On a Katuta-Junnila Problem

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Abstract. We consider the Katuta-Junnila problem.

1. Is a space metacompact if every directed open cover of the space has a cushioned refinement?

2. Is a space submetacompact if every directed open cover of the space has a σ-cushioned refinement?

   We summarize some previous known partial results and present an affirmative answer to problem 1 in the class of strongly first countable spaces.

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1 Introduction

In 1975 Katuta [16] asked the following

1.1 Problem

Is a space metacompact if every directed open cover of the space has a cushioned refinement?

1.2 Problem

Is a space submetacompact if every directed open cover of the space has a σ-cushioned refinement?

Junnila pointed out that these problems are equivalent to the following: Is a space metacompact if every directed open cover of the space has a semi-open point-star refinement? Is a space submetacompact if every directed open cover of the space has a point-star refining sequence by semi-open covers? [14].

Comparing with previous results (see [1] Theorem 3.5, Theorem 3.6), we see that this is another closure-preserving vs. cushioned phenomenon. As Davis [4] pointed out, it is closely related to a long-standing open problem, namely the M_3 implies M_1 problem.

Observe that since Katuta showed that if every directed open cover has a cushioned refinement then the space is almost expandable ([16] Theorem 2.2) and every submetacompact almost expandable space is metacompact, an affirmative answer to problem 1.2 would also yield an affirmative answer to problem 1.1.

These problems are still open. Burke [1] summarized very nice characterization theorems for metacompact and submetacompact spaces. Junnila [14] gives important lemma and nice partial solutions in the class of preorthcompact spaces. Gruenhage [6] Chaber [3] Jiang [10] Yajima [22] got partial solutions in the class of locally compact, orthocompact, suborthocompact spaces. Because these problems are related to covering properties, generalized metric spaces as well as quasi-uniform spaces, they were asked several times by different authors. Recently it is listed in the book "Recent progress in general topology" [15].

Section 2 discusses problem 1.1, section 3 discusses problem 1.2 and in section 4 we present some recent results.

We refer the reader to Burke's article [1] and Junnila's paper [14] for undefined notions and more related theorems.

2 Metacompactness

In 1978[13] Junnila proved

Theorem 2.1. ([13] *Theorem* 1.25) *A space is metacompact iff the space is η-doubly covered and every directed open cover has a semi-open point-star refinement.*

A cover $\mathcal{U}$ of X is a η-double cover if there exists a neighbornet U of X such that $U^2 \subset \cup\{\cup\{G|x \in G \in \mathcal{U}\}\ x \in X\}, X$ is η-doubly coverd if every open cover of X is η-double cover.

He showed that every orthocompact space is η-doubly covered, so he solved problem 1.1 for the orthocompact case. I rediscovered this fact in

1988[10]. He also solved the problem for preorthocompact case, the result appears in Fletcher and Lindgren's monograph [5].

Theorem 2.2. ([5] *Lemma* 5.39) *Let X be a preorthocompact space, then X is metacompact if and only if for every directed open cover $\mathcal{U}$ of X, there is a neighbornet V of X so that $\{V^{-1}(x)|x \in X\}$ refines $\mathcal{U}$.*

The second condition is equivalent to every directed open cover having a cushioned refinement. Recall that a space is called preorthocompact, if it has a cocushioned neighbornet.

Gruenhage solved the problem for locally compact spaces, by using Game Theory technique.

Theorem 2.3. ([6] *Theorem* 6) *The following are equivalent for a locally compact spaces X:*

1. X is metacompact,
2. for each open cover $\mathcal{U}$,$\mathcal{U}^{\mathcal{F}}$ has a cushioned refinement.
3. there exists $\sigma : X \to K(X)$ (the collection of all compact subsets of X) such that no sequence $< x_n >$ satisfying $x_{n+1} \in \cup\{\sigma(x_i)|i \leq n\}$ has a limit point.

Recently Yajima generalized Theorem 2.1.

Theorem 2.4. ([22] *Theorem* 3.1) *A space is metacompact if and only if X is suborthocompact and every open cover of X has a cushioned refinement.*

A space X is said to be suborthocompact, if for every open cover $\mathcal{U}$ of X, there is a sequence of open refinements $< \mathcal{V}_n >$ of $\mathcal{U}$ such that for each $x \in X$ there is some $n \in \omega$ such that $\cap\{V \in \mathcal{V}_n | x \in V\}$ is open.

The general question is

Question 2.5. Find a topological property P such that if X has P and every directed open conver of X has a cushioned refinement, then X is metacompact.

For example, it is easy to show that σ-orthocompactness is a property of this type.

Lemma 2.6. [12] *If every directed open cover of X has a point-star refinement by semi-open covers, then X is almost expandable and hence is countably metacompact.*

Theorem 2.7. [12] *Let X be a σ-orthocompact space. If every directed open cover of X has a cushioned refinement, then X is metacompact.*

Proof: By lemma 2.6 such an X is countably metacompact and σ-orthocompact, hence orthocompact. By theorem 2.1 it is metacompact.

3 Submetacompactness

Junnila extensively discussed submetacompact spaces, and had some partial solutions for problem 1.2.

Theorem 3.1. ([3] *Proposition* 2*) For a locally compact space* X*, if every directed open cover of* X *has a* σ*-cushioned refinement, then* X *is submetacompact.*

It is also known that the answer for problem 1.2 is positive for orthocompact spaces.

Theorem 3.2. ([10]) *For an orthocompact space* X*, if every directed open cover of* X *has a* σ*-cushioned refinement, then* X *is submetacompact.*

Moreover we have the following characterization theorem of submetacompact spaces.

Theorem 3.3. [10] *A space* X *is submetacompact iff every directed open cover of* X *has a* σ*-cushioned refinement, and every open cover of* X *has a sequence* $< \mathcal{W}_n >$ *of open refinements such that for each* $n \in \omega$*, there is a closed subset* F_n *of* X *such that if* $x \in F_n$ *then* $\cap\{W \in \mathcal{W}_n | x \in W\}$ *is open, and* $\cup F_n = X$*.*

The second condition is quite close to suborthocompactness, but we can't answer the following question:

Question 3.4. (Yajima) Can we get a result similar to Theorem 2.4 for submetacompact space? [22].

Kunzi and Fletcher modified Howes's definition for confinally $\triangle$-complete space by saying that a regular Hausdorff space X is cofinally θ-complete provided that every directed open cover of X has a point-star refining sequence of open covers. They asked the following.

Question 3.5. [18] Is every cofinally θ-complete space submetacompact? They provided some partial answers there.

Theorem 3.6. [18] *Every preorthocompact cofinally θ-complete space is submetacompact.*

Theorem 3.7. [18] *Let X be a cofinally θ-complete point-star orthocompact space. Then X is submetacompact.*

A space is point-star orthocompact provided that, if $\mathcal{U}$ is an open cover of X, there is an interior-preserving open refinement $\mathcal{V}$ of $\{st(x,\mathcal{U}) : x \in X\}$ so that, for each $x \in X$ there exist $V(x) \in \mathcal{V}$ so that $x \in V(x) \subset st(x,\mathcal{U})$.

In 1977 Liu [19] defined quasi-paracompact and strictly quasi-paracompact spaces while studying collectionwise normality. A space X is called (strictly) quasi-paracompact iff every open cover of X has a refinement $\cup\{\mathcal{F}_n : n \in \omega\}$ such that $\mathcal{F}_0$ is (closed) discrete family and $\mathcal{F}_n (n \geq 1)$ is (closed) discrete family relative to $X \setminus \cup_{i<n} (\cup \mathcal{F}_i)$. It is a common generalization of both metacompact and subparacompact spaces.

The strict quasi-paracompactness implies the following property b_1 defined by Chaber (see [21]).

A space X has property b_1 iff every open cover of X has a refinement $\cup\{\mathcal{F}_n : n \in w\}$ such that $\mathcal{F}_n$ is a locally finite closed family relative to $X \setminus \cup_{i<n} (\cup \mathcal{F}_i)$.

Jiang Jiguang proved the following.

Theorem 3.8. [9] *A space is submetacompact if it is almost discretely θ-expandable and strictly quasi-paracompact.*

Long Bing showed that strictly quasi-paracompact space is equivalent to bounded weakly $\overline{\theta}$-refinable space ([20] Theorem 1.5). Zhu ([23]) showed that quasi-paracompactness implies weakly θ-refinability, but not vice versa.

Since every directed open cover having a σ-cushioned refinement implies almost θ-expandability, we know that for bounded weakly $\overline{\theta}$-refinable spaces problem 1.2 has a positive answer.

Chaber has the following result:

Theorem 3.9. *(see* [21]*) A space is submetacompact if it is almost θ-expandable and has property b_1.*

The general question here is:

Question 3.10. Find a topological property Q, such that if a space has Q and every directed open cover of X has a σ-cushioned refinement, then X is submetacompact.

Many previous results suggest that we don't need any such property to get a positive answer for problem 3.11, but so far we don't even know if weakly θ-refinability or weakly $\overline{\theta}$-refinability is enough to get a positive answer. In this paper we have provided some characterization theorems which may lead to some interesting counterexamples.

4 Recent results

Yajima and Kemoto [17] recently showed the following

Theorem 4.1. [17] *A β-space X is submetacompact if and only if every monotone open cover of X has a σ-closure-preserving closed refinement. Recall that a space X is called a β-space if there is a function $g : X \times \omega \to$ Top (X), satisfying*

1. $x \in \cap_{n \in w} g(x, n)$
2. *if* $x \in g(x_n, n)$ *for each* $n \in \omega$*, then* $< x_n >$ *has a cluster point in* X*.*

In their paper the following lemma is quoted.

Lemma 4.2. [11, 14]. *Let X be a space and $\mathcal{U}$ an interior-preserving open cover of X. Then $\mathcal{U}$ has a (σ)-closure-preserving closed refinement if and only if it has a (σ)-cushioned refinement.*

Note that every monotone open cover is interior-preserving as well as directed, thus theorem 4.1 actually answers problems 1.1, 1.2 in the affirmative in the class of β-spaces. This brings generalized metric spaces into attention. Motivated by their work, in a joint work with Reilly and Cao we get the following:

Lemma 4.3. [2] *If space X is strongly first countable and every directed open cover $\mathcal{U}$ of X has a cushioned refinement, then X is suborthocompact.*

Recall that a space X is called strongly first countable if there is a function $g : X \times \omega \to$ Top (X) satisfying:

1. $< g(x, n) >$ is a neighborhood base for each $x \in X$.

2. if $y \in g(x,n)$, then $g(y,n) \subset g(x,n)$. Without loss of generality we may assume that $g(x,n+1) \subset g(x,n)$ for $n \in w$.

Proof: If every directed open cover of X has a cushioned refinement, then X is almost expandable, hence countably metacompact.

Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be any open cover for strongly first countable space X.

For any $x \in X$, let $n(x) = \min\{m \in \omega | g(x,m) \subset U_\alpha$ for some $\alpha < \kappa\}$. Let $V_n = \cup\{g(x,m)|n(x) \leq n\}$, then $\mathcal{V} =< V_n >$ is an increasing countable open cover, hence has a point-finite open refinement $\mathcal{W} =< W_n >$, without loss of generality, we may assume that $\mathcal{W}$ is precise, i.e. each $W_n \subset V_n$.

Let $F_n = \{x \in X|$ ord $(x, \mathcal{W}) \leq n\}$, then $F_n \subset F_{n+1}$ for any $n \in w$, F_n is closed and $X = \cup_{n \in w} F_n$.

We construct ι-sequence $\mathcal{H}$ as follows. for any $x \in F_n$, choose smallest $n(x)$ such that $g(x, n(x)) \subset U_\alpha$ for some $\alpha < k$, for any $x \notin F_n$ let $G_x = (x \backslash F_n) \cap U_\beta$ for some β with $x \in U\beta$.

Let

$$\mathcal{H}_n = \{g(x, n(x) + n)|x \in F_n\} \cup \{G_x | x \notin F_n\}$$

Let $\mathcal{H} =< \mathcal{H}_n >$.

For any $p \in x, p \in F_n$ for some n, let $m = \max\{k_i | p \in W_{k_i}, i \in n\}$, then we have

$$\cap\{H \in \mathcal{H}_m | p \in H\} = g(p, n(p) + m)$$

which is open, note that for any $y \in X, y \neq p$ and $n \in w$, either $p \notin g(y,n)$ or $g(p,n) \subset g(y,n)$. This completes the proof. $\square$

Corollary 4.4. *A strongly first countable space X is metacompact if and only if every directed open cover of X has a cushioned refinement.*

Proof: Combine theorem 4.2 and theorem 2.4. □

Remark 4.5. Corollary 4.4 answers problem 1.1 in the affirmative in the class of strongly first countable spaces.

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