On a Katuta-Junnila Problem

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Abstract. We consider the Katuta-Junnila problem.

1. Is a space metacompact if every directed open cover of the space has a cushioned refinement?

2. Is a space submetacompact if every directed open cover of the space has a \( \sigma \)-cushioned refinement?

We summarize some previous known partial results and present an affirmative answer to problem 1 in the class of strongly first countable spaces.

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1 Introduction

In 1975 Katuta [16] asked the following

1.1 Problem

Is a space metacompact if every directed open cover of the space has a cushioned refinement?

1.2 Problem

Is a space submetacompact if every directed open cover of the space has a $\sigma$-cushioned refinement?

Junnila pointed out that these problems are equivalent to the following: Is a space metacompact if every directed open cover of the space has a semi-open point-star refinement? Is a space submetacompact if every directed open cover of the space has a point-star refining sequence by semi-open covers? [14].

Comparing with previous results (see [1] Theorem 3.5, Theorem 3.6), we see that this is another closure-preserving vs. cushioned phenomenon. As Davis [4] pointed out, it is closely related to a long-standing open problem, namely the $M_3$ implies $M_1$ problem.

Observe that since Katuta showed that if every directed open cover has a cushioned refinement then the space is almost expandable ([16] Theorem 2.2) and every submetacompact almost expandable space is metacompact, an affirmative answer to problem 1.2 would also yield an affirmative answer to problem 1.1.

Section 2 discusses problem 1.1, section 3 discusses problem 1.2 and in section 4 we present some recent results.

We refer the reader to Burke’s article [1] and Junnila’s paper [14] for undefined notions and more related theorems.

2 Metacompactness


**Theorem 2.1.** ([13] Theorem 1.25) A space is metacompact iff the space is $\eta$-doubly covered and every directed open cover has a semi-open point-star refinement.

A cover $\mathcal{U}$ of $X$ is a $\eta$-double cover if there exists a neighbornet $U$ of $X$ such that $U^2 \subset \bigcup\{\bigcup\{G|x \in G \in \mathcal{U}\} | x \in X\}$, $X$ is $\eta$-doubly covered if every open cover of $X$ is $\eta$-double cover.

He showed that every orthocompact space is $\eta$-doubly covered, so he solved problem 1.1 for the orthocompact case. I rediscovered this fact in
1988[10]. He also solved the problem for preorthocompact case, the result appears in Fletcher and Lindgren’s monograph [5].

**Theorem 2.2.** ([5] Lemma 5.39) Let $X$ be a preorthocompact space, then $X$ is metacompact if and only if for every directed open cover $U$ of $X$, there is a neighbornet $V$ of $X$ so that $\{V^{-1}(x)\mid x \in X\}$ refines $U$.

The second condition is equivalent to every directed open cover having a cushioned refinement. Recall that a space is called preorthocompact, if it has a cocushioned neighbornet.

Gruenhage solved the problem for locally compact spaces, by using Game Theory technique.

**Theorem 2.3.** ([6] Theorem 6) The following are equivalent for a locally compact spaces $X$:

1. $X$ is metacompact,
2. for each open cover $U, U^F$ has a cushioned refinement.
3. there exists $\sigma: X \to K(X)$ (the collection of all compact subsets of $X$) such that no sequence $<x_n>$ satisfying $x_{n+1} \in \bigcup \{\sigma(x_i)\mid i \leq n\}$ has a limit point.

Recently Yajima generalized Theorem 2.1.

**Theorem 2.4.** ([22] Theorem 3.1) A space is metacompact if and only if $X$ is suborthocompact and every open cover of $X$ has a cushioned refinement.
A space $X$ is said to be suborthocompact, if for every open cover $\mathcal{U}$ of $X$, there is a sequence of open refinements $<\mathcal{V}_n>$ of $\mathcal{U}$ such that for each $x \in X$ there is some $n \in \omega$ such that $\cap\{V \in \mathcal{V}_n|x \in V\}$ is open.

The general question is

**Question 2.5.** Find a topological property $P$ such that if $X$ has $P$ and every directed open cover of $X$ has a cushioned refinement, then $X$ is metacompact.

For example, it is easy to show that $\sigma$-orthocompactness is a property of this type.

**Lemma 2.6.** [12] If every directed open cover of $X$ has a point-star refinement by semi-open covers, then $X$ is almost expandable and hence is countably metacompact.

**Theorem 2.7.** [12] Let $X$ be a $\sigma$-orthocompact space. If every directed open cover of $X$ has a cushioned refinement, then $X$ is metacompact.

**Proof:** By lemma 2.6 such an $X$ is countably metacompact and $\sigma$-orthocompact, hence orthocompact. By theorem 2.1 it is metacompact.

### 3 Submetacompactness

Junnila extensively discussed submetacompact spaces, and had some partial solutions for problem 1.2.
Theorem 3.1. ([3] Proposition 2) For a locally compact space $X$, if every directed open cover of $X$ has a $\sigma$-cushioned refinement, then $X$ is submetacompact.

It is also known that the answer for problem 1.2 is positive for orthocompact spaces.

Theorem 3.2. ([10]) For an orthocompact space $X$, if every directed open cover of $X$ has a $\sigma$-cushioned refinement, then $X$ is submetacompact.

Moreover we have the following characterization theorem of submetacompact spaces.

Theorem 3.3. [10] A space $X$ is submetacompact iff every directed open cover of $X$ has a $\sigma$-cushioned refinement, and every open cover of $X$ has a sequence $<W_n>$ of open refinements such that for each $n \in \omega$, there is a closed subset $F_n$ of $X$ such that if $x \in F_n$ then $\cap\{W \in W_n | x \in W\}$ is open, and $\cup F_n = X$.

The second condition is quite close to suborthocompactness, but we can’t answer the following question:

Question 3.4. (Yajima) Can we get a result similar to Theorem 2.4 for submetacompact space? [22].

Kunzi and Fletcher modified Howes’s definition for confinally $\triangle$-complete space by saying that a regular Hausdorff space $X$ is confinally $\theta$-complete provided that every directed open cover of $X$ has a point-star refining sequence of open covers. They asked the following.
Question 3.5. [18] Is every cofinally \( \theta \)-complete space submetacompact? They provided some partial answers there.

Theorem 3.6. [18] Every preorthocompact cofinally \( \theta \)-complete space is submetacompact.

Theorem 3.7. [18] Let \( X \) be a cofinally \( \theta \)-complete point-star orthocompact space. Then \( X \) is submetacompact.

A space is point-star orthocompact provided that, if \( \mathcal{U} \) is an open cover of \( X \), there is an interior-preserving open refinement \( \mathcal{V} \) of \( \{ st(x, \mathcal{U}) : x \in X \} \) so that, for each \( x \in X \) there exist \( V(x) \in \mathcal{V} \) so that \( x \in V(x) \subset st(x, \mathcal{U}) \).

In 1977 Liu [19] defined quasi-paracompact and strictly quasi-paracompact spaces while studying collectionwise normality. A space \( X \) is called (strictly) quasi-paracompact iff every open cover of \( X \) has a refinement \( \bigcup \{ \mathcal{F}_n : n \in \omega \} \) such that \( \mathcal{F}_0 \) is (closed) discrete family and \( \mathcal{F}_n (n \geq 1) \) is (closed) discrete family relative to \( X \cup \bigcup_{i<n} (\bigcup \mathcal{F}_i) \). It is a common generalization of both metacompact and subparacompact spaces.

The strict quasi-paracompactness implies the following property \( b_1 \) defined by Chaber (see [21]).

A space \( X \) has property \( b_1 \) iff every open cover of \( X \) has a refinement \( \bigcup \{ \mathcal{F}_n : n \in \omega \} \) such that \( \mathcal{F}_n \) is a locally finite closed family relative to \( X \setminus \bigcup_{i<n} (\bigcup \mathcal{F}_i) \).

Jiang Jiguang proved the following.

Theorem 3.8. [9] A space is submetacompact if it is almost discretely \( \theta \)-expandable and strictly quasi-paracompact.
Long Bing showed that strictly quasi-paracompact space is equivalent to bounded weakly $\vartheta$-refinable space ([20] Theorem 1.5). Zhu ([23]) showed that quasi-paracompactness implies weakly $\theta$-refinability, but not vice versa.

Since every directed open cover having a $\sigma$-cushioned refinement implies almost $\theta$-expandability, we know that for bounded weakly $\vartheta$-refinable spaces problem 1.2 has a positive answer.

Chaber has the following result:

**Theorem 3.9.** (see [21]) A space is submetacompact if it is almost $\theta$-expandable and has property $b_1$.

The general question here is:

**Question 3.10.** Find a topological property $Q$, such that if a space has $Q$ and every directed open cover of $X$ has a $\sigma$-cushioned refinement, then $X$ is submetacompact.

Many previous results suggest that we don’t need any such property to get a positive answer for problem 3.11, but so far we don’t even know if weakly $\theta$-refinability or weakly $\vartheta$-refinability is enough to get a positive answer. In this paper we have provided some characterization theorems which may lead to some interesting counterexamples.

4 Recent results

Yajima and Kemoto [17] recently showed the following
Theorem 4.1. [17] A $\beta$-space $X$ is submetacompact if and only if every monotone open cover of $X$ has a $\sigma$-closure-preserving closed refinement. Recall that a space $X$ is called a $\beta$-space if there is a function $g : X \times \omega \to \text{Top} (X)$, satisfying

1. $x \in \cap_{n \in \omega} g(x, n)$

2. if $x \in g(x_n, n)$ for each $n \in \omega$, then $<x_n>$ has a cluster point in $X$.

In their paper the following lemma is quoted.

Lemma 4.2. [11, 14]. Let $X$ be a space and $\mathcal{U}$ an interior-preserving open cover of $X$. Then $\mathcal{U}$ has a $\sigma$-closure-preserving closed refinement if and only if it has a $\sigma$-cushioned refinement.

Note that every monotone open cover is interior-preserving as well as directed, thus theorem 4.1 actually answers problems 1.1, 1.2 in the affirmative in the class of $\beta$-spaces. This brings generalized metric spaces into attention. Motivated by their work, in a joint work with Reilly and Cao we get the following:

Lemma 4.3. [2] If space $X$ is strongly first countable and every directed open cover $\mathcal{U}$ of $X$ has a cushioned refinement, then $X$ is suborthocompact.

Recall that a space $X$ is called strongly first countable if there is a function $g : X \times \omega \to \text{Top} (X)$ satisfying:

1. $<g(x, n)>$ is a neighborhood base for each $x \in X$. 

2. if \( y \in g(x, n) \), then \( g(y, n) \subset g(x, n) \). Without loss of generality we may assume that \( g(x, n+1) \subset g(x, n) \) for \( n \in w \).

**Proof:** If every directed open cover of \( X \) has a cushioned refinement, then \( X \) is almost expandable, hence countably metacompact.

Let \( U = \{ U_\alpha : \alpha < \kappa \} \) be any open cover for strongly first countable space \( X \).

For any \( x \in X \), let \( n(x) = \min\{ m \in \omega | g(x, m) \subset U_\alpha \text{ for some } \alpha < \kappa \} \). Let \( V_n = \cup\{ g(x, m) | n(x) \leq n \} \), then \( V = V_n \) is an increasing countable open cover, hence has a point-finite open refinement \( W = W_n \), without loss of generality, we may assume that \( W \) is precise, i.e. each \( W_n \subset V_n \).

Let \( F_n = \{ x \in X | \text{ord}(x, W) \leq n \} \), then \( F_n \subset F_{n+1} \) for any \( n \in w \), \( F_n \) is closed and \( X = \bigcup_{n \in w} F_n \).

We construct \( \kappa \)-sequence \( \mathcal{H} \) as follows. for any \( x \in F_n \), choose smallest \( n(x) \) such that \( g(x, n(x)) \subset U_\alpha \) for some \( \alpha < k \), for any \( x \notin F_n \) let \( G_x = (x \setminus F_n) \cap U_\beta \) for some \( \beta \) with \( x \in U_\beta \).

Let \( \mathcal{H}_n = \{ g(x, n(x) + n) | x \in F_n \} \cup \{ G_x | x \notin F_n \} \)

Let \( \mathcal{H} = \langle \mathcal{H}_n \rangle \).

For any \( p \in x, p \in F_n \) for some \( n \), let \( m = \max\{ k_i | p \in W_{k_i}, i \in n \} \), then we have

\[ \cap\{ H \in \mathcal{H}_m | p \in H \} = g(p, n(p) + m) \]

which is open, note that for any \( y \in X, y \neq p \) and \( n \in w \), either \( p \notin g(y, n) \) or \( g(p, n) \subset g(y, n) \). This completes the proof. \( \square \)
Corollary 4.4. A strongly first countable space $X$ is metacompact if and only if every directed open cover of $X$ has a cushioned refinement.

Proof: Combine theorem 4.2 and theorem 2.4. \qed

Remark 4.5. Corollary 4.4 answers problem 1.1 in the affirmative in the class of strongly first countable spaces.

References


12


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