# NOTE

# Inverse and direct theorems for best uniform approximation by polynomials

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**ABSTRACT**

It is shown that the standard method of obtaining direct (Jackson) theorems for the order of best uniform approximation by algebraic polynomials from those for trigonometric polynomials also provides inverse (Bernstein) theorems.

**Key Words:** Bernstein theorem, inverse theorem, Jackson theorem, direct theorem, Ditzian–Totik moduli of smoothness

**AMS (MOS) Subject Classifications:** primary G41A17, 41A25, 41A27, secondary 30E10, 42A10

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# 1. Introduction and main results

The purpose of this note is to show that the standard method of obtaining direct (Jackson) theorems for the order of best uniform approximation by algebraic polynomials from those for trigonometric polynomials also provides inverse (Bernstein) theorems. The simple proof offered for the inverse theorems for approximation order $\mathcal{O}(1/n^\alpha)$, $0<\alpha\le 1$ by algebraic polynomials (which were obtained only in the early 1980's) naturally leads to the Ditzian–Totik modulus of continuity (see Ditzian and Totik [DT87] for details). It is hoped that this elementary approach will give insight into the celebrated observation of Nikol'skii [N46] that "algebraic polynomial approximation to smooth functions is better towards the endpoints of the interval".

Let $C(\mathbb{T})$ denote the space of continuous $2\pi$–periodic functions (continuous functions on the circle group $\mathbb{T}$) with the uniform norm $\|\cdot\|_\infty$. We identify $g\in C(\mathbb{T})$ with a function from $C[-\pi,\pi]$ in the usual way, so, e.g., $g$ is even means $g(\theta)=g(-\theta)$, $-\pi\le\theta\le\pi$. The error in best uniform approximation to $g\in C(\mathbb{T})$ from $\mathrm{T}_n$ (the trigonometric polynomials of degree $\le n$) will be denoted by $E_n^*(g)$, and the error in best uniform approximation to $f\in C[-1,1]$ from $\Pi_n$ (the algebraic polynomials of degree $\le n$) by $E_n(f)$.

Direct (Jackson) theorems state that if $E_n^*(g)$, $E_n(f)$ converge to 0 at a certain rate then this implies $g$, $f$ belong to certain smoothness classes, while inverse (Bernstein) theorems give results in the opposite direction. The standard method for obtaining direct results for $E_n(f)$ is to make the substitution $f\mapsto g:=f\circ\cos$, and use known direct results for the (simpler) case of $E_n^*(g)$. We now prove that if this is done carefully enough, then inverse results for $E_n^*(g)$ also imply inverse results for $E_n(f)$.

**Theorem 1.1.** *Suppose that $r_n\to 0$, and $S\subset C(\mathbb{T})$. Then, the direct–inverse theorems*
(a) *For $g\in C(\mathbb{T})$,* $\quad E_n^*(g)=\mathcal{O}(r_n)\iff g\in S$
(b) *For even $g\in C(\mathbb{T})$,* $\quad E_n^*(g)=\mathcal{O}(r_n)\iff g\in S$
(c) *For $f\in C[-1,1]$,* $\quad E_n(f)=\mathcal{O}(r_n)\iff f\circ\cos\in S$
*satisfy the implications* (a) $\Longrightarrow$ (b) $\iff$ (c). *These implications remain valid if the $\mathcal{O}(r_n)$ is replaced by $o(r_n)$ throughout, or only the inverse (direct) part of* (a),(b),(c) *is considered.*

**Proof:** Since the 1–1 onto linear map

$$C[-1,1]\to\{g\in C(\mathbb{T}): g \text{ is even}\}: f\mapsto g:=f\circ\cos \tag{1.2}$$

is an isometry which maps $\Pi_n$ onto the even trigonometric polynomials of degree $\le n$, and the best approximation to an even $g\in C(T)$ from $\mathrm{T}_n$ is even, it follows that $E_n^*(g)=E_n(f)$. This establishes (b) $\iff$ (c). The implication (a) $\Longrightarrow$ (b) is immediate. $\square$

Theorem 1.1 says that direct–inverse theorems for algebraic polynomial approximation are in 1–1 correspondence with direct–inverse theorems for trigonometric polynomial approximation to *even* functions. The author is unaware whether, under conditions on $S$ necessary for (a) to hold ($S$ is translation and reflection invariant), these also correspond to direct–inverse theorems for trigonometric polynomial approximation to *all* functions in $C(\mathbb{T})$, i.e., the implication (a) $\Longleftarrow$ (b) holds. However, this is not a crucial question, since in light of the history of Jackson–Bernstein type results, it is unlikely that such a result would ever be needed to deduce direct–inverse theorems for trigonometric polynomial approximation from those for algebraic polynomial approximation.

The smoothness condition $f \circ \cos \in S$ ($S$ translation invariant) imposes less smoothness on $f$ at the endpoints of the interval $[-1,1]$ (cf Lemma 1.3). Thus, Theorem 1.1 can be interpreted as Nikol'skii's observation in the form: "the functions which can be approximated by algebraic polynomials to a given order are less smooth towards the endpoints of the interval".

We illustrate Theorem 1.1 with some examples. These are related to the 'Ditzian–Totik' modulus of continuity (see [DT87])

$$\omega_\varphi(f,t) := \omega_\varphi(f,t,[a,b]) := \sup_{0<h\le t} |\Delta_{h\varphi(x)} f(x)|, \qquad f \in C[a,b],$$

where $\varphi$ is a nonnegative function, with $\varphi := 1$ giving the usual modulus $\omega(f,t)$. For simplicity of exposition, we take $\Delta_{h\varphi(x)}$ to be the forward difference operator (instead of the symmetric difference operator), which makes no essential difference since the $\varphi$ we consider is continuous. Two moduli $\omega_1$, $\omega_2$ are said to be of the same order, written $\omega_1(t) \sim \omega_2(t)$, if there exist positive constants for which $C_1\omega_1(t) \le \omega_2 \le C_2\omega_2(t)$, for small $t > 0$. First we need a lemma.

**Lemma 1.3.** *For $f \in C[-1,1]$,*

$$\omega_\varphi(f,t,[-1,1]) \sim \omega(f\circ\cos,t,[-\pi,\pi]), \tag{1.4}$$

*where $\varphi(x) := \sqrt{1-x^2}$.*

**Proof:** In

$$\omega(f\circ\cos,t) := \sup_{\substack{0<h\le t \\ \theta,\theta+h\in[-\pi,\pi]}} |f\circ\cos(\theta+h) - f\circ\cos(\theta)|$$

we make the substitution $x = \min\{\cos\theta, \cos(\theta+h)\}$, $x+\delta = \max\{\cos\theta,\cos(\theta+h)\}$. Since cosine is continuously differentiable (with derivative $-\sin$),

$$\delta = \sqrt{1-x^2}(h+\varepsilon_x(h)) = \varphi(x)(h+\varepsilon_x(h)), \qquad |\varepsilon_x(h)| \le |o(h)|,$$

where $|o(h)|$ denotes some nonnegative function of $h$ which is independent of $x$ and is $o(h)$. Thus, for small $t$, we obtain

$$\sup_{\substack{0<h\le t-|o(t)| \\ x,x+\delta\in[-1,1] \\ \delta=\varphi(x)h}} |f(x+\delta)-f(x)| \le \sup_{\substack{0<h\le t \\ x,x+\delta\in[-1,1] \\ \delta=\varphi(x)(h+\varepsilon_x(h))}} |f(x+\delta)-f(x)| \le \sup_{\substack{0<h\le t+|o(t)| \\ x,x+\delta\in[-1,1] \\ \delta=\varphi(x)h}} |f(x+\delta)-f(x)|,$$

i.e.,

$$\omega_\varphi(f,t-|o(t)|) \le \omega(f\circ\cos,t) \le \omega_\varphi(f,t+|o(t)|). \qquad (1.5).$$

From (1.5), and the subadditivity property of $\omega$, we obtain

$$\frac{\omega_\varphi(f,t)}{\omega(f\circ\cos,t)} \le \frac{\omega(f\circ\cos,t+|o(t)|)}{\omega(f\circ\cos,t)} \le \left(1+\frac{t+|o(t)|}{t}\right)\frac{\omega(f\circ\cos,t)}{\omega(f\circ\cos,t)} \to 2, \quad t\to 0,$$

and similarly,

$$\frac{\omega(f\circ\cos,t)}{\omega_\varphi(f,t)} \le \frac{\omega(f\circ\cos,t)}{\omega(f\circ\cos,t-|o(t)|)} \le \left(1+\frac{t}{t-|o(t)|}\right) \to 2, \quad t\to 0,$$

which establishes (1.4). □

There are numerous variations of the above argument, e.g., if $g:[c,d]\to[a,b]$ is a differentiable bijection, then $\omega(f\circ g,t,[c,d])\sim\omega_\varphi(f,t,[a,b])$, $\varphi:=|Dg\circ g^{-1}|$.

We now translate the classical result of Bernstein (inverse part) and Jackson (direct part) that: for $g\in C(\mathbb{T})$, $0<\alpha<1$,

$$E_n^*(g)=\mathcal{O}(1/n^\alpha) \quad\Longleftrightarrow\quad g\in\operatorname{Lip}\alpha, \tag{1.6}$$

into a direct–inverse result for algebraic polynomial approximation.

**Corollary 1.7.** *For $f\in C[-1,1]$ and $0<\alpha<1$, the following are equivalent*
(a) $E_n(f)=\mathcal{O}(1/n^\alpha)$
(b) $f\circ\cos\in\operatorname{Lip}\alpha$.
(c) $\omega_\varphi(f,t)=\mathcal{O}(t^\alpha)$, *where* $\varphi(x):=\sqrt{1-x^2}$
(d) $\omega(f\circ\cos,t)=\mathcal{O}(t^\alpha)$
(e) *There exists a constant $C$ and a sequence of polynomials $p_n$ of degree $n$ satisfying*

$$|f(x)-p_n(x)|\le C/n^\alpha, \qquad -1\le x\le 1.$$

**Proof:** The equivalence of (a),(b),(d),(e) follows immediately from Theorem 1.1 and the result (1.6) for trigonometric approximation. The implications (c) $\Longleftrightarrow$ (d) follow from Lemma 1.3. □

In the same way, the classical results for trigonometric polynomial approximation order $\mathcal{O}(1/n^\alpha)$, $\alpha\ge 1$, in terms of the *generalised Lipschitz space* $\operatorname{Lip}^*(\alpha,\infty)$ (see DeVore and Lorentz [DL93:p51]) can be translated into results for algebraic polynomial approximation. We consider only the case $\alpha=1$, that: for $g\in C(\mathbb{T})$

$$E_n^*(g)=\mathcal{O}(1/n) \quad\Longleftrightarrow\quad g\in Z, \tag{1.8}$$

where $Z$ is the *Zygmund space*, which consists of those functions satisfying

$$|g(\theta)-2g(\theta+h)+g(\theta+2h)|\le M\,h, \tag{1.9}$$

for some constant $M$, i.e., $\omega_2(g,t)=\mathcal{O}(t)$, with $\omega_2$ the usual second order modulus of smoothness.

**Corollary 1.10.** *For $f \in C[-1,1]$, the following are equivalent*
(a) $E_n(f) = \mathcal{O}(1/n)$
(b) $f \circ \cos \in Z$
(c) *There exists a constant $M$ such that*

$$|f \circ \cos(\theta) - 2f \circ \cos(\theta + h) + f \circ \cos(\theta + 2h)| \le M\, h$$

(d) $\omega_2(f \circ \cos, t) = \mathcal{O}(t)$
(e) *There exists a constant $C$ and a sequence of polynomials $p_n$ of degree $n$ satisfying*

$$|f(x) - p_n(x)| \le C/n, \qquad -1 \le x \le 1.$$

**Proof:** Follows immediately from Theorem 1.1 and (1.8), (1.9). □

The isometry (1.2) can also be used to transfer sharp direct theorems to the algebraic case, e.g., Korneichuk's sharp form of Jackson's theorem (see [K62]),

$$E_n^*(g) \le \omega(g, \frac{\pi}{n+1}), \qquad g \in C(\mathbb{T}), \tag{1.11}$$

gives the following.

**Corollary 1.12 (Sharp Jackson Theorem).** *For $f \in C[-1,1]$,*

$$E_n(f) \le \omega(f \circ \cos, \frac{\pi}{n+1}) \le M\, \omega_\varphi(f, \frac{1}{n}) \le M\, \omega(f, \frac{1}{n}), \tag{1.13}$$

*where the constant 1 multiplying $\omega(f \circ \cos, \pi/(n+1))$ is the best possible one independent of $f$ and $n$, and $M$ is some constant which is independent of $f$ and $n$.*

**Proof:** Since (1.11) is sharp as an inequality for even $g \in C(T)$, using (1.2) we obtain the sharp inequality

$$E_n(f) = E_n^*(f \circ \cos) \le \omega(f \circ \cos, \frac{\pi}{n+1}).$$

By the subadditivity property of $\omega$, and Lemma 1.3,

$$\omega(f \circ \cos, \frac{\pi}{n+1}) \le \left(1 + \frac{n\pi}{n+1}\right) \omega(f \circ \cos, \frac{1}{n}) \le M\, \omega_\varphi(f, \frac{1}{n}) \le M\, \omega(f, \frac{1}{n}),$$

with the last inequality following immediately from the definition of $\omega_\varphi$ (since $\varphi \le 1$). □

The $L_p$–analogue ($1 \le p < \infty$) of Theorem 1.1 relates direct–inverse theorems for *weighted* $L_p[-1,1]$–approximation by algebraic polynomials (the weight being $1/\varphi$) to those for $L_p(\mathbb{T})$–approximation of even functions by trigonometric polynomials. The corresponding version of the equivalence (a)$\Longleftrightarrow$(c) of Corollary 1.7 is given in Felton [F97:Th.5.1].

## References


[DL93] R. A. DeVore and G. G. Lorentz, "Constructive approximation", Springer-Verlag, Berlin, 1993.

[DT87] Z. Ditzian and V. Totik, "Moduli of smoothness", Springer–Verlag, New York, 1987.

[F97] M. Felton, Characterisation of best algebraic approximation by an algebraic modulus of smoothness, *J. Approx. Theory* **89** (1997), 1–25.

[K62] N. P. Korneichuk, The exact constant in D. Jackson's theorem on best uniform approximation of continuous periodic functions, *Soviet Math. Dokl.* **3** (1962), 1040–1041.

[N46] S. M. Nikol'skii, On the best approximation of functions satisfying a Lipschitz condition by polynomials, *Izv. Akad. Nauk SSSR Ser. Math* **10** (1946), 295–322.