that $\mathsf{k}_b(v, u)$ and $\theta \in \delta(u, b)$. Then $\mathsf{k}'_b(v, u)$ and $\theta \in \delta'(u, b)$. Contradiction. It is similar to prove the case that $[v, b, \psi]$ is an @ defect.

♣

For any @ defect, a similar repair method can be adopted:

Lemma 3.4.7 *For any finite coherent network $\mathcal{N}$ and $w \in W$, $a \in A$, $[w, a, @_n\varphi]$ has a repair.*

Proof:

We just propose the structure: $\mathcal{N}' = \langle W, A', \mathsf{k}, \mathsf{s}, \delta'\rangle$ as follows:

$$A' = A \cup \{a^\dagger\}$$

$$\delta'(v, b) = \begin{cases} [\{n\} \cup @\delta(w, a)]^e & \text{if } v = w, b = a^\dagger \\ [@\delta(v, a) \cup \neg\mathsf{Nom}]^e & \text{if } v \neq w, b = a^\dagger \\ \delta(v, b) & \text{Otherwise} \end{cases}$$

The proof that $\mathcal{N}'$ is the repair we need is very similar to what we have done for Case Two in Lemma 3.4.6, so we will not repeat it here.

♣

For K defects, the repair method is just slightly different but the proof is still very similar.

Lemma 3.4.8 *For any coherent network $\mathcal{N}$ and $w \in W$, $a \in A$, $[w, a, \langle K\rangle\varphi]$ has a repair.*

Proof:

Define $\mathcal{N}' = \langle W', A, \mathsf{k}', \mathsf{s}, \delta' \rangle$ as follows:

$$W' = W \cup \{w^\dagger\}^8$$

$$\mathsf{k}'_b = \begin{cases} \mathsf{k}_b & \text{if } b \neq a \\ \mathsf{k}_b \cup \{\langle w, w^\dagger \rangle\} & \text{Otherwise} \end{cases}$$

$$\delta'(w^\dagger, a) = [\{\varphi\} \cup \{\psi | K\psi \in \delta(w, a)\}]^e$$

$$\text{For all } b \neq a,\ \delta'(v, b) = \begin{cases} [@\delta'(w^\dagger, a) \cup \{\neg n | n \in \mathsf{Nom}\}]^e & \text{if } v = w^\dagger \\ \delta(v, b) & \text{Otherwise} \end{cases}$$

Notice that we now add a row not a column. Moreover, we have to build $\delta'(w^\dagger, a)$ at first, then copy over the @-formulas from that node, add the negations of all names and extend it to the unique maximal consistent set. Finally we assign it to the rest nodes of the $w^\dagger$ row.

Similarly, we avoid presenting the proof that $\mathcal{N}'$ is a repair.

♣

Based on the lemmas of the three kinds of defects, we have Repair Lemma:

Lemma 3.4.9 (Repair Lemma) *Any potential defect $[w, a, \varphi]$ in a coherent network $\mathcal{N}$ can be repaired by a finite network.*

As we have discussed in the last section, the Repair Lemma is the line of demarcation for the step-by-step method. All we have done so far can be seen as an adaption of this method to the new settings brought by $\mathbf{BSEL}$. The rest of the procedures of the method, as we shown for the basic modal logic, could be taken somewhat more generally.

Lemma 3.4.10 *Let $\mathcal{N}'$ be an extension of $\mathcal{N}$. Then $\mathcal{N}'$ does not introduce the defects which have already been removed.*

Proof:

The proof of S-defects and K-defects are very similar to that of basic modal logic.

For contraposition, suppose $[w, a, @_n\varphi]$ is not an @-defect of $\mathcal{N}$. Then by Definition 3.4.5, we have: $@_n\varphi \in \delta(w, a)$ implies that there exists $b \in A$ such that $n, \varphi \in$

[8] $w^\dagger = |W|$ where $|W|$ is the successors of the largest number in W of $\mathcal{N}$. And $|W|$ exists since W is finite.

$\delta(w,b)$. Suppose $@_n\varphi \in \delta(w,a)$. By $\mathcal{N}'$ is an extension of $\mathcal{N}$, $b \in A'$. Also $@_n\varphi \in \delta'(w,a)$ and $n, \varphi \in \delta'(w,b)$. So $[w,a,@_n\varphi]$ is not a defect of $\mathcal{N}'$.

♣

Lemma 3.4.11 (Limit Coherence) *If $\mathcal{N}_0 \trianglelefteq \mathcal{N}_1 \trianglelefteq \ldots$ is a chain of coherent networks then the limit of the chain is also coherent.*

Proof:

Define the limit network $\mathcal{N}$ by taking unions of all the components of the networks. To show that $\mathcal{N}$ is coherent, we need to check each clause of coherency of the $\mathcal{N}$ of $\mathbf{BSEL}$. We just show the s clause as an example since the other clauses are very similar to prove. To make the notations read easily here, let $\mathcal{N}_i = \{W^i, A^i, \mathsf{k}^i, \mathsf{s}^i, \delta^i\}$ for each $i \in \mathbb{N}$. Suppose $\mathsf{s}_w^{\mathcal{N}}(a,b)$ and $S\varphi \in \delta^{\mathcal{N}}(w,a)$ for some $w \in W^{\mathcal{N}}$ and $a, b \in A^{\mathcal{N}}$. Then there must be an i such that $\mathsf{s}_w^i(a,b)$ and a j such that $S\varphi \in \delta^j(w,a)$. Let $k = max(i,j)$. Then $w \in W^k$ and $a, b \in A^k$. Also $\mathsf{s}_w^k(a,b)$ and $S\varphi \in \delta^k(w,a)$. So by the coherence of $\mathcal{N}_k$, $\varphi \in \delta^k(w,b)$ and so $\varphi \in \delta^{\mathcal{N}}(w,b)$.

♣

Lemma 3.4.12 (Perfect Extension) *If $\mathcal{N}$ is coherent then there is a perfect $\mathcal{N}'$ such that $\mathcal{N} \trianglelefteq \mathcal{N}'$.*

Proof:

The proof is similar to Lemma 3.3.6. We just adapt it to the two dimensional language.

Enumerate the set $\mathcal{L} \times \mathbb{N}$ of *potential defects*, so every defect of every network is in this set. Now let the chain of the networks $\mathcal{N}_0 \trianglelefteq \mathcal{N}_1 \trianglelefteq \ldots$ be defined by $\mathcal{N}_0 = \mathcal{N}$ and $\mathcal{N}_{i+1}$ is a repair of the first defect of $\mathcal{N}_i$, i.e., the one that occurs first in our enumeration of potential defects by Repair Lemma. If $\mathcal{N}_i$ has no defects, then $\mathcal{N}_{i+1} = \mathcal{N}_i$. Let $\mathcal{N}'$ be the limit of the chain of the networks. By Repair Lemma, every network in the sequence is coherent. And so $\mathcal{N}'$ is coherent by Lemma 3.4.11. Now suppose for contradiction that $\mathcal{N}'$ is not perfect, and so it is not saturated. Then it has a defect $[w,a,\varphi]$. Let D be the set of defects that occur before this one in the enumeration of potential defects. Since $[w,a,\varphi]$ is a defect of $\mathcal{N}'$, by Lemma 3.4.10, $[w,a,\varphi]$ is also a defect of $\mathcal{N}_i$ for some i. For each $j \in \mathbb{N}$, there is a defect of $\mathcal{N}_{i+j}$ that is repaired in $\mathcal{N}_{i+j+1}$ (by definition, since none of these networks is perfect) and so is not a defect of $\mathcal{N}_{i+k}$ for any $k > j$ by Lemma 3.4.10. Since it is the first defect of $\mathcal{N}_{i+j}$ in the enumeration of potential defects, it must occur before $[w,a,\varphi]$ and so is in the set D. Then there is an infinite sequence of distinct defects in D, but D is a finite set. Contradiction.

♣

Theorem 3.4.3 *Any consistent set of* $\mathbf{BSEL}$ *is satisfiable.*

Proof: Similar to Lemma 3.3.7. Instead, $\mathcal{N}_0$ is two dimensional in this case.

♣

Corollary 3.4.1 (Completeness) *If* φ *is a semantic consequence of a set of formulas* Γ*, then this can be proved in the axiomatic system* $\mathbf{BSEL}$*, i.e.,* $\mathbf{BSEL}$ *is strongly complete.*

3.5 Some Extensions of BSEL

We have proven that $\mathbf{BSEL}$ is sound and complete. However, $\mathbf{BSEL}$ is too weak to be an epistemic logic tool on either the epistemic dimension or social dimension. Neither K nor S can be specifically interpreted because of their semantic definition. To apply it for modeling knowledge or belief, we have to do more. For example, we'd like to interpret the operator K as "... know that". So $m \wedge Kp$ means that "I am m and I know that p is true to me." Recall that the formulas are agent-indexical, which means that the objective of each atomic proposition is the current agent, say "I". We also want to read S as "I see that". Then Sn can represent that "I can see n." and KSp says that "I know that p is true to all the agents I can see." Moreover, it is a bit weird that the name is non-rigid unless we are talking about descriptive definitions or some very special cases. Common sense such as "I know who I am" should be characterised in a social logic. That requires names to be rigid. In this section, we are going to present several extensions to either the epistemic dimension or social dimension. Then we discuss some interactions between them. Rigid names will be discussed after those.

3.5.1 Extending the Epistemic Dimension

The axiomatisation of the extensions of $\mathbf{BSEL}$ are based on the axiomatisation of $\mathbf{BSEL}$ with additional axioms. In particular, we will be interested in the following axioms:

(D) $Kp \rightarrow \langle K \rangle p$

(T) $p \rightarrow \langle K \rangle p$

(4) $\langle K \rangle \langle K \rangle p \rightarrow \langle K \rangle p$

(5) $\langle K \rangle Kp \rightarrow Kp$

If Γ is a set of such additional axioms, we write $\vdash_\Gamma \varphi$ to mean that φ is provable from the axioms and rules of $\mathbf{BSEL}$ together with additional axioms from Γ (notation: $\mathbf{BSEL} + \Gamma$). Then for any $\Sigma \subseteq \mathsf{L}$ we say that

Σ is inconsistent in $\mathbf{BSEL}$+Γ if there are formulas $\varphi, \psi_1, \ldots, \psi_j$ from Σ such that $\vdash_\Gamma \psi_1 \wedge \ldots \wedge \psi_j \rightarrow \neg\varphi$. Otherwise, Σ is consistent in $\mathbf{BSEL}$+Γ, abbreviated as Γ-consistent.

We are interested in the following extensions:

S4 $\mathbf{BSEL} + \{T, 4\}$

S5 $\mathbf{BSEL} + \{T, 4, 5\}$

D $\mathbf{BSEL} + \{D, 4, 5\}$

From the perspective of semantics, extensions correspond to more conditions on the frame of models. To discuss this, let:

$\mathcal{4}$ denote the class of models in which k is reflexive and transitive,

$\mathcal{D}$ denote the class of models in which k is serial transitive and Euclidean,

$\mathcal{5}$ denote the class of models in which k is reflexive, symmetric and transitive.

Knowledge can be modeled with $\mathbf{S5}$, in which k is the equivalence relation. Even more controversially, belief is often modeled with $\mathbf{D}$.

Theorem 3.5.1 (Soundness of extensions) *We have:*

$\vdash_{\mathbf{S4}} \varphi$ *implies* $\models_{\mathcal{4}} \varphi$

$\vdash_{\mathbf{D}} \varphi$ *implies* $\models_{\mathcal{D}} \varphi$

$\vdash_{\mathbf{S5}} \varphi$ *implies* $\models_{\mathcal{5}} \varphi$

Proof: By checking the axioms additional to $\mathbf{BSEL}$ in each case.

♣

With Theorem 3.5.1, it is easy to prove the following proposition:

Proposition 4 *For any* $\Sigma \subseteq L$*,*

$\Sigma \vdash_{\mathbf{S4}} \varphi$ *implies* $\Sigma \models_{\mathcal{4}} \varphi$

$\Sigma \vdash_{\mathbf{D}} \varphi$ *implies* $\Sigma \models_{\mathcal{D}} \varphi$

$\Sigma \vdash_{\mathbf{S5}} \varphi$ *implies* $\Sigma \models_{\mathcal{5}} \varphi$

Proof: Take the first clause as an example. Suppose $\Sigma \vdash_{\mathbf{S4}} \varphi$ and $M, w, a \models \Sigma$ where $M \in \mathcal{4}$, we prove $M, w, a \models \varphi$. By $\Sigma \vdash_{\mathbf{S4}} \varphi$, there exists $\varphi_1 \ldots \varphi_j \in \Sigma$ such that $\vdash_{\mathbf{S4}} \varphi_1 \wedge \ldots \wedge \varphi_j \rightarrow \varphi$. Then by the first clause of Theorem 3.5.1, we have $\models_{\mathcal{4}} \varphi_1 \wedge \ldots \wedge \varphi_j \rightarrow \varphi$ $(*)$. By $M, w, a \models \Sigma$, we have $M, w, a \models \varphi_1$, $M, w, a \models \varphi_2$,

... and $M, w, a \models \varphi_j$. So $M, w, a \models \varphi_1 \wedge \ldots \wedge \varphi_j$. Then by $M \in \mathit{4}$ and $(*)$, $M, w, a \models \varphi$.

The other two clauses are similar to prove.

♣

Lemma 3.5.1 *Let* $\mathbf{BSEL}$+Θ *be any extension of* $\mathbf{BSEL}$*, and* Γ*,* Δ*,* Σ *be any maximal* Θ*-consistent set. If* $\mathbf{K}(\Gamma, \Delta)$ *is defined as the relation between* Γ *and* Δ *such that* $K\varphi \in \Gamma$ *implies* $\varphi \in \Delta$*, then we have:*

1. *If* $\mathsf{T} \in \Theta$ *then* $\mathbf{K}(\Gamma, \Gamma)$.
2. *If* $\mathsf{4} \in \Theta$*, then* $\mathbf{K}(\Gamma, \Delta)$ *and* $\mathbf{K}(\Delta, \Sigma)$ *imply* $\mathbf{K}(\Gamma, \Sigma)$.
3. *If* $\mathsf{5} \in \Theta$*, then* $\mathbf{K}(\Gamma, \Delta)$ *and* $\mathbf{K}(\Gamma, \Sigma)$ *imply* $\mathbf{K}(\Delta, \Sigma)$.

Proof: Straightforward. ♣

Although we have proven that $\mathbf{BSEL}$ is complete, we have a more ambitious goal in mind: the completeness of $\mathbf{S4}$, $\mathbf{D}$ and $\mathbf{S5}$:

Theorem 3.5.2 (Strong Completeness for extensions) *For any* $\Gamma \subseteq \mathsf{L}$ *and* $\varphi \in \mathsf{L}$*,*

$$\begin{array}{lll} \Gamma \models_{\mathit{4}} \varphi & \textit{implies} & \Gamma \vdash_{\mathbf{S4}} \varphi \\ \Gamma \models_{\mathcal{D}} \varphi & \textit{implies} & \Gamma \vdash_{\mathbf{D}} \varphi \\ \Gamma \models_{\mathit{5}} \varphi & \textit{implies} & \Gamma \vdash_{\mathbf{S5}} \varphi \end{array}$$

In fact, we only need to adapt our proof for $\mathbf{BSEL}$ a little here. To shorten the length of this chapter, we just take $\mathbf{S4}$ as an example.

Inspired by the proof of $\mathbf{BSEL}$, we want to show that for each $\mathbf{S4}$-consistent set Γ, there is a network of Γ such that it is perfect and k is reflexive and transitive. Then Γ will be satisfied by the model induced by this network by the Truth Lemma. k is reflexive and transitive, so is k. Done. We need to find a proper way to construct a perfect network such that an $S4$ model can be induced from that.

In the case of $\mathbf{S4}$, an extra clause of a coherent network is required to define:

$$(S4)\ \mathsf{k} \text{ is reflexive and transitive}$$

One more clause needs to be satisfied: we have to make sure that the new network, say $\mathcal{N}''$, is still coherent. Then the question is: how should we construct $\mathcal{N}''$?

Here is an answer. Given any coherent (satisfies the new definition) network $\mathcal{N}$, let $\mathcal{N}'$ be the repair built by the method for $\mathbf{BSEL}$. Let $\mathcal{N}'' = \mathcal{N}'$ but $\mathsf{k}'' \neq \mathsf{k}'$.

Repair method: let k'' be the reflexive and transitive closure of k' of $\mathcal{N}'$.

Now we check $\mathcal{N}''$ is coherent. By the $\mathbf{BSEL}$ result, we do not need to check (n), (s) or $(@)$. And $(S4)$ is automatically satisfied by the Limit Coherence Lemma.[9] So (k) is the only clause we need to check. We need to consider the following two cases:

Case One: (k) is destroyed when we repair an S-defect or $@$-defect.

For example, there exists ψ such that $K\psi \in \delta''(w,a)$ but $\psi \notin \delta''(v,a)$ for some $\mathsf{k}''_a(w,v)$ where $\mathcal{N}''$ is a repair of a coherent (new definition) $\mathcal{N}$ for $[v,b,\varphi]$. But this is impossible:

For any $\mathsf{k}''_a(w,v)$, $\mathsf{k}_a(w,v)$. Also, we have $\delta''(w,a) = \delta(w,a)$ and $\delta''(v,a) = \delta(v,a)$ by the repair methods of S-defect or $@$-defect.

Case Two: (k) is destroyed when we repair a K-defect, say $[w,a,\langle K\rangle\varphi]$.

But it is impossible. We added $\mathsf{k}'_a(w,w^\dagger)$ since we adopt the repair method of $\mathbf{BSEL}$. So we have to check whether (k) is also preserved between $(w^\dagger,a)$ and (v,a) for any $v \in W$ (old worlds) such that $\mathsf{k}'_a(v,w)$ because of the reflexive and transitive closure of k'. By Lemma 3.5.1.2, this holds. And the reflexivity of (k) for "all" nodes in $\mathcal{N}''$ is guaranteed by Lemma 3.5.1.1.

So $\mathcal{N}''$ is coherent, and we have the repair lemma. It is not difficult to see that the Limit Coherence Lemma still holds in $\mathbf{S4}$ case. So the completeness can be proven very similarly.

Also, we can prove the completeness of $\mathbf{S5}$ and $\mathbf{D}$. We just propose the extra clauses for the coherency and the repair method for k'':

$(S5)$ k is reflexive, symmetric and transitive.

Repair method: let k'' be the equivalence closure of k' of $\mathcal{N}'$.

$(D45)$ k is serial, transitive and Euclidean.

Repair method: k'' is the transitive and Euclidean closure of k' of $\mathcal{N}'$.

We need to say a little more about $\mathbf{D}$. The "serial" property cannot be achieved by producing a closure. What it requires is that all K-defects are repaired, but this is obviously what we have already done.

The completeness of $\mathbf{S5}$ is also discussed in [18] and [54], especially [54] also discusses $\mathbf{T}$ and $\mathbf{S4}$.

All the extensions we discussed above are at the epistemic level: different properties of k relation. But we have another operator S. One of the goals of this thesis is to

[9] Any node that destroys $(S4)$ will be included by some coherent network in X. Then by directedness, this network is not coherent.

model agents' vision so that we can discussion the interactions between epistemic and vision. Before we introduce the related extensions, we discuss rigid names first.

3.5.2 When names are rigid

We have introduced the definition of rigid when we proposed the semantics of $\mathbf{BSEL}$ in the previous chapter. Obviously, $\mathbf{BSEL}$ is non-rigid. Do we have any logic complete with respect to the class of rigid models?

The answer is No. It is easy to check that $\{n, \langle K\rangle\neg n\}$ is $\mathbf{S5}$-consistent. But any model satisfying this set is not rigid.

The reason is that we need Rig axioms:

Rig	$n \to Kn$
Rig¬	$\neg n \to K\neg n$
Rig@	$@_n m \to K@_n m$
Rig¬@	$\neg @_n m \to K\neg @_n m$

Given an extension of $\mathbf{BSEL}$, the rigid extension (denotation: $\mathbf{R}$) requires the above four R axioms. For simplicity, if we prove the rigid $\mathbf{BSEL_R}$ is complete, we need (1) more conditions for coherency and (2) to improve the current method for repairing each defect to guarantee the new coherency can be kept.

The extra coherent conditions: For any $n, m \in \mathsf{Nom}$, $w, v \in W$ and $a, b \in A$

$(\mathsf{Rig_n})$	$n \in \delta(w, a)$ iff $n \in \delta(v, a)$
$(\mathsf{Rig_@})$	$@_n m \in \delta(w, a)$ iff $@_n m \in \delta(v, b)$

Then for any perfect network $\mathcal{N}$ under this definition, $M_{\mathcal{N}}$ is a rigid social epistemic model.

For (2), we improve the repair method of $\mathbf{BSEL}$ only by changing the definition of δ'. We discuss it by cases:

Case One: $[w, a, \langle S\rangle\varphi]$ or $[w, a, @_n\varphi]$

At first we build $\delta'(w, a^\dagger)$ with the Repair Lemma of $\mathbf{BSEL}$:

$$\delta'(w, a^\dagger) = [\{\varphi\} \cup \{\psi | S\psi \in \delta(w, a)\}]^e$$

.

Then for any other nodes of $a^\dagger$ column, we define $\delta'(v, a^\dagger)$ based on $\delta(v, a^\dagger)$ of the original network as follows:

$$\delta'(v, a^\dagger) = [@\delta(v, a) \cup \mathsf{Nom}(\delta'(w, a^\dagger))]^e$$

Recall $\mathsf{Nom}(\Sigma) = \{n | n \in \Sigma\} \cup \{\neg n | \neg n \in \Sigma\}$ which is defined by Definition 3.4.1.

What we are doing here is copying all names and their negations from $\delta'(w, a^\dagger)$. So ($\mathsf{Rig_n}$) holds at $a^\dagger$ column. And all $@_m n$ and $@_m \neg n$ are also copied at the same time because of Intro. So ($\mathsf{Rig_@}$) holds at $a^\dagger$ too. By $\mathcal{N}$ is coherent, the other columns satisfy $\mathsf{Rig_n}$ and $\mathsf{Rig_@}$. All results for checking other clauses of coherency can be borrowed from the Repair Lemma of $\mathbf{BSEL}$. So $\mathcal{N}'$ is coherent.

Case Two: $[w, a, \langle K \rangle \varphi]$

First, we build $\delta'(w^\dagger, a)$ by the same repair method of $\mathbf{BSEL}$. Then for the rest of nodes of the $w^\dagger$ row:

$$\delta'(w^\dagger, b) = [@\delta'(w^\dagger, a) \cup \mathsf{Nom}(\delta(w, b))]^e$$

Instead, we are now copying the name information from each corresponding node of the w row. The proof to show $\mathcal{N}'$ coherent is very similar to the first case. So we avoid repeating it.

3.5.3 Social dimension extensions and the interactions between the two dimensions

The inspiration of introducing S is to model social relations and communities. When S is a modal operator for "mutual friendship" of a community, the corresponding relation s in the model could be assumed irreflexive and symmetric. Then $S\neg p$ says that p is false to all of my friends. Symmetric is obvious for friendship, the irreflexivity here means you can not be your own friend. The formula $S\neg p$ could also be interpreted as "for all the people I can see, they are not p". Then the symmetric shouldn't be a necessary condition of the corresponding relation s in the model. In this thesis, we assume that s is only irreflexive.

When we consider the interaction between seeing and knowing, then we realise there would be some interesting features. For any "visual" property φ, we have the following pattern:

If you see φ, you know you see φ, and if you cannot see φ, then you know you cannot see φ.

We assume that names have the visual property. And it seems only make sense when the logic is rigid. Otherwise, you see the same agent with two different names, the above pattern cannot hold. If we let names are the only thing that has visual property, then the pattern could be expressed by the following SK axioms:

$(\mathsf{SK_I})$	$Sn \to KSn$
$(\mathsf{SK_{I'}})$	$\langle S\rangle n \to K\langle S\rangle n$
$(\mathsf{SK_{II}})$	$\neg Sn \to K\neg Sn$
$(\mathsf{SK_{II'}})$	$\neg\langle S\rangle n \to K\neg\langle S\rangle n$
$(\mathsf{SK_{III}})$	$S@_n m \to KS@_n m$
$(\mathsf{SK_{III'}})$	$\langle S\rangle @_n m \to K\langle S\rangle @_n m$
$(\mathsf{SK_{IV}})$	$\neg S@_n m \to K\neg S@_n m$
$(\mathsf{SK_{IV'}})$	$\neg\langle S\rangle @_n m \to K\neg\langle S\rangle @_n m$

And $\mathsf{Irreflexive}$ expresses the irreflexivity:

$(\mathsf{Irreflexive})\quad n \to \neg\langle S\rangle n$

Then let $\mathbf{BSEL_{SK}}$ denotes $\{\mathsf{Irreflexive}, \mathsf{SK}\}+$ $\mathbf{S5}$ + Rig. Formulas $\langle S\rangle Kn \leftrightarrow K\langle S\rangle n$ and $\langle S\rangle K@_n m \leftrightarrow K\langle S\rangle @_n m$ are theorems of $\mathbf{BSEL_{SK}}$.

To prove $\mathbf{BSEL_{SK}}$ is complete, the networks have to satisfy the extra coherent conditions as follows:

(sk)	If $\mathsf{k}_a(w,v)$, then $\mathsf{s}_w(a,b)$ iff $\mathsf{s}_v(a,b)$ for any $b \in A$
(Irre)	s is irreflexive

The repair methods are very similar to those of $\mathbf{S5}+\mathbf{R}$. We just need to copy all seeing information $\mathsf{s}_w(a,b)$ ((w,a) is the node under repairing) to the rest of the rows, or to the new world $w^\dagger$ when a K-defect is being repaired, e.g.

$[w,a,\langle S\rangle\varphi]$	$\mathsf{s}'_v = \begin{cases} \mathsf{s}_v \cup \{\langle a, a^\dagger\rangle\} & \text{if } v = w \text{ or } \mathsf{k}_a(w,v) \\ \mathsf{s}_v & \text{Otherwise} \end{cases}$
$[w,a,@_n\varphi]$	$\mathsf{s}'_v = \mathsf{s}_v$
$[w,a,\langle K\rangle\varphi]$	$\mathsf{s}'_v = \begin{cases} \mathsf{s}_v & \text{if } v \neq w^\dagger \\ \{\langle a,b\rangle \mid \mathsf{s}_w(a,b)\} & \text{Otherwise} \end{cases}$

3.6 A Comparison with Sano's Work

At Section Two, we introduced Sano's Hilbert system which has also rigid names. We now give a brief comparison between his system and mine.

The following figure shows the difference.

Sano's	$\mathsf{BSEL_{Rig}}$
$\mathsf{DCom@}K$	Rigid
Name Rule	Rigid$\neg$
L(BG) Rule	@Name Rule

Actually, $\mathsf{DCom@}K$ Axiom and Kn Axiom are mutually replaceable.

Proposition 5 $\mathsf{DCom@}K$ *can be derived from* $\mathsf{BSEL_{Rig}}$.

Proof:

1. $\vdash n \to (p \leftrightarrow @_n p)$ (Intro)
2. $\vdash Kn \to K(p \leftrightarrow @_n p)$ (2, $\mathsf{Nec_K}$)
3. $\vdash n \to Kn$ (Rigid)
4. $\vdash n \to K(p \leftrightarrow @_n p)$ (2,3, MP)
5. $\vdash @_n(n \to K(p \leftrightarrow @_n p))$ (4, $\mathsf{Nec_@}$)
6. $\vdash @_n n \to @_n K(p \leftrightarrow @_n p)$ (5, $\mathsf{K_@}$, MP)
7. $\vdash @_n n$ (Ref@)
8. $\vdash @_n K(p \leftrightarrow @_n p)$ (6,7 MP)
9. $\vdash @_n Kp \leftrightarrow @_n K @_n p$ (8. $\mathsf{Nec_@}$, $\mathsf{Nec_K}$, MP)

♣

Proposition 6 Rigid *and* Rigid$\neg$ *can be derived from Sano's system.*

Proof:

Rigid is derived as follows:

1. $\vdash Kn \to (n \to Kn)$ (Taut)
2. $\vdash @_n(Kn \to (n \to Kn))$ (2, $\mathsf{Nec_@}$)
3. $\vdash @_n Kn \to @_n(n \to Kn)$ (2, $\mathsf{K_@}$, MP)
4. $\vdash @_n K @_n n \to @_n Kn$ ($\mathsf{DCom@}K$)
5. $\vdash @_n K @_n n \to @_n(n \to Kn)$ (3,4, MP)
6. $\vdash @_n K @_n n$ (Ref@, $\mathsf{Nec_K}$, $\mathsf{Nec_@}$)
7. $\vdash @_n n \to @_n Kn$ (5,6, MP, $\mathsf{K_@}$)
8. $\vdash @_n Kn$ (7, Ref@, MP)
9. $\vdash n \to Kn$ (8, Intro, MP)

And for Rigid¬:

1. $\neg @_n m \to K \neg @_n m$ (Rigid¬@)
2. $@_n \neg m \leftrightarrow \neg @_n m$ (Selfdual@, Taut)
3. $K @_n \neg m \leftrightarrow K \neg @_n m$ (2, $\mathsf{K_K}$, MP)
4. $@_n \neg m \to K @_n \neg m$ (1,2,3, Taut)
5. $@_n @_n \neg m \to @_n K @_n \neg m$ (4, $\mathsf{Nec_@}$, $\mathsf{K_@}$, MP)
6. $@_n @_n \neg m \to @_n \neg m$ (Agree)
7. $@_n K @_m \neg m \leftrightarrow @_n K \neg m$ (DCom@K)
8. $@_n \neg m \to @_n K \neg m$ (5,6,7, Taut)
9. $n \to (\neg m \leftrightarrow @_n \neg m)$ (Intro)
10. $n \to (K \neg m \leftrightarrow @_n K \neg m)$ (Intro)
11. $n \to (\neg m \leftrightarrow K \neg m)$ (8,9,10, Taut)
12. $\neg m \leftrightarrow K \neg m$ (11, Name)

♣

Either Sano's system or ours is very similar to the hybrid logic Hilbert system. But both systems have an extra rule than the hybrid logic. As to **BSEL**, it is the @Name rule. As to Sano's system, they are the Name rule and the L(BG) rule. The reason for the @Name Rule has nothing to do with syntax. Since we only use the rule to prove that any consistent set of formulas will still be consistent when a set of negation names is added. So it is a methodology issue. It is an open question whether the @Name rule is admissible or not.

It is no surprised that Sano has the Name rule since the pure extensions of hybrid logic also need it so as to generate a named model, which is required for proving completeness. But the L(BG) rule is different. The L(BG) rule is defined using necessity forms (see Definition 3.2.1). Sano uses necessity forms to track the unique path from a label of a tree sequent to the root label of the tree because **BSEL** doesn't refer names to the worlds, whereas the hybrid logic does. The names of **BSEL** refer to agents. This requires us to use the methodology of adding witnesses, as is done in predicate logic, and this cause huge problems, as we will see in the next chapter. We cannot directly apply the canonical model, the standard methodology from modal logic[11], either.

3.7 Why not Canonical Model?

I first tried to prove the completeness of **BSEL** via the canonical model method. Recall the canonical model construction for basic modal logic. Worlds are taken to be maximal consistent sets of formulas. We can show that a world w satisfies a formula φ if and only if φ is in w. Then the canonical model M is constructed from all maximal consistent sets. We show that every consistent set Γ is satisfied at some world of M. That is because any consistent set can be extended to a maximal consistent set by the Lindenbaum lemma.

To adapt this to our logic, we need to construct not just a world but a world-agent pair. Why? Because our models have two sets: agents and worlds and formulas are evaluated at world-agent pairs.

A natural idea is to define two equivalence classes on the set of all maximal consistent sets. Let $\sim_1$ denote "the same" worlds and $\sim_2$ denote "the same" agents. Then the worlds and agents could be defined as the equivalence classes of $\sim_1$ and $\sim_2$. Now what we need is for there to be exactly one maximal consistent set in $w \cap a$ and it is the exactly the set that can be satisfied by M at w, a.

Although world-agent pairs with the same world need not satisfy all the same formulas, for example $w, a \models p$ but $w, b \not\models p$, @ formulas (formulas beginning with the @ operator) have the property:

$$w, a \models @_n\varphi \quad \text{iff} \quad w, b \models @_n\varphi$$

This suggests defining $w \sim_1 v$ by:

$$@_n\varphi \in w \quad \text{iff} \quad @_n\varphi \in v$$

Defining $\sim_2$ is harder. The sameness of agents doesn't mean the sameness of names. In the case that some agent doesn't have a name, $\sim_2$ cannot be defined even if the names are restricted to be rigid.

The only possible solution is to add new "witnesses" to the language to keep track of the sameness of agents. This idea is from predicate logic and also used by hybrid logic. These witnesses should be rigid. Then the maximal consistent sets we work with should be "named", i.e each maximal consistent set contains at least one new witness. Then we define $w \sim_2 v$ if and only if

$$n \in w \quad \text{iff} \quad n \in v$$

for every witness name n. Now we can define W and A of the proposed canonical model. Given a named maximal consistent set u, let $[u]_1$ and $[u]_2$ be the equivalence classes of u with respect to $\sim_1$ and $\sim_2$.

$$W = \{[u]_1 | u \text{ is a named maximal consistent set }\}$$

$$A = \{[u]_2 | u \text{ is a named maximal consistent set }\}$$

This guarantees that $w \cap a$ is a singleton. Let $w \cdot a$ denote the unique element in $w \cap a$. Then the canonical relations k and s can be defined as usual: for every formula φ,

$k_a(w, v)$ iff $K\varphi \in w \cdot a$ implies $\varphi \in v \cdot a$

$s_w(a, b)$ iff $S\varphi \in w \cdot a$ implies $\varphi \in w \cdot b$

Then g and V are defined as:

$g_w(n) = \{\varphi | @_n\varphi \in u \text{ for every } u \in w\}$

$V(p) = \{\langle w, a\rangle | p \in w \cdot a\}$

What remains is to show the Truth Lemma holds in the canonical model M, i.e.,

$M, w, a \models \varphi$ iff $\varphi \in w \cdot a$

In the inductive proof of this lemma, the clauses for K and S deserve a discussion.

The clause for K still follows a familiar pattern from modal logic. We show the directions that if $M, w, a \models K\varphi$ then $K\varphi \in w \cdot a$. Suppose $K\varphi \notin w \cdot a$. Then $\neg K\varphi \in w \cdot a$ by $w \cdot a$ is maximal. Let Γ be the set containing $\neg\varphi$ and all ψ that $K\psi \in w{\cdot}a$. By standard modal logic reasoning, Γ is consistent. So it can be extended to a maximal consistent set u. Moreover, u is named because $w \cdot a$ is named by some witness n. Then by Axiom $\mathsf{Rig_n}$, $n \to Kn$, $Kn \in w \cdot a$. So $n \in v$. Let $v = [u]_1$. We have $k_a(w, v)$ and $\neg\varphi \in v \cdot a$. By induction, $M, v, a \not\models \varphi$. Contradiction.

We prove the S clause similarly. Suppose $M, w, a \models S\varphi$. By the standard argument, we define a set Γ containing $\neg\varphi$ and every formula ψ such that $S\psi \in w \cdot a$. It is consistent and can be extended to a maximal consistent set u. By the axiom $@_n\varphi \to S@_n\varphi$, u contains the same @-formulas. So $[u]_1 = w$. But a problem arises when we consider $[u]_2$. There is no guarantee that u contains a name. We ensure to construct a maximal consistent set but not a named maximal consistent set. Moreover, there is also no guarantee that we can add one because Γ may already contains the negations of all names.

Again, we look for a solution from predicate logic. If φ can be seen, then there must be some witness with property φ. Any witnessing condition should ensure that for every formula φ, the set contains the formula

$$\langle S\rangle\varphi \to (\langle S\rangle n \wedge @_n\varphi)$$

for some witness name n.

By adding the following rule, we can consistently add witnessing formulas. And this is enough to ensure that $[u]_2$ is named.

If $\vdash (\langle S\rangle\varphi \to (\langle S\rangle n \wedge @_n\varphi)) \to \psi$ and n does not occur in φ or ψ, then $\vdash \psi$.

However, there is a further problem. The requirement that we use witnessed maximal consistent sets is too demanding for the K-clause of the definition. Recall that we were able to use the rigidity of witness names to ensure that the generated maximal consistent set was named. But there is no way of ensuring it is witnessed, which is a stronger condition.

How about strengthening the witnessing condition for this difficulty? To ensure that the generated consistent set is witnessed, we need to ensure that the original set contains all formulas of the form

$$K(\langle S\rangle\varphi \to (\langle S\rangle n \wedge @_n\varphi))$$

i.e., witnessing formulas with K prefixes. Of course, this has to be a recursive condition, so we need arbitrary prefixes of Ks and @s, i.e.,

$$K^*(\langle S\rangle\varphi \to (\langle S\rangle n \wedge @_n\varphi))$$

where K^* refers any length of @ and K operators as follows: $@_{n_1}K@_{n_2}K\ldots$

Then the corresponding rule to add should be:

(Wit) If $\vdash K^*(\langle S\rangle\varphi \to (\langle S\rangle n \wedge @_n\varphi)) \to \psi$ and n does not occur in $K^*\varphi$ or ψ, then $\vdash \psi$.

This should be enough to get the Truth Lemma. Unfortunately, Wit is not sound. Figure 3.5 shows a counter example.

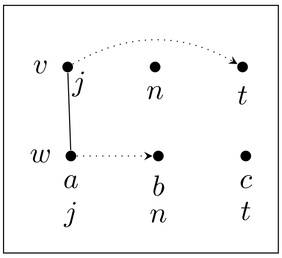


Figure 3.5: A counter example of the soundness of Wit

In this example, $k_a(w,v), k_a(w,w), s_w(a,b)$ and $s_v(a,c)$. So we have $(K\langle S\rangle\top \wedge \langle K\rangle Sn \wedge \langle K\rangle S\neg n)$ is satisfied at $\langle w,a\rangle$, but we claim that $K(\langle S\rangle\top \to (\langle S\rangle m \wedge @_m\top)) \to \psi$ is provable in $\mathsf{BSEL}_{\mathsf{Rig}}$, where ψ is the formula $\neg(K\langle S\rangle\top \wedge \langle K\rangle Sn \wedge \langle K\rangle S\neg n)$. So ψ is not valid. Then Wit is not sound.

Proof: Note that the following are theorems of $\mathsf{BSEL}_{\mathsf{Rig}}$:

(1) $\quad K\langle S\rangle m \to (\langle K\rangle Sn \to \langle K\rangle @_n m)$
(2) $\quad K\langle S\rangle m \to (\langle K\rangle S\neg n \to \neg K @_n m)$

But with the addition of rigid axioms, we can also prove

(3) $\quad \langle K\rangle @_n m \to @_n m$
(4) $\quad \neg K @_n m \to \neg @_n m$

So, combining (1) to (4), we get

$K\langle S\rangle m \to \neg(\langle K\rangle Sn \wedge \langle K\rangle S\neg n)$

and hence

$(K\langle S\rangle\top \to K\langle S\rangle m) \to \neg(K\langle S\rangle\top \wedge \langle K\rangle Sn \wedge \langle K\rangle S\neg n)$

Now we can also easily prove

$K(\langle S\rangle\top \to (\langle S\rangle m \wedge @_m\top)) \to (K\langle S\rangle\top \to K\langle S\rangle m)$

Then our claim has been proven.

3.8 Future Directions

The @Name rule deserves further discussion. Without this rule, the axiomatisation of $\mathbf{BSEL}$ is the same as *basic hybrid logic* [10]. The completeness of basic hybrid logic can be proven by building a canonical model. Nevertheless, in order to get a more powerful result, two admissible rules are introduced. The Name rule[10] and the Paste rule[11] work together to guarantee that the canonical model is a "named" model, in which every point has a name, and there are enough named maximal consistent sets to support the existence lemma. By ensuring that the model is "named", the construction enables a more powerful completeness theorem for any extension of the basic logic with axioms containing only nominals, i.e., no propositional variables. In Chapter 4, the Name rule is included in $\mathbf{SEL}$ and the Paste rule is just a special case of the $\langle$K$\rangle$Paste rule of $\mathbf{SEL}$. However, the @Name rule in our thesis has nothing to do with pure formula extensions. It results from the need to show the consistency of sets of formulas involved in the repair, and so it is a special requirement of our use of the step-by-step method. Therefore, an important question is whether it is really needed for completeness. So far, I am not aware of any theorem that uses it in a non-trivial way. In the next chapter, the axiomatisation of $\mathbf{SEL}$ doesn't have the @Name rule. I suspect it is because we use named model instead.

The last section discussed my failed attempt to adapt the canonical model method to prove completeness. But I have not shown it impossible. This deserves to be investigated more.

Common knowledge, distributed knowledge and group knowledge are introduced in Chapter 2. So far, they haven't been considered as additions to $\mathbf{BSEL}$. The semantics of these operators in standard epistemic logic can be modified for social epistemic models as follows:

Let $M = \langle W, A, k, s, g, V\rangle$ and $B \subseteq A$. Define:

[10] From $\vdash n \to \varphi$, infer φ where n doesn't occur in φ.

[11] From $\vdash @_n \diamond m \wedge @_m\varphi \to \psi$, infer $\vdash @_n \diamond \varphi \to \psi$ where $m \neq n$ and m doesn't occur in φ or ψ.

$K_{EB} = \bigcup_{a \in B} k_a$	The group knowledge relation is the union of the k relations of each member of the group.
$K_{DB} = \bigcap_{a \in B} k_a$	The group distributive knowledge relation is the intersection of the k relations of each member of the group.
$K_{CB} = K^*_{EB}$	The common knowledge relation of group B is the reflexive transitive closure of K_{EB}

Then given a finite set X of agent names we can define general knowledge (E_X), distributed knowledge (D_X) and common knowledge (C_X) of the group named by X as follows:

$$M, w, a \models E_X\varphi \quad \text{iff} \quad M, v, b \models \varphi \text{ for every } v \in W \text{ s.t } K_{Eg_w(X)}(w, v)$$

$$M, w, a \models D_X\varphi \quad \text{iff} \quad M, v, b \models \varphi \text{ for every } v \in W \text{ s.t } K_{Dg_w(X)}(w, v)$$

$$M, w, a \models C_X\varphi \quad \text{iff} \quad M, v, b \models \varphi \text{ for every } v \in W \text{ s.t } K^*_{Eg_w(X)}(w, v)$$

where $g_w(X) = \{g_w(n) | n \in X\}$.

Note that non-rigid and/or rigid names in X can be used to define a range of semantically distinct operators that are not in epistemic logic.

We predict that an axiomatisation based on $\mathbf{5}$ would consists of $\mathbf{5}$ and the axioms and rules listed in Figure 2.5 adapted to **BSEL** as depicted in Figure 3.6. We conjecture that this is a sound and complete axiomatisation.

$\vdash C_X(\varphi \to \psi) \to (C_X\varphi \to C_X\psi)$	$\mathsf{K_C}$
$\vdash D_X(\varphi \to \psi) \to (D_X\varphi \to D_X\psi)$	$\mathsf{K_D}$
$\vdash E_X(\varphi \to \psi) \to (E_X\varphi \to E_X\psi)$	$\mathsf{K_E}$
$\vdash C_X\varphi \to \varphi$	$\mathsf{T_C}$
$\vdash D_X\varphi \to \varphi$	$\mathsf{T_D}$
$\vdash D_X\varphi \to D_XD_X\varphi$	$\mathsf{4_D}$
$\vdash \neg D_X\varphi \to D_X\neg D_X\varphi$	$\mathsf{5_D}$
$\vdash E_X\varphi \leftrightarrow \bigwedge_{n \in X} @_nK\varphi$	KE
$\vdash C_X\varphi \to E_XC_X\varphi$	EC
$\vdash C_X(\varphi \to E_B\varphi) \to (\varphi \to C_X\varphi)$	Cind
from φ infer $D_X\varphi$	$\mathsf{Nec_D}$
from φ infer $C_X\varphi$	$\mathsf{Nec_C}$

Figure 3.6:

Another interesting direction is to look for further axioms about the interaction of seeing and knowing. For example,

1. For any two agents you can see, you must know who can see whom:

$$(\langle S\rangle n \wedge \langle S\rangle m) \wedge @_n\langle S\rangle m) \rightarrow K@_n\langle S\rangle m$$

$$(\langle S\rangle n \wedge \langle S\rangle m) \wedge \neg @_n\langle S\rangle m) \rightarrow K\neg @_n\langle S\rangle m$$

2. Any two agents who can see each other commonly know that they can see each other.

$$(@_n\langle S\rangle m \wedge @_m\langle S\rangle n) \rightarrow C_{\{n,m\}}(@_n\langle S\rangle m \wedge @_m\langle S\rangle n)$$

Chapter 4

Axiomatisation of Social Epistemic Logic

In this chapter, we introduce $\mathbf{SEL}$, an extension of $\mathbf{BSEL}$, with one more operator: $\downarrow$. The operator $\downarrow$ works as a variable binder so that in $\downarrow_x \varphi$, occurrences of x are bound in φ. For example, $\downarrow_x @_n \neg\langle S\rangle x$ says that n cannot see me. Here "x" is a new kind of variable that we use for agents. It is assumed to be rigid. So "$\downarrow_n \varphi$" would be ill-formed in general; n may be a non-rigid name. The addition of the downarrow operator $\downarrow$, however, brings huge complexity for completeness. When we prove it with step-by-step method, we add a new node to the original network to repair S or $@$ defects. Now the downarrow operator requests adding a fresh rigid name (as a witness) to the new node. However, we may use up all fresh rigid names in a repair. Our solution is to rename all bounded names in φ. We also need a variant of maximal consistency that is sensitive to the distinction between free and bound names.

Here is the outline of the completeness proof. We will first introduce the full definition of the new language and its semantics. The substitution requires us to prove a number of results (Lemma 4.1.1, 4.1.2, etc) showing the relationship between its syntactic form and semantic form. The axiomatisation will be an extension of the previous axiomatisation but with a complicated new rule called $\langle\mathsf{K}\rangle\mathsf{Paste}$. Stating the axioms will require some preparatory definitions and results. These are given in Section 4.2. Then we propose a network for $\mathbf{SEL}$ and prove Extended Truth Lemma. Similar we have to discuss how to repair each potential defect. After that, we finally have the completeness result.

4.1 Language and Semantics

Definition 4.1.1 *The social epistemic logic language* $\mathsf{L}_\downarrow$ *is defined using a countably infinite set* Prop *of propositional variables whose elements are usually denoted* p, q, r, *and so on, a countably infinite set* Nom *of agent names whose elements are denoted* n, m, *and so on, a countably infinite set of rigid agent names* $\mathsf{RNom} \subseteq \mathsf{Nom}$

whose elements are denoted x, y and so on, and two unary modal operators: K and S. The well-formed formulas φ of this language are given by the rule

$$\varphi \quad ::= \quad p \mid n \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K\varphi \mid S\varphi \mid @_n\varphi \mid \downarrow_x \varphi$$

The formulas of $\mathsf{L}_\downarrow$ are also agent-indexical: Formulas must be evaluated at a world-agent pair, which means they are also evaluated from the perspective of an agent. Since we have explained formulas in $\mathbf{BSEL}$ in Chapter Two, we only discuss the meaning of downarrow formulas. The downarrow operator $\downarrow_x$ binds x to the current agent. So the formula "$\downarrow_x \neg\langle S\rangle x$" says that I don't see myself, and "$\downarrow_x @_n\langle S\rangle x$" says that n sees me.

The semantics of $\mathbf{SEL}$ is based on $\mathbf{BSEL}$ with the additional semantic definition for downarrow.

A model $M = \langle W, A, k, s, g, V\rangle$ where g_w is a function for each world w assigning an agent $g_w(n)$ to each agent name n and $g_u(x) = g_v(x)$ for all worlds u, v and each rigid agent name x. Defining $\models$ for the downarrow operator, we have:

$$M, w, a \models \downarrow_x \varphi \quad \text{iff} \quad [x/a]M, w, a \models \varphi$$

where $[x/a]M = \langle W, A, k, s, [x/a]g, V\rangle$ is the result of changing the model M by reassigning a as the denotation of x in every world w, i.e.,

$$[x/a]g_v(m) = \begin{cases} a & \text{if } m = x \\ g_v(m) & \text{Otherwise} \end{cases}$$

This model change requires syntactical correspondences, which are defined as follows:

Definition 4.1.2 *The substitution of y for x in formula φ is written $\varphi[x/y]$. This is defined inductively as follows:*

$$n[x/y] = \begin{cases} y & \text{if } n = x \\ n & \text{Otherwise} \end{cases}$$

$$p[x/y] = p$$

$$(\neg\varphi)[x/y] = \neg\varphi[x/y]$$

$$(\varphi \wedge \psi)[x/y] = (\varphi[x/y] \wedge \psi[x/y])$$

$$(@_n\varphi)[x/y] = @_{n[x/y]}\varphi[x/y]$$

$$(S\varphi)[x/y] = S\varphi[x/y]$$

$$(K\varphi)[x/y] = K\varphi[x/y]$$

$$\downarrow_z \varphi[x/y] = \begin{cases} \downarrow_x \varphi & \text{if } z = x \\ \downarrow_z \varphi[x/y] & \text{Otherwise} \end{cases}$$

Similar to the first-order logic, there are cases in which there is an occurrence of x inside the scope of $\downarrow_y$, and y will get captured when it is substituted for x. We will say x is free for y in φ if y avoids being captured.

Then we can prove an Agreement lemma.

Lemma 4.1.1 (Agreement) *If either x is not free in φ or x is not in φ, then*

$$[x/b]M, w, a \models \varphi \quad \textit{iff} \quad M, w, a \models \varphi$$

Proof: By induction on φ.

♣

With the help of the Agreement Lemma, the following lemma can be proven so that the substitution has a nice semantic correspondence:

Lemma 4.1.2 (Semantic Substitution) *If x is free for y in φ, then*

$$M, w, a \models \varphi[x/y] \quad \textit{iff} \quad [x/g_w(y)]M, w, a \models \varphi$$

Proof: By induction on φ. We only require investigating the base case for x and the inductive case for downarrow:

1. Case $\varphi = x$. By $x[x/y] = y$, we have

   $M, w, a \models x \quad \text{iff} \quad g_w(y) = a$

   And we also have

   $[x/g_w(y)]M, w, a \models x \quad \text{iff} \quad g_w(y) = a$

2. Case $\varphi = \downarrow_z \psi$. Then $\varphi[x/y] = (\downarrow_z \psi)[x/y]$.

   (a) Subcase $z = x$. Then x is not free in φ. By Agreement Lemma, we have

   $[x/g_w(y)]M, w, a \models \varphi \quad \text{iff} \quad M, w, a \models \varphi$

   Also we have $\varphi[x/y] = (\downarrow_x \psi)[x/y] = \downarrow_x \psi = \varphi$ and so we are done.

(b) Subcase $z \neq x$. By the semantic definition for downarrow,

$$[x/g_w(y)]M, w, a \models \downarrow_z \psi \quad \text{iff} \quad [z/a][x/g_w(y)]M, w, a \models \psi$$

But we have $[z/a][x/g_w(y)]M = [x/g_w(y)][z/a]M$ since $z \neq x$, so

$$[z/a][x/g_w(y)]M, w, a \models \psi \quad \text{iff} \quad [x/g_w(y)][z/a]M, w, a \models \psi$$

By inductive hypothesis,

$$[x/g_w(y)][z/a]M, w, a \models \psi \quad \text{iff} \quad [z/a]M, w, a \models \psi[x/y]$$

And again by the semantic definition for downarrow,

$$[z/a]M, w, a \models \psi[x/y] \quad \text{iff} \quad M, w, a \models \downarrow_z \psi[x/y]$$

Then by $z \neq x$, we have $(\downarrow_z \psi)[x/y] = \downarrow_z \psi[x/y]$, so we are done.

♣

Let X be a set of rigid names. Given φ, how can we guarantee that each bounded name occurring in φ is free for x? The method we adopt is to rename all bounded names in φ with the names not in X. Specially, define φ^X inductively, for example, $n^X = n$ and $(\neg\varphi)^X = \neg\varphi^X$, with the clause for downarrow:

$$(\downarrow_x \varphi)^X = \downarrow_y \varphi^X[x/y]$$

where y is the first rigid name neither in X nor in φ^X. Here "first" refers to a standard enumeration of formulas.

In order to make this definition work, X cannot be RNom, the set of all rigid names. There must be at least as many rigid names not in X as how many downarrows are in φ. And this motivates us to define the modest set as follows:

Definition 4.1.3 *A set of formulas is modest if there is an infinite number of rigid names that do not occur in any formula in the set.*

Since φ^X is obtained by renaming all bound names of φ, the following lemma shouldn't be surprisingly.

Lemma 4.1.3 *1. φ^X is the same length as φ*

2. φ^X and φ have the same free names

3. φ^X is logically equivalent to φ

4. Any bound name in φ^X not in X.

Proof:

The clause 1,2 and 4 are obvious by the definition of φ^X. So we only prove (3) by induction on φ and only discuss the case $\varphi =\downarrow_x \psi$. By semantic definition for downarrow, we have

$$M, w, a \models \varphi \quad \text{iff} \quad [x/a]M, w, a \models \psi$$

Obviously, $\varphi^X =\downarrow_y \psi^X$ where y is in neither X nor ψ^X. By the semantics for downarrow again,

$$M, w, a \models \varphi^X \quad \text{iff} \quad [y/a]M, w, a \models \psi^X[x/y]$$

By the fact that the assignment function in $[y/a]M$ is $[y/a]g$ and the Semantic Substitution Lemma, we have

$$[y/a]M, w, a \models \psi^X[x/y] \quad \text{iff} \quad [x/[y/a]g_w(y)][y/a]M, w, a \models \psi^X$$

It is easy to see that x is free for y in ψ^X because y is not in ψ^X. Now we have $[y/a]g_w(y) = a$ and so

$[x/[y/a]g_w(y)][y/a]M = [x/a][y/a]M = [y/a][x/a]M$

By the fact that y is not free in ψ^X and Agreement Lemma 4.1.1, we have

$$[y/a][x/a]M, w, a \models \psi^X \quad \text{iff} \quad [x/a]M, w, a \models \psi^X$$

By the induction hypothesis,

$$[x/a]M, w, a \models \psi^X \quad \text{iff} \quad [x/a]M, w, a \models \psi$$

By the above chain of equivalence, we are done.

♣

4.2 Axiomatisation

The axiomatisation has a rule called $\langle \mathsf{K} \rangle$Paste, which is complicated to state. In order to present it more simply and warm-up to prove it is sound, we propose the following definitions and lemmas before we provide the axiomatisation.

Definition 4.2.1 *Let* $\varphi, \psi \in \mathsf{L}_{\downarrow}$, $x, z \in \mathsf{RNom}$ *and* $n \in \mathsf{Nom}$. *We define a* <u>$\langle K \rangle$*-formula*</u>*, the* <u>*paste*</u> *of this* $\langle K \rangle$*-formula, an* <u>$\langle @ \rangle$*-formula*</u> *and the* <u>*paste*</u> *of this* $\langle @ \rangle$*-formula by induction as follows:*

Base Case:

$\langle\mathsf{K}\rangle$*-formula:* $@_x(\langle S\rangle y \wedge @_y\varphi)$ *is a* $\langle K\rangle$*-formula where* $y \in \mathsf{RNom}$ *is the witness of it such that* $y \neq x$ *and* y *doesn't occur in* φ.

And $@_x\langle S\rangle\varphi$ *is the paste of this* $\langle K\rangle$*-formula.*

$\langle @\rangle$*-formula:* $(@_x y \wedge @_y\varphi)$ *is an* $\langle @\rangle$*-formula where* $y \in \mathsf{RNom}$ *is the witness of it such that* $y \neq x$ *and* y *doesn't occur in* φ.

And $@_x\varphi$ *is the paste of this* $\langle @\rangle$*-formula.*

Inductive Case:

If θ *is a* $\langle K\rangle$*-formula (or an* $\langle @\rangle$*-formula) with* ρ *as its paste, then*

$@_z\langle K\rangle(@_n\psi \wedge \theta)$ *is a* $\langle K\rangle$*-formula (or an* $\langle @\rangle$*-formula) if the witness of* θ *is not* n *or* z *and doesn't occur in* ψ.

And $@_z\langle K\rangle(@_n\psi \wedge \rho)$ *is the paste of this* $\langle K\rangle$*-formula (or* $\langle @\rangle$*-formula).*

Now we can state the $\langle\mathsf{K}\rangle\mathsf{Paste}$ rule. Given a $\langle K\rangle$-formula or an $\langle @\rangle$ formula φ, for any ψ where the witness of φ doesn't occur, from $\vdash \varphi \to \psi$, we can infer $\vdash \varphi' \to \psi$, where φ' is the paste of φ. For example, $@_z\langle K\rangle(@_n\theta \wedge @_x(\langle S\rangle y \wedge @_y\varphi))$ is a $\langle K\rangle$-formula. The witness is y and the paste is $@_z\langle K\rangle(@_n\theta \wedge @_x\langle S\rangle\varphi)$. Then suppose

$$\vdash @_z\langle K\rangle(@_n\theta \wedge @_x(\langle S\rangle y \wedge @_y\varphi)) \to \psi$$

for some ψ such that y doesn't occur in ψ. By $\langle\mathsf{K}\rangle\mathsf{Paste}$ Rule, we have:

$$\vdash @_z\langle K\rangle(@_n\theta \wedge @_x\langle S\rangle\varphi) \to \psi$$

Definition 4.2.2 *A formula is a witnessed formula if it is of the form* $@_n\varphi \wedge \psi$ *where* $\varphi \in \mathsf{L}_\downarrow$, $n \in \mathsf{Nom}$ *and* ψ *is a* $\langle K\rangle$*-formula or an* $\langle @\rangle$*-formula.*

The logic $\mathbf{SEL}$ consists of all the axioms and rules of $\mathbf{BSEL}$ and some additional axioms and rules. See Figure 4.1.

The first four additional axioms are exactly the axioms for rigid basic social epistemic logic. Down captures the meaning of $\downarrow$ as a variable binder. The $\mathsf{Leftname}$ Rule is to prove consistency when a fresh name is added.

Recall the necessity form of [54] introduced in Chapter 3. In fact, we can rewrite $\langle\mathsf{K}\rangle\mathsf{Paste}$ Rule with necessity form.

Definition 4.2.3 *Let* $\#$ *be a symbol not occurring in the language. We define necessity forms as follows:*

1. $\#$ *is a necessity form;*

$\vdash \varphi$ if φ is a tautology	Taut
$\vdash S(p \to q) \to (Sp \to Sq)$	$\mathsf{K_S}$
$\vdash K(p \to q) \to (Kp \to Kq)$	$\mathsf{K_K}$
$\vdash @_n(p \to q) \to (@_n p \to @_n q)$	$\mathsf{K_@}$
$\vdash @_n p \leftrightarrow \neg @_n \neg p$	Selfdual@
$\vdash @_n n$	Ref@
$\vdash @_n @_m p \leftrightarrow @_m p$	Agree
$\vdash n \to (p \leftrightarrow @_n p)$	Intro
$\vdash @_n p \to S@_n p$	Back
from $\vdash \varphi$ infer $\vdash \varphi[p/\psi]$ or $\vdash \varphi[n/m]$	Sub
from $\vdash \varphi$ and $\vdash \varphi \to \psi$, infer $\vdash \psi$	MP
from $\vdash \varphi$ infer $\vdash S\varphi$	$\mathsf{Nec_S}$
from $\vdash \varphi$ infer $\vdash @_n\varphi$	$\mathsf{Nec_@}$
from $\vdash \varphi$ infer $\vdash K\varphi$	$\mathsf{Nec_K}$

$\vdash x \to Kx$	Rigid
$\vdash \neg x \to K\neg x$	Rigid¬
$\vdash @_x y \to K@_x y$	Rigid@
$\vdash \neg @_x y \to K\neg @_x y$	Rigid¬@
$\vdash y \to (\downarrow_x \varphi \leftrightarrow \varphi[x/y])$ where x is free for y in φ	Down
from $\vdash n \to \varphi$, infer $\vdash \varphi$ where n is free in φ	LeftName
from $\vdash \varphi \to \psi$, infer $\vdash \varphi' \to \psi$, where φ is a $\langle K\rangle$-formula or an $\langle @\rangle$-formula such that the witness doesn't occur in ψ, and φ' is the paste of φ	$\langle\mathsf{K}\rangle$Paste

Figure 4.1: An Axiomatisation of **SEL**.

2. *if L is a necessity form n is a name, then $@_n\varphi \wedge L$ is also a necessity form;*
3. *if L is a necessity form and n is a name, then $@_n\langle K\rangle L$ is also a necessity form; and*
4. *nothing else is a necessity form.*

The $\langle\mathsf{K}\rangle$Paste Rule can be rewritten as follows:

from $\vdash L(@_x(\langle S\rangle y \wedge @_y\varphi)) \to \psi$, infer $\vdash L(@_x\langle S\rangle\varphi) \to \psi$,
where $y \neq x$ and doesn't occur in φ or ψ
from $\vdash L(@_x y \wedge @_y\varphi)) \to \psi$, infer $\vdash L(@_x y) \to \psi$,
where $y \neq x$ and doesn't occur in φ or ψ

There is a similarity between the $\langle\mathsf{K}\rangle$Paste rule and the $\mathsf{L(BG)}$ rule. Both rules allow us to track the new information. However, the $\mathsf{L(BG)}$ rule aims to construct a tree-shaped model, but the step-by-step method guarantees a tree-shaped model by the tree like network that we will discuss later in this chapter.

In hybrid logic, it is well-known that we can turn the BG rule into an axiom. However, we don't know if it can be done for $\mathbf{SEL}$.

Lemma 4.2.1 *If Γ is consistent and n is not free in Γ, then $\Gamma \cup \{n\}$ is consistent.*

Proof: Suppose Γ is consistent and n is not free in Γ. For contradiction, suppose $\Gamma\cup\{n\}$ is inconsistent. There is a conjunction φ of formulas in Γ such that $\vdash \neg(\varphi\wedge n)$. So $\vdash n \to \neg\varphi$. And by $\mathsf{LeftName}$, we have $\vdash \neg\varphi$, which contradicts the consistency of Γ. So $\Gamma \cup \{n\}$ is consistent.

♣

Recall that we have $@\mathsf{Name}$ Rule in the axiomatisation of $\mathbf{BSEL}$. Its only purpose is to show that any consistent set is still consistent when it unions a set of negative (negation) names. When we repair a defect by adding a new node to the original network, we actually add a row (or a column) of nodes instead. It may be the simplest way to construct maximal consistent sets for the rest of nodes in the row (or column) that we use this kind of union. However, the rule is redundant in the axiomatisation of $\mathbf{SEL}$. The reason is simple: we need a named model where each world has a rigid name. The readers will see that Lemma 4.2.1 takes place of Lemma 3.4.2.

The $\langle\mathsf{K}\rangle$Paste rule is to prove an addition of a witness without destroying the consistency.

Lemma 4.2.2 *Let φ be a $\langle K\rangle$-formula where y is the witness and φ' be the paste of it. Then we have:*

$M, w, a \models \varphi'$ *implies* $[y/b]M, w, a \models \varphi$ *for some* $b \in A$.

Proof: We prove it by induction on φ.

Base Case: $\varphi = @_x(\langle S \rangle y \wedge @_y \psi)$

Then the paste $\varphi' = @_x \langle S \rangle \psi$. Suppose $M, w, a \models @_x \langle S \rangle \psi$.

By semantic definition for $@$, $M, w, g_w(x) \models \langle S \rangle \psi$. So there exists $b \in A$ such that $s_w(g_w(x), b)$ and $M, w, b \models \psi$.

By y is the witness of φ, y is not in ψ.

So by Agreement Lemma, $[y/b]M, w, b \models \psi$ (1)

Also, $[y/b]M, w, b \models y$. By $s'_w(g_w(x), b)$, $[y/b]M, w, g_w(x) \models \langle S \rangle y$.

Then by $[y/b]g_w(y) = b$ and (1), $[y/b]M, w, g_w(x) \models @_y \psi$.

So $[y/b]M, w, g_w(x) \models \langle S \rangle y \wedge @_y \psi$.

And then, $[y/b]M, w, a \models @_x(\langle S \rangle y \wedge @_y \psi)$

Inductive Case: $\varphi = @_x \langle K \rangle (@_n \psi \wedge \theta)$.

Then the paste φ' is $@_x \langle K \rangle (@_n \psi \wedge \theta')$. Suppose $M, w, a \models \varphi'$. Then $M, w, g_w(x) \models \langle K \rangle (@_n \psi \wedge \theta')$. So there exists $v \in W$ such that $k_{g_w(x)}(w, v)$ and $M, v, g_w(x) \models @_n \psi \wedge \theta'$. Then we have

$$M, v, g_w(x) \models @_n \psi (1)$$

and

$$M, v, g_w(x) \models \theta' (2)$$

By inductive hypothesis and (2), $[y/b]M, v, g_w(x) \models \theta$ for some $b \in A$.

By the definition of $\langle K \rangle$-formula, the witness of θ, y, doesn't occur in $@_n \psi$. Then by Agreement Lemma and (1), $[y/b]M, v, g_w(x) \models @_n \psi$.

So $[y/b]M, v, g_w(x) \models @_n \psi \wedge \theta$.

By $k_{g_w(x)}(w, v)$, $[y/b]M, w, g_w(x) \models \langle K \rangle (@_n \psi \wedge \theta)$.

By $y \neq x$, $g_w(x) = [y/b]g_w(x)$. So $[y/b]M, w, [y/b]g_w(x) \models \langle K \rangle (@_n \psi \wedge \theta)$.

Then $[y/b]M, w, a \models @_x \langle K \rangle (@_n \psi \wedge \theta)$.

♣

Lemma 4.2.3 *Let φ be an $\langle @ \rangle$-formula where y is the witness and φ' be the paste of it. Then we have:*

$M, w, a \models \varphi'$ *implies* $[y/b]M, w, a \models \varphi$ *for some* $b \in A$.

Proof: Similar to the proof of Lemma 4.2.2. ♣

Lemma 4.2.4 $\langle\mathsf{K}\rangle\mathsf{Paste}$ *is sound.*

Proof:

For contradiction, suppose it is not sound. Then there exists M, $w \in W$ and $a \in A$ such that

$$M, w, a \not\models \varphi' \to \psi$$

where $M, w, a \models \varphi'$. but $M, w, a \not\models \psi$ (1).

If φ is a $\langle K\rangle$-formula whose witness is y, then by Lemma 4.2.2, we have $[y/b]M, w, a \models \varphi$ for some $b \in A$. Then $[y/b]M, w, a \models \psi$ since we suppose $\models \varphi \to \psi$. Since y is not in ψ, by Agreement Lemma, we have

$$[y/b]M, w, a \models \psi \quad \text{iff} \quad M, w, a \models \psi.$$

So $M, w, a \models \psi$, which contradicts to (1).

If φ is an $\langle @\rangle$-formula, then by Lemma 4.2.3 with the same argument, we are done.

♣

Theorem 4.2.1 (Soundness) *If* $\vdash \varphi$ *then* φ *is valid.*

Proof: This is routine. As in the first order logic, Lemma 4.1.1 and 4.1.2 are required by Down and $\mathsf{LeftName}$, and Lemma 4.2.4 is for $\langle\mathsf{K}\rangle\mathsf{Paste}$.

♣

SEL needs a variant of maximal consistency that is sensitive to the distinction between free and bound names.

Definition 4.2.4 *We say that a consistent set* Γ *is locally maximal consistent if and only if it has no strictly larger superset with the same free names.*

Every maximal consistent set is locally maximal consistent, but not every locally maximal consistent is maximal consistent. For example, if n does not occur free in Γ, then Γ is not maximal consistent since we can add either it or its negation.

Here are the properties that any locally maximal consistent set has.

Lemma 4.2.5 *If* Γ *is locally maximal consistent, then*

1. *If* $\vdash \varphi$ *and every free name of* φ *is free in* Γ*, then* $\varphi \in \Gamma$.

2a. *If* $\neg\varphi \in \Gamma$*, then* $\varphi \notin \Gamma$.

2b. *If* $\varphi \notin \Gamma$ *and every free name of* φ *is free in* Γ*, then* $\neg\varphi \in \Gamma$.

2. $(\varphi \wedge \psi) \in \Gamma$ *iff* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$.

3. *If* $\vdash \varphi \to \psi$ *and* $\varphi \in \Gamma$ *and every free name of* ψ *is free in* Γ*, then* $\psi \in \Gamma$.

To adapt to the setting of local maximal consistency, we need a slightly different Lindenbaum construction.

We now define a function l from any consistent set Γ to a locally maximal consistent extension of Γ. We use the Lindenbaum method to construct $l(\Gamma)$. For the standard enumeration of all formulas, pick up $\varphi_0, \varphi_1, \ldots$ from the enumeration such that their free names are all free in some formula of Γ and $\varphi_0, \varphi_1, \ldots$ are in the order they appear in the standard enumeration, i.e., φ_0 is the first formula to have all its free names free in Γ, and φ_{i+1} is next one after φ_i. Now define $\Gamma_0, \Gamma_1, \ldots$ by

1. $\Gamma_0 = \Gamma$

2. $\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\varphi_i\} & \text{if consistent} \\ \Gamma_i & \text{Otherwise} \end{cases}$

and let $l(\Gamma)$ be the union of the Γ_i for $i \in \mathbb{N}$. Clearly, $\Gamma \subseteq l(\Gamma)$ and if Γ is consistent, then $l(\Gamma)$ is locally maximal consistent by a standard argument.

4.3 Substitution and Equivalence Theorems

Lemma 4.3.1 *If* $\vdash \varphi$ *and* y *is not in* φ*, then* $\vdash \varphi[x/y]$

Proof: This is because every axiom and rule schema is closed under substitution by a refresh rigid name. We just give a brief explanation for the Down rule: if x' is free for y' in φ, and y is not in φ, then x' is also free for y' in $\varphi[x/y]$.

♣

Lemma 4.3.2 *The downarrow operator* $\downarrow_x$ *is a normal modal operator, i.e.,*

$\vdash \downarrow_x (\varphi \to \psi) \to (\downarrow_x \varphi \to \downarrow_x \psi)$	$(\mathsf{K}_\downarrow)$
if $\vdash \varphi$ *then* $\vdash \downarrow_x \varphi$	$(\mathsf{Nec}_\downarrow)$

Proof:

Let y be a rigid name not in $\varphi \to \psi$. Then x is free for y in φ, ψ and $\varphi \to \psi$. Then by Down,

$\vdash y \to (\downarrow_x \varphi \leftrightarrow \varphi[x/y])$
$\vdash y \to (\downarrow_x \psi \leftrightarrow \psi[x/y])$
$\vdash y \to (\downarrow_x (\varphi \to \psi) \leftrightarrow (\varphi \to \psi)[x/y])$

But $(\varphi \to \psi)[x/y]$ is just $(\varphi[x/y] \to \psi[x/y])$. So by Taut, we have

$\vdash y \to (\downarrow_x (\varphi \to \psi) \to (\downarrow_x \varphi \to \downarrow_x \psi))$

Then we prove it by LeftName.

For ($\mathsf{Nec}_\downarrow$), suppose $\vdash \varphi$ and let y be a rigid name not in φ. Then by Lemma 4.3.1, $\vdash \varphi[x/y]$. Then by Down, we have $\vdash y \to (\downarrow_x \varphi \leftrightarrow \varphi[x/y])$. By Taut, we have $\vdash y \to \downarrow_x \varphi$ and then have $\vdash \downarrow_x \varphi$ by LeftName.

♣

Lemma 4.3.3 (Replacement of Logical Equivalents) *Let φ^p_ψ result from replacing every occurrence of property symbol p in φ with ψ. If $\vdash \psi_1 \leftrightarrow \psi_2$ then $\vdash \varphi^p_{\psi_1} \leftrightarrow \varphi^p_{\psi_2}$.*

Proof: By induction on the structure of φ with the help of Lemma 4.3.2.

♣

Recall that we defined φ^X for renaming every bound name of φ with other names not in X. The following lemma guarantees that the renamed formula is logical equivalent to the original one.

Lemma 4.3.4 (Renaming) *Given a finite set X of rigid names, for every formula φ there is a formula φ^X such that:*

$$\vdash (\varphi^X \leftrightarrow \varphi)$$

where

1. *φ^X is of the same length as φ.*
2. *φ^X and φ have the same free names.*
3. *All bound names in φ^X are not in X*

Proof: We only prove the equivalence by induction on φ by only discussing the case of downarrow. Other cases are straightforward by the induction hypothesis and Lemma 4.3.3.

Suppose $\varphi = \downarrow_x \psi$. By induction hypothesis $\vdash \psi \leftrightarrow \psi^X$ and Lemma 4.3.3,

$$\vdash \downarrow_x \psi \leftrightarrow \downarrow_x \psi^X$$

It is clear that $\varphi^X = \downarrow_y \psi^X[x/y]$ and y does not occur in ψ^X. By Down, we have

$$\vdash y \rightarrow (\downarrow_x \psi^X \leftrightarrow \psi^X[x/y])$$

And by $\varphi = \downarrow_x \psi$ and the above equivalences,

$$\vdash y \rightarrow (\varphi \leftrightarrow \psi^X[x/y])$$

It is clear that y is free for y in $\psi^X[x/y]$. Why? It is because y does not occur in ψ^X and the substitution $[x/y]$ only replaces free occurrences of x by free occurrences of y in ψ^X. By the fact that y is free for y in $\psi^X[x/y]$ and Down, we have

$$\vdash y \rightarrow (\downarrow_y \psi^X[x/y] \leftrightarrow \psi^X[x/y][y/y])$$

But $\psi^X[x/y][y/y]$ is just $\psi^X[x/y]$. So by Taut, we have

$$\vdash y \rightarrow (\varphi \leftrightarrow \downarrow_y \psi^X[x/y])$$

Since y is not free here, we can imply LeftName to have

$$\vdash (\varphi \leftrightarrow \downarrow_y \psi^X[x/y])$$

And the right part of the equivalence is just φ^X. So we are done.

♣

4.4 Networks and Models

The step-by-step method we use in this chapter is slightly different from what we used in Chapter Two. We will highlight the difference.

A network of **SEL** $\langle W, A, \mathsf{k}, \mathsf{s}, \delta\rangle$ consists of a set A of rigid names, a set W of natural numbers, a binary relation s_w on A for each $w \in W$, a binary relation k_a on W for each $a \in A$, and a locally maximally consistent set $\delta(w, a)$ of formulas, for each $a \in A$ and $w \in W$, such that $a \in \delta(w, a)$.

The set A now is a set of rigid names and δ now associates with locally maximally consistent sets. We need to pay close attention to the formulas that are settled by the network. So we define the language of (w, a), denoted by $L(w, a)$, to be the set of formulas φ whose property symbols and free names occur in some formula of $\delta(w, a)$. So by local maximal consistency, it contains either φ or $\neg\varphi$.

Recall the definition "modest". A set is modest if there are an infinite number of rigid names that do not occur in any formula in the set. For any network, we say that a network is modest if all its languages are modest; and we define it to be finite if both W and A are finite.

The model induced by $\mathcal{N}$ is defined to be the same as the one in the last chapter, e.g. $M_{\mathcal{N}} = \langle W, A, \mathsf{k}, \mathsf{s}, g, V\rangle$ where

$\langle w, a\rangle \in V_p$	iff	$p \in \delta(w, a)$
$g_w(n) = a$	iff	a is the first element of A such that either $n \in \delta(w, a)$ or $n \notin \delta(w, b)$ for every b

In the case of $\mathbf{BSEL}$, the structural properties of a perfect network are designed to ensure coordination between the formulas in the sets $\delta(w, a)$ and those satisfied by the model at w, a. For $\mathbf{SEL}$, these properties are also necessary, but not enough since we have both rigid names and normal ones. Now we list all we need

Definition 4.4.1 *A network $\mathcal{N} = \langle W, A, \mathsf{s}, \mathsf{k}, \delta\rangle$ is perfect*[1] *if the following conditions are satisfied for all $a \in A$ and n, x and φ in $L(w, a)$:*

(n) $n \in \delta(w, a)$	*iff*	$\neg n \in \delta(w, b)$ *for every* $b \in A$ *such that* $b \neq a$
(s) $S\varphi \in \delta(w, a)$	*iff*	$\varphi \in \delta(w, b)$ *for every* $b \in A$ *such that* $\mathsf{s}_w(a, b)$
(k) $K\varphi \in \delta(w, a)$	*iff*	$\varphi \in \delta(v, a)$ *for every* $v \in W$ *such that* $\mathsf{k}_a(w, v)$
(@) $@_n\varphi \in \delta(w, a)$	*iff*	$(n \to \varphi) \in \delta(w, b)$ *for every* $b \in A$
(x) $x \in \delta(w, a)$	*iff*	$x \in \delta(v, a)$ *for every* $v \in W$

As before, that the coherency of $\mathcal{N}$ is the condition from left to right and the saturation of $\mathcal{N}$ is the condition from the right to left.

Lemma 4.4.1 (World) *In any perfect network, we have:*

1. $L(w, a) = L(w, b)$ *for all* $a, b \in A$, *and*
2. $L(w, a) = L(v, a)$ *for all* w, v *such that* $\mathsf{k}_a(w, v)$.

Proof: By the definition of $\mathcal{N}$, $a \in L(w, a)$. Let s be a property symbol or free name.

[1] The first four clauses are defined slightly different from those of Chapter 2, but equivalent.

1. Suppose s is in $L(w,a)$. Then s occurs in some $\varphi \in \delta(w,a)$. But then $@_a\varphi$ is in $\delta(w,a)$ since $a \in \delta(w,a)$ and $\vdash a \rightarrow (\varphi \leftrightarrow @_a\varphi)$ by Intro, and $\delta(w,a)$ is logically closed by Lemma 4.2.5. Then for any $b \in A$, we have $(a \rightarrow \varphi) \in \delta(w,b)$, by the perfect condition for $(@)$.

2. Suppose $\mathsf{k}_a(w,v)$ and s is in $L(w,a)$. So there is some $\varphi \in \delta(w,a)$ containing s. By modal logic, $\vdash K(\varphi \rightarrow \varphi)$ and so $K(\varphi \rightarrow \varphi)$ is also in $\delta(w,a)$. Then by the perfection condition for (k), $(\varphi \rightarrow \varphi) \in \delta(v,a)$ and s is also a symbol of $L(v,a)$. So we have $L(w,a) = L(v,a)$.

♣

Lemma 4.4.2 (Hintikka) *If $\mathcal{N}$ is perfect then $M_{\mathcal{N}}$ satisfies the following conditions, which mirror the semantic conditions for $\models$:*

$n \in \delta(w,a)$	*iff*	$g_w(n) = a$
$p \in \delta(w,a)$	*iff*	$\langle w,a\rangle \in V_p$
$\neg\varphi \in \delta(w,a)$	*iff*	$\varphi \notin \delta(w,a)$
$(\varphi \wedge \psi) \in \delta(w,a)$	*iff*	$\varphi \in \delta(w,a)$ *and* $\psi \in \delta(w,a)$
$S\varphi \in \delta(w,a)$	*iff*	$\varphi \in \delta(w,b)$ *for every* $b \in A$ *such that* $s_w(a,b)$
$K\varphi \in \delta(w,a)$	*iff*	$\varphi \in \delta(v,a)$ *for every* $v \in W$ *such that* $k_a(w,v)$
$@_n\varphi \in \delta(w,a)$	*iff*	$\varphi \in \delta(w, g_w(n))$
$\downarrow_x \varphi \in \delta(w,a)$	*iff*	$\varphi^{\{a\}}[x/a] \in \delta(w,a)$

for any φ, ψ, p and $n, x \in L(w,a)$.

Proof:

1. By Lemma 4.4.1 (World Lemma), we have $L(w,a) = L(w,b)$ for any $b \in A$. Then by $n \in L(w,a)$, either n or $\neg n$ is in $\delta(w,b)$ but not both. By the Definition 4.4.1 condition for (n), we have $n \in \delta(w,b)$ for exactly one $b \in A$. By the definition of g_w, we are done.

2. The second clause is proven just by the definition of V_p. The Boolean cases are routine by Lemma 4.2.5 and the assumption that φ is in $L(w,a)$.

3. By the corresponding perfect conditions, it is easy to prove the Hintikka properties for S and K.

4. The Hintikka property for @ is straightforward from $(@)$ of Definition 4.4.1 and the Hintikka property for n.

5. For downarrow, by x is free for a in $\varphi^{\{a\}}$ and Down, we have

$$\vdash a \to (\downarrow_x \varphi^{\{a\}} \leftrightarrow \varphi^{\{a\}}[x/a])$$

Then by Lemma 4.3.4 (Renaming), we have $\vdash \varphi \leftrightarrow \varphi^{\{a\}}$ and so by Lemma 4.3.3 (Replacement of Logical Equivalents), we have

$$\vdash a \to (\downarrow_x \varphi \leftrightarrow \varphi^{\{a\}}[x/a])$$

Then by Renaming again, $\varphi^{\{a\}}$ has the same free names as φ, and so $\varphi^{\{a\}}[x/a]$ has the same as $\downarrow_x \varphi$ (with the possible addition of a) and so is in $L(w, a)$. By $a \in \delta(w, a)$, we proved the last clause.

♣

We need a slight strengthening of the Hintikka properties of n and $@_n$.

Lemma 4.4.3 *If $\mathcal{N}$ is perfect, then for every $a \in A$, $w \in W$, every $[\sigma] = [x_1/a_1] \ldots [x_k/a_k]$ such that $a_1, \ldots, a_k \in A$ and $x_1, \ldots, x_k \notin A$, and every name n such that $n[\sigma] \in L(w, a)$, we have*

1. $n[\sigma] \in \delta(w, a)$ *iff* $[\sigma]g_w(n) = a$, *and*
2. *for every φ such that $\varphi[\sigma] \in L(w, a)$,*

   $@_{n[\sigma]}\varphi[\sigma] \in \delta(w, a)$ *iff* $\varphi[\sigma] \in \delta([\sigma]g_w(n), w)$

Proof: We prove it by induction on the length of $[\sigma]$. For the base case, whose length is 0, both clauses are just the Hintikka properties for n and $@_n$.

For the inductive case $[x/b][\sigma]$, we have to discuss two sub-cases: $n = x$ or not.

If $n = x$,

we have $[x/b][\sigma]g_w(n) = b$, and so

$[x/b][\sigma]g_w(n) = a$ iff $b = a$

But by the Hintikka property of b, we have

$b \in \delta(w, b)$ iff $g_w(b) = b$

$b \in \delta(w, a)$ iff $g_w(b) = a$

Since $b \in \delta(w,b)$ in any network, $g_w(b) = b$ and so

$$b = a \quad \text{iff} \quad b \in \delta(w,a)$$

Finally, $n[x/b][\sigma] = b$ because $n[x/b] = b$ and $b[\sigma] = b$ since $b \in A$ and so not one of the $x_1, \ldots, x_k$ will be replaced by $[\sigma]$, which are all not in A. So we have

$$b \in \delta(w,a) \quad \text{iff} \quad n[x/b][\sigma] \in \delta(w,a)$$

So we got the inductive clause for 1:

$$n[x/b][\sigma] \in \delta(w,a) \quad \text{iff} \quad [x/b][\sigma]g_w(n) = a$$

Now we prove the second. First, we have the result $[x/b][\sigma]g_w(n) = b$. So we have

$$\psi[x/b][\sigma] \in \delta(w, [x/b][\sigma]g_w(n)) \quad \text{iff} \quad \psi[x/b][\sigma] \in \delta(w,b)$$

But $b \in \delta(w,b)$. So $g_w(b) = b$ and by the Hintikka property for @, we have

$$\psi[x/b][\sigma] \in \delta(w,b) \quad \text{iff} \quad @_b(\psi[x/b][\sigma]) \in \delta(w,a)$$

And since $n[x/b][\sigma] = b$, we have

$$@_b(\psi[x/b][\sigma]) \in \delta(w,a) \quad \text{iff} \quad @_{n[x/b][\sigma]}(\psi[x/b][\sigma]) \in \delta(w,a)$$

So we have the inductive clause for 2.

If $n \neq x$,

we have $[x/b][\sigma]g_w(n) = [\sigma]g_w(n)$ and $n[x/b][\sigma] = n[\sigma]$, so $n[\sigma] \in L(w,a)$ and we can use the inductive hypothesis for 1.

For 2, suppose $\varphi[x/b][\sigma] \in L(w,a)$. By inductive hypothesis, we have

$$@_{n[\sigma]}\psi[\sigma] \in \delta(w,a) \quad \text{iff} \quad \psi[\sigma] \in \delta(w, [\sigma]g_w(n))$$

Lwt ψ be $\varphi[x/b]$. Then we have

$$@_{n[\sigma]}(\varphi[x/b][\sigma]) \in \delta(w,a) \quad \text{iff} \quad \varphi[x/b][\sigma] \in \delta(w, [\sigma]g_w(n))$$

But $[x/b][\sigma]g_w(n) = [\sigma]g_w(n)$ and $n[x/b][\sigma]) = n[\sigma]$, so we are done.

♣

Now we can prove the Extended Truth Lemma. Because of the downarrow operator, we have to consider a substitution case.

Lemma 4.4.4 (Extended Truth) *If $\mathcal{N}$ is perfect then for any formula φ, every $a \in A$, $w \in W$, and every $[\sigma] = [x_1/a_1] \ldots [x_k/a_k]$ such that $a_1, \ldots, a_k \in A$ and $x_1, \ldots, x_k \notin A$, if $\varphi[\sigma] \in L(w, a)$ then*

$$[\sigma]M_{\mathcal{N}}, w, a \models \varphi \quad \textit{iff} \quad \varphi[\sigma] \in \delta(w, a)$$

Proof: By induction on φ.

1. Case $\varphi = p$. Straightforward by the Hintikka property of p and the fact that $p[\sigma] = p$.

2. Case $\varphi = n$. By the semantic definition,

   $$[\sigma]M_{\mathcal{N}}, w, a \models n \quad \text{iff} \quad [\sigma]g_w(n) = a$$

   The result then follows from Item 1 of Lemma 4.4.3, which shows an extended Hintikka property.

3. Modal cases: S and K. Straightforward by the induction hypothesis and the Hintikka properties of S and K. For the S case, Lemma 4.4.1 (World Lemma) ensures that $\varphi[\sigma] \in L(w, b)$ when $\mathsf{s}_w(a, b)$. For the K case, it also ensures that $\varphi[\sigma] \in L(v, a)$ when $\mathsf{k}_a(w, v)$.

4. Case $\varphi = @_n\psi$. By the semantic definition,

   $$[\sigma]M_{\mathcal{N}}, w, a \models @_n\psi \quad \text{iff} \quad [\sigma]M_{\mathcal{N}}, w, [\sigma]g_w(n) \models \psi$$

   Assume $\varphi[\sigma] \in L(w, a)$. So $\psi[\sigma] \in L(w, a) = L(w, [\sigma]g_w(n))$ by World Lemma 4.4.1. Then by inductive hypothesis,

   $$[\sigma]M_{\mathcal{N}}, w, [\sigma]g_w(n) \models \psi \quad \text{iff} \quad \psi[\sigma] \in \delta(w, [\sigma]g_w(n))$$

   Then by Item 2 of Lemma 4.4.3, the extended Hintikka property of $@_n$, we have

   $$\psi[\sigma] \in \delta(w, [\sigma]g_w(n)) \quad \text{iff} \quad @_{n[\sigma]}\psi[\sigma] \in \delta(w, a)$$

   since $\varphi[\sigma] = @_{n[\sigma]}\psi[\sigma]$ is assumed to be in $L(w, a)$. We are done.

5. Case $\varphi = \downarrow_x \psi$. By Lemma 4.1.3, ψ is logically equivalent to $\psi^{\{a\}}$. Then by our semantic definition, we have

   $$[\sigma]M_{\mathcal{N}}, w, a \models \downarrow_x \psi \quad \text{iff} \quad [x/a][\sigma]M_{\mathcal{N}}, w, a \models \psi^{\{a\}}$$

   Now any free name in $\psi^{\{a\}}[x/a][\sigma]$ is either a or is free in $(\downarrow_x \psi^{\{a\}})[\sigma]$, which means it is also free in $(\downarrow_x \psi)[\sigma]$, since ψ and $\psi^{\{a\}}$ share free names by Lemma 4.1.3. But $(\downarrow_x \psi)[\sigma] = \varphi[\sigma]$, which is assumed to be in $L(w, a)$. Also $a \in L(w, a)$, so $\psi[x/a][\sigma]$ is in $L(w, a)$, so we can use the inductive hypothesis,

$$[x/a][\sigma]M_{\mathcal{N}}, w, a \models \psi^{\{a\}} \quad \text{iff} \quad \psi^{\{a\}}[x/a][\sigma] \in \delta(w, a)$$

Now let $[\sigma']$ be the result of removing all substitutions of x from $[\sigma]$. Then $\psi^{\{a\}}[x/a][\sigma] = \psi^{\{a\}}[\sigma'][x/a]$. So we have

$$\psi^{\{a\}}[x/a][\sigma] \in \delta(w, a) \quad \text{iff} \quad \psi^{\{a\}}[\sigma'][x/a] \in \delta(w, a)$$

Since renaming just changed bound names, we have $\psi^{\{a\}}[\sigma'] = (\psi[\sigma'])^{\{a\}}$. So by Hintikka property for downarrow,

$$\psi^{\{a\}}[\sigma'][x/a] \in \delta(w, a) \quad \text{iff} \quad \downarrow_x \psi[\sigma'] \in \delta(w, a)$$

Finally, we have $(\downarrow_x \psi)[\sigma] = \downarrow_x \psi[\sigma']$ by definition of substitution. Based on the above equivalences, we get the desired result:

$$[\sigma]M_{\mathcal{N}}, w, a \models \downarrow_x \psi \quad \text{iff} \quad (\downarrow_x \psi)[\sigma] \in \delta(w, a)$$

♣

4.5 Completeness

Similar to **BSEL**, we have to show how to obtain a perfect network. First, we are interested in a special kind of network called tree network.

Definition 4.5.1 *A tree network is a tree network with additional conditions for* k *as follows:*

1. *The element* $0 \in W$ *(called the root) does not have any* k*-predecessor.*[2]
2. *Every element of* W *distinct from* 0 *has a unique* k*-predecessor.*
3. *If* $\mathsf{k}_a(w, v)$ *and* $\mathsf{k}_b(w, v)$ *then* $a = b$.
4. *If* $\mathsf{k}_a(w, v)$ *then* $v > w$.

Figure 4.2 represents a very simple tree network where $W = \{0, 1, 2, 3\}$ and $A = \{a, b, c\}$ and the root is 0.

Definition 4.5.2 *Given a tree network* $\mathcal{N}$, $w, v \in W$ *and* $b, c \in A$, *a* k *path from* w *to* v *is a sequence* $w_0 a_0 w_2 \ldots a_{k-1} w_k$ *for some* $w_0, \ldots, w_k \in W$, $a_0, \ldots, a_{k-1} \in A$ *such that* $w_0 = w$, $w_k = v$, $w_m \neq w_n$ *if* $m \neq n$, *and* $\mathsf{k}_{a_j}(w_j, w_{j+1})$ *for each* $1 \leq j < k$. *And the length of this* k *path is* k, *the number of possible worlds in the*

[2] w is a k-predecessor of v iff $\mathsf{k}_a(w, v)$ for some $a \in A$.

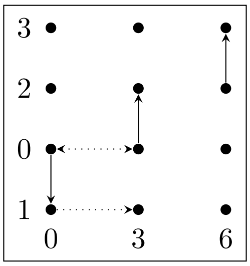


Figure 4.2: A simple network

sequence. Specially, there is a k *path from* w *to* w *for any* $w \in W$ *where its length is* 0*. If the* k *path from* w *to* v *exists, then we say that* <u>*w and v are related by a k path,*</u> *and denote the path as* $\vec{\mathsf{k}}(w,v)$*. The length of* $\vec{\mathsf{k}}(w,v)$ *is denoted as* $|\vec{\mathsf{k}}(w,v)|$*.*

Lemma 4.5.1 *If* $\mathcal{N}$ *is a tree network, then there is at most one* k *path from* w *to* v *for any* $w, v \in W$*.*

Especially, for any $v \in W$ *there is one* k *path from* 0 *to* v*.*

Proof: Straightforward by Definition 4.5.1.

♣

By Lemma 4.5.1, $\vec{\mathsf{k}}(w,v)$ is unique for any $w, v \in W$. By Definition 4.5.2, any $\vec{\mathsf{k}}(w,v) = w_0 a_0 w_2 \dots a_{k-1} w_k$, corresponds to a unique sequence of modal operators, i.e., $@_{a_0}\langle K\rangle @_{a_1}\langle K\rangle \dots @_{a_{k-1}}\langle K\rangle$. We use $\overset{w\to v}{@\langle K\rangle}$ to denote this sequence. In particular, $\overset{w\to v}{@\langle K\rangle}$ is the empty string if $w = v$ or not $\vec{\mathsf{k}}(w,v)$. And we use $\overrightarrow{@\langle K\rangle}$ to denote an arbitrary such a sequence. Moreover, $\overset{w\to v}{@K}$ denotes the dual of $\overset{w\to v}{@\langle K\rangle}$.

Why are we interested in tree networks? It is because a tree network allows us to access any node from its root by sequences linked by the k relation. This property will play a very important role when we reconstruct the original locally maximal consistent sets in our repair. We will track from the node where we repair a defect back to the root along the unique k path, collecting suitable witnessed formulas for each node on this path. When we arrive at the root, we will apply a general method to extend locally maximal consistent sets for all nodes to an expanded language.

The following definitions are the same as those we give for $\mathbf{BSEL}$. We repeat them here for the readers' convenience of reading.

Definition 4.5.3 *We say that* $\mathcal{N}' = \langle W', A', \mathsf{k}', \mathsf{s}', \delta'\rangle$ *is an* <u>*extension*</u> *of network* $\mathcal{N} = \langle W, A, \mathsf{k}, \mathsf{s}, \delta\rangle$ *and write* $\mathcal{N} \trianglelefteq \mathcal{N}'$ *if* $W \subseteq W'$*,* $A \subseteq A'$*,* $\mathsf{k}_a \subseteq \mathsf{k}'_a$*,* $\mathsf{s}_w \subseteq \mathsf{s}'_w$*, and* $\delta(w,a) \subseteq \delta'(w,a)$ *for each* $w \in W$ *and* $a \in A$*.*

Definition 4.5.4 *Given $\mathcal{N}$, the following are potential defects of $\mathcal{N}$:*

$[w, a, \varphi]$ *is a K-defect* *iff* $\mathcal{N}$ *fails the (K) clause of saturation for φ at w, a.*

$[w, a, \varphi]$ *is an S-defect* *iff* $\mathcal{N}$ *fails the (S) clause of saturation for φ at w, a.*

$[w, a, \varphi]$ *is an @-defect* *iff* $\mathcal{N}$ *fails the $(@_n)$ clause of saturation for φ at w, a.*

Note that the downarrow operator doesn't create a defect.

Definition 4.5.5 *If D is a defect of $\mathcal{N}$ then $\mathcal{N}'$ is a repair of D in $\mathcal{N}$ iff*
(1) $\mathcal{N}'$ is a coherent extension of $\mathcal{N}$
(2) D is not a defect of $\mathcal{N}'$
(3) $\mathcal{N}'$ is finite if $\mathcal{N}$ is.

We also have some abbreviations, lemmas and theorems to help with the proof of repair.

For a set of formulas Σ,

$$@\Sigma = \{@_n\varphi | @_n\varphi \in \Sigma, n \in \mathsf{Nom}\}$$

$$@_n^+\Sigma = \{@_n\varphi | \varphi \in \Sigma\}$$

$$@_nK^-\Sigma = \{@_m\varphi | @_nK@_m\varphi \in \Sigma, m \in \mathsf{Nom}\}$$

We need all the theorems of Theorem 3.4.2 together with the following:

1. $\vdash @_xy \to @_zK@_xy$
2. $\vdash @_x\neg y \to @_zK@_x\neg y$
3. $\vdash @_xy \to @\vec{K}@_xy$
4. $\vdash @_x\neg y \to @\vec{K}@_x\neg y$
5. $\vdash @\langle\vec{K}\rangle\ @_x(\neg x_1 \wedge \ldots \wedge \neg x_k) \to @_x(\neg x_1 \wedge \ldots \wedge \neg x_k)$
6. $\vdash @_x(\neg x_1 \wedge \ldots \wedge \neg x_k) \to @\vec{K}@_x(\neg x_1 \wedge \ldots \wedge \neg x_k)$
7. $\vdash @_yK\varphi \to @_yK@_y\varphi$
8. $\vdash @_nm \to (@_m\varphi \to @_n\varphi)$
9. $\vdash @_n\varphi \wedge @_m\psi \leftrightarrow @_j(@_n\varphi \wedge @_m\psi)$

Lemma 4.5.2 *Lemma 3.4.1, which says that any conjunction of @ formulas will be equivalent to an @ formula, still works for* **SEL**.

Lemma 4.5.3 *If Σ is consistent, then $@\Sigma$ is consistent.*

Proof: For contradiction suppose not. Then there exists a conjunction θ of $@\Sigma$ such that $\vdash \neg\theta$. Since θ is an @-formula, we have $\vdash @_n\varphi \leftrightarrow \varphi$ by Lemma 3.4.1. So $\vdash @_n\neg\varphi$. Then $@_n\neg\varphi \in \Sigma$. Contradiction to the consistency of Σ by $@_n\varphi \in \Sigma$.

♣

We are now going to discuss the repair methods. We first focus on the construction of a network we need to repair an S defect. It shows all techniques we will use for repairing other defects. We will explain explicitly what the network is and prove all properties that are necessary for it to be a repair.

Definition 4.5.6 *Given any finite modest coherent tree network $\mathcal{N} = \langle W, A, \mathsf{k}, \mathsf{s}, \delta\rangle$, with a defect $[w, a, \langle S\rangle\varphi]$ such that for any $m \in \mathsf{Nom}$ and $b \in A$, $\langle S\rangle(m \wedge \varphi) \in \delta(w, a)$ implies $m \notin \delta(w, b)$, we define the network $[w, a, \langle S\rangle\varphi]\mathcal{N}$ as follows:*

$[w, a, \langle S\rangle\varphi]\mathcal{N} = \langle W, A \cup \{i\}, \mathsf{k}, \mathsf{s}', \delta'\rangle$

where $\mathsf{s}'_w = \begin{cases} \mathsf{s}_v \cup \{\langle a, i\rangle\} & \text{if } v = w \\ \mathsf{s}_v & \text{Otherwise} \end{cases}$

Recall that A is a set of rigid names. Here i is a fresh rigid name for each $L(v, b)(v \in W, b \in A)$. So $i \notin A$. By the fact that $\mathcal{N}$ is modest, $L(v, b)$ is modest. So there are an infinite number of rigid names that do not occur in any formula in $L(v, b)$. By Lemma 4.4.1 (World Lemma), each $L(v, b)$ shares the same rigid names that do not occur in any formula in it. So such i exists.

To define δ', we will first construct local maximal consistent sets Δ_v[3] for each $v \in W$. But we have to pay special attention to all v in the path from the root 0 to the world w. We present our method with the following three steps.

> ***Step One:***
>
> *Let $w_k a_{k-1} \ldots w_1 a_0 w_0$ be the unique k path from 0 to w such that $w_k = 0$ and $w_0 = w$. By Lemma 4.5.1, such k path exists. For each w_j in this path, we inductively define Γ_{w_j} as follows:*
>
> $\Gamma_{w_0} = \{\psi \wedge @_a(\langle S\rangle i \wedge @_i(\varphi \wedge \bigwedge_{b\in A} \neg b)) \mid \psi \in @\delta(w_0, a)\}$, *and*
>
> $\Gamma_{w_{j+1}} = \{\psi \wedge @_{a_j}\langle K\rangle\theta \mid \psi \in @\delta(w_{j+1}, a), \theta \in \Gamma_{w_j}\}$.
>
> *Notice that each Γ_{w_j} is a set of witnessed formulas.*
>
> ***Step Two:***
>
> *For each $v \in W$, Δ_v is defined by induction on the length of $\vec{\mathsf{k}}(0, v)$ as follows:*

[3] We will prove Δ_v is local maximal consistent after this definition.

Base Case:

$$\Delta_0 = l(\Gamma_{w_k})$$

Inductive Case:

$$\Delta_{j+1} = \begin{cases} l(\Gamma_{w_u} \cup @_b K^- \Delta_t) & \text{if } \vec{\mathsf{k}}(j+1, w) \\ l(@\delta(j+1, a) \cup @_b K^- \Delta_t) & \text{Otherwise} \end{cases}$$

where w_u is in the k path such that $w_u = j+1$, and t is the k-predecessor of $j+1$ in $\mathcal{N}$ such that $\mathsf{k}_b(t, j+1)$ for some $b \in A$. By the fact that $\mathcal{N}$ is a tree network and $j+1$ is not the root, both t and b uniquely exist. Specially in the k path, $b = a_u$ and $t = w_{u+1}$.

Notice that the definition of Δ_v covers every $v \in W$. This is because $\mathcal{N}$ is a tree network so any node can be accessed from the root 0 by links along the k relation.

Step Three:

For every $b \in A \cup \{i\}$, we define δ' as follows:

$$\delta'(v, b) = \{\psi \mid @_b\psi \in \Delta_v\}$$

After giving such a long definition, we now discuss the properties of

$[w, a, \langle S\rangle\varphi]\mathcal{N}$

Lemma 4.5.4 *Suppose Γ is a locally maximal consistent set, then let $\Sigma = \{\varphi | @_n\varphi \in \Gamma\}$ for some $n \in \mathsf{Nom}$. Σ is also a locally maximal consistent set with the same language, e.g. $L(\Gamma) = L(\Sigma)$.*

Proof: We first prove Σ is consistent. For contradiction, suppose not. Then there exists a conjunction θ in Σ such that $\vdash \neg\theta$. Then by $\mathsf{Nec}_@$, we have $\vdash @_n\neg\theta$. So $@_n\neg\theta$ is in Γ. However, $@_n\theta$ has to be in Γ since θ is in Σ. This contradicts the fact that Γ is consistent.

Now we prove it is locally maximal. By the definition of Σ, every property symbol and free name in Σ should be contained by Γ. So Σ is local and $L(\Sigma) = L(\Gamma)$. Suppose it is not locally maximal, then there exists $\varphi \in L(\Sigma)$ such that $\varphi \notin \Sigma$ and $\neg\varphi \notin \Sigma$. So $@_n\varphi \notin \Gamma$ and $@_n\neg\varphi \notin \Gamma$, which means $\neg @_n\varphi \notin \Gamma$. Then Γ is not locally maximal. Contradiction.

♣

To make sure that the structure we defined in Definition 4.5.6 is indeed a network, we have to show $\delta'(v, b)$ is locally maximal consistent for any $v \in W$ and $b \in A'$. Since we have Lemma 4.5.4, we only need to prove each Δ_v is locally maximal consistent. Before that, notice the following fact:

Fact 1: Under the assumption of Definition 4.5.6, $\langle S\rangle(\varphi \wedge \bigwedge_{b \in A} \neg b) \in \delta(w, a)$.

Proof: Suppose it is not the case. By $\delta(w, a)$ is locally maximal consistent and $\langle S\rangle(\varphi \wedge \bigwedge_{b \in A} \neg b)$ is in the language $L(w, a)$, we have

$$\neg\langle S\rangle(\varphi \wedge \bigwedge_{b \in A} \neg b) \in \delta(w, a)$$

.

And so

$$S(\varphi \rightarrow \neg \bigwedge_{b \in A} \neg b) \in \delta(w, a)$$

.

By $\langle S\rangle\varphi \in \delta(w, a)$ and the theorem $\vdash S(\varphi \rightarrow \psi) \wedge \langle S\rangle\varphi \rightarrow \langle S\rangle(\varphi \wedge \psi)$, we have

$$\langle S\rangle(\varphi \wedge \neg \bigwedge_{b \in A} \neg b) \in \delta(w, a)$$

which is equivalent to $\langle S\rangle(\varphi \wedge \bigvee_{b \in A} b) \in \delta(w, a)$.

By $\delta(w, a)$ is locally maximal and $\langle S\rangle(\varphi \wedge \bigvee_{b \in A} b)$ is in $L(w, a)$, there exists some $c \in A$ such that $\langle S\rangle(\varphi \wedge c) \in \delta(w, a)$. However, Definition 4.5.6 says that for any $m \in \mathsf{Nom}$ and $b \in A$, $\langle S\rangle(m \wedge \varphi) \in \delta(w, a)$ implies $m \notin \delta(w, b)$. So we have $c \notin \delta(w, a)$. Contradiction.

♣

Fact 2: For each Γ_{w_j}, any conjunction θ of Γ_{w_j} can be logically implied by some witnessed formula $(@_m\rho \wedge \psi) \in \Gamma_{w_j}$ where ψ', the paste of ψ, is in $\delta(w_j, b)$ and $@_m\rho$ is also in $\delta(w_j, b)$ for any $b \in A$. E.g.

$$\vdash @_m\rho \wedge \psi \rightarrow \theta$$

Proof: We prove it by induction on Γ_{w_j}.

Base Case: Γ_{w_0}

Let $@_{n_1}\varphi_1 \wedge @_a(\langle S\rangle i \wedge @_i(\varphi \wedge \bigwedge_{b\in A} \neg b))), \ldots, @_{n_k}\varphi_k \wedge @_a(\langle S\rangle i \wedge @_i(\varphi \wedge \bigwedge_{b\in A} \neg b)))$ be in Γ_{w_0}. Let γ denote the conjunction of these formulas. By Taut, we have

$$\vdash (@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \wedge @_a(\langle S\rangle i \wedge @_i(\varphi \wedge \bigwedge_{b\in A} \neg b)) \to \gamma$$

By Lemma 3.4.1, the conjunction of these @ formulas is equivalent to an @ formula. So we have

$$\vdash @_m(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \wedge @_a(\langle S\rangle i \wedge @_i(\varphi \wedge \bigwedge_{b\in A} \neg b)) \to \gamma$$

Obviously, $@_m(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \wedge @_a(\langle S\rangle i \wedge @_i(\varphi \wedge \bigwedge_{b\in A} \neg b))$ is a witnessed formula.

To prove that the first conjunct is in Γ_{w_0}, we just need to prove

$$@_m(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \in \delta(w_0, a)$$

Since $@_{n_t}\varphi_t \in @\delta(w_0, a)$ for each $1 \leq t \leq k$ and $@\delta(w_0, a) \subseteq \delta(w_0, a)$, $@_{n_t}\varphi_t \in \delta(w_0, a)$ and so

$$(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \in \delta(w_0, a)$$

So we have $@_m(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \in \delta(w_0, a)$ by Lemma 3.4.1.

Now we discuss the paste, which is $@_a\langle S\rangle(\varphi \wedge \bigwedge_{b\in A} \neg b))$. By Fact 1, $a \in \delta(w, a)$ and Intro, $@_a\langle S\rangle(\varphi \wedge \bigwedge_{b\in A} \neg b)) \in \delta(w, a)$. By $\mathcal{N}$ satisfies @ clause of the coherency definition, $@_a\langle S\rangle(\varphi \wedge \bigwedge_{b\in A} \neg b)) \in \delta(w, b)$ for any $b \in A$.

By $@_{n_1}\varphi_1 \ldots @_{n_k}\varphi_k$ are all in $@\delta(w, a)$, they are all in $\delta(w, a)$ and so is the conjunction. Let θ denote this conjunction.

Then $@_m(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \in \delta(w, a)$ by the above theorem. By $\mathcal{N}$ is coherent, $@_m(@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k) \in \delta(w, b)$.

Inductive Case: $\Gamma_{w_{j+1}}$

Let $@_{n_1}\varphi_1 \ldots @_{n_k}\varphi_k$ be in $@\delta(w_{j+1}, a)$ and $\psi_1 \ldots \psi_k$ be in Γ_{w_j}.

So $\bigwedge_{1\leq l\leq k} (@_{n_l}\varphi_l \wedge @_{a_j}\langle K\rangle\psi_l)$ is a conjunction in $\Gamma_{w_{j+1}}$.

Let $\theta = (@_{n_1}\varphi_1 \wedge \ldots \wedge @_{n_k}\varphi_k)$. By Taut, we have

$$\vdash \theta \land (@_{a_j}\langle K\rangle\psi_1 \land \ldots \land @_{a_j}\langle K\rangle\psi_j) \to \bigwedge_{1\leq l\leq k}(@_{n_l}\varphi_l \land @_{a_j}\langle K\rangle\psi_l)$$

Similarly to the base case,

$$\vdash @_m\theta \land (@_{a_j}\langle K\rangle\psi_1 \land \ldots \land @_{a_j}\langle K\rangle\psi_j) \to \bigwedge_{1\leq l\leq k}(@_{n_l}\varphi_l \land @_{a_j}\langle K\rangle\psi_l)(1)$$

By inductive hypothesis, $\vdash @_t\psi \land \gamma \to (\psi_1 \land \ldots \land \psi_j)$ for some witnessed formula $(@_t\psi \land \gamma) \in \Gamma_{w_j}$ where $@_t\psi \in \delta(w_j, a_j)$ and γ', the paste of γ, is also in $\delta(w_j, a_j)$. By Theorem 3.4.2, we have

$$\vdash @_{a_j}\langle K\rangle(@_t\psi \land \gamma) \to @_{a_j}\langle K\rangle(\psi_1 \land \ldots \land \psi_j)$$

Since $\vdash @_{a_j}\langle K\rangle(\psi_1 \land \ldots \land \psi_j) \to @_{a_j}\langle K\rangle\psi_1 \land \ldots \land @_{a_j}\langle K\rangle\psi_k$, we have

$$\vdash @_{a_j}\langle K\rangle(@_t\psi \land \gamma) \to @_{a_j}\langle K\rangle\psi_1 \land \ldots \land @_{a_j}\langle K\rangle\psi_k$$

Then by Taut,

$$\vdash @_m\theta \land @_{a_j}\langle K\rangle(@_t\psi \land \gamma) \to @_m\theta \land @_{a_j}\langle K\rangle\psi_1 \land \ldots \land @_{a_j}\langle K\rangle\psi_k$$

By (1) and Taut,

$$\vdash @_m\theta \land @_{a_j}\langle K\rangle(@_t\psi \land \gamma) \to \bigwedge_{1\leq l\leq k}(@_{n_l}\varphi_l \land @_{a_j}\langle K\rangle\psi_l)$$

Since θ is a conjunction of $@\delta(w_{j+1}, a)$, θ is in $\delta(w_{j+1}, a)$ and so is $@_m\theta$. Then $@_m\theta \in \delta(w_{j+1}, b)$ for any $b \in A$ by $\mathcal{N}$ is coherent. By inductive hypothesis, $(@_t\psi \land \gamma) \in \Gamma_{w_j}$. Then $@_m\theta \land @_{a_j}\langle K\rangle(@_t\psi \land \gamma) \in \Gamma_{w_{j+1}}$ by the definition of $\Gamma_{w_{j+1}}$.

Now we prove $@_{a_j}\langle K\rangle(@_t\psi \land \gamma')$, the paste of $@_{a_j}\langle K\rangle(@_t\psi \land \gamma)$, is in $\delta(w_{j+1}, b)$.

By $\gamma' \in \delta(w_j, a_j)$ and $@_t\psi \in \delta(w_j, a_j)$, $@_t\psi \land \gamma' \in \delta(w_j, a_j)$. Since $\mathcal{N}$ is coherent and $\mathsf{k}_{a_j}(w_{j+1}, w_j)$, $@_{a_j}\langle K\rangle(@_t\psi \land \gamma') \in \delta(w_{j+1}, a_j)$. But it is an $@$ formula, so it is in $\delta(w_{j+1}, b)$ by $\mathcal{N}$ is coherent.

♣

Fact 3: Each Γ_{w_j} is consistent.

Proof:

We prove it by induction on j.

Base Case: Γ_{w_0}

For contradiction suppose it is not. Then there exists a conjunction θ of $@\delta(w_0, a)$ such that

$$\vdash \neg(\theta \wedge @_a(\langle S\rangle i \wedge @_i(\varphi \wedge \bigwedge_{b\in A} \neg b)))$$

Then we have

$$\vdash @_a(\langle S\rangle i \wedge @_i(\varphi \wedge \bigwedge_{b\in A} \neg b)) \rightarrow \neg\theta$$

The formula $@_a(\langle S\rangle i \wedge @_i(\varphi \wedge \bigwedge_{b\in A} \neg b))$ is a $\langle K\rangle$-formula because the witness i doesn't occur in θ and $i \neq a$. Then by $\langle \mathsf{K}\rangle$paste,

$$\vdash @_a\langle S\rangle(\varphi \wedge \bigwedge_{b\in A} \neg b) \rightarrow \neg\theta$$

By Fact 1, $\langle S\rangle(\varphi \wedge \bigwedge_{b\in A} \neg b) \in \delta(w, a)$. By $a \in \delta(w, a)$ and Intro, $@_a\langle S\rangle(\varphi \wedge \bigwedge_{b\in A} \neg b) \in \delta(w, a)$. So $\neg\theta \in \delta(w, a)$. But it contradicts the fact that θ is a conjunction of $@\delta(w_0, a)$, which is a subset of $\delta(w_0, a)$ where $w_0 = w$.

Inductive Case: $\Gamma_{w_{j+1}}$

For contradiction, suppose it is not. Then there exists $\psi_1 \ldots \psi_l \in @\delta(w_{j+1}, a)$ and $\theta_1 \ldots \theta_l \in \Gamma_{w_j}$ such that

$$\vdash \neg(\psi_1 \wedge @_{a_j}\langle K\rangle\theta_1 \ldots \wedge \psi_l \wedge @_{a_j}\langle K\rangle\theta_l)$$

By Taut, we have

$$\vdash (@_{a_j}\langle K\rangle\theta_1 \ldots \wedge @_{a_j}\langle K\rangle\theta_l) \rightarrow \neg(\psi_1 \wedge \ldots \wedge \psi_l)$$

Then by repeatedly applying the theorem $\vdash @_n\langle K\rangle(\varphi \wedge \psi) \rightarrow @_n\langle K\rangle\varphi \wedge @_n\langle K\rangle\psi$,

$$\vdash @_{a_j}\langle K\rangle(\theta_1 \wedge \ldots \wedge \theta_l) \rightarrow (@_{a_j}\langle K\rangle\theta_1 \ldots \wedge @_{a_j}\langle K\rangle\theta_l)$$

So

$$\vdash @_{a_j}\langle K\rangle(\theta_1 \wedge \ldots \wedge \theta_l) \to \neg(\psi_1 \wedge \ldots \wedge \psi_l)$$

Since $\theta_1 \wedge \ldots \wedge \theta_l$ is a conjunction of Γ_{w_j}, by Fact 2, we have

$$\vdash @_n\theta \wedge \gamma \to (\theta_1 \wedge \ldots \wedge \theta_l)$$

where $@_n\theta \wedge \gamma$ is a witnessed formula such that $@_n\theta \in \delta(w_j, a_j)$ and γ', the paste of γ, is in $\delta(w_j, a_j)$ too. So

$$\vdash @_j\langle K\rangle(@_n\theta \wedge \gamma) \to @_j\langle K\rangle(\theta_1 \wedge \ldots \wedge \theta_l)$$

Then we have

$$\vdash @_j\langle K\rangle(@_n\theta \wedge \gamma) \to \neg(\psi_1 \wedge \ldots \wedge \psi_l)$$

Obviously, $@_j\langle K\rangle(@_n\theta \wedge \gamma)$ is a $\langle K\rangle$-formula. Then by $\langle \mathsf{K}\rangle$Paste,

$$\vdash @_j\langle K\rangle(@_n\theta \wedge \gamma') \to \neg(\psi_1 \wedge \ldots \wedge \psi_l)$$

By the fact that $(@_n\theta \wedge \gamma') \in \delta(w_j, a_j)$, $\mathsf{k}_{a_j}(w_{j+1}, w_j)$ and $\mathcal{N}$ satisfies k and $@$ clauses of coherency, $@_j\langle K\rangle(@_n\theta \wedge \gamma') \in \delta(w_{j+1}, a_j)$ and then

$@_j\langle K\rangle(@_n\theta \wedge \gamma') \in \delta(w_{j+1}, a)$.

So $\neg(\psi_1 \wedge \ldots \wedge \psi_l) \in \delta(w_{j+1}, a)$. Then $(\psi_1 \wedge \ldots \wedge \psi_l) \notin \delta(w_{j+1}, a)$. By $@\delta(w_{j+1}, a) \subseteq \delta(w_{j+1}, a)$, $(\psi_1 \wedge \ldots \wedge \psi_l) \notin @\delta(w_{j+1}, a)$. Contradiction.

♣

Now we are ready to discuss the properties of Δ_v.

Fact 4: For any $v \in W$, Δ_v is locally maximal consistent.

Proof:

Prove by induction on v.

Base case: $v = 0$

$\Delta_0 = l(\Gamma_{w_k})$. By Fact 3, Γ_{w_k} is consistent. So Lindenbuam construction can be applied.

Inductive case: $v = j + 1$

By the definition, we have two cases to discuss:

Case One: $\Delta_{j+1} = l(\Gamma_{w_u} \cup @_b K^- \Delta_t)$ where $\vec{\mathsf{k}}(j+1, w)$, $w_u = j+1$ and $t \in W$ is the k-predecessor of $j+1$ in $\mathcal{N}$ such that $\mathsf{k}_b(t, j+1)$ for some $b \in A$.

For contradiction, suppose it is not consistent. Let θ be a conjunction of Γ_{w_u} and $@_b K\psi_1, \ldots, @_b K\psi_k \in \Delta_t$. Since Γ_{w_u} is consistent, we have

$$\vdash \neg(\psi_1 \wedge \ldots \wedge \psi_k \wedge \theta)$$

By Taut,

$$\vdash (\psi_1 \wedge \ldots \wedge \psi_k) \to \neg\theta$$

Then we apply $\mathsf{Nec_K}$, $\mathsf{Nec_@}$, $\mathsf{K_K}$ and MP to get

$$\vdash @_b K(\psi_1 \wedge \ldots \wedge \psi_k) \to @_b K\neg\theta$$

By repeatedly applying theorem $\vdash @_b K\varphi \wedge @_b K\psi \to @_b K(\varphi \wedge \psi)$, we have

$$\vdash @_b K\psi_1 \wedge \ldots \wedge @_b K\psi_k \to @_b K(\psi_1 \wedge \ldots \wedge \psi_k)$$

So

$$\vdash @_b K\psi_1 \wedge \ldots \wedge @_b K\psi_k \to @_b K\neg\theta$$

By $@_b K\psi_l \in \Delta_t$ for each $1 \leq l \leq k$ and Δ_t is maximal by inductive hypothesis, $@_b K\psi_1 \wedge \ldots \wedge @_b K\psi_k \in \Delta_t$. So $@_b K\neg\theta \in \Delta_t$. (1)

By Fact 2, there exists a witness formula $(@_m\rho \wedge \psi) \in \Gamma_{w_u}$ such that

$$\vdash (@_m\rho \wedge \psi) \to \theta$$

Similar to the above, we have

$$\vdash @_b\langle K\rangle(@_m\rho \wedge \psi) \to @_b\langle K\rangle\theta$$

By (1), $@_b\langle K\rangle(@_m\rho\wedge\psi) \notin \Delta_t$. So $(@_m\rho\wedge\psi) \notin \Gamma_{w_u}$ and contradiction. And we are done. Why? If $(@_m\rho \wedge \psi) \in \Gamma_{w_u}$, then we have $@_m\top \wedge @_b\langle K\rangle(@_m\rho \wedge \psi) \in \Gamma_t$ by the definition of Γ_t. And $\Gamma_t \subseteq \Delta_t$ since Δ_t was constructed by the first clause of the definition by $\vec{\mathsf{k}}(t, w)$. So $@_b\langle K\rangle(@_m\rho \wedge \psi) \in \Delta_t$, which is a contradiction.

Case Two: $\Delta_{j+1} = l(@\delta(j+1, a) \cup @_b K^- \Delta_t)$ where $\mathsf{k}_b(t, j+1)$.

Suppose it is not consistent. Then there exists a conjunction θ in $@\delta(j+1, a)$ and $@_b K\psi_1, \ldots, @_b K\psi_k \in \Delta_t$ such that

$$\vdash \neg(\psi_1 \wedge \ldots \wedge \psi_k \wedge \theta)$$

With the same proof to Case One, we have $@_b K\neg\theta \in \Delta_t$.
We also have $\vdash @_b\langle K\rangle\theta \to @_b\langle K\rangle\neg(\psi_1 \wedge \ldots \wedge \psi_k)$. (2)
By $\theta \in @\delta(j+1,a)$ and $\mathcal{N}$ is coherent, $\theta \in @\delta(j+1,b)$.
So $\theta \in \delta(j+1,b)$. By $\mathsf{k}_b(t,j+1)$ and the fact that $\mathcal{N}$ satisfies k clause of the coherency, $@_b\langle K\rangle\theta \in \delta(t,b)$. It is an @-formula, so it is in $@\delta(t,a)$.
If $\vec{\mathsf{k}}(t,w)$, then $@\delta(t,a) \subseteq \Gamma_t \subseteq \Delta_t$. Otherwise, $@\delta(t,a) \subseteq \Delta_t$.
So $@\delta(t,a) \subseteq \Delta_t$. Then $@_b\langle K\rangle\theta \in \Delta_t$.
By (2), we have $@_b\langle K\rangle\neg(\psi_1 \wedge \ldots \wedge \psi_k) \in \Delta_t$. This is to say $@_b K(\psi_1 \wedge \psi_k) \notin \Delta_t$. So $(@_b K\psi_1 \wedge \ldots \wedge @_b K\psi_k) \notin \Delta_t$. But this means $@_b K\psi_l \notin \Delta_t$ for some $1 \leq l \leq k$. Contradiction.

♣

Fact 5: If Δ_v is locally maximal consistent, then $\delta'(v,b)$ is also locally maximal consistent.

Proof:

Straightforward by Lemma 4.5.4.

♣

Fact 6: For any $v \in W$ and $b \in A$, $\delta(v,b) \subseteq \delta'(v,b)$.

Proof: Let $\psi \in \delta(v,b)$ for any $v \in W$ and $b \in A$. Then we have $@_b\psi \in \delta(v,b)$ by $b \in \delta(v,b)$ and Intro Axiom. So $@_b\psi \in \delta(v,a)$ by $\mathcal{N}$ is coherent. Then $@_b\psi \in @\delta(v,0) \subseteq \Delta_v$. Also, $@\delta(v,a) \subseteq \Gamma_v \subseteq \Delta_v$. So $@_b\psi \in \Delta_v$ in either case of the definition of Δ_v. So $\psi \in \delta'(v,b)$.

♣

Fact 7: For any $v \in W$ and $c \in A \cup \{i\}$, we have $@_i \bigwedge\limits_{b\in A} \neg b \in \delta'(v,c)$.

Proof:

If we can prove $@_i \bigwedge\limits_{b\in A} \neg b \in \Delta_v$ for all $v \in W$ such that $\vec{\mathsf{k}}(v,w)$, then we have $@_i \bigwedge\limits_{b\in A} \neg b \in \Delta_0$. Then by the fourth of the theorems listed previously, we have $@[K]@_i \bigwedge\limits_{b\in A} \neg b \in \Delta_0$ for any modal operator sequences $@[K]$.

So for any $v \in W$, $@_i \bigwedge\limits_{b\in A} \neg b \in \Delta_v$ since $\vec{\mathsf{k}}(0,v)$. Then $@_c@_i \bigwedge\limits_{b\in A} \neg b \in \Delta_v$ by Agree Axiom. So $@_i \bigwedge\limits_{b\in A} \neg b \in \delta'(v,c)$.

Here is an induction proof of $@_i \bigwedge_{b \in A} \neg b \in \Delta_v$ for all v such that $\vec{\mathsf{k}}(v, w)$.

Base Case: $|\vec{\mathsf{k}}(v, w)| = 0$

So $v = w$. By the definition of Γ_w and the fact that $\Gamma_w \subseteq \Delta_w$,

$\psi \wedge @_a(\langle S \rangle i \wedge @_i \bigwedge_{b \in A} \neg b) \in \Delta_w$ for some $\psi \in @\delta(w, a)$.

So $@_a(\langle S \rangle i \wedge @_i \bigwedge_{b \in A} \neg b) \in \Delta_w$.

By the theorem $\vdash @_n(\varphi \wedge \psi) \to @_n \varphi \wedge @_n \psi$, $@_a @_i \bigwedge_{b \in A} \neg b \in \Delta_w$.

Then by Agree, $@_i \bigwedge_{b \in A} \neg b \in \Delta_w$.

Inductive Case: $|\vec{\mathsf{k}}(v, w)| = j + 1$

By Inductive Hypothesis, $@_i \bigwedge_{b \in A} \neg b \in \Delta_u$, where $\mathsf{k}_c(v, u)$ for some $c \in A$ and $|\vec{\mathsf{k}}(u, w)| = j$. Suppose $@_i \bigwedge_{b \in A} \neg b \notin \Delta_v$. Then $@_i b_j \in \Delta_v$ for some $b_j \in A$. By $\mathsf{Rig}_@$, $@_c K @_i b_j \in \Delta_v$. So $@_i b_j \in \Delta_u$ by the construction method of Δ_u. Then it contradicts the Inductive Hypothesis.

♣

Fact 8: $[w, a, \langle S \rangle \varphi]\mathcal{N}$ is a finite modest tree network.

Proof: Since $\mathcal{N}$ is finite, W and $A \cup \{i\}$ are finite. So $[w, a, \langle S \rangle \varphi]\mathcal{N}$ is finite.

By $\mathcal{N}$ is modest, every $L(v, b)$ of $\mathcal{N}$ is modest. Since we only add i to the languages of $[w, a, \langle S \rangle \varphi]\mathcal{N}$, we do not destroy the modest property. So $[w, a, \langle S \rangle \varphi]\mathcal{N}$ is modest.

$[w, a, \langle S \rangle \varphi]\mathcal{N}$ is a tree network because it shares W and k with $\mathcal{N}$.

By $\mathcal{N}$ is a network, to show $[w, a, \langle S \rangle \varphi]\mathcal{N}$ is also a network only requires: (1) $\delta'(v, b)$ is locally maximal consistent for every $v \in W$ and $b \in A \cup \{i\}$;

(2) $b \in \delta'(v, b)$ for any $v \in W$ and $b \in A \cup \{i\}$.

Obviously, (1) is just the result of Fact 4 and 5. Since $\mathcal{N}$ is a network, $b \in \delta(v, b)$ for each $v \in W$ and $b \in A$. By Fact 6, this is kept by δ'. So we only need to show the addition i satisfies this, e.g. $i \in \delta'(v, i)$. But it is obvious since $\vdash @_i i$, so $@_i i \in \Delta_v$ for any $v \in W$.

♣

Fact 9: $[w, a, \langle S \rangle \varphi]\mathcal{N}$ is coherent.

Proof: We check each clause of coherency.

(k)

Suppose $\mathsf{k}_b(u,v)$ and $K\psi \in \delta'(u,b)$ for any $b \in A \cup \{i\}$.
Then $@_b\ K\psi \in \Delta_u$. By the theorem $\vdash @_xK\varphi \to @_xK@_x\varphi$ and Δ_u is maximal, $@_bK@_b\psi \in \Delta_u$. Then by the definition of Δ_v, we have $@_b\psi \in \Delta_v$. So $\psi \in \delta'(v,b)$.

(n)

By $\mathcal{N}$ is coherent, we only need to guarantee that $i \notin \delta'(v,b)$ and $b \notin \delta'(v,i)$ for any $v \in W$ and $b \in A$. But it is obvious by Fact 7 and the Symmetric Axiom.

(s)

By $\mathcal{N}$ satisfies (s), we only need to check that $S\psi \in \delta'(w,a)$ implies

$\psi \in \delta'(w,i)$. Suppose not. Then $S\psi \in \delta'(w,a)$ but $\psi \notin \delta'(w,i)$ for some ψ.

By $\delta'(w,i)$ is maximal, $\neg\psi \in \delta'(w,i)$. So $@_i\neg\psi \in \Delta_w$. Then $@_a@_i\neg\psi \in \Delta_w$ by Intro Axiom. So $@_i\neg\psi \in \delta'(w,a)$. By $@_a\langle S\rangle i \in \Gamma_{w_0}$, we have $@_a\langle S\rangle i \in \Delta_w$. So $\langle S\rangle i \in \delta'(w,a)$. By the theorem $\vdash S\varphi \to (\langle S\rangle n \to @_n\psi)$, $\langle S\rangle\neg\psi \in \delta'(w,a)$. So $\neg S\psi \in \delta'(w,a)$. The $S\psi \notin \delta'(w,a)$. Contradiction.

(@)

Suppose $@_n\psi \in \delta'(v,b)$, we prove $(n \to \psi) \in \delta'(v,c)$ for any $c \in A \cup \{i\}$. We have $@_b@_n\psi \in \Delta_v$. Then by Agree, $@_n\psi \in \Delta_v$. Again by Agree, we have $@_c@_n\psi \in \Delta_v$ and so $@_n\psi \in \delta'(v,c)$. By Intro and $\delta'(v,c)$ is locally maximal consistent, $(n \to \psi) \in \delta'(v,c)$.

(x)

Suppose $x \in \delta'(v,b)$, we prove $x \in \delta'(u,b)$ for any $u \in W$.

For contradiction, suppose $x \notin \delta'(u,b)$ for some u. Then $\neg x \in \delta'(u,b)$. By $b \in \delta'(u,b)$ and Intro, $@_b\neg x \in \delta'(u,b)$ (1)

Similarly by $x \in \delta'(v,b)$, we have $@_bx \in \delta'(v,b)$. (2)

Either $@_bx \in \delta'(0,b)$ or $@_bx \notin \delta'(0,b)$. We discuss it by cases.

Case One: $@_bx \in \delta'(0,b)$. Then by the theorem $\vdash @_bx \to @\vec{K}@_bx$ where $@\vec{K}$ is an arbitrary model operator sequence, we have $\overset{0\to u}{@K}\ @_bx \in \delta'(0,b)$.

By $[w,a,\langle S\rangle\varphi]\mathcal{N}$ satisfies (k), $(@)$ and $\vec{\mathsf{k}}(0,u)$, we have $@_bx \in \delta'(u,b)$, which contradicts to (1).

Case Two: $@_bx \notin \delta'(0,b)$

Similarly, $\overset{0\to v}{@K}\ @_b\neg x \in \delta'(0,b)$.

So again by $\vec{\mathsf{k}}(0,v)$, $@_b\neg x \in \delta'(v,b)$, which contradicts to (2).

♣

Now we discuss how to repair each kind of potential defect.

Lemma 4.5.5 *Any S defect* $[w, a, \langle S\rangle\varphi]$ *in a finite modest coherent tree network* $\mathcal{N}$ *has a repair which is also modest and a tree network.*

Proof:

Suppose $[w, a, \langle S\rangle\varphi]$ is a defect of a finite modest coherent tree network

$\mathcal{N} = \langle W, A, \mathsf{k}, \mathsf{s}, \delta\rangle$. We discuss $[w, a, \langle S\rangle\varphi]$ by two cases, very similar to what we have done for repairing an S defect in $\mathbf{BSEL}$.

Case One: $\langle S\rangle(m \wedge \varphi) \in \delta(w, a)$ and $m \in \delta(w, b)$ for some $m \in \mathsf{Nom}$ and $b \in A$.

It is possible that there are more than one m satisfying the above condition. So we pick up the smallest b such that $m \in \delta(w, b)$.

Then we define $[w, a, \langle S\rangle\varphi]\mathcal{N} = \langle W, A, \mathsf{k}, \mathsf{s}', \delta\rangle$

where $\mathsf{s}'_w = \begin{cases} \mathsf{s}_v \cup \{\langle a, b\rangle\} & \text{if } v = w \\ \mathsf{s}_v & \text{Otherwise} \end{cases}$.

Obviously, $\mathcal{N} \trianglelefteq [w, a, \langle S\rangle\varphi]\mathcal{N}$ and $[w, a, \langle S\rangle\varphi]\mathcal{N}$ doesn't have the defect $[w, a, \langle S\rangle\varphi]$.

Since $\mathcal{N}$ is coherent and the only upgrade is s', we only need to show that $S\psi \in \delta(w, a)$ implies that $\psi \in \delta(w, b)$ for any $\psi \in L(w, a)$. Recall that we have proven that $L(w, a) = L(w, b)$. For contradiction, suppose not. Then $S\psi \in \delta(w, a)$ but $\psi \notin \delta(w, b)$. Then $\neg\psi \in \delta(w, b)$. So $@_m\neg\psi \in \delta(w, b)$ by $m \in \delta(w, b)$. By $\mathcal{N}$ is coherent, $@_m\neg\psi \in \delta(w, a)$. By $\langle S\rangle(\varphi \wedge m) \in \delta(w, a)$, $\langle S\rangle m \in \delta(w, a)$. So by $\vdash (\langle S\rangle m \wedge @_m\neg\psi) \to \langle S\rangle\neg\psi$, $\langle S\rangle\neg\psi \in \delta(w, a)$. Then $\neg S\psi \in \delta(w, a)$. So $S\psi \notin \delta(w, a)$. Contradiction.

Since $[w, a, \langle S\rangle\varphi]\mathcal{N}$ is coherent, it is a repair of $[w, a, \langle S\rangle\varphi]$ in $\mathcal{N}$. Also it is a modest tree network.

Case Two: Otherwise. So for any $m \in \mathsf{Nom}$ and $b \in A$, $\langle S\rangle(m \wedge \varphi) \in \delta(w, a)$ implies $m \notin \delta(w, b)$.

Then we define $[w, a, \langle S\rangle\varphi]\mathcal{N}$ with Definition 4.5.6.

Now we check whether it still has the defect.

We have $@_i\varphi \in \Gamma_w$ and $\Gamma_w \subseteq \Delta_w$. So $@_i\varphi \in \Delta_w$. So $\varphi \in \delta'(w, i)$. Also $\mathsf{s}'_w(a, i)$. So $[w, a, \langle S\rangle\varphi]$ is now not a defect.

By Fact 8 and 9, $[w, a, \langle S\rangle\varphi]\mathcal{N}$ is a repair of $[w, a, \langle S\rangle\varphi]$ and a modest tree network.

♣

Lemma 4.5.6 *Any $@_n$ defect $[w, a, @_n\varphi]$ in a finite modest coherent tree network $\mathcal{N}$ has a repair which is also modest and a tree network.*

Proof:

Define $[w, a, @_n\varphi]\mathcal{N} = \langle W, A \cup \{i\}, \mathsf{k}, \mathsf{s}, \delta' \rangle$, where i is a fresh rigid name for each $L(v, b)(v \in W, b \in A)$. Similar to the discussion we made before, such i exists.

To define δ', we will follow a very similar method to that we used in Definition 4.5.6:

> **Step One:**
>
> Let $w_k a_{k-1} \dots w_1 a_0 w_0$ be the unique k path from 0 to w such that $w_k = 0$ and $w_0 = w$. By Lemma 4.5.1, such k path exists. For each w_j in this path, we inductively define Γ_{w_j} as follows:
>
> $\Gamma_{w_0} = \{\psi \wedge (@_n i \wedge @_i \varphi) \mid \psi \in @\delta(w_0, a)\}$, and
>
> $\Gamma_{w_{j+1}} = \{\psi \wedge @_{a_j} \langle K \rangle \theta \mid \psi \in @\delta(w_{j+1}, a), \theta \in \Gamma_{w_j}\}$.
>
> Each Γ_{w_j} is a set of witnessed formulas.
>
> **Step Two:**
>
> For each $v \in W$, Δ_v is defined the same as in Definition 4.5.6.
>
> **Step Three:**
>
> For every $b \in A \cup \{i\}$, we define δ' as in Definition 4.5.6.

Fact 1: $[w, a, @_n\varphi]\mathcal{N}$ is a finite modest coherent tree network.

Proof of this fact:

Here we collect the witnesses for each Γ with $\langle @ \rangle$-formula instead of $\langle K \rangle$-formula in Defintion 4.5.6. Since both of them share $\langle \mathsf{K} \rangle \mathsf{Paste\ Rule}$, every result for Γ of Step One in Definition 4.5.6 will be kept here. Also, this is the only difference from Definition 4.5.6. So we can borrow every result of Definition 4.5.6.

Fact 2: $[w, a, @_n\varphi]\mathcal{N}$ doesn't have the fact $[w, a, @_n\varphi]$.

Proof of this fact:

By $(@_n i \wedge @_i \varphi) \in \Gamma_{w_0}$ and $\Gamma_{w_0} \subseteq \Delta_w$, $(@_n i \wedge @_i \varphi) \in \Delta_w$. So $@_n i \in \Delta_w$ and $@_i \varphi \in \Delta_w$. By $\mathsf{Ref@}$, $@_i n \in \Delta_w$. So $n \in \delta'(w, i)$ and $\varphi \in \delta'(w, i)$.

♣

Lemma 4.5.7 *Any K defect $[w, a, \langle K \rangle \varphi]$ in a finite modest coherent tree network $\mathcal{N}$ has a repair which is also modest and a tree network.*

Proof:

Define $[w, a, \langle K\rangle\varphi]\mathcal{N} = \langle W \cup \{w^\dagger\}, A, \mathsf{k}', \mathsf{s}, \delta'\rangle$ where

$\mathsf{k}'_b = \begin{cases} \mathsf{k}_b \cup \{\langle w, w^\dagger\rangle\} & \text{if } b = a \\ \mathsf{k}_b & \text{Otherwise} \end{cases}$

$\Delta_{w^\dagger} = l(\{@_a\varphi\} \cup \{@_a\psi \mid K\psi \in \delta(w, a)\})$

Then for all $b \in A$, define

$$\delta'(v, b) = \begin{cases} \{\psi \mid @_b\psi \in \Delta_{w^\dagger}\} & \text{if } v = w^\dagger \\ \delta(v, b) & \text{Otherwise} \end{cases}$$

where $w^\dagger$ is the successor of the largest number in W.

In **BSEL**, we have proven that $\{@_a\varphi\} \cup \{@_a\psi \mid K\psi \in \delta(w, a)\}$ is consistent. So $\Delta_{w^\dagger}$ exists, as a local maximal consistent set because of the different version of Lindenbaum Lemma here.

To prove it lacks the defect, we only need $\varphi \in \delta'(w^\dagger, a)$. But it is obvious since $@_a\varphi \in \Delta_{w^\dagger}$.

Now we prove $[w, a, \langle K\rangle\varphi]\mathcal{N}$ is coherent. Since $\mathcal{N}$ is coherent and $\delta'(v, b) = \delta(v, b)$ for $v \neq w'$, we only need to check whether the $w^\dagger$ world has destroyed the coherency. We only show $w^\dagger$ world keeps (n) since the other clauses are easily to check.

Suppose $n \in \delta'(w^\dagger, b)$. For contradiction, suppose $\neg n \notin \delta'(w^\dagger, c)$ for some $c \neq b$. So $n \in \delta'(w^\dagger, c)$. Then $@_cn \in \Delta_{w^\dagger}$. By $n \in \delta'(w^\dagger, b)$, $@_bn \in \delta'(w^\dagger, b)$. By $\mathsf{Ref@}$, $@_nb \in \delta'(w^\dagger, b)$. Since we have the theorem $\vdash @_nm \wedge @_m\varphi \rightarrow @_n\varphi$, $@_cb \in \delta'(w^\dagger, b)$. So $@_b@_cb \in \Delta_{w^\dagger}$. Then by Agree, $@_a@_cb \in \Delta_{w^\dagger}$. Then $K\neg@_cb \notin \delta(w, a)$. So $\langle K\rangle@_cb \in \delta(w, a)$. So $@_cb \in \delta(w, a)$ by $\mathsf{Rig\neg@}$. Then $b \in \delta(c, b)$ and $c \in \delta(c, b)$. But $c \neq b$. Contradicts to the fact that $\mathcal{N}$ satisfies (x).

♣

Lemma 4.5.8 *Any finite modest coherent tree network with a defect has a repair which is also modest and a tree network.*

Proof: By Lemma 4.5.5, 4.5.6 and 4.5.7.

♣

Recall that we defined the extension of a network in Definition 4.5.3. Now we discuss the properties of $\trianglelefteq$.

The next lemma is to say that the formula will not be lost during the extension.

Lemma 4.5.9 *If $\mathcal{N} \trianglelefteq \mathcal{N}'$, then for every $\varphi \in L(w,a)$ where (w,a) is a node of $\mathcal{N}$ we have*

$\varphi \in \delta(w,a)$ *iff* $\varphi \in \delta'(w,a)$

Proof: The left to right direction is directly from the definition of $\trianglelefteq$. The other direction follows from the fact that $\delta(w,a)$ is a locally maximal consistent set.

♣

The following lemma is to say that any defect in the extension is definitely in the original network.

Lemma 4.5.10 *If $\mathcal{N} \trianglelefteq \mathcal{N}'$ and $\varphi \in L(w,a)$ where (w,a) is a node of $\mathcal{N}$, if $[w,a,\varphi]$ is a defect of $\mathcal{N}'$, then it must be a defect of $\mathcal{N}$.*

Proof: This lemma expresses the same idea as expressed by Lemma 3.4.10. So the proof is also the same.

♣

Now we repeat the Limit Coherence Lemma and Perfect Extension Lemma of $\mathbf{BSEL}$.

Lemma 4.5.11 (Limit Coherence) *If $\mathcal{N}_0 \trianglelefteq \mathcal{N}_1 \trianglelefteq \ldots$ is a chain of coherent networks then the limit of the chain is also coherent.*

Proof: A similar proof to Lemma 3.4.11.

♣

Lemma 4.5.12 (Perfect Extension) *If $\mathcal{N}$ is coherent then there is a perfect $\mathcal{N}'$ such that $\mathcal{N} \trianglelefteq \mathcal{N}'$.*

Proof: A similar proof to Lemma 3.4.12.

♣

With the Extended Truth Lemma we have proved, we have the following nice lemma:

Lemma 4.5.13 *Every modest consistent set of formulas is satisfiable.*

Proof: Suppose Γ is modest consistent. We pick up a rigid name a that not in any $\varphi \in \Gamma$. We define a network $\mathcal{N}_\Gamma = \langle W, A, \mathsf{k}, \mathsf{s}, \delta \rangle$ as follows:

$W = \{0\}$

$A = \{a\}$

$\mathsf{s} = \mathsf{k} = \emptyset$

$\delta(0, a) = l(\{a\} \cup \Gamma)$

Notice that $\{a\} \cup \Gamma$ is consistent since Γ is consistent and $\mathsf{LeftName}$. Also $a \in \delta(0, a)$. So $\mathcal{N}_\Gamma$ is a finite network. Also it is coherent. Since Γ is modest, $\{a\} \cup \Gamma$ is modest. So $\mathcal{N}_\Gamma$ is modest. Finally, $\mathcal{N}_\Gamma$ is tree network. Then by the Perfect Extension Lemma, we extend $\mathcal{N}_\Gamma$ to a perfect network N'. Then $\Gamma \subseteq \delta_{N'}(0, a)$. By the Extended Truth Lemma, $M_{N'}, 0, a \models \varphi$ for every $\varphi \in \Gamma$.

♣

So far, we have just assumed that the modest set of formula exists. We have not yet shown how to obtain it. The next theorem tells us how.

Theorem 4.5.1 (Model Existence) *Every consistent set of formulas is satisfiable.*

Proof: Suppose Γ is consistent. We now show how to make it be modest. Let B be an infinite and co-infinite subset of RNom. So both B and $\mathsf{RNom}\backslash B$ are infinite. Now define $f : \mathsf{RNom} \to B$ be a bijection. For each φ, we define $f[\varphi]$ to be the result of replacing every rigid name x in φ by $f(x)$. Both free and rigid names are replaced at the same time. Then let $f[\Gamma]$ denote the set of $f[\varphi]$ for each $\varphi \in \Gamma$.

To show $f[\Gamma]$ is consistent, for contradiction we suppose there is a proof of contradiction from some formulas in $f[\Gamma]$. Since our axioms and rules is schematic, they are invariant under renaming. So if we replace each formula ψ of each line of the proof of contradiction with $f^{-1}[\psi]$, the obtained proof will be still a proof of contradiction. However, $f^{-1}[f[\Gamma]] = \Gamma$. But this means we have a proof of contradiction from Γ which is consistent. Contradiction. So $f[\Gamma]$ is consistent.

Now we check whether $f[\Gamma]$ is modest. Since every bound name of each formula in $f[\Gamma]$ has been renamed by some rigid name in B, which is disjointed with $\mathsf{RNom}\backslash B$, each formula in $f[\Gamma]$ contains no name of $\mathsf{RNom}\backslash B$. But $\mathsf{RNom}\backslash B$ is an infinite rigid name set. So $f[\Gamma]$ is modest. Then by Lemma 4.5.13, $f[\varphi]$ is satisfiable, i.e., $M, w, a \models f[\varphi]$ for each $\varphi \in \Gamma$.

To jump from the fact that $f[\varphi]$ is satisfiable to the desired result that φ is satisfiable, we define $M' = \langle W, A, k, s, g', V \rangle$ where

$$g'_v(n) = \begin{cases} g_v(n) & \text{if } n \notin \mathsf{RNom} \\ g_v(f(n)) & \text{Otherwise} \end{cases}$$

Then it is easy to show by induction that

$$M, w, a \models f[\varphi] \quad \text{iff} \quad M', w, a \models \varphi$$

And we are done.

♣

Corollary 4.5.1 (Completeness) *If φ is a semantic consequence of a set of formulas Γ, then this can be proved in the axiomatic system* $\mathbf{SEL}$*, i.e.,* $\mathbf{SEL}$ *is strongly complete.*

4.6 Further Directions

The above is an axiomatisation of $\mathbf{SEL}$, the minimal version with downarrow. We still need to investigate the axiomatisations of the $\mathbf{S5}$, $\mathbf{KD45}$ and $\mathbf{SK}$[4] extensions. The main challenge is the repair methods for each extension. In the minimal case, the repair method is complicated because $\mathbf{SEL}$ is too weak to control the witness we add for a repair. I expect that repair will be more simple in the extensions. For example, with $\mathbf{S5}$, the Axiom 4 and 5 may offer better control of witnesses so that we don't need to define Γ_{w_j}.

An interesting subclass of $\mathbf{SEL}$ is the class of pure formulas. A pure formula is one that doesn't contain any atomic property symbols. As mentioned in the last section of Chapter 3, every pure extension of basic hybrid logic is complete. A similar result for $\mathbf{BSEL}$ (and $\mathbf{SEL}$) would give us axiomatisations of a number of extensions, without needing to modify the construction used for completeness. These include the SK and irreflexivity axioms discussed in Chapter 3.

[4]Recall that the $\mathbf{SK}$ extension is to characterise the interactions of seeing and knowing. The details can be found in Section 3.5.3

Chapter 5

Dynamic Social Epistemic Logic

In real social life, people reason according to what they see, hear and so on. Behind this, people are processing information that changes all the time. The possible-worlds framework used in previous chapters was initially only capable of handling a fairly static view of knowledge and belief. But in the last thirty years, logicians have gone on to study more dynamic features of communication and subsequent information change.

Aimed at investigating the behaviour of computer programmes, dynamic modal logic was developed in the early 1980s. Dynamic modal logic contains formulas of the form $[\pi]\varphi$, interpreted as "successfully executing programme π yields a φ state" [30]. By enriching the programmes for describing information change and adding epistemic operators to the language, we are ready for dynamic epistemic logic.

Van Benthem suggested an application of dynamic modal logic to model information change in [58]. The dynamic operators could be used to describe factual change at the level of propositions. He also suggested to improve the methodology of *belief revision* by the replacement of "theory change operators" with dynamic modal operators.

This was started by Plaza [46] who defined a logic for public announcements, characterising the logical consequences of making an announcement received by everyone with a common knowledge that the announcement has occurred. The operator $[!p]$ means "after publicly announcing p" with the condition that the announcement must be true. There is a general methodology to this approach: first, we choose a class of models to represent the relevant information structures together with some appropriate static language for describing them, such as standard epistemic logic. Next, an update mechanism is defined to represent actions that transform the static models. Usually these update mechanisms have presuppositions. For public announcement, this is that the announcement must be true. Then an elimination of all possible worlds that do not satisfy the announcement results in a definable sub-model. Subsequently, many papers have addressed more complicated forms of epistemic action such as private announcement [7], public suggestion [42], [7] and so on.

In this chapter, we firstly formally introduce *public announcement logic* ($\mathbf{PAL}$). Then we discuss the dynamic operators of dynamic social epistemic logic ($\mathbf{DSEL}$). We assume that all logics we are discussing in this chapter are based on $\mathsf{S5}$ frames. To explain them vividly, we present models of several scenarios from Chapter One.

5.1 Public Announcement

Recall my story about the All Blacks in Chapter Two. As a fan of the All Blacks I was disappointed because I had no way to know the outcome of the game while flying from China to New Zealand. Then came a surprise. The captain broadcast the result: the All Blacks just won! Recall the model "Win or lose" which is as the left part of Figure 5.1. It shows the situation before the captain's announcement. How to depict that I knew the All Blacks won (p)? Just by the fact that the $\neg p$ world is no longer possible after the update. The right-hand part of Figure 5.1 shows the result.

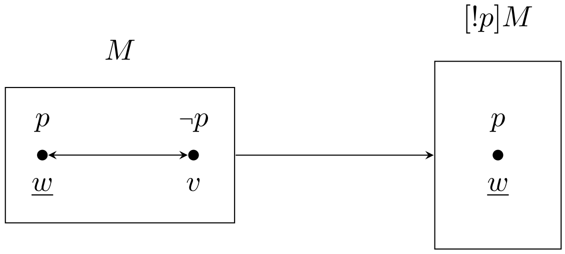


Figure 5.1: The result of publicly announcing p.

Let's explain it briefly. The left-hand model is just Figure 2.1, which shows my epistemic state before the captain's announcement. We use $[!p]$ to denote an announcement of p. Then $[!p]M$ is the model after the announcement. Because I learned that p, the $\neg p$ world is no longer considered. This leads to Kp at the actual world w. This example shows the methodology of $\mathbf{PAL}$. The class of formulas is defined in BNF as

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid [!\varphi]\varphi$$

where everything is the same as multi-agent epistemic logic except $[!\varphi]\psi$ which means "after a truthful announcement of φ, it holds that ψ." The dual of $[!\varphi]$ is $\langle !\varphi \rangle$. Then $\langle !\varphi \rangle\psi$ stands for "φ is truthfully announced and after that, ψ." Everything announced has to be true. Lies are not allowed when we accept the T axiom.

Given a Kripke model $M = \langle W, R, V \rangle$, the semantic definition of $[!\varphi]\psi$ is as follows:

$$M, w \models [!\varphi]\psi \quad \text{iff} \quad M, w \models \varphi \text{ implies } [!\varphi]M, w \models \psi$$

where $[!\varphi]M = \langle W', R', V' \rangle$ is defined as:

$$W' = \{v | M, v \models \varphi\}$$

$$R'_a = \{\langle w, v\rangle | R_a(w, v) \text{ and } w, v \in W'\}$$

$$V'(p) = \{w | w \in V(p) \text{ and } w \in W'\}$$

In other words, the model $[!\varphi]M$ is the restriction of M to those worlds that satisfy φ (in M), assuming that w is one of them.

We divide the axiomatisation of $\mathbf{PAL}$ into three parts. The system $\mathbf{S5}$ of basic epistemic logic that we introduced in Chapter 2 (Figure 2.2) is a sub-system of $\mathbf{PAL}$. The second part, which comprises all reduction axioms is shown in Figure 5.2. The reduction axioms state that what is the case after an announcement can be expressed by saying what is the case before the announcement. What are these reduction axioms for? They are introduced in order to "reduce" the announcement operators from a formula. By repeatedly applying the reduction axioms, any formula with an announcement operator can be proved to be equivalent to a formula of the basic epistemic language. But the second part cannot handle the iteration of announcement operators. That is why we need the third part, which consists of the axiom schema:

$\vdash [!\varphi][!\psi]\theta \leftrightarrow [!(\varphi \wedge [!\varphi]\psi)]\theta$ announcement$_{\mathsf{COM}}$

Adding this axiom schema is one of many possible ways of obtaining reduction in $\mathbf{PAL}$. A detailed investigation has been given by [62]. We basically have two strategies to prove the reduction: "inside-out" or "outside-in". The difference between them is the order of reducing dynamic operators from a formula. Take $[!p]Kp \rightarrow (q \wedge [!r]Kq)$ as an example. The inside-out method will reduce $[!r]$ first, then $[!p]$, while the outside-in method uses the reverse order. The main technical difference is that the outside-in method requires axioms, e.g. announcement$_{\mathsf{C}}$OM, to deal with the iteration of operators (e.g. $[!p][!r]$) whereas inside-out will require the rule of Replacement of Logical Equivalents:

From $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \theta \leftrightarrow \theta[\varphi/\psi]$

where $\theta[\varphi/\psi]$ denotes any formula obtained by replacing one or more occurrences of φ in θ with ψ. In this way, the reduction axioms provide as recursive definition of a translation of each formula φ in the language of $\mathbf{PAL}$ to a formula $T(\varphi)$ in the language of basic epistemic logic, such that the following lemma holds.

Lemma 5.1.1 (Reduction) *For all φ in the language of* $\mathbf{PAL}$*,*

$$\vdash \varphi \leftrightarrow T(\varphi)$$

$\vdash [!\varphi]p \leftrightarrow (\varphi \rightarrow p)$	atomic
$\vdash [!\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[!\varphi]\psi)$	$\mathsf{announcement}_\neg$
$\vdash [!\varphi](\psi \wedge \theta) \leftrightarrow ([!\varphi]\psi \wedge [!\varphi]\theta)$	$\mathsf{announcement}_\wedge$
$\vdash [!\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[!\varphi]\psi)$	$\mathsf{announcement}_\mathsf{K}$

Figure 5.2: Reduction theorems of $\mathbf{PAL}$

Also $\mathbf{PAL}$ is sound.

Theorem 5.1.1 (Soundness) *$\vdash \varphi$ implies $\models \varphi$, for every φ in the language of* $\mathbf{PAL}$.

Then we have the completeness of $\mathbf{PAL}$.

Theorem 5.1.2 (Completeness) *$\models \varphi$ implies $\vdash \varphi$, for each φ in the language of* $\mathbf{PAL}$.

Proof: Suppose $\models \varphi$. By the Soundness Theorem and of Reduction Lemma, $\models T(\varphi)$. Then by completeness of $\mathbf{S5}$, $T(\varphi)$ is a theorem of $\mathbf{S5}$. Since $\mathbf{S5}$ is a subsystem of $\mathbf{PAL}$, $T(\varphi)$ is also a theorem of $\mathbf{PAL}$. Then by Reduction Lemma, $\vdash \varphi$.

♣

We will borrow this strategy to prove some completeness result later in this Chapter. Now we only focus on the semantics of PAL and adapt it to $\mathbf{BSEL}$, which is indexical and two-dimensional.

Given a model $M = \langle W, A, k, s, g, V \rangle$, a formula φ and an agent name n for the announcer, we define $[n!\varphi]M = \langle W', A, k', s', g', V' \rangle$ where:

W'	is the set of worlds w such that $M, w, g_w(n) \models \varphi$.
$k'_b(w, v)$	iff $k_b(w, v)$ and $w, v \in W'$.
$s'_w(b, c)$	iff $s_w(b, c)$ and $w \in W'$.
$g'_w(m)$	$= g_w(m)$ for all $m \in \mathsf{Nom}$ and $w \in W'$.
$V'(p)$	$= V(p) \cap (W' \times A)$.

So $[n!\varphi]\psi$ means "ψ is true after n announces φ". As in public announcement logic, the model $[n!\varphi]M$ is a restriction of M, this time to those worlds that satisfy φ, for the agent named "n" in M.

Then we have:

$$M, w, a \models [n!\varphi]\psi \quad \text{iff} \quad \text{either } M, w, g_w(n) \not\models \varphi \text{ or } [n!\varphi]M, w, a \models \psi$$

We define $?K\varphi$, to mean "I don't know whether φ or not", as an abbreviation of $(\neg K\varphi \wedge \neg K\neg\varphi)$.

We will now present models of the Muddy Monk scenarios introduced in Chapter One. For Scenario 1, we only discuss Case Two of PV+PA. We repeat it briefly here. This case assumes public vision and public announcement. Monks a, b and c stand in a line such that a is at the front of the line and c is at the back. So a sees nobody, b only sees a and c can see both a and b. Only a is clean. As a result, b knows he is muddy on round 2 and a knows he is clean on round 3. Figure 5.2 shows the successive models. We use $\underline{a}$, $\underline{b}$, $\underline{c}$ as names for a, b and c, respectively. We use a triple of binary numbers to denote possible worlds so that in world $d_1d_2d_3$ monk a is muddy if $d_1 = 1$ and clean if $d_1 = 0$, and similarly for d_2 and d_3. For example, in the world denoted 001 only c is muddy and in world 110 only c is clean. The left column depicts the situation after the Abbot's first statement. So there is no world 000. According to the story, no one speaks on round one. The silence means that every monk effectively announces that he doesn't know that he is muddy, e.g. $(@_{\underline{a}}\neg Kp \wedge @_{\underline{b}}\neg Kp \wedge @_{\underline{c}}\neg Kp)$. Then by the semantic definition of the announcement operator, the world 001 has to be deleted since $@_{\underline{c}}Kp$ (c knows he is muddy) is true at that world. As a result, in the real world 011, monk b now knows he is muddy since 010 and 011 are the only possible worlds he can access from 011. So b publically announces $@_{\underline{b}}Kp$ (b knows he is muddy). Then all the possible worlds have to be deleted except 010 and 011. Monk a then knows that he is clean on round three. But monk c still has no idea since he is muddy at 011 but clean at 010.

5.2 Observable Announcements

A public announcement is "transparent", meaning that every agent's update is known to the other agents. A "private" update is one that not every agent knows has occurred. Now we introduce my friend Jeremy to the All Blacks story. If he and I were on the same plane that night, he would also have received the captain's announcement about the outcome of the match. Since the announcement was public, the outcome would have been common knowledge to us. See Figure 5.4 where a is me, b is Jeremy and c is the captain. After the captain made his announcement, p would be common knowledge between us.

But Jeremy was not on the plane. Since he doesn't care about rugby, he didn't watch the midnight match nor look up the outcome. So after the captain's announcement, I know p but Jeremy does not. This difference cannot be modeled using $\mathbf{PAL}$ alone.

[1]The order of these announcements does not make any difference in this example.

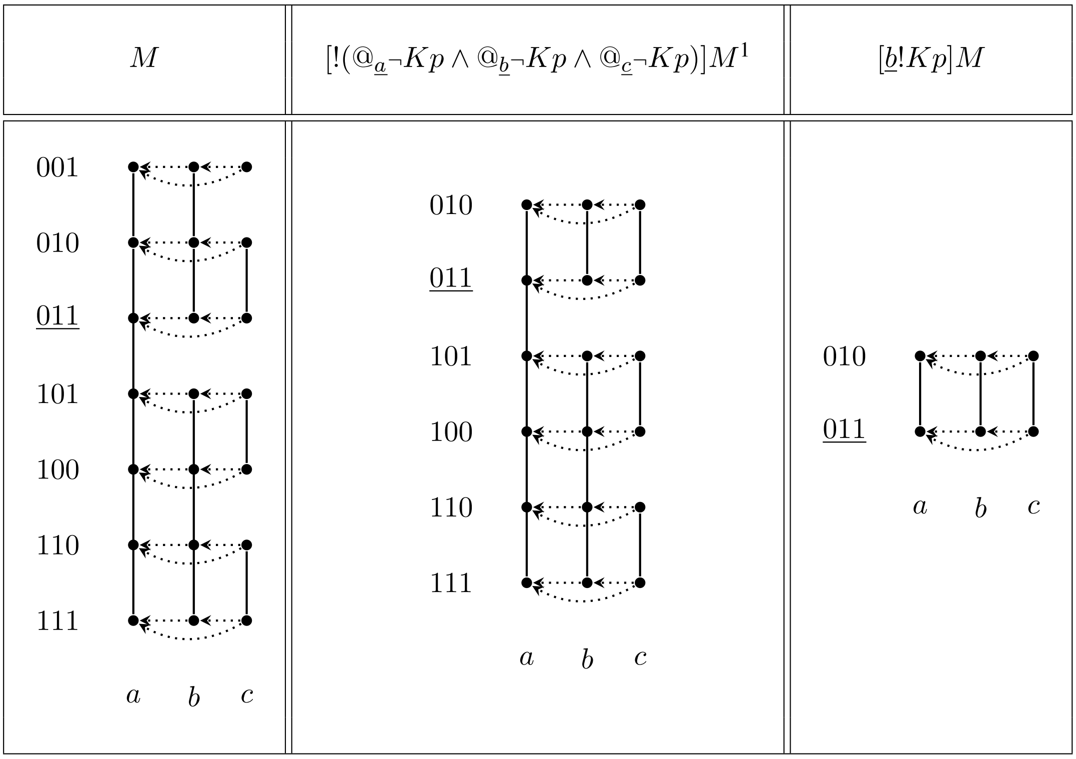


Figure 5.3: Case Two of PA+PV of Scenario 1

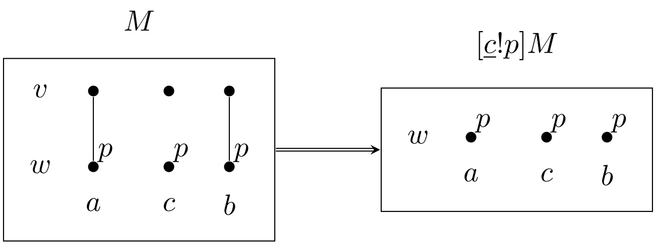


Figure 5.4: Public announcement is transparent.

A solution to the above is to characterise a *private* announcement. We are going to adapt the method of Baltag, Moss and Solecki [7] to our setting. This logical system incorporated various group-level epistemic actions into the multi-agent basic epistemic language. We will adapt this to the two-dimensional setting of $\mathbf{BSEL}$.

5.2.1 Private Announcements For BSEL

A private announcement occurs when some people are uncertain as to whether the announcement occurred or not. More generally, assume that there are some possible actions to consider, from n's perspective; for some of them, n will be aware when they happen, and this should result in some update for n, but n may be unaware of some of them, then when they happen, n will remain ignorant.

A Kripke frame is used to characterise people's uncertainty about possible actions

Definition 5.2.1 *An action structure to be a finite Kripke frame* $\mathbf{S} = \langle S, N, n, R, \mathsf{pre}\rangle$ *consisting of:*

- *a finite set S of action tokens to denote all possible information update from the perspective of n,*
- *a finite set N of agent names, and this represents whose uncertainty is characterised by the action structure,*
- *an agent name $n \in N$, shows from whose perspective,*
- *an indistinguishability relation R_m for each agent m between any two action tokens in S, and*
- *a precondition formula* $\mathsf{pre}(s)$ *for each action token $s \in S$. Intuitively, if an action token s can happen in some possible world w, then* $\mathsf{pre}(s)$ *must be satisfied at w. Let $s, t \in S$. $R_m(s,t)$ means that if s is the action token that really happens, then the agent named m thinks it is possible that t happens.*

The purpose for introducing the action structure is to discuss how the agent's uncertainty about the actions epistemically influences people. So we need to relate the action structure to the epistemic model in some proper way. Here is the method adopted from [7].

Definition 5.2.2 *Given a* $\mathbf{BSEL}$ *model* $M = \langle W, A, k, s, g, V\rangle$ *and an action structure* $\mathbf{S} = \langle S, N, n, R, \mathsf{pre}\rangle$ *where* $N \subseteq \mathsf{Nom}$ *we then define the model* $M^{\mathbf{S}} = \langle W^{\mathbf{S}}, A, k^{\mathbf{S}}, s^{\mathbf{S}}, g^{\mathbf{S}}, V^{\mathbf{S}}\rangle$ *as follows:*

- $W^{\mathbf{S}}$ *is the set of pairs (w, s) such that $w \in W$, $s \in \mathbf{S}$ and $M, w, g_w(n) \models$* $\mathsf{pre}(s)$

  So this representation is from n's perspective to illustrate the world and action.
- $k^{\mathbf{S}}_a((w,s),(v,t))$ *iff $k_a(w,v)$ and $R_m(s,t)$ for $m \in N$ such that $g_w(m) = a$.*

  The agents in M may be not in N. In this case, these agents know that one of the actions occurred but do not know which one. The agents in N may know which action has occurred.
- $s^{\mathbf{S}}_{(w,s)}(a,b)$ *iff $s_w(a,b)$*

  The action doesn't change who sees whom.

- $g^{\mathbf{S}}_{(w,s)}(n) = g_w(n)$
  The action doesn't change who is who.
- $V^{\mathbf{S}}(p) = \{(w,s) | w \in V(p)\}$
  The action doesn't change the truth value of propositional variables.

For all $n \in N$ of $\mathbf{S}$, we presuppose that n is rigid in M such that: if $g(m) = g(n)$ then $R_m = R_n$. If so, we say that M is suitable for $\mathbf{S}$.

Like the public announcement, we define dynamic operators so that the language can express the corresponding update.

Definition 5.2.3 *Given an action structure* $\mathbf{S}$ *and some* $s \in S$*, we define the dynamic operators* $[\mathbf{S}, s]$ *as follows:*

$$\begin{array}{lll} M, w, a \models [\mathbf{S}, s]\varphi & \textit{iff} & \textit{if } M \textit{ is suitable for } \mathbf{S} \textit{ and } M, w, g_w(n) \models \mathsf{pre}(s) \\ & & \textit{then } M^{\mathbf{S}}, (w,s), a \models \varphi \end{array}$$

Our semantics follows that of BMS closely. We will use this method to model the captain's announcement when Jeremy is *not* on the plane. Figure 5.5 shows the action structure of the example of All Blacks. The symbols $+$, $-$ and $=$ denote the action tokens such that $+$ represents c announces p, $-$ represents that c announces $\neg p$ and $=$ represents c doesn't announce anything.

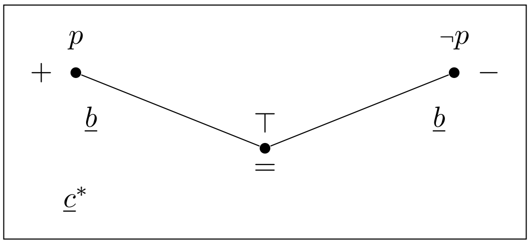


Figure 5.5: The indexical action structure of the All Blacks

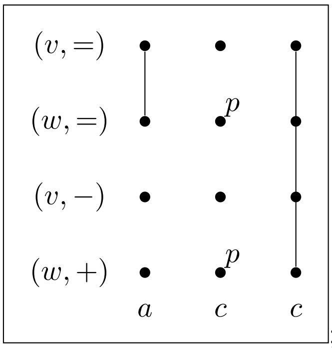
So we have $R_{\underline{b}}(+,-)$, $R_{\underline{b}}(+,=)$ and $R_{\underline{b}}(-,=)$, which means only $\underline{b}$ (Jeremy) cannot distinguish any actions of captain's. Then $M^{\mathbf{S}}$, is depicted by Figure 5.6. We have:

$$M, (w, =), c \models K(p \wedge @_{\underline{a}}?Kp)$$

Before the captains announcement, only the captain knows the outcome of the match, and I do not know it.

$$[\underline{c}!p]M, (w,+), a \models K(@_{\underline{c}}p \wedge @_{\underline{b}}?K@_{\underline{c}}p \wedge @_{\underline{b}}?K@_{\underline{a}}K@_{\underline{c}}p)$$

After captain's announcement p, I know the outcome of the match and I know that Jeremy doesn't know the outcome. Also he doesn't know that I know the outcome.



;

Figure 5.6: $M^{\mathbf{S}}$ of All Blacks

Now consider another example. You are bald so you were wearing a hat. Suddenly the hat was blown away by the wind and you caught it before it flew into the sky. Then you put it on quickly. Suppose that a stranger a was standing behind you, which you are aware of. Before the wind came, the epistemic situation can be represented by Figure 5.7. There are four possible worlds since you (b) are uncertain whether the stranger (a) could see you or not. Since you were wearing your hat at this stage, a doesn't know that you are bald (p). Then the action s (wind blew away the hat) happened. Obviously, a would't distinguish the action with the situation in which nothing happened when a couldn't see you at w_0 or w_1 while a is able to do at w_2 and w_3 where a saw you. However, either $R_{\underline{a}}(s, \top)$ or not $R_{\underline{a}}(s, \top)$ in an action structure. This seems to suggest that the current method is not suitable to denote the *observable announcements*, e.g. announcements that are only received by those who can see the announcer. Facts regarding who sees whom are decided by the model not the action structure. Agents may receive the announcement in some worlds but not in others. So there is no way of encoding this in the relation R_m of an action model.

Since our logic is talking about seeing and knowing, observable announcements deserves our focus. So we have to do more to fix this incompatibility.

5.2.2 Indexical Actions

Definition 5.2.4 *An indexical action structure* $\mathbf{S} = \langle S, \mathsf{ind}, \mathsf{pre}\rangle$ *consists of a set* S *of action tokens, a precondition formula* $\mathsf{pre}(s)$ *for each action token* $s \in S$ *and an indistinguishability condition formula* $\mathsf{ind}(s,t)$ *for* $s, t \in S$.

Specially, if: (1) $\mathsf{ind}(s,s) = \top$ (tautology); (2) $\vdash_{\mathbf{BSEL_{S5}}} \mathsf{ind}(s,t) \leftrightarrow \mathsf{ind}(t,s)$; and (3) $\vdash_{\mathbf{BSEL_{S5}}} \mathsf{ind}(s,u) \wedge \mathsf{ind}(u,t) \to \mathsf{ind}(t,s)$ then we say that $\mathbf{S}$ is a **BSEL** action structure. The clauses (1), (2) and (3) correspond to reflexive, symmetry and

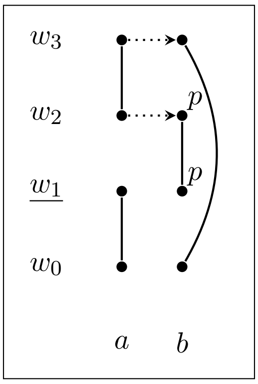


Figure 5.7: Bald head with a hat

transitivity respectively. Also, an agent is uncertain whether the action token s or t has occurred when the agent satisfies the condition $\mathsf{ind}(s, t)$.

We draw the graphs of indexical action structures in a very similar way to those of section 5.2.1 structures except that we label the edge between s and t with $\mathsf{ind}(s, t)$ instead of the name of the agent. We also omit drawing the reflexive and transitive edges if they exist. The following is a simple indexical action structure.

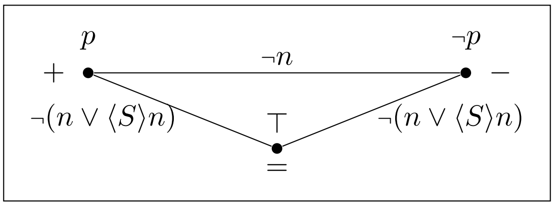


Figure 5.8: An example of an indexical action structure

There are three action tokens: $+, -, =$. The edge between $+$ and $-$ is labelled $\neg n$. This means that no agents other than n can distinguish between actions $+$ and $-$. But the edge between $+$ and $=$ (and also between $-$ and $-$) is labelled by $\neg(n \vee \langle S \rangle n)$, meaning that both n and any agent that can see n can distinguish between $=$ and the other nodes, but agents who cannot see n cannot. Since $+$ has precondition p and $-$ precondition $\neg p$, this means that n is the only agent to receive the announcement.

Definition 5.2.5 *Given a model $M = \langle W, A, k, s, g, V \rangle$ of* $\mathbf{BSEL}$ *and an indexical action structure* $\mathbf{S}$*, we define the* <u>*product*</u> $M^{\mathbf{S}} = \langle W^{\mathbf{S}}, A, k^{\mathbf{S}}, s^{\mathbf{S}}, g^{\mathbf{S}}, V^{\mathbf{S}} \rangle$ *as follows:*

- $W^{\mathbf{S}}$ *is the set of pairs* (w, s) *such that* $w \in W$*,* $s \in \mathbf{S}$ *and* $M, w, a \models \mathsf{pre}(s)$
- $k^{\mathbf{S}}_a((w, s), (v, t))$ *iff* $k_a(w, v)$ *and* $M, w, a \models \mathsf{ind}(s, t)$*.*

- $s^{\mathbf{S}}_{(w,s)}(a,b)$ *iff* $s_w(a,b)$
- $g^{\mathbf{S}}_{(w,s)}(n) = g_w(n)$
- $V^{\mathbf{S}}(p) = \{(w,s)|w \in V(p)\}$

Since the settings here are different from Definition 5.2.1 (the structure of action structure), we need a new semantic definition for the operators defined by the indexical action structure. Then we won't repeat the semantics for each operator we introduce later.

Definition 5.2.6 *Given an indexical action structure* $\mathbf{S}$ *and* $s \in S$*, the semantic definition of the operator* $[\mathbf{S}, s]$ *is given as follows:*

$M, w, a \models [\mathbf{S}, s]\varphi$ *iff if* $M, w, a \models \mathsf{pre}(s)$ *then* $M^{\mathbf{S}}, (w, s), a \models \varphi$

$M, w, a \models [\mathbf{S}]\varphi$ *iff* $M, w, a \models [\mathbf{S}, s]\varphi$ *for every* $s \in S$.

If M is a model of $\mathbf{BSEL_{S5}}$, what about $M^{\mathbf{S}}$? If yes, what conditions? The answer is the suitability.

A $\mathbf{BSEL}$ model M is <u>suitable</u> for $\mathbf{S}$ if $k_a(w, v)$, then $M, w, a \models \mathsf{ind}(s, t)$ iff $M, v, a \models \mathsf{ind}(s, t)$.

Lemma 5.2.1 *If* M *is a model of* $\mathbf{BSEL_{S5}}$*,* $\mathbf{S}$ *is a* $\mathbf{BSEL}$ *action structure and* M *is suitable for* $\mathbf{S}$*, then* $M^{\mathbf{S}}$ *is also a model of* $\mathbf{BSEL_{S5}}$*.*

Proof:

Recall that a model of $\mathbf{BSEL_{S5}}$ is an S5 model. We check that $k^{\mathbf{S}}_a$ is the equivalence relation. Let $(w, s), (v, t), (u, r) \in W^{\mathbf{S}}$.

First, $k^{\mathbf{S}}_a$ is reflexive, i.e. $k^{\mathbf{S}}_a((w, s), (w, s))$ since $k_a(w, w)$ and $M, w, a \models \top$.

Second, $k^{\mathbf{S}}_a$ is symmetric, e.g. $k^{\mathbf{S}}_a((w, s), (v, t))$ implies $k^{\mathbf{S}}_a((v, t), (w, s))$.

Suppose $k^{\mathbf{S}}_a((w, s), (v, t))$. Then we have $k_a(w, v)$ and $M, w, a \models \mathsf{ind}(s, t)$. Since M is an S5 model, $k_a(v, w)$. Since M is suitable for $\mathbf{S}$, $M, v, a \models \mathsf{ind}(s, t)$. $\mathbf{BSEL_{S5}}$ is sound, so $M, v, a \models \mathsf{ind}(s, t) \to \mathsf{ind}(t, s)$ by Clause (2) of Definition 5.2.4. So $M, v, a \models \mathsf{ind}(t, s)$, and then $k^{\mathbf{S}}_a((v, t), (w, s))$.

Third, $k^{\mathbf{S}}_a$ is transitive. And it can be proven similarly.

♣

But we need to go further to discuss the bald and hat story we discussed at the end of the last section. Recall the visual property pattern we discussed in Chapter 3: if you see n, you know you see n, and if you cannot see n, then you know you cannot see n. This is an exactly underlying assumption of that story. Now we want to categorise a set of formulas. We call this set VP and define it inductively as follows:

- $n \in \mathsf{VP}$ for any $n \in \mathsf{Nom}$;
- if $\varphi \in \mathsf{VP}$, then $\neg\varphi \in \mathsf{VP}$;
- if $\varphi \in \mathsf{VP}$, then $\langle S\rangle\varphi \in \mathsf{VP}$;
- if $\varphi, \psi \in \mathsf{VP}$, then $\varphi \wedge \psi \in \mathsf{VP}$;

Especially, $\top \in \mathsf{VP}$.

Then define a model M is an $\underline{\mathsf{SK\ model}}$, if it is a rigid model of $\mathbf{BSEL_{S5}}$ and satisfies:

if $k_a(w, v)$, then $s_w(a, b)$ iff $s_v(a, b)$ for every $w, v \in W$ and $a, b \in A$.

Lemma 5.2.2 *Given an* SK *model* M*, if* $k_a(w, v)$*, then we have*

$M, w, a \models \varphi$ *iff* $M, v, a \models \varphi$ *for any* $\varphi \in \mathsf{VP}$.

Proof: By induction on φ. We only show the case in which $\varphi = \langle S\rangle\psi$.

$M, w, a \models \langle S\rangle\psi$ iff there is b such that $s_w(a, b)$ and $M, w, b \models \psi$ (1)

By $k_a(w, v)$ and M is an SK model,

$s_w(a, b)$ iff $s_v(a, b)$

Then by inductive hypothesis,

$M, w, b \models \psi$ iff $M, v, b \models \psi$

So (1) iff $M, v, a \models \langle S\rangle\psi$.

♣

Lemma 5.2.3 *An* SK *model* M *is suitable for* $\mathbf{S}$ *if* $\mathsf{ind}(s, t) \in \mathsf{VP}$ *for every* $s, t \in S$.

Proof: By definition of suitability and Lemma 5.2.2.

♣

Lemma 5.2.4 *If M is an* SK *model,* $\mathbf{S}$ *is a* $\mathbf{BSEL}$ *action structure and M is suitable for* $\mathbf{S}$*, then $M^{\mathbf{S}}$ is also an* SK *model.*

Proof: By Definition 5.2.5, Lemma 5.2.1, 5.2.3.

♣

Lemma 5.2.5 *If M is a model of* $\mathbf{BSEL_{SK}}$*,* $\mathbf{S}$ *is a* $\mathbf{BSEL}$ *action structure where* $\mathsf{ind}(s,t) \in \mathsf{VP}$ *for $s, t \in S$, then $M^{\mathbf{S}}$ is also a model of* $\mathbf{BSEL_{SK}}$.

Proof: Recall that a $\mathbf{BSEL_{SK}}$ model is not only an SK model, but also has irreflexive s.

The relation $s^{\mathbf{S}}$ is still irreflexive by the definition of $M^{\mathbf{S}}$: $s^{\mathbf{S}}_{(w,s)}(a,b)$ iff $s_w(a,b)$.

$M^{\mathbf{S}}$ is an SK model by Lemma 5.2.4.

♣

5.2.3 Observable Announcements

With indexical action structures, we can characterise various kinds of observable announcements. An announcement is "observable" if only those who see the announcer receive its content. The announcer may or may not be "aware" of this. Both cases will be considered.

First, we characterise the *aware observable announcement*. This is when agent n announces φ, and: (1) n is aware of both the content of this announcement and of having revealed it to any agent who sees n, and (2) any agent who is not n and cannot see n does not know whether n has done so or not.

The notation $[n \lhd \varphi]$ stands for "after agent n is aware of having revealed φ", and as usual, this requires that φ is true. Then $[n \lhd \varphi]\psi$ means that ψ is true after n is aware of having revealed that φ. $[n \lhd \varphi] = [\mathbf{S}, +]$ where $\mathbf{S}$ is an $\mathbf{BSEL}$ action structure: $S = \{+, -, =\}$; $\mathsf{pre}(+) = @_n\varphi$, $\mathsf{pre}(-) = @_n\neg\varphi$ and $\mathsf{pre}(=) = \top$;

$$\mathsf{ind}(s,t) = \begin{cases} \neg(n \vee \langle S \rangle n) & \text{if } s \neq t \\ \top & \text{Otherwise} \end{cases}.$$

See Figure 5.9.

Second, we characterise the *unaware observable announcement*. Agent n is unaware of observably announcing φ, then n reveals φ to any one who sees but without knowing he has done so.

The notation $[n \bar{\lhd} \varphi]$ stands for "after agent n is unaware of revealing φ", which also requires φ to be true. Then $[n \bar{\lhd} \varphi]$ is the dynamic operator $[\mathbf{S}, +]$ where $\mathbf{S}$ is an $\mathbf{BSEL}$ action structure defined below.

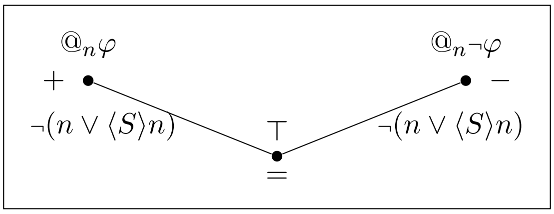


Figure 5.9: The $\mathbf{BSEL}$ action structure of aware observable announcement

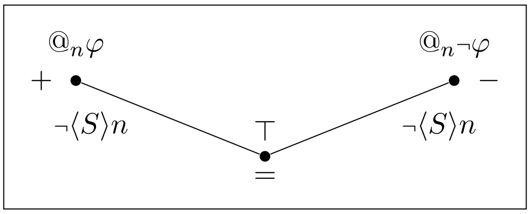


Figure 5.10: The $\mathbf{BSEL}$ action structure of unaware observable announcement

Lemma 5.2.6 *Let* $\mathbf{S}$ *be an* $\mathbf{BSEL}$ *action structure of* $[n\lhd\varphi]$ *or* $[n\bar{\lhd}\varphi]$ *and* M *be a model of* $\mathbf{BSEL_{SK}}$. *Then* $M^{\mathbf{S}}$ *is also a model of* $\mathbf{BSEL_{SK}}$.

Proof: By the fact that $\mathsf{ind}(s,t) \in \mathsf{VP}$ for every $s, t \in S$ and Lemma 5.2.5.

♣

Now we go back to the example that you are bald with a hat which is suddenly blown away by the wind. In this case, you "show" the people who can see you that you are bald. And this observable announcement is one you are aware of. Applying the indexical action structure for $[\underline{b}\lhd p]$ to the model shown in Figure 5.7, we get Figure 5.11. Recall that in world w_1, a cannot see b. Then we have:

(1) $M, (w_1, +), b \models [\underline{b}\lhd p]@_{\underline{a}}?K@_{\underline{b}}p$

After you ($\underline{b}$) are aware of revealing that you are bald (p), the stranger ($\underline{a}$) doesn't know whether you are bald or not because he didn't see you.

(2) $M, (w_2, +), a \models [\underline{b}\lhd p](@_{\underline{a}}K@_{\underline{b}}p \wedge @_{\underline{b}}K@_{\underline{a}}K@_{\underline{b}}p)$

After you are aware of revealing that you are bald, the stranger knows that you are bald because he can see you, and you know it because you are aware of revealing you are bald.

(3) $M \models [\underline{b}\lhd p]@_{\underline{a}}K@_{\underline{b}}p \leftrightarrow @_{\underline{a}}\langle S\rangle\underline{b}$

After you are aware of revealing that you are bald, the stranger knew that if and only if he could see you. This is valid in M, i.e. true in every world.

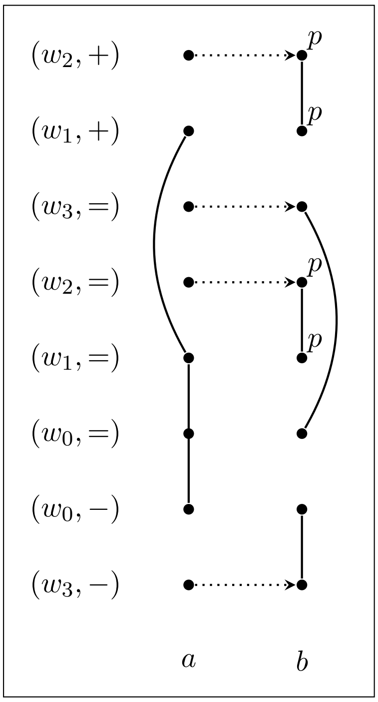


Figure 5.11: Agent b's aware observable announcement p

Here is another example of unaware observable announcement. Agents a and b are playing poker. The situation was that a got "A,A,2,3" and b got "6,7,8,10". These cards are face up on the table, so everyone can see them. Each player has one more card face down. It is obvious that b wins if and only if b's face-down card is "9". Before this competition, player a had watched lots of b's previous poker videos very carefully. And he found that b always touched his ring inadvertently when he bluffed in a losing situation. Unfortunately for b, this is true and he is unaware of the "tell". Now, on the River Round (the last round of betting), b touches his ring after checking his face-down card again, and went all-in (bet all his chips). Seeing the tell, a also went all-in. Naturally, the result was that a won.

Let p be an indexical proposition "I have a losing hand." Then the model M in Figure 5.12 depicts the situation before bluffing. Player a didn't know whether b had a losing hand, so for him w and v are indistinguishable. Player b already knew he had a losing hand, because he had seen his final card. He also knew that player a didn't know this. Then b started to bluff. When b touched his ring, this was an unaware observable announcement to a, because a knows this reveals b's bluffing. So we have $[\underline{b}\bar{\lhd}p]M$. In this model, player a understands what is really happening, but b doesn't. As a result, in the actual world, a knows he won't lose, w, after the tell, a knows both that he has the best cards ($\neg p$), and that b doesn't know he knows. Expressing these by formulas, we have:

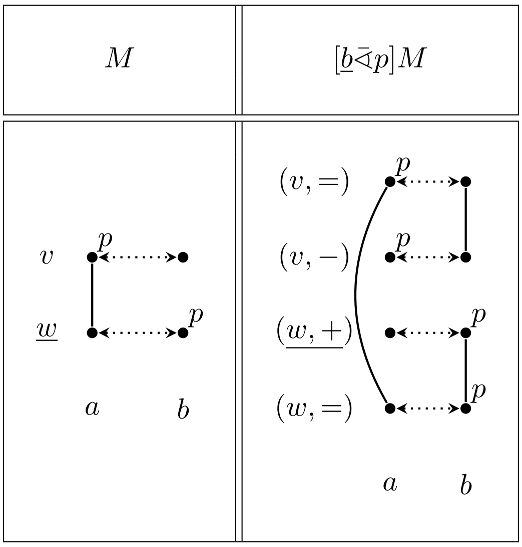


Figure 5.12: An example of unaware observable announcement

$M, w, a \models [\underline{b}\bar{\lhd}p]K@_{\underline{b}}p$

$M, w, b \models [\underline{b}\bar{\lhd}p]?K@_{\underline{a}}K@_{\underline{b}}p$

Definition 5.2.7 *The language of* <u>*observable announcement basic social epistemic logic*</u> *(*$\mathbf{BSEL_{OA}}$*) is defined from a set* Prop *of propositional atoms, a set* Nom *of agent names. The well-formed formulas* φ *of this language are given by the rule*

$\varphi \quad ::= \quad p \mid n \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K\varphi \mid S\varphi \mid @_n\varphi \mid X\varphi$

$X ::= [n\lhd\varphi] \mid [n\bar{\lhd}\varphi]$

We will show that formulas containing dynamic operators are equivalent to formulas without them. In other words, we need reduction axioms:

$$[n\lhd\varphi]p \leftrightarrow (@_n\varphi \to p)$$
$$[n\bar{\lhd}\varphi]p \leftrightarrow (@_n\varphi \to p)$$
$$[n\lhd\varphi]m \leftrightarrow (@_n\varphi \to m)$$
$$[n\bar{\lhd}\varphi]m \leftrightarrow (@_n\varphi \to m)$$
$$[n\lhd\varphi]\neg\psi \leftrightarrow (@_n\varphi \to \neg[n\lhd\varphi]\psi)$$
$$[n\bar{\lhd}\varphi]\neg\psi \leftrightarrow (@_n\varphi \to \neg[n\bar{\lhd}\varphi]\psi)$$
$$[n\lhd\varphi](\psi \wedge \theta) \leftrightarrow ([n\lhd\varphi]\psi \wedge [n\lhd\varphi]\theta)$$
$$[n\bar{\lhd}\varphi](\psi \wedge \theta) \leftrightarrow ([n\bar{\lhd}\varphi]\psi \wedge [n\bar{\lhd}\varphi]\theta)$$
$$[n\lhd\varphi]@_m\psi \leftrightarrow (@_n\varphi \to @_m[n\lhd\varphi]\psi)$$
$$[n\bar{\lhd}\varphi]@_m\psi \leftrightarrow (@_n\varphi \to @_m[n\bar{\lhd}\varphi]\psi)$$
$$[n\lhd\varphi]S\psi \leftrightarrow (@_n\varphi \to S[n\lhd\varphi]\psi)$$
$$[n\bar{\lhd}\varphi]S\psi \leftrightarrow (@_n\varphi \to S[n\bar{\lhd}\varphi]\psi)$$
$$[n\lhd\varphi]K\psi \leftrightarrow ((@_n\varphi \wedge (n \vee \langle S\rangle n)) \to K[n\lhd\varphi]\psi)$$
$$[n\bar{\lhd}\varphi]K\psi \leftrightarrow ((@_n\varphi \wedge \langle S\rangle n) \to K[n\bar{\lhd}\varphi]\psi)$$
$$[n\lhd\varphi](\psi \to \theta) \to ([n\lhd\varphi]\psi \to [n\lhd\varphi]\theta)$$
$$[n\bar{\lhd}\varphi](\psi \to \theta) \to ([n\bar{\lhd}\varphi]\psi \to [n\bar{\lhd}\varphi]\theta)$$

We only discuss the reduction axiom for $[n\bar{\lhd}\varphi]K\psi$. After n is unaware of observably announcing φ, in the case that φ is true for n, which is the precondition for the announcement, the agents who get influenced are exactly those who can see n (n cannot see n). So if they know ψ after being informed: for n, φ, then in the original model they should potentially know that ψ is the consequence of n's such a announcement.

(Choice One:)

As to the other direction, for any agent who can see n when n is φ, if any his (or her) accessible world looks ψ as a consequence of n's announcement, then these worlds will be kept in the updated model from the announcement and also accessible by him, then he will know ψ. That is exactly what $[n\lhd\varphi]K\psi$ means.

(Choice Two:)

For the other direction, given n has φ property, in the case that you can see n, at all your accessible worlds, that n is unaware of observably announcing φ implies ψ is true, then these accessible worlds are exactly those that you still can access even if n makes the announcement first, so you will know ψ;

The axiomatisation of the observable announcement social epistemic logic consists of the axioms and rules of $\mathbf{BSEL_{SK}}$, the above reduction axioms and the following :

From $\vdash \varphi$	infer $\vdash [n\triangleleft\psi]\varphi$
From $\vdash \varphi$	infer $\vdash [n\bar{\triangleleft}\psi]\varphi$

Since our announcement operators are standard, by [62], the rules of Replacement of Logical Equivalents (RE Rule) is admissible in the observable announcement social epistemic logic.

Recall the strategy of the completeness proof that we introduced when we discussed $\mathbf{PAL}$. Since we have the RE rule, we will apply the inside-out strategy here with the Lemma 5.1.1 and prove the completeness of $\mathbf{BSEL_{OA}}$. Notice that we already have reduction axioms. So a translation can be defined to have the following lemma:

Lemma 5.2.7 (Reduction) *Every formula of the language of* $\mathbf{BSEL_{OA}}$ *is provably equivalent to a formula of* $\mathbf{BSEL}$.

Theorem 5.2.1 (Soundness) $\vdash \varphi$ *implies* φ *is valid.*

Proof: By checking the reduction axioms and additional rules.

♣

Theorem 5.2.2 (Completeness) *If* φ *is valid, then* $\vdash \varphi$.

Proof: similar to Theorem 5.1.2.

♣

We have so far discussed the observable announcement of a single agent. Now consider what happens when two agents each make observable announcements. The announcements may occur sequentially or simultaneously. A sequence of observable announcement operators can deal with the sequential case. But simultaneous announcement is a little different. Recall the scenarios we discussed in the first chapter. When the Abbot asks the monks whether they know they are muddy or not, they make their observable announcements by either raising their hands or keeping silent. But their announcements are made simultaneously. And if we regard them as a sequence of announcements, we cannot give an accurate description. Why? Take Figure 5.3 (on page 104) as an example. In M, after Bob is aware of observably announcing that $?Kp$, whose token is denoted as $\emptyset$, so $\mathsf{pre}(\emptyset) = ?Kp$), then Bob cannot distinguish between $(011, \emptyset)$ and $(001, \emptyset)$ since $k_b((011), (001))$ in M. But Charlie knows he is muddy at both worlds now ($@_{\underline{c}}Kp$). Even if Charlie makes an observable announcement $?Kp$ now, we won't get $((011, \emptyset), \emptyset)$ and $((001, \emptyset), \emptyset)$ (not satisfy the precondition of $\mathsf{pre}(\emptyset)$), but only get $((011, \emptyset), +)$ and $((001, \emptyset), +)$, where $+$ is the token for the observable announcement that I know I am muddy. However, we have $k_b(((011, \emptyset), +), ((001, \emptyset), +))$, which means Bob doesn't know why Charlie knows he is muddy. Charlie may know it because Bob just made an announcement or he sees two clean monks. In fact, a similar problem will occur even if we apply this method to Scenario 0.

The solution is using simultaneous observable announcements. But there are various specific situations. We discuss a very simple case in which it is commonly known that n_i $(1 \leq i \leq k)$ is either aware of having revealed φ_i or does nothing, we define the following indexical action structure $\mathbf{S} = \langle S, \mathsf{ind}, \mathsf{pre}\rangle$ as follows:

- $S = \mathrm{p}ow\{n_1, \ldots, n_k\}$
- $\mathsf{pre}(s) = \bigwedge\limits_{n_i \in s} @_{n_i}\varphi_i$
- $\mathsf{ind}(s, t) = \begin{cases} \bigwedge\limits_{n \in s \bowtie t} \neg(n \vee \langle S\rangle n)^2 & \text{if } s \neq t \\ \top & \text{Otherwise} \end{cases}$

  where $s \bowtie t = \{n \in \{n_1, \ldots, n_k\} | n \in s\backslash t \text{ or } n \in t\backslash s\}$

  e.g. $s \bowtie t = (s\backslash t) \cup (t\backslash s)$

For any action token $s \subseteq \{n_1, \ldots, n_k\}$, n_i is aware of observably announcing φ_i iff $n_i \in s$.

The set $s \bowtie t$ collects all names of the agents who act differently between s and t.

For example, suppose n, m simultaneously announces p, q in this way, we have the following action structure:

where $S = \mathrm{p}ow\{n, m\}$, $\mathsf{pre}(\{n, m\}) = @_n p \wedge @_m q$, $\mathsf{pre}(\{n\}) = @_n p$, $\mathsf{pre}(\{m\}) = @_m q$, $\mathsf{pre}(\emptyset) = \top$ and ind is shown by Figure 5.13.

Now we define an operator: $[(n_1, \ldots, n_k)\lhd(\varphi_1, \ldots, \varphi_k)]$, which means that agents $n_1, \ldots, n_k$ make a simultaneous aware observable announcement $\varphi_1, \ldots \varphi_k$ respectively. $[(n_1, \ldots, n_k)\lhd(\varphi_1, \ldots, \varphi_k)] = [\mathbf{S}, \{n_1 ... n_k\}]$ where $\mathbf{S} = \langle S, \mathsf{pre}, \mathsf{ind}\rangle$ is just defined.

[2] For an unaware observable announcement operator, we only need $\bigwedge\limits_{n \in s \bowtie t} \neg\langle S\rangle n$.

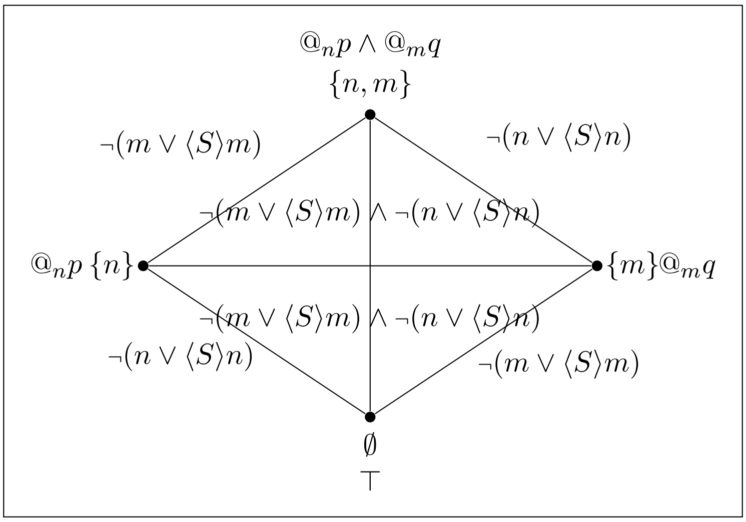


Figure 5.13: An example of a simultaneous indexical action structure

5.3 Models of Scenarios in Chapter One

Recall the scenarios we proposed in Chapter One, in which all announcements are simultaneous. In each round, the monks either raise their hands, which is an observable announcement of "I know I am muddy" (Kp), or keep silent, which means "I don't know I am muddy" ($\neg Kp$). But it means that we need to adapt the indexical action structure we just defined by $[n_1, \ldots, n_k \lhd \varphi_1, \ldots, \varphi_k]$ to fit this new settings.

In each round, the monks either raise their hand, which is an observable announcement of I know I am muddy (Kp), or keep silent. How to model this? A first attempt would be to use the simultaneous aware observable announcement $[\underline{a}, \underline{b}, \underline{c} \lhd \varphi_1, \varphi_2, \varphi_3]$ but what to choose for $\varphi_1, \varphi_2, \varphi_3$? The problem is that this is determined by the model not the action structure. If Andrew knows he is muddy, he'll raise his hand, but if he doesn't he won't. We define a new announcement operator $[n_1, \ldots, n_k \lhd K\varphi?]$ to model the action of $n_1, \ldots, n_k$ simultaneously (and truthfully) answering the question"φ?". This is defined as $\mathbf{S}$, where $\mathbf{S} = \langle S, pre, ind \rangle$ is the action structure defined by

- $S = \mathrm{p}ow\{n_1, \ldots, n_k\}$
- $\mathsf{pre}(s) = \bigwedge_{n_i \in s} @_{n_i} K\varphi \wedge \bigwedge_{n_i \notin s} @_{n_i} \neg K\varphi$
- $\mathsf{ind}(s, t) = \begin{cases} \bigwedge_{n \in s \bowtie t} \neg(n \vee \langle S \rangle n) & \text{if } s \neq t \\ \top & \text{Otherwise} \end{cases}$

Note that the definition of pre ensures that for any model M, each world satisfies exactly one of the action token preconditions. This means that $M^{\mathbf{S}}$ has the same number of worlds and the same structure as M, except possibly for the k relation.

Also, we have the following fact:

$$(s \bowtie t) = (t \bowtie s)$$
$$(s \bowtie t) \subseteq (s \bowtie u) \cup (u \bowtie t)$$

So, $\mathbf{S}$ is a $\mathbf{BSEL}$ action structure. By $\mathsf{ind}(s,t) \in \mathsf{VP}$ for $s, t \in S$ and Lemma 5.2.5, we have:

Fact: If M is a model of $\mathbf{BSEL_{SK}}$ then $M^{\mathbf{S}}$ is also a model of $\mathbf{BSEL_{SK}}$.

Notice that the semantic definition here should be the second clause of Definition 5.2.6.i.e.

$M, w, a \models [\mathbf{S}]\varphi$ iff $M, w, a \models [\mathbf{S}, s]\varphi$ for every $s \in S$.

The reason is simple: $[n_1, \ldots, n_k \lhd K\varphi?]$ expresses that agents are simultaneously answering the question "φ?". It doesn't mean a specified token.

For modeling the scenarios of Chapter One, we define the following action structure for the operator $[\underline{a}, \underline{b}, \underline{c} \lhd Kp?]$ where p denotes "I am muddy", depicted by Figure 5.13:

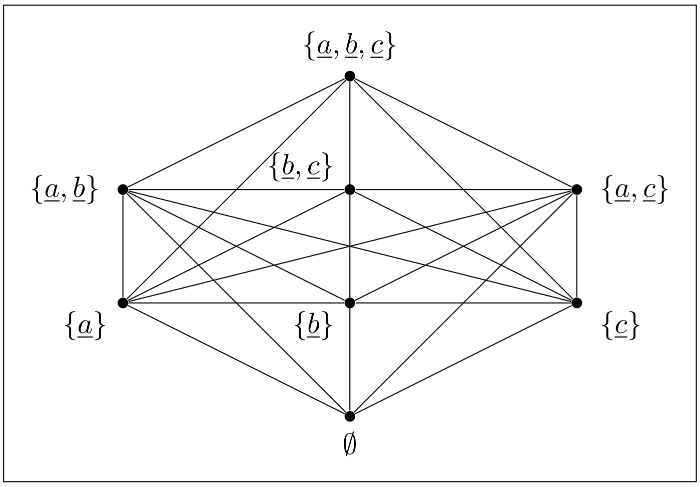


Figure 5.14: The simultaneous indexical action structure for scenarios of Chapter One

First, $S = pow\{\underline{a}, \underline{b}, \underline{c}\}$. To make the graph neat, we avoid representing ind for each edge and the pre formulas for each node. But the rule is simple: A token has a name iff the agent who has the name knows p. So $\mathsf{pre}(\{\underline{a}\}) = @_{\underline{a}}Kp \wedge @_{\underline{b}}\neg Kp \wedge @_{\underline{c}}\neg Kp$, $\mathsf{pre}(\{\underline{a}, \underline{c}\}) = @_{\underline{a}}Kp \wedge @_{\underline{b}}\neg Kp \wedge @_{\underline{c}}Kp$ and $\mathsf{pre}(\emptyset) = @_{\underline{a}}\neg Kp \wedge @_{\underline{b}}\neg Kp \wedge @_{\underline{c}}\neg Kp$. There is an edge between any two nodes with the conditions defined by ind. For example, $\mathsf{ind}(\{\underline{a}, \underline{c}\}, \{\underline{c}\}) = \neg(\underline{a} \vee \langle S\rangle\underline{a})$ and

$$\mathsf{ind}(\{\underline{b}, \underline{c}\}, \{\underline{a}\}) = \bigwedge_{n\in\{\underline{a},\underline{b},\underline{c}\}} \neg(n \vee \langle S\rangle n).$$

We now show model presentations of some scenarios in Chapter 1. We assume that all original models (before announcements) are $\mathbf{BSEL_{SK}}$ models as discussed in Chapter 3. They are rigid $\mathbf{S5}$ models satisfying SK properties: if you see n, you know you see n, and if you cannot see n, then you know you cannot see n.

The first proposed model presentation is Case Two in OA+PV of Scenario One in which Andrew (a) is clean, and Bob (b) and Charlie (c) are muddy. They stand in a line such that c can see both b and a, b can see a and a sees nothing. The vision is public. But they can only raise their hands when they know they are muddy. Otherwise, they keep silent. Before the Abbot says: "At least one of you is muddy. If you know you are, raise your hands.", the model M is depicted by the left part of Figure 5.15. After the Abbot's request, we have the right part of the figure. Except for world (001) in M, every world just satisfies the $\mathsf{pre}(\emptyset)$, which every monk keeps silent (observable announcement of $\neg Kp$). In 001, c knows he is muddy. By Figure 5.13, $\mathsf{ind}(\emptyset, \{\underline{c}\}) = \neg(\underline{c} \vee \langle S\rangle\underline{c})$. However, neither b nor a can see c, so they cannot distinguish whether c raises his hands or not. This makes the updated model the same as the original model and no monk will know whether he is muddy or not. This result agrees with Figure 1.2.

We are ready to discuss the last case of Scenario 2 (Figure 1.3). Recall that a is still the only clean monk. It is publicly known that b and c see each other, a and b see each other and a cannot see c. So the only question is whether c sees a, which is not common knowledge. In fact, c sees a, and only a doesn't know it by the definition of semi-private vision. All monks finally know whether they are muddy or not. Moreover, a gets the new knowledge that c can see him. The original model M, expressing the situation of the beginning, was depicted by Figure 5.16. Monk a's uncertainty about whether c can see him is shown by a's accessible relations between the two parts of the model.

After the Abbot's request, we have $[(\underline{a}, \underline{b}, \underline{c})\lhd Kp?]M$, shown by Figure 5.17. We only explain why it is not $k_a((011, \emptyset), (010, \{\underline{b}\}))$ now. Since the token $\{\underline{b}\} \neq \emptyset$, $\mathsf{ind}(\{\underline{b}\}, \emptyset) = \neg(\underline{b} \vee \langle S\rangle\underline{b})$. By $M, 011, a \models \langle S\rangle\underline{b}$, not $k_a((011, \emptyset), (010, \{\underline{b}\}))$. For the current world $(011, \emptyset)$, we have:

$$M^{\mathbf{S}}, (011, \emptyset), a \models \neg Kp \wedge \neg K @_{\underline{c}}\langle S\rangle\underline{a}$$
$$M^{\mathbf{S}}, (011, \emptyset), b \models Kp$$
$$M^{\mathbf{S}}, (011, \emptyset), c \models Kp$$

Now b and c raise their hands since they know they are muddy at the current world

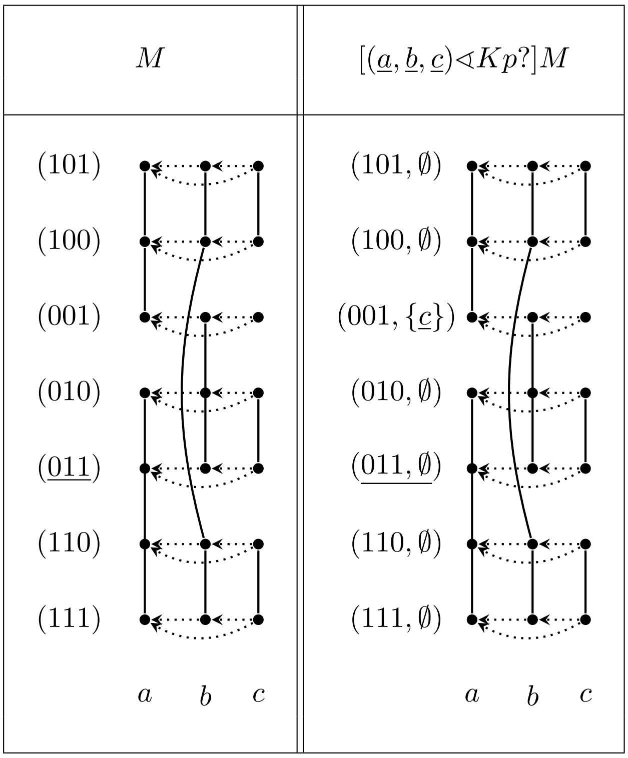


Figure 5.15: Case Three of PV+OA of Scenario 1.2

$(011, \emptyset)$. We then have the next model: $[(\underline{a}, \underline{b}, \underline{c}) \lhd Kp?]M'$ where $M' = [(\underline{a}, \underline{b}, \underline{c}) \lhd Kp?]M$. Check Figure 5.18. Now there is no k_a relation between $(011, \emptyset), \{\underline{b}, \underline{c}\})$ and $(011, \emptyset), \emptyset)$. It is because $M', (011, \emptyset), a \models \langle S \rangle \underline{b}$. In other words, a can see b so a receives b's observable announcement. So we have

$$M'^{\mathbf{S}}, ((011, \emptyset), \{\underline{b}, \underline{c}\}), a \models K\neg p \wedge K @_{\underline{c}} \langle S \rangle \underline{a}$$

We now briefly discuss Scenario 3. In the first case, every monk knows b is blind. After two rounds of public announcement $[!(@_{\underline{a}} \neg Kp \wedge @_{\underline{c}} \neg Kp)]$, the possible worlds $(101), (100), (001)$ have been deleted from the original model. Then at world (011) b knows he is muddy, but neither a nor c knows whether they are muddy or not.

The final case is about belief rather than knowledge. That's because a and c mistakenly believe that b can see them. Although $\mathbf{BSEL}$ is available as a base for a logic of belief, the epistemic operator would need other properties, such as D45 rather than S5. But also, we would have to modify the dynamic operators so as to perform belief revision rather than epistemic update. This will be a matter for future research.

We could also consider dynamic operators that change not only the epistemic states of agents but also their relationship to each other. This has been studied by Seligman,

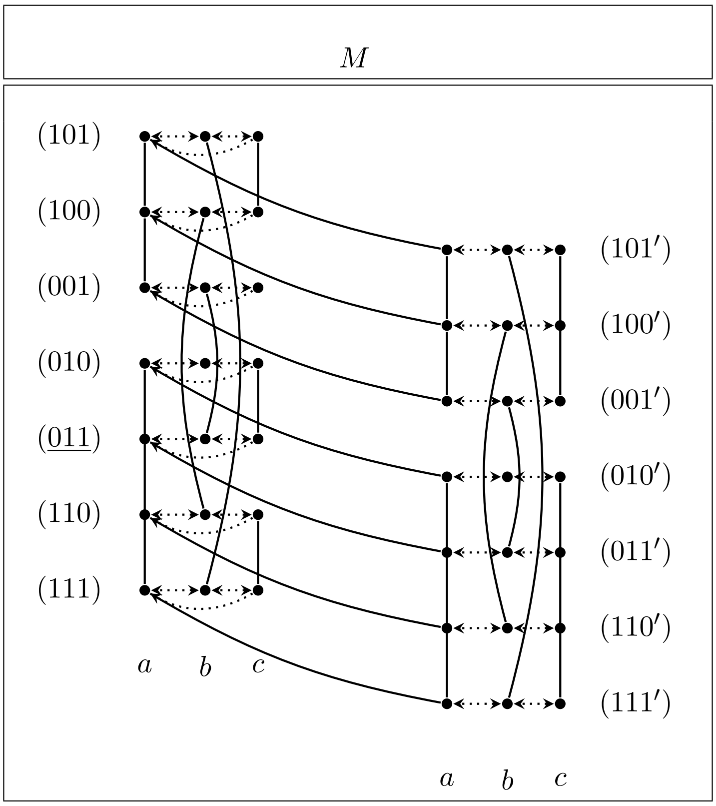


Figure 5.16: Last case of Scenario2 (Round One)

Liu and Girard [56], and in Gasquet, Goranko, and Schwarzentruber [22]. [56] addresses the issue of how to change social relations in general, but with few technical details. [22] examines a particular kind of relation change that is directly relevant to that of SEL, and so we will discuss it in more detail. with our logic as below.

5.4 A Comparison with Big Brother Logic

The logic that [22] developed is a multi-agent system. It has the following setting. First, different agents have different positions. They are stationary, but can rotate freely. Second, the observation ability of each agent is reflected by his position, the direction of his vision and the angle of the vision. Each agent can see everything of the agents in his visual field. This includes the positions, directions and angles of these agents. Third, each agent knows all the other agents' positions, even the positions of those agents that he can never see. All of the above is common knowledge of all agents. The motivation of [22] is to reason about not only the agents' observation abilities, but also their knowledge. For example, if agent b is in a's visual field, a will see b's position and direction of vision. Since a also knows each agent's position, he

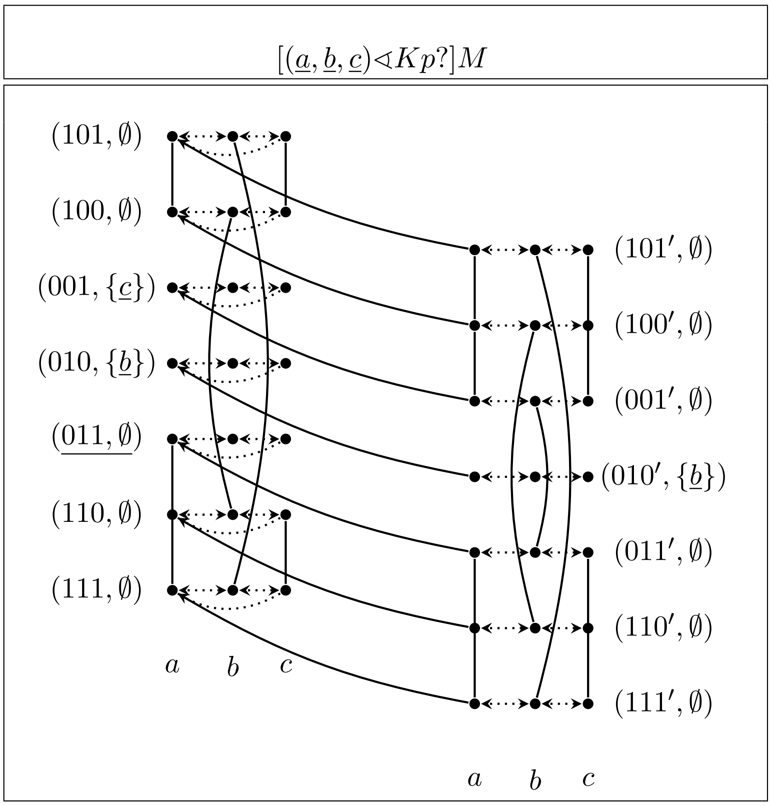


Figure 5.17: Last case of Scenario2 (Round Two)

knows who can be seen by b. So if b can turn and see a, then a should know that before b turns around.

The basic language of $\mathbf{BBL}$ is defined by a finite set of agents $Agt = \{a_1, \ldots, a_m\}$ as follows:

$$\varphi \quad ::= \quad a \rhd b \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid \overrightarrow{\langle a \rangle}\varphi$$

Then $\mathbf{BBL_C}$ and $\mathbf{BBL_D}$ are obtained by adding C_A and D_A to $\mathbf{BBL}$ respectively for every group of agents A. C_A and D_A are operators for common knowledge and distributed knowledge respectively, which we have discussed in Chapter Two.

The atomic proposition $a \rhd b$ reads: "agent a sees agent b." $K_a\varphi$ means "agent a knows φ." And $\overrightarrow{\langle a \rangle}$ is the *turning* (diamond) operator where $\overrightarrow{\langle a \rangle}\varphi$ means "agent a can turn so that φ holds."

Now we introduce the basic idea of the semantics. The geometric model in [22] is a tuple (pos, dir, ang) defined in the Euclidean plane. The quantities pos, dir and ang means the positions, the directions and the visual angles of each agent. The figure

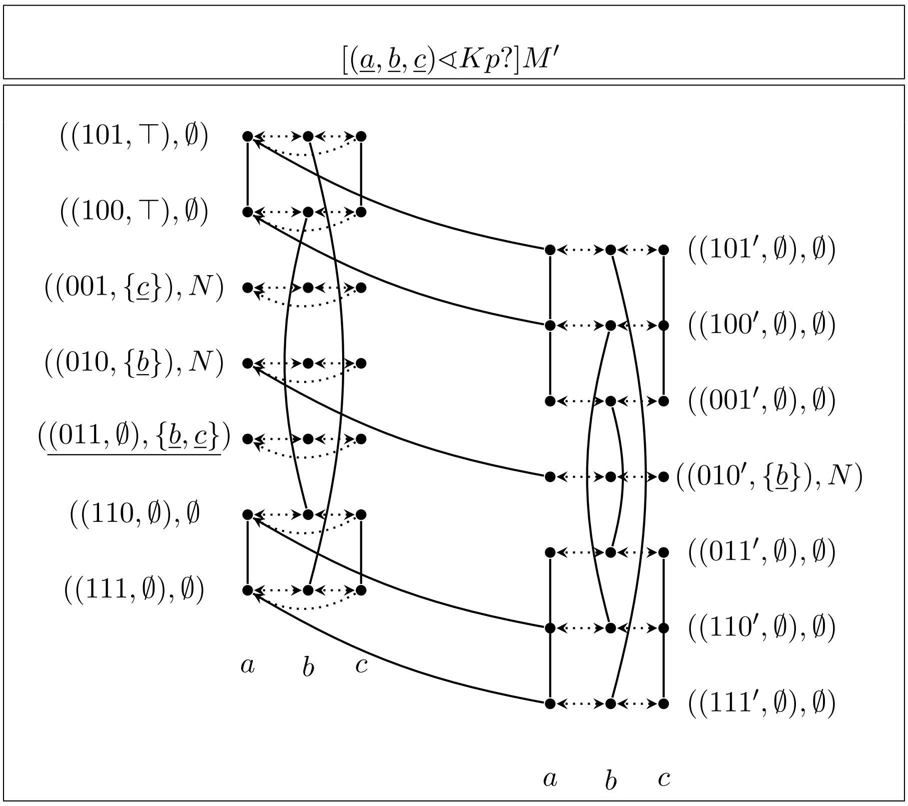
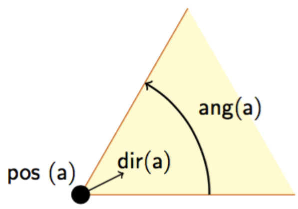


Figure 5.18: Last case of Scenario 2 (Round Three)

below shows how these quantities describe the visual field of an agent a.



However, the geometric models are not Kripke models. So they cannot model the actions of turning around, which are characterised by the turning operator $\overrightarrow{\langle a \rangle}$ for each $a \in Agt$. Given a geometric model, for any $a \in Agt$, define the set of the agents who are in a's visual field to be the *vision set of* a (notation: Γ_a). Take Figure 5.19 as example. The agent a sees nothing, b only sees a and c only sees b. So we have:

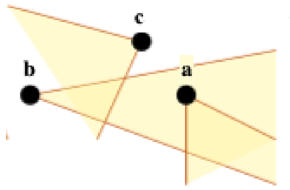


Figure 5.19: A geometric model

$$\Gamma_a = \emptyset, \Gamma_b = \{a\}, \Gamma_c = \{b\}$$

Since a can turn around in a full circle, his vision set may keep changing. So we have to consider all possible Γ_a. Call the collection of these sets *the spectrum of a's vision* (notation: $\mathcal{S}_a$). Back to Figure 5.19, we have:

$\mathcal{S}_a = \{\emptyset, \{b\}, \{c\}\}$

$\mathcal{S}_b = \{\emptyset, \{a\}, \{a, c\}, \{c\}\}$

$\mathcal{S}_c = \{\emptyset, \{a\}, \{b\}\}$

We only explain $\mathcal{S}_c$ briefly. For agent c, the spectrum of c's vision is a set of vision sets of c when c turns around. The agent c sees nobody when he stands with his back to a and b. In this case, his vision set is the empty set. Then he turns counterclockwise, he will see b first. Keep turning, he will see a instead, as depicted by the figure. That is why $\{a\}$ and $\{b\}$ are elements of $\mathcal{S}_c$. You may ask why $\{a, b\}$ is not in $\mathcal{S}_c$. This is because the angle of c's vision is not big enough to see both of them, given where they are standing in Figure 5.19. More generally, for any geometric model we can specify a function $\mathcal{S}$ mapping each agent a to its spectrum of vision $\mathcal{S}_a$. Given $\mathcal{S}$, [22] shows how to define a Kripke model $\mathcal{V}(\mathcal{S}) = \langle \mathbf{\Gamma}, \tau, \varepsilon \rangle$ where $\mathbf{\Gamma}$ is a set of possible worlds, τ_a and ε_a are binary relations on $\mathbf{\Gamma}$ for each $a \in Agt$. Here are the details.

$$\mathbf{\Gamma} = \{(\Gamma_a)_{a \in Agt} | \Gamma_a \in \mathcal{S}_a \text{ for all } a \in Agt\}$$

Then we have

$$\mathcal{V}(\mathcal{S}), (\Gamma_a)_{a \in Agt} \models a \triangleright b \quad \text{iff} \quad b \in \Gamma_a$$

For example, the world Γ of Figure 5.19 is as given earlier: $\Gamma_a = \emptyset, \Gamma_b = \{a\}, \Gamma_c = \{b\}$. This clearly does not satisfy $a \triangleright b$ because b is not in Γ_a, which is the empty set.

For each $a \in Agt$, define τ_a of $\mathcal{V}(\mathcal{S})$ as follows:

$$\tau_a = \{((\Gamma_b)_{b\in Agt}, (\Gamma'_b)_{b\in Agt}) | \Gamma_b = \Gamma'_b \text{ for all } b \neq a\}$$

Then we have

$\mathcal{V}(\mathcal{S}), (\Gamma_b)_{b\in Agt} \models \overrightarrow{\langle a\rangle}\varphi$ iff $\mathcal{V}(\mathcal{S}), (\Gamma'_b)_{b\in Agt} \models \varphi$ for some $(\Gamma'_b)_{b\in Agt} \in \mathbf{\Gamma}$ such that $\tau_a((\Gamma_b)_{b\in Agt}, (\Gamma'_b)_{b\in Agt})$

Back to Figure 5.19, we now consider another possible world Γ' where everything keeps unchanged except that b turns around so that he sees nothing, i.e.,

$$\Gamma_a = \emptyset, \Gamma_b = \emptyset, \Gamma_c = \{b\}$$

Then by the definition of τ, we have $\tau_b(\Gamma, \Gamma')$, but not $\tau_a(\Gamma, \Gamma')$. So we have $\mathcal{V}(\mathcal{S}), \Gamma \models \overrightarrow{\langle b\rangle}\neg b \triangleright a$ saying that b can turn in Γ so that b cannot see a.

Finally, a should not be able to distinguish between Γ and Γ' in Figure 5.19 because a sees the same agents in Γ and Γ'. But this condition is not sufficient to define the indistinguishability. Why? Take c as example. Since c sees b in both worlds, c realises that b's vision direction has changed. Since a's position is common knowledge, c realises that b cannot see a in Γ'. So c is distinguishable between Γ and Γ' although he sees the same agents in both worlds. Hence, an agent is indistinguishable between two worlds if he sees the same agents and all the agents he sees see the same agents in both worlds. Formally, for each $a \in Agt$, the indistinguishable relation ε_a of $\mathcal{V}(\mathcal{S})$ is defined as follows:

$$\varepsilon_a = \{((\Gamma_b)_{b\in Agt}, (\Gamma'_b)_{b\in Agt}) | \Gamma_b = \Gamma'_b \text{ for all } b \in \{a\} \cup \Gamma_a\}$$

Then we have

$\mathcal{V}(\mathcal{S}), (\Gamma_b)_{b\in Agt} \models K_a\varphi$ iff $\mathcal{V}(\mathcal{S}), (\Gamma'_b)_{b\in Agt} \models \varphi$ for all $(\Gamma'_b)_{b\in Agt}$ such that $\varepsilon_a((\Gamma_b)_{b\in Agt}, (\Gamma'_b)_{b\in Agt})$.

The standard public announcement operator, as we have discussed in the previous section, can be added to $\mathbf{BBL}$. This may give the agents additional information to obtain knowledge. Back to Figure 5.19, if b public announces that he cannot see c (denoted by $[!\neg b \triangleright c]$), then Γ' will be deleted in the updated model, then we do not have $\varepsilon_a(\Gamma, \Gamma')$.

$\mathbf{BBL}$ and $\mathbf{DSEL}$ clearly focus on different aspects of the logic of seeing and knowing.

$\mathbf{BBL}$ studies an interesting class of dynamic modalities of motion. But the movement studied in $\mathbf{BBL}$ is limited. As we discussed at the beginning, [22] assumes that every agent's position is common knowledge. This requires that agents are stationary. If the agents are mobile, then it is no longer true that the truth of formulas depends on what agents see. The construction of $\mathcal{V}(\mathcal{S})$ also doesn't support mobile agents. For example, in Figure 5.19, however, c will now be indistinguishable between Γ and Γ' since a may be still in b's visual field by moving.

$\mathbf{DSEL}$ considers a wider range of communications such as observable announcements which are semi-private. Even on the level of public announcements, $[n!p]$ (n public announces that he is muddy) cannot expressed by $\mathbf{BBL}$ due to the syntactic limitation.

Another way of comparing the two logics is to ask their points of view on the relationship between seeing and knowing. For $\mathbf{DSEL}$, we have listed a number of axioms that related to this issue (See Chapter 3, section 3.5.3). $\mathbf{BBL}$ can realise SK_{I}, $\mathsf{SK}_{\mathsf{I}'}$, $\mathsf{SK}_{\mathsf{II}}$ and $\mathsf{SK}_{\mathsf{II}'}$. The failure to realise the rest of axioms is also due to the syntactic limitation of $\mathbf{BBL}$.

Another property that we haven't so far considered is the relationship between seeing and common knowledge. Both $\mathbf{BBL}$ and $\mathbf{DSEL}$ can express that seeing each other is common knowledge: $(a \triangleright b \wedge b \triangleright a) \to C_{a,b}(a \triangleright b \wedge b \triangleright a)$. However, the $\mathbf{BBL}$ formula $a \triangleright b \to K_a[\overrightarrow{b}](a \triangleright b)$ states that if a can see b, then a knows that no matter how b turns around, this will still be true. But this is only approximated by SK_{I} saying that if a can see b, then a knows it. Another interesting example is c's distinguishability of Γ and Γ' in the above example. This is under the assumption that each agent's position is common knowledge. $\mathbf{DSEL}$ doesn't have this assumption, so c will be indistinguishable to Γ and Γ' because c cannot see a and then he cannot realise that b cannot see a in Γ' too.

On a purely technical level, $\mathbf{BBL}$ can define betweenness and collinearity while $\mathbf{DSEL}$ can do neither. For example, under the assumption that the agents' angles of vision are strictly less than 2π, the betweenness relation $B(acb)$ (c lies strictly between a and b) can be defined as follows:

$$B(acb) := [\overrightarrow{a}](a \triangleright b \to a \triangleright c) \wedge [\overrightarrow{b}](b \triangleright a \to b \triangleright c)$$

In all, it seems that $\mathbf{BBL}$ could be a good complement to $\mathbf{DSEL}$. The expressivity of agents' observation abilities could be embedded into $\mathbf{DSEL}$. Consider a logic combining the turning operator of $\mathbf{BBL}$ with observable announcement operators of $\mathbf{DSEL}$. Take Scenario I OA+SV as example (See Chapter One, Section 1.1). If b can turn and sees a and c, b will know not only both a and c can see him, but also whether a and c are muddy or not. Also b will see c's observable announcement saying that c doesn't know if c is muddy or not. Then b will know he is muddy. Basically, the combination makes agents be more powerful on reasoning.

5.5 Further Directions

This chapter provides a semantic treatment of dynamic operators to axiomatise $\mathbf{DSEL}$ with the methods of dynamic epistemic logic. I proposed an indexical action structure in which the epistemic accessibility relations between actions are defined indirectly, by the formulas $\mathsf{ind}(s, t)$ which describe those agents who cannot distinguish between action tokens s and t. Then I defined two kinds of dynamic operator, $[\mathbf{S}, s]$, where s is a specific action token in indexical action structure $\mathbf{S}$, and $[\mathbf{S}]$, which quantifies over all its action tokens. The semantics of these operators is closely based on the product construction of DEL.

I then defined $[n \lhd \varphi]$ and $[n \bar{\lhd} \varphi]$ to model the act of agent n observably announcing φ. Although useful, these operators are not sufficient to analyse most of the scenarios from Chapter 1, which require simultaneous announcements by a number of agents. I introduced the simultaneous announcement operators. I defined $[n_1, \ldots, n_k \lhd \varphi?]$ to model the effect of asking the question "φ?" to the agents $n_1, \ldots, n_k$. Then I gave model presentations of some scenarios of Chapter 1.

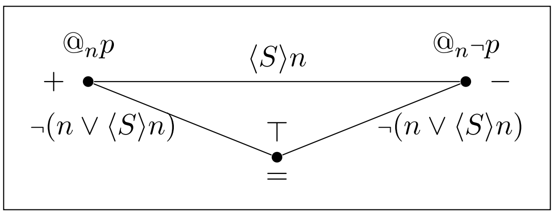
Here are some further developments:

1. propose an axiomatisation for the logic with the indexical action structure operators and prove the completeness. There are recursion axioms for public announcement logic and DEL in [7] and [61] and so on. I hope a similar strategy can be followed but the two-dimensional nature of the operators makes it a bit harder.

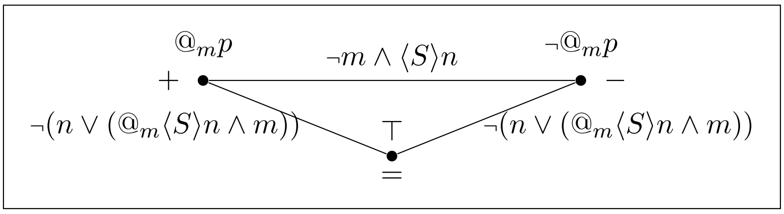
2. investigate the frame properties in the product construction.

3. investigate the dynamic operators, which are discussed informally by Seligman, Liu and Girard in [56].

For example, agent a reads his medical report, which has either a positive result p or negative result $\neg p$. All agents who can see a will know that a knows the result, without knowing the result itself. And those that cannot see a don't know the fact that a even knows the result. This can be represented by the following action structure:



Another example is a second-person indexical announcement, such as a gesture or wink by n as an indication of something to m, e.g. you are in danger ($@_m p$). Other agents who can see n only know that n made a gesture or winked. Only m and n know the meaning behind it. This is represented by the following action structure:


$@_m p$
$\neg m \wedge \langle S \rangle n$
$\neg @_m p$
$+$
$-$
$\neg(n \vee (@_m \langle S \rangle n \wedge m))$
$\top$
$\neg(n \vee (@_m \langle S \rangle n \wedge m))$
$=$

Chapter 6

Conclusions and Future Work

This thesis is a technical contribution to the development of **SEL**. The main result is an axiomatisation of social epistemic logic and proof of its completeness.

We started with a number of scenarios of seeing and knowing to motivate and provide illustrative examples of the development of the logic. Chapter 2 reviewed the semantics and axiomatisation of standard epistemic logic and gave the language and semantics of Basic Social Epistemic Logic (**BSEL**).

Chapter 3 firstly proved completeness of the basic modal logic with the step-by-step method, which we formulated in a general way, to allow applications to the more complicated settings of **BSEL** and **SEL**. Then we proposed an axiomatisation of **BSEL** and several extensions along both dimensions with a proof of completeness for each of them using the step-by-step method. We discussed extensions with rigid names, **KD45** and **DSEL**, and the SK axioms as an example of the interaction of seeing and knowing. We proposed a tableau system based on Christoff, Hansen and Proietti's results. We also compared our results with Sano's.

Chapter 4 discussed an axiomatisation and completeness proof of **SEL**, which is an extension of **BSEL** with the downarrow operator. The proof is long and complex.

Chapter 5 covered a brief discussion of **DSEL**, the result of adding dynamic operators to **SEL**. We adapted dynamic epistemic logic to our two dimensional settings and gave a semantic definition of different kinds of observable announcements. A semantic treatment of dynamic operators was given with the methods of dynamic epistemic logic.

In Chapter 3, 4 and 5, I discussed further directions for research on the technical development of the logic. There is also potential for philosophical extensions. The **S5** extension of **KD45** provides "social doxastic logic" for analysing the diffusion of possibly false beliefs in a community. Such a logic would make comparison with the social psychology literature much easier. Take a study by Miller and Macfarland [45] for example. Readers who recall their own study experience, will find this very familiar. Several years ago, 132 undergraduates of Princeton University were randomly chosen to respond to a questionnaire about alcohol use on the Princeton campus. They answered some questions about their own opinions on alcohol consumption, and what they thought the "average student" thinks ([47]). The results indicated a sharp divergence between their own comfort level and the comfort level they estimated for others. In other words, most students believed that other students were more comfortable with the use of alcohol on campus than they were themselves. This phenomenon is called "pluralistic ignorance" and was first coined by Floyd Allport [1]. It refers to a situation where the majority of a community has an opinion but assumes wrongly that most others believe the negation of what they believe. Prentice and Miller [47] also discussed the consequences of pluralistic ignorance. They think there are three strategies for people involved pluralistic ignorance: (1) change their own opinion to make it compatible with those of others; (2) change the opinion of the group to match their own opinion; and (3) alienate themselves from the group. They think that (1) is the easiest or cheapest to adopt. This helps to keep the phenomenon robust and leads to a collective irrationality.

Recently, Hendicks and Hanson, in their book *Infostorms* [32], introduced the topic to logicians but without giving technical details.

Christoff and Hansen [17] gave an account of pluralistic ignorance using a logical framework inspired by *Logic in the Community*. So it closely relates to $\mathbf{SEL}$. A more direct influence was an earlier work of mine. Liang and Seligman [41] used a two-dimensional logic to give a model of "peer pressure". We proposed an account of preference change in a community, which has a network of "friends" with an operator F meaning "all my friends" and an operator P meaning "prefers". Social influence by an issue φ was modeled by a dynamic operator $\#_\varphi$. It changed every agent's preference depending on the extent that he is influenced by his friends. If he is strongly influenced, he will prefer φ no matter what he preferred before. If he is weakly influenced, he will at least not prefer $\neg\varphi$. And if he is not influenced at all, he will maintain his original preference. Strong and weak influence were defined as:

$$SI^1\varphi = FP\varphi \wedge \langle F\rangle P\varphi$$
$$WI^1\varphi = \langle F\rangle P\varphi \wedge F\neg P\neg\varphi$$

In other words, an agent is strongly influenced (SI^1) if all his friends prefer φ (and he has at least one friend) and weakly influenced (WI^1) if none of his friends prefer $\neg\varphi$ and at least one prefers φ.

We showed how this operator describes a number of patterns of behavior in social groups and gave a logical analysis of the conditions under the stablisation of a group's preferences.

Our approach was then extended by Liu, Seligman and Girard [43] to model social influence concerning beliefs.

Christoff and Hansen [17] applied this approach to model pluralistic ignorance by introducing a description between two kinds of belief. Inner belief (I_B) is the belief that the agent has in his mind, which can only be accessed by the agent himself. Expressed belief (E_B) is other agents' interpretation of his action. Only the expressed belief can be accessed by other agents, and only the expressed belief can influence other agents. They proposed an analysis of pluralistic ignorance as:

$$PI_\varphi := G(I_B\varphi \wedge E_B\neg\varphi)$$

where G is a universal operator interpreted as "for all people". In other words, pluralistic ignorance about φ is the state in which everyone has an inner belief φ but expresses the belief $\neg\varphi$.

They suggested that social influence works only on the level of expressed belief and they defined strong and weak influence as:

$$SI^2\varphi = FE_B\varphi \wedge \langle F\rangle E_B\varphi$$
$$WI^2\varphi = \langle F\rangle E_B\varphi \wedge F\neg E_B\neg\varphi$$

For example, an agent who believes φ but all his friends all believe $\neg\varphi$ will end up expressing a belief that $\neg\varphi$, even though he still has the inner belief that φ. It is easy to see how PI_φ can result. One problem with Christoff and Hanson's approach is that they didn't analyse the relationship between expressed belief and inner belief.

I suggest that a doxastic version of **SEL**, i.e. $\mathbf{SEL} + \mathbf{D45}$, can describe pluralistic ignorance similarly to Christoff and Hanson's account, but use only one kind of belief (B). The basic idea is that pluralistic ignorance occurs when everyone believe both φ and anyone he can see believes $\neg\varphi$, i.e.

$$PI^2_\varphi := GB(\varphi \wedge SB\neg\varphi)$$

"Expressed belief" is replaced by a (higher order) belief in those who can see me. The corresponding strong and weak influence are as follows:

$$SI^3\varphi = B(SB\varphi \wedge \langle S\rangle B\varphi)$$
$$WI^3\varphi = B(\langle S\rangle B\varphi \wedge S\neg B\neg\varphi)$$

However, this is only the beginning of a solution. In future work, I propose to study dynamic operators for modeling social influence, using the indexical action structures from Chapter 5.

Nonetheless, I think that the link between perception and belief is an important part of many cases that have been considered to be relevant to pluralistic ignorance.

To model the dynamic properties of pluralistic ignorance, we need some plausible revisited operators like $\downarrow_x [\downarrow_y @_x\langle S\rangle y{\lhd}B\varphi]\psi$ expressing "after all the people I can see show that they believe φ, ψ."

Back to the campus alcohol example. When agent a believes that people shouldn't drink too much alcohol at campus (p), he looks around, and only sees others enjoying their drinking. Let b be one of them. The observation that b drinks like a fish can be regarded by a as b's observable announcement of $\neg p$ without being aware of this. Since most of the students a sees are making the same announcements, a has a strong influence by $\neg p$ and ends up believing $\neg p$. This response is the first of the strategies for pluralistic ignorance that we have just introduced by Prentice and Miller [47].

The method can also be applied to modeling the Bystander Effect, another well-studied phenomenon of social psychology. Someone falls to the ground in a crowded street and obviously needs help. Many people have seen him but no one helps. The more witnesses there are, the less likely it is that anyone will offer help. Explanations for this result are that the bystanders either are [47] unsure of the seriousness of the victim's condition, looking at the reaction of others who are also not helping, then concluding that the situation is not serious, or [20] believe that one of the other bystanders should offer help to them. In either case, the result is the same: no one helps.

I believe that who sees what will affect the outcome of the Bystander Effect. And these details are exactly how $\mathbf{SEL}$ can be used to reason. Consider the situation in which an agent falls at a street corner. There are only two bystanders walking towards the agent such that one is walking on one street and the other is walking on another. They are in such a situation that both of them can see the agent but neither of them can see each other. So after the agent shows that he believes he needs help, either bystander will be strongly influenced and as such believe he needs help. Since they cannot see anyone else, they also believe that no one else can offer help. So seeing "dissolves" the bystander effect.

From an epistemological point of view, pluralistic ignorance and the bystander effect involve false belief about the beliefs of others and end up with the collective belief error because people don't have enough information. Observation is a primary way for people to access information about others, but observation alone is usually not enough.

There are are also some purely technical problems that could be investigated. For example, hybrid logic has various powerful completeness theorems for extensions of the base logic with axioms of various syntactic forms: the so-called "pure" formulas, which contain no propositional variables. It would be interesting to investigate pure extensions of $\mathbf{SEL}$ and $\mathbf{BSEL}$. The existence of Sahlqvist forms, known from modal logic, is another point to consider. It is a fairly strong conjecture that similar results could be given for $\mathbf{SEL}$. However, more central to the concerns of this thesis is the open problem of proving completeness for $\mathbf{SEL}$ or $\mathbf{BSEL}$ using a canonical model.

As described in Section 3.7, we were unable to do this.

Finally, an important issue that we have not addressed at all in this thesis is that of computational complexity. The obvious questions to address are whether $\mathbf{BSEL}$ and $\mathbf{SEL}$ are decidable, i.e., whether the satisfaction problem for these logics is decidable. By the tableau system in Section 3.1, we know that $\mathbf{BSEL}$ is decidable. The extensions $\mathbf{T}$ and $\mathbf{S4}$ could be easily applied with the same method to show they are also decidable. But we don't know about $\mathbf{S5}$ and $\mathbf{BSEL_{SK}}$. The undecidability of hybrid logic $\mathcal{H}(@, \downarrow)$ is proved in [2, 12], based on an encoding of the $\aleph \times \aleph$ tiling problem. So $\mathbf{SEL}$ is likely to be undecidable, but there may still be useful to investigate the tableau system for it.

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