Available online at www.sciencedirect.com
SCIENCE DIRECT
JOURNAL OF Algebra
Journal of Algebra 292 (2005) 110–121
www.elsevier.com/locate/jalgebra

Dade's invariant conjecture for the Chevalley groups of type G_2 in the defining characteristic

Shih-chang Huang

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand

Received 12 September 2004
Available online 2 March 2005
Communicated by Gerhard Hiss

Abstract

This paper is part of a program to study the conjecture of E.C. Dade on counting characters in blocks for several finite groups. In this paper, we verify Dade's invariant conjecture for the Chevalley groups $G_2(p^a)$ in the defining characteristic when $p \neq 2$ or 3. This implies Dade's final conjecture when $p \neq 2$ or 3.

1. Introduction

Let G be a finite group, p a prime dividing the order of G and B a p-block of G. Dade [7] generalized the Knörr–Robinson version [13] of the Alperin weight conjecture (see [1]) and presented his ordinary conjecture exhibiting the number of ordinary irreducible characters with a fixed defect in a given p-block B in terms of an alternating sum of related values for p-blocks of certain p-local subgroups (i.e. the p-subgroups and their normalizers) of G. He also announced that his final conjecture can be confirmed by verifying it for all non-abelian finite simple groups [8]; in addition, the invariant form of the conjecture is equivalent to the final conjecture if a finite group has both trivial Schur multiplier and cyclic outer automorphism group. Dade's invariant conjecture has been verified for

E-mail address: shua003@math.auckland.ac.nz.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.11.023

the groups $G_2(q)$ in the defining characteristic when $q = 2^a$ or 3^a [2] and non-defining characteristics [3]. Our goal in this paper is to verify Dade's invariant conjecture for the group $G_2(p^a)$ in the defining characteristic when $p \neq 2$ or 3, using the character tables of $G_2(q)$ and its Borel and parabolic subgroups [4,6]. Together with [12], this completes the verification of the conjecture for any blocks with positive defect of these groups.

The outline of this paper is as follows. In Section 2, we fix some notation and state Dade's invariant conjecture. In Section 3, we prove two lemmas on the parameter sets. In Section 4, we verify Dade's invariant conjecture for $G_2(p^a)$ when $p \neq 2$ or 3. Tables 2–5 provides details of characters with a fixed defect.

2. Dade's invariant conjecture

Let R be a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and by $\mathrm{Blk}(G)$ the set of p-blocks. We denote the principal block of G by B_0. If $H \leqslant G$, $B \in \mathrm{Blk}(G)$, and d is an integer, we denote by $\mathrm{Irr}(H, B, d)$ the set of characters $\chi \in \mathrm{Irr}(H)$ satisfying $\mathrm{d}(\chi) = d$ and $B(\chi)^G = B$ (in the sense of Brauer), where $\mathrm{d}(\chi) = \log_p(|H|_p) - \log_p(\chi(1)_p)$ and $B(\chi)$ is the block of H containing χ.

Given a p-subgroup chain

$$C: \quad P_0 < P_1 < \cdots < P_n$$

of G, define the length $|C| = n$, C_k: $P_0 < P_1 < \cdots < P_k$, $C(C) = C_G(P_n)$, and

$$N(C) = N_G(C) = N_G(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_n).$$

The chain C is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and
(b) $P_k = O_p(N(C_k))$ for $1 \leqslant k \leqslant n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G.

Suppose $1 \to G \to E \to \overline{E} \to 1$ is an exact sequence, so that E is an extension of G by $\overline{E}$. Then E acts on $\mathcal{R}$ by conjugation. Given $C \in \mathcal{R}$ and $\varphi \in \mathrm{Irr}(N_G(C))$, let $N_E(C, \varphi)$ be the stabilizer of (C, φ) in E, and

$$N_{\overline{E}}(C, \varphi) = N_E(C, \varphi)/N_G(C).$$

For $B \in \mathrm{Blk}(G)$, an integer $d \geqslant 0$ and $U \leqslant \overline{E}$, let $\mathrm{k}(N_G(C), B, d, U)$ be the number of characters in the set

$$\mathrm{Irr}\big(N_G(C), B, d, U\big) = \big\{\varphi \in \mathrm{Irr}\big(N_G(C), B, d\big)\colon\ N_{\overline{E}}(C, \varphi) = U\big\}.$$

Dade's invariant conjecture is stated as follows.

Dade's Invariant Conjecture [8]. *If $O_p(G) = 1$ and B is a p-block of G with defect group $D(B) \neq 1$, then*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}\big(N_G(C), B, d, U\big) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

Let $A = \mathrm{Aut}(G)$ and $O = \mathrm{Out}(G)$ be the automorphism and outer automorphism groups of G, respectively. Then we may suppose $\overline{E} = \mathrm{Out}(G)$. If moreover, $\mathrm{Out}(G)$ is cyclic, then we set

$$\mathrm{k}\big(N_G(C), B, d, |U|\big) = \mathrm{k}\big(N_G(C), B, d, U\big).$$

Dade's invariant conjecture is equivalent to his final one (see [8]) if G has a trivial Schur multiplier and a cyclic outer automorphism group. If $G = G_2(q)$, then $\mathrm{Out}(G)$ is cyclic and the Schur multiplier of G is trivial except when $q = 3$ or 4, in which case $G_2(3)$ and $G_2(4)$ have Schur multipliers 3 and 2, respectively.

3. Two lemmas

Let $G = G_2(p^a)$ with $p \neq 2$ or 3, let W be a Borel subgroup, P, Q the parabolic subgroups of G. Let $O = \mathrm{Out}(G)$. If $p \neq 2$ or 3, then $O = \langle \alpha \rangle$ and $A = G \rtimes \langle \alpha \rangle$, where α is a field automorphism of order a.

If $L \in \{G, W, P, Q\}$, then the character table of L is given in [4,6]. We will follow the same notation. Let $X = X(q)$ be a parameter set given by [4], and let $U \leqslant O$. Denote by $C_X(U)$ the set of fixed-points of X under the action of U.

Lemma 3.1. *In the notation of* [4] (*cf. also* [9,10]), *suppose $p \neq 2$ or 3, $q = p^a$, and $t \mid a$. Let $X(p^a)$ be one of*

$$R_0\big(p^a\big), \quad {}^2R_2\big(p^a\big) \cup {}^2T_1\big(p^a\big), \quad {}^2R_3\big(p^a\big) \cup {}^2T_1\big(p^a\big), \quad {}^2R_1\big(p^a\big) \cup {}^2S_1\big(p^a\big),$$
$${}^2R_1^*\big(p^a\big) \cup {}^2S_1^*\big(p^a\big).$$

Suppose $\langle \alpha \rangle$ acts on $X(p^a)$ by $x^\alpha = px$ and $H = \langle \alpha^t \rangle$. Then

$$C_{X(p^a)}(H) \simeq X\big(p^t\big)$$

as H-sets, where α acts similarly on $X(p^t)$.

Proof. (i) Suppose $k \in R_0(p^a)$, and

$$C_{R_0(p^a)}(H) = \big\{x \in R_0\big(p^a\big)\colon\ x^{\alpha^t} = x\big\} = \big\{x \in R_0\big(p^a\big)\colon\ p^t x = x\big\}.$$

We identify $R_0(p^a)$ with $\mathbb{Z}_{p^a-1}$. Then $k \in C_{R_0(p^a)}(H)$ if and only if $(p^t-1)k = 0$ in $\mathbb{Z}_{p^a-1}$ (i.e. $(p^t-1)k \equiv 0 \bmod \mathbb{Z}_{p^a-1}$). Let

$$L_t = \langle (p^a-1)/(p^t-1) \rangle \leqslant \mathbb{Z}_{q-1}.$$

Then $k \in C_{R_0(p^a)}(H)$ if and only if $k \in L_t$. But $L_t \simeq \mathbb{Z}_{p^t-1} = R_0(p^t)$ as H-sets, so $C_{R_0(p^a)}(H) \simeq R_0(p^t)$ as H-sets.

(ii) Let $f = (p^{2a}-1)/(p^{2t}-1)$ and $K_{2t} = \langle f \rangle \leqslant \mathbb{Z}_{q^2-1} = T_0(p^a)$, so that $K_{2t} \simeq T_0(p^t) = \mathbb{Z}_{p^{2t}-1}$. Suppose $x = \{i, qi\} \in {}^2T_1(p^a)$, and

$$C_{{}^2T_1(p^a)}(H) = \{x \in {}^2T_1(p^a)\colon x^{\alpha^t} = x\} = \{x \in {}^2T_1(p^a)\colon (p^t i, p^t qi) = (i, qi)\}.$$

Then $x \in C_{{}^2T_1(p^a)}(H)$ if and only if $p^t i = i$ or $p^t i = qi$ since $p^t qi = qi$ is equivalent to $p^t i = i$, and $p^t qi = i$ is equivalent to $p^t i = qi$. In both cases, $i \in K_{2t}$.

If $2t \mid a$, then $(p^t+1) \mid (p^a-1)$, so that

$$f = \frac{p^a-1}{p^{2t}-1}(q+1)$$

and by definition, $i \notin T_1(p^a)$. This is impossible, so that $2t \nmid a$.

Suppose $x = \{i, qi\}$ such that $i \in K_{2t}$ and $i = (p^t+1)k$ for some $k \in K_{2t}$. Then $i \in \langle (p^t+1)f \rangle$. Since

$$(p^t+1)f = \frac{p^a-1}{p^t-1}(q+1),$$

it follows that $i \notin T_1(p^a)$, which is impossible. Thus, $i \in T_1(p^t)$ and $x = \{i, qi\} = \{i, p^t i\} \in {}^2T_1(p^t)$. Here we identify K_{2t} with $T_0(p^t)$. Conversely, suppose $x = \{i, p^t i\} \in {}^2T_1(p^t)$ and $2t \nmid a$. Identify K_{2t} with $T_0(p^t)$. Then $x = \{i, p^a i\} \in C_{{}^2T_1(p^a)}(H)$. It follows that

$$C_{{}^2T_1(p^a)}(H) \simeq \begin{cases} {}^2T_1(p^t) & \text{if } 2t \nmid a, \\ \emptyset & \text{if } 2t \mid a \end{cases}$$

as H-sets.

(iii) Suppose $Y(p^a) = {}^2R_2(p^a)$, and $x = \{(i,j),(j,i)\} \in Y(p^a)$. Then $x \in C_{Y(p^a)}(H)$ if and only if $p^t i = i$ and $p^t j = j$ or $p^t i = j$ and $p^t j = i$ in $R_0(p^a) = \mathbb{Z}_{q-1}$. Let

$$Y_\ell(p^a) = \begin{cases} \{\{(i,j),(j,i)\} \in C_{Y(p^a)}(H)\colon p^t i = i,\ p^t j = j\} & \text{if } \ell = 1, \\ \{\{(i,j),(j,i)\} \in C_{Y(p^a)}(H)\colon p^t i = j,\ p^t j = i\} & \text{if } \ell = 2. \end{cases}$$

If $x \in Y_1(p^a)$, then $i, j \in L_t \simeq \mathbb{Z}_{p^t-1} = R_0(p^t)$; and if $i - j = 0$ in L_t, then $i - j = 0$ in $R_0(p^a)$, which is impossible, so that $(i,j) \in R_2(p^t)$, and hence $x \in Y(p^t)$. Conversely, each element of $Y(p^t)$ is an element of $C_{Y(p^a)}(H)$ after identifying L_t with $R_0(p^t)$. Thus

$$Y_1\big(p^a\big) \simeq Y\big(p^t\big)$$

as H-sets.

Suppose $x = \{(i,j),(j,i)\} \in Y_2(p^a)$. Then $(p^{2t}-1)i = 0$ in $\mathbb{Z}_{p^a-1}$ and $(p^{2t}-1) \mid (p^a-1)$, so $2t \mid a$. In particular, $i \in L_{2t} = \langle (p^a-1)/(p^{2t}-1)\rangle \simeq T_0(p^t) = \mathbb{Z}_{p^{2t}-1}$. If $i = (p^t+1)k$ for some $k \in L_{2t}$, then $j = p^t i = p^t(p^t+1)k = (p^{2t}+p^t)k = (p^t+1)k = i$ and $(i,j) \notin R_2(p^a)$, which is impossible. Thus $i \notin (p^t+1)\mathbb{Z}_{p^{2t}-1}$ and $i \in T_1(p^t)$, $\{i,j\} = \{j,i\} = \{i, p^t i\} \in {}^2T_1(p^t)$. Conversely, suppose $\{i, p^t i\} \in {}^2T_1(p^t)$ and $2t \mid a$. Identify L_{2t} with $T_0(p^t)$. Then $x = \{(i,j),(j,i)\} \in Y_2(p^a)$, where $j = p^t i$. It follows that

$$Y_2\big(p^a\big) \simeq \begin{cases} {}^2T_1\big(p^t\big) & \text{if } 2t \mid a, \\ \emptyset & \text{if } 2t \nmid a \end{cases}$$

as H-sets.

(iv) Suppose $U(p^a) = {}^2R_3(p^a)$, and $x = \{(i,j),(i,i-j)\} \in U(p^a)$. Then $x \in C_{U(p^a)}(H)$ if and only if $p^t i = i$ and $p^t j = j$ or $p^t i = i$ and $p^t j = i - j$ in $R_0(p^a) = \mathbb{Z}_{q-1}$.

Let

$$U_\ell\big(p^a\big) = \begin{cases} \big\{\big\{(i,j),(i,i-j)\big\} \in C_{U(p^a)}(H)\colon\ p^t i = i,\ p^t j = j\big\} & \text{if } \ell = 1, \\ \big\{\big\{(i,j),(i,i-j)\big\} \in C_{U(p^a)}(H)\colon\ p^t i = i,\ p^t j = i - j\big\} & \text{if } \ell = 2. \end{cases}$$

If $x \in U_1(p^a)$, then $i, j \in L_t \simeq \mathbb{Z}_{p^t-1} = R_0(p^t)$; and if $i - 2j = 0$ in L_t, then $i - 2j = 0$ in $R_0(p^a)$, which is impossible, so that $(i,j) \in R_3(p^t)$, and hence $x \in U(p^t)$. Conversely, suppose $x = \{(i,j),(i,i-j)\} \in U(p^t)$. Identify L_t with $R_0(p^t)$. Then $x \in U_1(p^a)$. It follows that

$$U_1\big(p^a\big) \simeq U\big(p^t\big)$$

as H-sets.

Suppose $x = \{(i,j),(i,i-j)\} \in U_2(p^a)$. Then $p^t i = i$ and $p^t j = i - j = p^t i - j$, so that $i \in L_t$, $p^t(i-j) = j$, $p^{2t} j = p^t(i-j) = j$ and $j \in L_{2t} \simeq T_0(p^t) = \mathbb{Z}_{p^{2t}-1}$. In particular, $(p^{2t}-1) \mid (p^a-1)$ and so $2t \mid a$. If $j = (p^t+1)k$ for some $k \in L_{2t}$, then $i - j = p^t j = (p^{2t}+p^t)k = (1+p^t)k = j$ and $(i,j) \notin R_3(p^a)$, which is impossible. Thus $j \notin (p^t+1)\mathbb{Z}_{p^{2t}-1}$ and $j \in T_1(p^t)$. Thus $\{j, p^t j\} = \{j, i-j\} \in {}^2T_1(p^t)$. Conversely, suppose $\{j, p^t j\} \in {}^2T_1(p^t)$ and $2t \mid a$. Identify L_{2t} with $T_0(p^t)$. Then $x = \{((p^t+1)j, j), ((p^t+1)j, p^t j)\} \in U_2(p^a)$. It follows that

$$U_2\big(p^a\big) \simeq \begin{cases} {}^2T_1\big(p^t\big) & \text{if } 2t \mid a, \\ \emptyset & \text{if } 2t \nmid a \end{cases}$$

as H-sets.

(v) Let $(E(p^a), F(p^a)) \in \{({}^2R_1(p^a), {}^2S_1(p^a)), ({}^2R_1^*(p^a), {}^2S_1^*(p^a))\}$, and let $X(p^a) = E(p^a) \cup F(p^a)$. If $x = \{k, -k\} \in X(p^a)$, then $x \in C_{X(p^a)}(H)$ if and only if $(p^t - 1)k = 0$ or $(p^t + 1)k = 0$. Let

$$X_\pm(p^a) = \{\{k, -k\} \in C_{X(p^a)}(H)\colon (p^t \pm 1)k = 0\},$$

so that $X_+(p^a) \cap X_-(p^a) = \emptyset$. Suppose $x = \{k, -k\} \in X_-(p^a)$, so that $k \in L_t \simeq \mathbb{Z}_{p^t-1} = R_0(p^t)$. Thus, if $E(p^a) = {}^2R_1(p^a)$, then $k \in R_1(p^t)$, and if $E(p^a) = {}^2R_1^*(p^a)$, then $k \in R_1^*(p^t)$. So $x \in E(p^t)$. Conversely, if $x \in E(p^t)$, then $x \in X_-(p^a)$ after identifying L_t with $R_0(p^t)$. It follows that

$$X_-(p^a) \simeq E(p^t)$$

as H-sets.

Suppose $x = \{k, -k\} \in X_+(p^a)$. Then $(p^t + 1) \mid (p^a - 1)$ or $(p^t + 1) \mid (p^a + 1)$ according as $2t \mid a$ or $2t \nmid a$. In the former case, let $J_t = \langle (p^a - 1)/(p^t + 1)\rangle \simeq \mathbb{Z}_{p^t+1}$, and in the later case let J_t be the subgroup $\langle (p^a + 1)/(p^t + 1)\rangle \leqslant \mathbb{Z}_{q+1}$. Then $k \in J_t$ and $J_t \simeq S_0(p^t) = \mathbb{Z}_{p^t+1}$ as H-sets. Thus, if $F(p^a) = {}^2S_1(p^a)$, then $k \in S_1(p^t)$, and if $F(p^a) = {}^2S_1^*(p^a)$, then $k \in S_1^*(p^t)$. So $x \in F(p^t)$. Conversely, if $x \in F(p^t)$, then $x \in X_+(p^a)$ after identifying J_t with $S_0(p^t)$. It follows that

$$X_+(p^a) \simeq F(p^t)$$

as H-sets. □

Lemma 3.2. *Suppose $p \neq 2$ or 3, $q = p^a$, and $t \mid a$. Let*

$$X(p^a) = {}^4T_3(p^a) \cup {}^4T_3(p^a) \cup {}^6V_1^*(p^a) \cup {}^6W_1^*(p^a) \cup {}^{12}R_6(p^a) \cup {}^{12}S_6(p^a)$$
(disjoint union).

Suppose $\langle\alpha\rangle$ acts on $X(p^a)$ by $x^\alpha = px$ and $H = \langle\alpha^t\rangle$. Then

$$C_{X(p^a)}(H) \simeq X(p^t)$$

as H-sets, where α acts similarly on $X(p^t)$.

Proof. Let $\mathbb{F}_q$ be a finite field with q elements and $\overline{\mathbb{F}}_q$ an algebraic closure of $\mathbb{F}_q$. Let $\overline{T}$ the maximal torus of $\overline{G} = G_2(\overline{\mathbb{F}}_q)$ and $\overline{W} = N_{\overline{G}}(\overline{T})/\overline{T}$ the Weyl group of $\overline{G}$.

Let $\chi = \chi(x)$ be an irreducible character of $G_2(q)$ labeled by the parameter x [4]. Then $\chi(x)^\alpha = \chi(px)$ (which will be verified later). In addition, let (s, μ) be the semisimple and unipotent labels of $\chi(x)$. Then $x \in X(p^a)$ if and only if $(s, \mu) = (s, 1)$ with s regular (cf. [11, p. 359]), so that $C_{\overline{G}}(s) = \overline{T}$. Thus $\chi^{\alpha^t} = \chi$ if and only if $(s)_G^{\alpha^t} = (s)_G$, namely, $s^{\alpha^t} = s^w$ for some $w \in \overline{W}$, where $(s)_G$ is the conjugacy class of G containing s. Thus $\chi^{\alpha^t} = \chi$ if and only if $s \in C_{\overline{T}}(\alpha^t w^{-1})$, namely, s is a regular element of $G_2(p^t)$, since

$C_{\overline{T}}(\alpha^t w^{-1})$ is a maximal torus of $G_2(p^t)$. But a regular element s of $G_2(p^t)$ labels an irreducible character $\psi = \psi_{s,1}$ of $G_2(p^t)$ such that its parameter y (see [4]) lies in $X(p^t)$. It follows that

$$C_{X(p^a)}(H) \simeq X\big(p^t\big)$$

as H-sets. □

Remark. Lemma 3.2 can also be proved by a direct calculation as that of Lemma 3.1, and part of Lemma 3.1 can be proved using a similar idea to that of Lemma 3.2.

4. Proof for $G_2(p^a)$

In this section, we prove Dade's invariant conjecture for $G = G_2(p^a)$ in the defining characteristic when $p \neq 2$ or 3.

Suppose $p \neq 2, 3$, $O = \mathrm{Out}(G) = \langle\alpha\rangle$, where α is a field automorphism of G with order a. We may assume that α stabilizes W, P and Q. According to the Borel–Tits theorem [5], the normalizers of radical p-subgroups are parabolic subgroups. The radical p-chains of G (up to G-conjugacy) are given in Table 1.

Since $C(5)$: $1 < O_p(Q) < O_p(W)$ and $C(6)$: $1 < O_p(W)$ have the same normalizers $N_G(C(5)) = N_G(C(6))$ and $N_A(C(5)) = N_A(C(6))$, it follows that

$$\mathrm{k}\big(N_G\big(C(5)\big), B_0, d, u\big) = \mathrm{k}\big(N_G\big(C(6)\big), B_0, d, u\big).$$

Thus, the contribution of $C(5)$ and $C(6)$ in the alternating sum of Dade's invariant conjecture is zero. Dade's invariant conjecture for G is equivalent to

$$\mathrm{k}(G, B_0, d, u) + \mathrm{k}(W, B_0, d, u) = \mathrm{k}(P, B_0, d, u) + \mathrm{k}(Q, B_0, d, u) \tag{1}$$

for any $u \mid a$.

Theorem 4.1. *Let B be a p-block of $G = G_2(p^a)$ with a positive defect. Then B satisfies the invariant conjecture of Dade.*

Table 1
Radical p-chains of G

C		$N_G(C)$	$N_A(C)$
$C(1)$	1	G	A
$C(2)$	$1 < O_p(P)$	P	$P \rtimes \langle\alpha\rangle$
$C(3)$	$1 < O_p(P) < O_p(W)$	W	$W \rtimes \langle\alpha\rangle$
$C(4)$	$1 < O_p(Q)$	Q	$Q \rtimes \langle\alpha\rangle$
$C(5)$	$1 < O_p(Q) < O_p(W)$	W	$W \rtimes \langle\alpha\rangle$
$C(6)$	$1 < O_p(W)$	W	$W \rtimes \langle\alpha\rangle$

Proof. Since $D(B) \neq 1$, it follows that $B = B_0 = B_0(G)$ is the principal block. Let $L \in \{G, W, P, Q\}$. By [4,6], $k(L, B_0, d) = 0$ when $d \notin \{3a, 4a, 5a, 6a\}$.

The action of O on the conjugacy L-classes induces an action of O on the set $\mathrm{Irr}(L, B_0, d)$, and then an action on the parameter sets. Using the values of characters in $\mathrm{Irr}(L, B_0, d)$ acting on the classes listed in the last column of Tables 2–5, we can describe the action of O on the parameter sets.

(i) If $d = 3a$, then $\mathrm{Irr}(G, B_0, 3a) = \{X_{31}\}$, $\mathrm{Irr}(P, B_0, 3a) = \{\theta_8\}$, $\mathrm{Irr}(W, B_0, 3a) = \mathrm{Irr}(Q, B_0, 3a) = \emptyset$. Thus (1) holds.

(ii) Suppose $d = 4a$, so that $\mathrm{Irr}(L, B_0, d)$ is given by Table 3. If $\theta \in \{X_{21} \in \mathrm{Irr}(G),\ \theta_i(k) \in \mathrm{Irr}(W)\colon\ i = 7, 8\}$, then by the degrees or values of θ on the classes given in the last column of Table 3, it follows that $\theta^\alpha = \theta$. Similarly, if $\theta \in \{\theta_i \in \mathrm{Irr}(P),\ \theta_{12} \in \mathrm{Irr}(Q)\colon\ 9 \leqslant i \leqslant 12\}$, then by the degrees or values of θ on the classes given in the last column of Table 3, it follows that $\theta^\alpha = \theta$.

Let $A(p^a) = R_0(p^a)$ and $B(p^a) = {}^2R_1(p^a) \cup {}^2S_1(p^a) \cup \{1\}$ with $1^\alpha = 1$. Here, 1 denotes the parameter of $\theta_7 \in \mathrm{Irr}(P)$. Then

$$\left|A\left(p^a\right)\right| = \left|B\left(p^a\right)\right| = p^a - 1.$$

Let $U(p^a) = A(p^a) \cup B(p^a)$. Then using the values of characters on the classes listed in the last column of Table 3, we know that the action of α on $U(p^a)$ is given by $x^\alpha = px$ for

Table 2
The characters of defect $3a$

Group	Character	Degree	Parameter	Number	Class
$G_2(p^a)$	X_{31}	$q^3(q^3+\epsilon)$		1	
P	θ_8	$q^3(q-1)$		1	

Table 3
The characters of defect $4a$

Group	Character	Degree	Parameter	Number	Class
$G_2(p^a)$	X_{21}	$q^2(q^4+q^2+1)$		1	
W	$\chi_7(k)$	$q^2(q-1)$	R_0	$q-1$	$C_{11}(i)$
	$\theta_7(k)$	$q^2(q-1)^2/2$		2	B_{34}
	$\theta_8(k)$	$q^2(q-1)^2/2$		2	B_{34}
P	$\chi_7(k)$	$q^2(q^2-1)$	${}^2R_1$	$(q-3)/2$	$C_{11}(i)$
	$\chi_8(k)$	$q^2(q-1)^2$	${}^2S_1$	$(q-1)/2$	$D_{11}(i)$
	θ_7	$q^2(q-1)$		1	
	θ_9	$q^2(q-1)^2/2$		1	A_{61}
	θ_{10}	$q^2(q-1)^2/2$		1	A_{61}
	θ_{11}	$q^2(q^2-1)/2$		1	A_{61}
	θ_{12}	$q^2(q^2-1)/2$		1	A_{61}
Q	θ_{12}	$q^2(q-1)$		1	

any $x \in U(p^a)\backslash\{1\}$. If $t \mid a$ and $H = \langle \alpha^t \rangle \leqslant O$, then by Lemma 3.1, $C_{A(p^a)}(H) \simeq A(p^t)$ and $C_{B(p^a)}(H) \simeq B(p^t)$, so that

$$A\left(p^a\right) \simeq B\left(p^a\right) \tag{2}$$

as O-sets. It follows that (1) holds.

(iii) Suppose $d = 5a$, so that $\mathrm{Irr}(L, B_0, d)$ is given by Table 4. Since $\theta_2(k,\ell) = \mathrm{Ind}_W^Q(\theta_2(k,\ell))$ for each pair (k,ℓ), it follows that

$$\theta_2(k,\ell)^\alpha = \mathrm{Ind}_W^Q\left(\theta_2(k,\ell)^\alpha\right),$$

and $\{{}_Q\theta_2(k,\ell)\} \simeq \{{}_W\theta_2(k,\ell)\}$ as O-sets.

Similarly, since $\theta_3(x) = \mathrm{Ind}_W^Q(\theta(x))$ for each $x \in K^*$ and $\epsilon = 1$, it follows that

$$\theta_3(x)^\alpha = \mathrm{Ind}_W^Q\left(\theta(x)^\alpha\right),$$

and $\{{}_Q\theta_3(x)\} \simeq \{{}_W\theta(x)\}$ as O-sets.

Moreover, since $\theta_4(x) = \mathrm{Ind}_W^Q(\theta(x))$ for each $x \in K$ and $\epsilon = -1$, it follows that

$$\theta_4(x)^\alpha = \mathrm{Ind}_W^Q\left(\theta(x)^\alpha\right),$$

and $\{{}_Q\theta_4(x)\} \simeq \{{}_W\theta(x)\}$ as O-sets.

Let

$$\begin{aligned}
\Phi_G &= \{X_{23}, X_{24}, X_{13}, X_{14}, X_{17}, X_{18}, X_{19}, \overline{X}_{19}\} \subseteq \mathrm{Irr}(G),\\
\Phi_P &= \left\{\theta_2(k), \theta_3(k), \theta_4, \theta_5, \theta_6(k)\right\} \subseteq \mathrm{Irr}(P),\\
\Phi_Q &= \left\{\theta_5(k), \theta_6(k), \theta_7, \theta_8, \theta_9, \theta_{10}\right\} \subseteq \mathrm{Irr}(Q).
\end{aligned}$$

If $\theta \in \Phi_G \cup \{\theta_i(k) \in \mathrm{Irr}(W)\colon 3 \leqslant i \leqslant 6\}$, then by the degrees or values of θ on the classes given in the last column of Table 4, it follows that $\theta^\alpha = \theta$. Similarly, if $\theta \in \Phi_P \cup \Phi_Q$, then by the degrees or values of θ on the classes given in the last column of Table 4, it follows that $\theta^\alpha = \theta$.

Let $C(p^a) = {}^2R_1^*(p^a) \cup {}^2S_1^*(p^a) \cup \{1\} \cup \{2\}$ with $1^\alpha = 1$ and $2^\alpha = 2$. Here, 1, 2 denote the parameters of $X_{15}, X_{16} \in \mathrm{Irr}(G)$. Using the values of characters on the classes listed in the last column of Table 4, we know that the action of α on $C(p^a)$ is given by $x^\alpha = px$ for any $x \in C(p^a) \setminus (\{1\} \cup \{2\})$. If $t \mid a$ and $H = \langle \alpha^t \rangle \leqslant O$, then by Lemma 3.1, $C_{C(p^a)}(H) \simeq C(p^t)$. Since $|C(p^a)| = p^a - 1$, it follows that

$$C\left(p^a\right) \simeq A\left(p^a\right) \tag{3}$$

as O-sets, where $A(p^a)$ is defined as in the proof of (ii). Similarly, (ii) (where $B(p^a)$ with 1 as parameter of $X_{33} \in \mathrm{Irr}(G)$) still holds when $d = 5a$. It follows by (2) and (3) that (1) holds.

Table 4
The characters of defect 5a

Group	Character	Degree	Parameter	Number	Class
$G_2(p^a)$	X_{13}	$q(q^4+q^2+1)/3$		1	A_1
	X_{14}	$q(q^4+q^2+1)/3$		1	A_1
	X_{15}	$q(q+1)^2(q^2-q+1)/2$		1	
	X_{16}	$q(q+1)^2(q^2+q+1)/6$		1	
	X_{17}	$q(q-1)^2(q^2+q+1)/2$		1	
	X_{18}	$q(q-1)^2(q^2-q+1)/6$		1	
	X_{19}	$q(q^2-1)^2/3$		1	B_0
	$\overline{X}_{19}$	$q(q^2-1)^2/3$		1	B_0
	X_{23}	$q(q^4+q^2+1)$		1	A_2
	X_{24}	$q(q^4+q^2+1)$		1	A_{31}
	X_{33}	$q(q+\epsilon)(q^3+\epsilon)$		1	
	X_{1a}	$q(q+1)(q^4+q^2+1)$	${}^2R_1^*$	$(q-4-\epsilon)/2$	$C_{22}(i)$
	X_{1b}	$q(q+1)(q^4+q^2+1)$	${}^2R_1$	$(q-3)/2$	$C_{12}(i)$
	X_{2a}	$q(q-1)(q^4+q^2+1)$	${}^2S_1$	$(q-1)/2$	$D_{12}(i)$
	X_{2b}	$q(q-1)(q^4+q^2+1)$	${}^2S_1^*$	$(q-2+\epsilon)/2$	$D_{22}(i)$
W	$\chi_4(k)$	$q(q-1)$	R_0	$q-1$	$C_{51}(i)$
	$\chi_5(k)$	$q(q-1)$	R_0	$q-1$	$C_{41}(i)$
	$\chi_6(k)$	$q(q-1)$	R_0	$q-1$	$C_{21}(i)$
	$\theta(x)(\epsilon=1)$	$q(q-1)^2$	K^*	$q-1$	
	$\theta(x)(\epsilon=-1)$	$q(q-1)^2$	K	q	
	$\theta_2(k,\ell)(\epsilon=+1)$	$q(q-1)^2/3$		9	
	$\theta_3(k)$	$q(q-1)^2/2$		2	B_{25}
	$\theta_4(k)$	$q(q-1)^2/2$		2	B_{25}
	$\theta_5(k)$	$q(q-1)^2/2$		2	B_{11}
	$\theta_6(k)$	$q(q-1)^2/2$		2	B_{11}
P	$\chi_3(k)$	q	R_0	$q-1$	$C_{31}(i)$
	$\chi_6(k)$	$q(q^2-1)$	R_0	$q-1$	$C_{31}(i)$
	$\theta_2(k)$	$q(q-1)(q^2-1)/2$		2	B_{11}
	$\theta_3(k)$	$q(q-1)(q^2-1)/6$		2	B_{11}
	θ_4	$q(q-1)(q^2-1)/3$		1	$B_2(1), B_2(2)$
	θ_5	$q(q-1)(q^2-1)/3$		1	$B_2(1), B_2(2)$
	$\theta_6(k)$	$q(q-1)(q^2-1)/3$		2	$B_2(1), B_2(2)$
Q	$\chi_3(k)$	q	R_0	$q-1$	$C_{31}(i)$
	$\chi_6(k)$	$q(q^2-1)$	${}^2R_1$	$(q-3)/2$	$C_{21}(i)$
	$\chi_7(k)$	$q(q^2-1)$	R_0	$q-1$	$C_{11}(i)$
	$\chi_8(k)$	$q(q-1)^2$	${}^2S_1$	$(q-1)/2$	$D_{11}(i)$
	$\theta_2(k,\ell)(\epsilon=+1)$	$q(q-1)(q^2-1)/3$		9	
	$\theta_3(x)(\epsilon=+1)$	$q(q-1)(q^2-1)$	K^*	$q-1$	
	$\theta_4(x)(\epsilon=-1)$	$q(q-1)(q^2-1)$	K	q	
	$\theta_5(k)$	$q(q-1)(q^2-1)/2$		2	B_{11}
	$\theta_6(k)$	$q(q-1)(q^2-1)/2$		2	B_{11}
	θ_7	$q(q-1)^2/2$		1	A_{41}
	θ_8	$q(q-1)^2/2$		1	A_{41}
	θ_9	$q(q^2-1)/2$		1	A_{41}
	θ_{10}	$q(q^2-1)/2$		1	A_{41}
	θ_{11}	$q(q-1)$		1	

Table 5
The characters of defect $6a$

Group	Character	Degree	Parameter	Number	Class
$G_2(p^a)$	X_{11}	1		1	
	X_{32}	$q^3+\epsilon$		1	
	X_{22}	q^4+q^2+1		1	
	X'_{1a}	$(q+1)(q^4+q^2+1)$	${}^2R_1^*$	$(q-4-\epsilon)/2$	$E_1(i)$
	X'_{2b}	$(q-1)(q^4+q^2+1)$	${}^2S_1^*$	$(q-2+\epsilon)/2$	$E_2(i)$
	X'_{1b}	$(q+1)(q^4+q^2+1)$	${}^2R_1$	$(q-3)/2$	$E_2(i)$
	X'_{2a}	$(q-1)(q^4+q^2+1)$	${}^2S_1$	$(q-1)/2$	$E_1(i)$
	X_1	$(q+1)^2(q^4+q^2+1)$	${}^{12}R_6$	$(q^2-8q+17+2\epsilon)/12$	$C_{11}(i)$
	X_2	$(q-1)^2(q^4+q^2+1)$	${}^{12}S_6$	$(q^2-4q+5-2\epsilon)/12$	$D_{11}(i)$
	X_a	q^6-1	${}^4T_3$	$(q-1)^2/4$	$C_{22}(i)$
	X_b	q^6-1	${}^4T_3$	$(q-1)^2/4$	$C_{12}(i)$
	X_3	$(q^2-1)^2(q^2-q+1)$	${}^6V_1^*$	$(q^2+q-1-\epsilon)/6$	$E_3(i)$
	X_6	$(q^2-1)^2(q^2+q+1)$	${}^6W_1^*$	$(q^2+q-1+\epsilon)/6$	$E_4(i)$
W	$\chi_1(k,\ell)$	1	$R_0\times R_0$	$(q-1)^2$	$C_{41}(i)$
	$\chi_2(k)$	$q-1$	R_0	$q-1$	$C_{61}(i)$
	$\chi_3(k)$	$q-1$	R_0	$q-1$	$C_{31}(i)$
	θ_1	$(q-1)^2$		1	
P	$\chi_1(k)$	1	R_0	$q-1$	$C_{31}(i)$
	$\chi_2(k,\ell)$	$q+1$	${}^2R_3$	$(q-1)(q-2)/2$	$C_{31}(i)$
	$\chi_4(k)$	$q-1$	${}^2T_1$	$q(q-1)/2$	$C_{41}(i)$
	$\chi_5(k)$	q^2-1	R_0	$q-1$	$C_{21}(i)$
	θ_1	$(q-1)(q^2-1)$		1	
Q	$\chi_1(k)$	1	R_0	$q-1$	$C_{31}(i)$
	$\chi_2(k,\ell)$	$q+1$	${}^2R_2$	$(q-1)(q-2)/2$	$C_{31}(i)$
	$\chi_4(k)$	$q-1$	${}^2T_1$	$q(q-1)/2$	$C_{41}(i)$
	$\chi_5(k)$	q^2-1	R_0	$q-1$	$C_{31}(i)$
	θ_1	$(q-1)(q^2-1)$		1	

(iv) Suppose $d=6a$, so that $\mathrm{Irr}(L,B_0,d)$ is given by Table 5. Let $X(p^a)$ be given as in Lemma 3.2 and $D(p^a)={}^2R_2(p^a)\cup{}^2T_1(p^a)\cup\{1\}$ with $1^\alpha=1$. Here, 1 denotes the parameter of $\theta_1\in\mathrm{Irr}(Q)$. Then

$$\left|D\left(p^a\right)\right|=\left|X\left(p^a\right)\right|=\left(p^a-1\right)^2+1.$$

Using the values of characters on the classes listed in the last column of Table 5, we know that the action of α on $D(p^a)\cup X(p^a)$ is given by $x^\alpha=px$ for any $x\in(D(p^a)\cup X(p^a))\backslash\{1\}$. If $t\mid a$ and $H=\langle\alpha^t\rangle\leqslant O$, then by Lemmas 3.1 and 3.2, $C_{D(p^a)}(H)\simeq D(p^t)$ and $C_{X(p^a)}(H)\simeq X(p^t)$ so that

$$D\left(p^a\right)\simeq X\left(p^a\right)$$

as O-sets.

Let $E(p^a) = {}^2R_3(p^a) \cup {}^2T_1(p^a)$ and $F(p^a) = R_0(p^a) \times R_0(p^a)$. Then

$$\left|E\left(p^a\right)\right| = \left|F\left(p^a\right)\right| = \left(p^a - 1\right)^2. \tag{4}$$

Using the values of characters on the classes listed in the last column of Table 5, we know that the action of α on $E(p^a) \cup F(p^a)$ is given by $x^\alpha = px$ for any $x \in (E(p^a) \cup F(p^a))$. If $t \mid a$ and $H = \langle \alpha^t \rangle \leqslant O$, then by Lemmas 3.1, $C_{E(p^a)}(H) \simeq E(p^t)$ and $C_{F(p^a)}(H) \simeq F(p^t)$ so that

$$E\left(p^a\right) \simeq F\left(p^a\right)$$

as O-sets. Similarly, (2) (with 1 as the parameter of $X_{11} \in \mathrm{Irr}(G)$) and (3) (with 1, 2 as the parameters of $X_{22}, X_{32} \in \mathrm{Irr}(G)$) still hold. It follows by (2)–(4) that (1) holds. □

Acknowledgments

I express my appreciation to Associate Professors Jianbei An and Eamonn O'Brien for valuable discussions and suggestions. In addition, I thank Professor Marston Conder for scholarship support and great hospitality.

References

[1] J.L. Alperin, Weights for finite groups, in: Proc. Sympos. Pure Math., vol. 47, Amer. Math. Soc., Providence, RI, 1987, pp. 369–379.
[2] J. An, Dade's invariant conjecture for the Chevalley groups $G_2(q)$ in the defining characteristics, $q = 2^a$, 3^a, Algebra Colloq. 10 (2003) 519–533.
[3] J. An, Uno's invariant conjecture for the Chevalley groups $G_2(q)$ in non-defining characteristics, submitted for publication.
[4] J. An, S. Huang, The character tables of the parabolic subgroups of the Chevalley groups of type (G_2), submitted for publication.
[5] N. Burgoyne, C. Williamson, On a theorem of Borel and Tits for finite Chevalley groups, Arch. Math. (Basel) 27 (1976) 489–491.
[6] B. Chang, R. Ree, The characters of $G_2(q)$, in: Sympos. Math., vol. 13, Cambridge Univ. Press, 1974, pp. 395–413.
[7] E.C. Dade, Counting characters in blocks, I, Invent. Math. 109 (1992) 187–210.
[8] E.C. Dade, Counting characters in blocks, II.9, in: Representation Theory of Finite Groups, Columbus, OH, 1995, Ohio State Univ. Math. Res. Inst. Publ., vol. 6, de Gruyter, Berlin, 1997, pp. 45–59.
[9] H. Enomoto, The characters of the finite group $G_2(q)$, $q = 3^f$, Japan J. Math. 2 (1976) 191–248.
[10] H. Enomoto, H. Yamada, The characters of $G_2(2^n)$, Japan J. Math. 12 (1986) 325–377.
[11] G. Hiss, On the decomposition numbers of $G_2(q)$, J. Algebra 120 (1989) 339–360.
[12] S. Huang, Conjecture on the character degrees of the Chevalley simple groups $G_2(q)$, $q = 3$ or 4, submitted for publication.
[13] R. Knörr, G.R. Robinson, Some remarks on a conjecture of Alperin, J. London Math. Soc. 39 (2) (1989) 48–60.