Geometry of Pseudospheres II.

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Abstract

We investigate finite sequences of hyperplanes in a pseudosphere. To each such sequence we associate a square symmetric matrix, the Gram matrix, which gives information about angle and incidence properties of the hyperplanes. We find when a given matrix is the Gram matrix of some sequence of hyperplanes, and when a sequence is determined up to isometry by its Gram matrix.

We also consider subspaces of pseudospheres and projections onto them. This leads to an n-dimensional cosine rule for spherical and hyperbolic simplices.

1 Introduction

The first part of this paper [M] dealt with the *pseudospheres*–the surfaces in $\mathbf{R}^{n+1}$ given by the equations,

$$x_1{}^2 + x_2{}^2 + \ldots + x_k{}^2 - x_{k+1}{}^2 - \cdots - x_{n+1}{}^2 = 1$$

$(1 \leq k \leq n+1)$.

These surfaces have a natural metric (not generally positive definite) from which we can define isometries, angles, geodesics, and hyperlanes. The pseudospheres include as special cases the ordinary Euclidean sphere ($k = n+1$) and (after changing the sign of the metric and deleting one component) hyperbolic space ($k = 1$).

We consider the pseudospheres as surfaces in $E_{n+1,k}$, where $E_{m,k} = \mathbf{R}^k \times (i\mathbf{R})^{m-k}$. The main result of [M] (Theorem 4) is that every matrix with

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columns in $E_{m,k}$ can be put into a unique canonical form by premultiplying by an orthogonal matrix [M, Section 4]. We term this *bitriangular* form. For real square matrices this is simply upper triangular form.

In this paper we apply these ideas. We assume familiarity with the results and notation of the first paper.

The main objects of investigation here are finite sequences of oriented hyperplanes $\mathbf{P}$. Each such sequence has an associated *normal matrix* $\mathrm{N}(\mathbf{P})$ whose column vectors are normals of the hyperplanes in $\mathbf{P}$. Two sequences, $\mathbf{P}$ and $\mathbf{P}'$ will be said to be isometric if there is an isometry of the space which maps each oriented hyperplane of $\mathbf{P}$ to the corresponding oriented hyperplane of $\mathbf{P}'$. Isometries on $\mathbf{P}$ correspond to premultiplying its normal matrix by an orthogonal matrix. This means that, when considering isometric invariants of a sequence $\mathbf{P}$, we may always assume that the normal matrix is in bitriangular form, a fact which we continually use.

The *Gram matrix* of a sequence with normal matrix N is defined to be $\mathrm{N}^t\mathrm{N}$. The Gram matrix is thus symmetric and we generally consider cases where it is also real. The Gram matrix is also given by $(\cos\theta_{ij})$ where θ_{ij} is the angle between the ith and the jth hyperplane in $\mathbf{P}$, whenever these angles are defined. Section **??** describes the relationship between the incidence properties of $\mathbf{P}$ and the properties of its Gram and normal matrices. In particular, we show that the intersection of hyperplanes in a sequence is isometric to a pseudosphere if and only if its normal matrix is regular as defined in [M] (i.e. its bitriangular form has no paired rows). A sequence $\mathbf{P}$ is not generally determined up to isometry by its Gram matrix. However in Section **??** we prove this to be true for regular sequences.

The Gram matrix also plays an important role in Section **??**, which investigates subspaces of $S_{n,k}$ and how the angle between two hyperplanes is related to the angle between their intersections with a subspace. When the hyperplanes of $\mathbf{P}$ form the faces of a polytope this is the relationship between the dihedral angles and the face angles. This leads to a matrix formula that relates the dihedral angles and the edge lengths of a simplex in spherical or hyperbolic space. In the plane, this is simply the familiar cosine rule.

2 The Gram Matrix

For each finite sequence, $\mathbf{P} \equiv P_1, P_2 \ldots P_j$, of non null oriented hyperplanes in $S_{n,k}$ we define the *normal matrix* $N(\mathbf{P})$ as the $(n+1) \times j$ matrix whose pth column is $\mathbf{n}_p/N(\mathbf{n}_p)$, where $\mathbf{n}_p$ is the outward unit normal vector of P_p. That is the pth column of $N(\mathbf{P})$ is just $\mathbf{n}_p$ itself when P_p is spacelike, and $\mathbf{n}_p/i$ when P_p is timelike.

We define $\mathbf{P}$ to be *spacelike* (resp. *timelike*) if each of its constituent hyperplanes is spacelike (resp. timelike). Recall that we have defined a matrix to be regular if its bitriangular form has no paired rows [M] . We now define a sequence $\mathbf{P}$ to be *regular* if $N(\mathbf{P})$ is regular.

The *Gram matrix*, $G(\mathbf{P})$ of the sequence is defined by,

$$G(\mathbf{P}) = N(\mathbf{P})^t N(\mathbf{P})$$

It is immediate from the definition that $G(\mathbf{P})$ is symmetric with entries of 1 on the leading diagonal. If $\mathbf{P}$ is spacelike or timelike then $G(\mathbf{P})$ is also real and the well known theorems about real symmetric matrices apply. For this reason we will deal mostly with spacelike or timelike $\mathbf{P}$ in what follows.

We will refer to two sequences $\mathbf{P}$ and $\mathbf{Q}$ as isometric if they are of the same length j and there is an isometry ϕ which maps each oriented hyperplane P_i to the oriented hyperplane Q_i for $1 \leq i \leq j$. (Equivalently, ϕ maps the outward normal to P_i to the outward normal to Q_i) Evidently two isometric sequences have the same Gram matrix. We will see in the next section that the converse of this is false. However if the two sequences are both required to be regular then the converse does hold (Theorem **??**).

3 Incidence

In this section we will characterize those matrices that occur as Gram matrices of some sequence $\mathbf{P}$ and relate the properties of $G(\mathbf{P})$ to the incidence properties of $\mathbf{P}$. The following terminology is used by Beardon [B] in plane hyperbolic geometry and adapts naturally to our more general context.

Definition. A sequence $\mathbf{P}$ of hyperplanes in $S_{n,k}$ is *intersecting*, *parallel* or *disjoint* according as the hyperplanes of $\mathbf{P}$ meet in $S_{n,k}$, at infinity only, or not at all.

Lemma 1 *A set of nonzero mutually orthogonal spacelike (resp. timelike) vectors in $E_{m,k}$ has cardinality not exceeding k (resp. $m-k$).*

Proof:- Let $\mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_p$ be a set of nonzero mutually orthogonal spacelike vectors in $E_{m,k}$. The column vectors of the bitriangular form of $(\mathbf{v}_1; \mathbf{v}_2; \ldots; \mathbf{v}_p)$ are also nonzero, mutually orthogonal and spacelike. An easy induction shows the jth column vector of this matrix to be a multiple of the jth unit vector in $\mathbf{R}^m$ so that, in particular, $p \leq k$. The timelike case is proved similarly.□

Proposition 2 *Let M be bitriangular with columns in $E_{m,k}$ and have α zero rows among its first k rows, γ zero rows among its remaining $m-k$ rows and β paired rows, then* $\mathrm{G} = \mathrm{M}^t\mathrm{M}$ *has $k-\alpha-\beta$ positive eigenvalues and $m-k-\gamma-\beta$ negative eigenvalues*

Proof:- By [M, Lemma 8], M has rank $m-\alpha-\gamma-\beta$. The matrix G is real symmetric and there is a real orthogonal Q for which $\mathrm{D} = \mathrm{Q}^t G\mathrm{Q} = (\mathrm{MQ})^t(\mathrm{MQ})$ is diagonal. The matrix MQ thus has mutually orthogonal columns and the same number of zero and paired rows as M. Using [M, Lemma 7] we may pre-multiply MQ by an orthogonal matrix P which leaves these zero and paired rows unchanged and so that the matrix M_1 obtained by deleting these rows from PMQ is bitriangular. The matrix M_1 can have no zero or paired rows since this would mean rank(PMQ) $<$ rank(M). Since deleting zero and paired rows leaves dot products between columns unchanged, we also have $\mathrm{M}_1^t\mathrm{M}_1 = (\mathrm{PMQ})^t(\mathrm{PMQ}) = D$. We have shown that M_1 is a regular bitriangular matrix with no zero rows and mutually orthogonal columns in $E_{m',k'}$, where $m' = m-\alpha-\gamma-2\beta$, $k' = k-\alpha-\beta$. A simple induction shows that such a matrix must have k' nonzero spacelike, and $m'-k'$ timelike, column vectors. Since the eigenvalues of G are the diagonal entries of D, the proposition follows.□

If M and G are as in the above proposition, the ranks of these two matrices are respectively $m-\alpha-\gamma-\beta$ and $m-\alpha-\gamma-2\beta$. (the rank of G being obtained by counting nonzero eigenvalues using the above proposition). The proof thus gives another way of characterizing regular matrices.

Corollary 3 *A matrix M with columns in $E_{m,k}$ is regular if and only if* $\mathrm{Rank}(M) = \mathrm{Rank}(\mathrm{M}^t\mathrm{M})$

The next theorem shows that, when Proposition **??** is applied to the normal matrix $N(\mathbf{P})$ of some sequence, the values of α, γ, and β determine, whether, in the first place, **P** is disjoint, parallel or intersecting and, when **P** is intersecting, what this intersection is up to isometry.

We have defined a matrix to be regular if its bitriangular form has no paired rows, and a sequence **P** to be regular if its matrix of normals $N(\mathbf{P})$ is regular. We now extend this definition further and say that an intersection of hyperplanes is *regular* if it can be obtained as the intersection of hyperplanes in some regular sequence **P**.

The next theorem characterizes geometrically the regular intersections of hyperplanes.

Theorem 4 *Let* $\mathbf{P} \equiv P_1, P_2 \ldots P_r$ *be a sequence of hyperplanes in* $S_{n,k}$*. Let* M *be the bitriangular form of* N($\mathbf{P}$) *and let* α*,* γ*, and* β *be as in the previous proposition.*

Let $P = \bigcap_{i=1}^{r} P_i$*. If* $\alpha > 0$ *then* P *is isometric to* $S_{\alpha+\gamma-1,\alpha} \times I^{\beta}$*, where* I *represents the real line with the identically zero " metric". If* $\alpha = 0$ *and* $\beta > 0$ *then* **P** *is parallel. If* $\alpha = \beta = 0$ *then* **P** *is disjoint.*

The intersection P *is isometric to a pseudosphere if and only if* $\alpha > 0$ *and* **P** *is regular.*

Proof:- By applying an isometry if neccesary, we may assume that N($\mathbf{P}$) itself is bitriangular. Using [M, Lemma 8] and the notation following [M, Lemma 7], the set underlying P comprises all the points

$$(x_1, x_2, \ldots x_k, iy_{k+1}, \ldots, iy_{n+1}) \in E_{n+1,k}$$

for which

$$x_i = 0, y_i = 0 \qquad i \in F \tag{1}$$

$$y_{\sigma(k)} = x_k \qquad k \in R \tag{2}$$

and

$$x_1^2 + x_2^2 + \cdots + x_k^2 - y_{k+1}^2 - \cdots - y_{n+1}^2 = 1 \tag{3}$$

The terms x_k^2 $(k \in R)$ in (**??**) are cancelled by the corresponding terms $y_{\sigma(k)}^2$ so that these x_k may take any values. After deleting these and the zero terms from (**??**), we obtain, when $\alpha > 0$, the equation of the pseudosphere,

$S_{\alpha+\gamma-1,\alpha}$. Clearly P is the Cartesian product of this pseudosphere with β copies of I. By definition, $\beta = 0$ precisely when N is regular. It is easy to see that, when $\beta > 0$, the dot product is degenerate, so that, in particular, P is not a pseudosphere.

When $\alpha = 0$ (**??**) has no solution, and, if the right hand side is replaced by zero, then it has a solution when $\alpha = 0$ if and only if $\beta \neq 0$. The latter case corresponds to an intersection at infinity, that is when **P** is parallel. □

Corollary 5 *Every sequence of fewer than k hyperplanes in $S_{n,k}$ is intersecting. Every sequence of exactly k hyperplanes in $S_{n,k}$, at least one of which is timelike, is intersecting.*

Proof:- If we assume that N(**P**) is bitriangular then, in the first case, clearly, $\alpha > 0$ and the required conclusion follows from the theorem. In the second case, we may also assume the first hyperplane in **P** is timelike so that again $\alpha > 0$.□

Theorem **??** and **??** also give,

Corollary 6 *If* **P** *is parallel then $G(\mathbf{P})$ is singular.*

Theorem 7 *A $j \times j$ real symmetric matrix is the Gram matrix of some sequence* **P** *of j oriented spacelike hyperplanes in $S_{n,k}$ if and only if it has entries of 1 on the leading diagonal, and at most k positive, and $n + 1 - k$ negative, eigenvalues. The sequence* **P** *is disjoint if and only if its Gram matrix has exactly k positive eigenvalues.*

Proof:- The last statement of the theorem and the upper bounds for the number of positive and negative eigenvalues for a Gram matrix, follow from Proposition **??** and Theorem **??**. It remains to show that any real symmetric matrix, M, with entries of 1 on the leading diagonal, and at most k positive, and $n+1-k$ negative, eigenvalues, is G(**P**) for some sequence **P** of j spacelike hyperplanes in $S_{n,k}$.

If M has these properties then there is a real orthogonal Q for which $\mathrm{D} = \mathrm{Q}^t M \mathrm{Q}$ is diagonal, and we may suppose that the positive eigenvalues are listed first along the diagonal of D followed by the zero, and then the negative eigenvalues. Let E be a diagonal matrix satisfying $\mathrm{E}^2 = \mathrm{D}$. By adding or removing some zero rows of EQ^t if neccesary, we obtain a matrix R with columns in $E_{n+1,k}$ and $\mathrm{R}^t\mathrm{R} = (\mathrm{EQ}^t)^t(\mathrm{EQ}^t) = \mathrm{M}$. Since the diagonal entries of M are all 1 by assumption, the columns of R are spacelike unit vectors. Hence, R $= N(\mathbf{P})$ for some sequence $\mathbf{P}$ of spacelike hyperplanes and M $= G(\mathbf{P})$.□

The above theorem adapts readily to the case where $\mathbf{P}$ is timelike. If $\mathbf{v}$ is timelike in $S_{m,k}$ then $\mathbf{v}/i$ is, after an obvious reordering of entries, spacelike in $S_{m,m-k}$. It follows that, if M is timelike, Proposition **??** holds with α interchanged with γ and k interchanged with $m-k$. That is we conclude that G has $m-k-\gamma-\beta$ positive, and $k-\alpha-\beta$ negative, eigenvalues in this case. Using this, we now easily establish the equivalent of Theorem **??** for timelike sequences.

Theorem 8 *A $j \times j$ real symmetric matrix is the Gram matrix of some sequence* $\mathbf{P}$ *of j oriented timelike hyperplanes in $S_{n,k}$ if and only if it has entries of 1 on the leading diagonal, and at most $n+1-k$ positive, and k negative, eigenvalues. The sequence* $\mathbf{P}$ *is disjoint if and only if its Gram matrix has exactly k negative eigenvalues.*

Observe that the above two theorems show that *any* real symmetric matrix with entries of 1 on the leading diagonal is the Gram matrix of both a spacelike and a timelike sequence of hyperplanes in some pseudosphere.

When the number of hyperplanes in $\mathbf{P}$ is equal to the dimension of the space, corollary **??** and the above two theorems give the following characterization of incidence properties in terms of Gram determinants.

Corollary 9 *If* $\mathbf{P}$ *is a spacelike or timelike sequence of n hyperplanes in $S_{n,k}$ with nonsingular Gram matrix then* $\mathbf{P}$ *is intersecting if and only if*

$$(-1)^{n+1-k}\mathrm{Det}(\mathrm{G}) > 0 \quad (\mathbf{P} \text{ spacelike})$$

$$(-1)^{k-1}\mathrm{Det}(\mathrm{G}) > 0 \quad (\mathbf{P} \text{ timelike})$$

Remark:- The incidence properties of a sequence of hyperplanes are, of course, independent of the ordering or the orientation of the hyperplanes in $\mathbf{P}$. Correspondingly, changing the orientation of any of the hyperplanes in $\mathbf{P}$, or reordering them, effects a similarity transformation on G($\mathbf{P}$), so that the eigenvalues remain unchanged.

If P is a hyperplane with unit normal $\mathbf{n}$, we denote by $\tilde{\mathrm{P}}$ the halfspace $\{\mathbf{x}|\mathbf{x}\cdot\mathbf{n} < 0\}$, which has the boundary P. A *convex polytope* is an intersection of the form,

$$\cap_{i\in I}\tilde{\mathrm{P}}_i$$

which has nonempty interior and for which the set of boundaries $\{\mathrm{P}_i|i \in I\}$ is locally finite. Given such a polytope we define its Gram matrix to be that of its bounding hyperplanes, taken in some particular order and oriented outwards.

In the sequel we will confine ourselves to the case where the index set I is finite so that the local finiteness condition will be satisfied automatically. There remains the question of whether the intersection of a given set of half-spaces has nonempty interior. When the halfspaces have timelike boundaries we have the following result, given for hyperbolic spaces in Vinberg [V Prop. 2.1]. The proof is essentially the same in the general case.

We begin with a definition. A square matrix is *decomposable* if by some permutation of the rows and the same permutation of the columns it can be brought to the form

$$\begin{pmatrix} \boxed{\mathrm{A}} & \boxed{0} \\ \boxed{0} & \boxed{\mathrm{B}} \end{pmatrix}$$

where the matrices A and B are square. Otherwise it is *indecomposable*

Proposition 10 *Let* $\mathbf{P} \equiv \mathrm{P}_1, \mathrm{P}_2 \ldots \mathrm{P}_l$ *be a disjoint sequence of timelike hyperplanes in* $S_{n,k}$ *with respective outward normals* $\mathbf{n}_1, \mathbf{n}_2 \ldots \mathbf{n}_l$ *Suppose that the* $\mathrm{G}(\mathbf{P}) = (g_{ij})$ *has nonpositive entries off the diagonal and is indecomposable,then,*

$$C = \cap_{i=1}^{l}\tilde{\mathrm{P}}_i$$

has nonempty interior.

Proof:- We may write,

$$\mathrm{G}(\mathbf{P}) = \mathrm{I}_l - \mathrm{B}$$

where B is indecomposable and has nonnegative entries. By the Perron-Frobenius theorem applied to B there is then an eigenvector $\mathbf{c} = (c_1, c_2 \ldots c_l)$ with all positive coordinates, corresponding to the least eigenvalue λ of $G(\mathbf{P})$. By Theorem **??** $\lambda < 0$. We set,

$$\mathbf{v} = -\sum_j c_j \mathbf{n}_j$$

We have,

$$\mathbf{v} \cdot \mathbf{n}_i = \sum_j g_{ij} c_j = \lambda c_i < 0 \qquad i = 1, 2 \ldots k \tag{4}$$

while

$$\mathbf{v} \cdot \mathbf{v} = -\sum_j c_j \mathbf{v} \cdot \mathbf{n}_j > 0$$

so $\mathbf{v}$ is spacelike. The unit vector $\mathbf{v}/(\mathbf{v} \cdot \mathbf{v})$ is thus in the interior of C.□

Theorem **??** allows us to tell, using the Gram matrix, whether or not a sequence of hyperplanes is disjoint, but not to distinguish between the intersecting and parallel cases. In fact it is possible for an intersecting and a parallel sequence to have the same Gram matrix. A simple example is given by the two sequences of hyperplanes in $S_{2,2}$ with respective normal matrices,

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & i \end{pmatrix}$$

which are, respectively, intersecting and parallel, and both have Gram matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

We will show however that a disjoint sequence of hyperplanes *is* determined up to isometry by its Gram matrix (Corollary **??**).

4 Disjoint Hyperplanes and the Dot Product

Let P_1 and P_2 be non-null oriented hyperplanes in $S_{n,k}$, with respective outward unit normals $\mathbf{n}_1$ and $\mathbf{n}_2$ and bounding halfspaces $\tilde{\mathrm{P}}_1$ and $\tilde{\mathrm{P}}_2$. We have noted in [M, Section 4] that, when $k \geq 2$, $\mathbf{n}_1 \cdot \mathbf{n}_2$ depends only on the intrinsic geometry of the hyperplanes and not on the particular choice of coordinates. If P_1 and P_2 intersect then it is clear from [M, Lemma 2] that $\mathbf{n}_1 \cdot \mathbf{n}_2$, is again an isometric invariant. In fact we have used $\mathbf{n}_1 \cdot \mathbf{n}_2$, in this case, to define the angle between P_1 and P_2 when these hyperplanes are either both spacelike or both timelike.

In this section we characterize $\mathbf{n}_1 \cdot \mathbf{n}_2$ geometrically, when P_1 and P_2 do not intersect. In particular, it will follow that the dot product is an isometric invariant in the case $k = 1$.

From Corollary **??**, P_1 and P_2 can fail to intersect only if $k \leq 2$. Suppose $k = 2$. As usual, we may assume $\mathrm{N}(\mathrm{P}_1,\mathrm{P}_2)$ is bitriangular, and, since neither of its top two rows can be zero, we must have $\mathbf{n}_1 = (1, 0, 0, \ldots, 0)^t$ and either $\mathbf{n}_2 = (a, b, 0, 0, \ldots, 0)^t$, where $a^2 + b^2 = 1$ or $\mathbf{n}_2 = (\pm 1, 1, 0, 0, \ldots, 0, i)^t$. In either case, both hyperplanes are spacelike. In the first case, the $n - 1$ hyperplanes with normal vectors $i\mathbf{e}_3$, $i\mathbf{e}_4$, …, $i\mathbf{e}_{n+1}$ are all perpendicular to P_1 and P_2. The intersection of these $n-1$ hyperplanes is the unique geodesic perpendicular to $\mathbf{n}_1$ and $\mathbf{n}_2$, which is given explicitly as the unit circle C in the x_1x_2-plane of $E_{n+1,k}$. Now $-\mathbf{n}_1 \cdot \mathbf{n}_2 = a$ is the cosine of the length of the geodesic arc $C \cap \tilde{\mathrm{P}}_1 \cap \tilde{\tilde{\mathrm{P}}}_2$.

In the second case, we see that P_1 and P_2 are parallel and $\mathbf{n}_1 \cdot \mathbf{n}_2 = \pm 1$, the sign depending on the orientation of the two hyperplanes.

If $k = 1$ every hyperplane in $S_{n,k}$ is timelike and P_1 and P_2 fail to intersect only if $\mathbf{n}_1 = (0, 0, \ldots, 0, i)^t$ and either $\mathbf{n}_2 = (a, 0, 0, \ldots, 0, bi)^t$, where $a > 0$ and $b^2 - a^2 = 1$ or $\mathbf{n}_2 = (1, 0, 0, \ldots, 0, i, \pm i)^t$. In the first case we may, as in the case $k = 2$, find, in each component of $S_{n,1}$, a unique geodesic perpendicular to P_1 and P_2. A routine calculation shows that the arc from this geodesic joining P_1 and P_2 has imaginary length, il and $\mathbf{n}_1 \cdot \mathbf{n}_2 = \pm \cosh l$. In the second case, P_1 and P_2 are parallel and $\mathbf{n}_1 \cdot \mathbf{n}_2 = \pm 1$. The sign of the dot product is determined by the relationship between $\tilde{\mathrm{P}}_1$ and $\tilde{\mathrm{P}}_2$ in each component of $S_{n,1}$. For example suppose that $\mathbf{n}_1 \cdot \mathbf{n}_2 < 0$, so that the last entry of $\mathbf{n}_2$ is a positive multiple of i. Suppose further that $\mathbf{x} \in Q_n \cap \tilde{\mathrm{P}}_2$ then x_{n+1} must be a nonnegative multiple of i, whence $\mathbf{x} \in \tilde{\mathrm{P}}_1$. We have shown in this case that $Q_n \cap \tilde{\mathrm{P}}_2 \subset Q_n \cap \tilde{\mathrm{P}}_1$.The same basic argument shows that this inclusion is reversed if we replace Q_n by $-Q_n$.

Similar arguments when $\mathbf{n}_1 \cdot \mathbf{n}_2 > 0$, show that, in this case, in each component of $S_{n,1}$, either the halfspaces or their complements are disjoint. We summarize these results in a theorem.

Theorem 11 *Let* P_1 *and* P_2 *be two non-intersecting oriented hyperplanes in* $S_{n,1}$ *with respective unit normals* $\mathbf{n}_1$ *and* $\mathbf{n}_2$*. When the hyperplanes are disjoint there is a unique geodesic arc joining them that is perpendicular to each. This arc has imaginary length il and* $\mathbf{n}_1 \cdot \mathbf{n}_2 = \pm \cosh l$*.*

In each component of $S_{n,1}$*, one of the halfspaces contains the other when*

$\mathbf{n}_1 \cdot \mathbf{n}_2 < 0$, *and the halfspaces or their complements are disjoint when* $\mathbf{n}_1 \cdot \mathbf{n}_2 > 0$.

5 Subspaces and Projections

Each linear subspace of codimension r can be written as

$$S_{n,k} \bigcap P_1 \bigcap P_2 \ldots \bigcap P_r$$

where the $\mathbf{P} \equiv P_1, P_2 \ldots P_r$ is a sequence of hyperplanes in $E_{n+1,k}$ with linearly independent normals. Recall from Theorem **??** that a linear subspace, P, in $S_{n,k}$ is isometric to a pseudosphere if and only if it is regular.

Definition. If P is a regular linear subspace of codimension r and P_1 is an oriented hyperplane in $S_{n,k}$ then $\mathrm{P}_1 \bigcap \mathrm{P}$, with the orientation inherited from P_1 in the obvious way, is the *projection* of P_1 onto P.

In the remainder of this section we consider angles and distances between the projections of two hyperplanes onto a given regular subspace. We recall the following result from linear algebra (see e.g.[S, Appendix B3])

Proposition 12 *If L is the subspace of* $\mathbf{R}^n$ *orthogonal to each vector in the linearly independent set* $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_k$*, then the projection of a vector* $\mathbf{v} \in \mathbf{R}^n$ *onto L is given by* $\mathrm{P_L}\mathbf{v}$*, where, if* $\mathrm{M} = [\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_k]$, $\mathrm{P_L}$ *is defined by*

$$\mathrm{P_L} = \mathrm{I} - \mathrm{M}(\mathrm{M}^t\mathrm{M})^{-1}\mathrm{M}^t$$

If Q is a hyperplane in $\mathbf{R}^n$*, which intersects L, and has normal* $\mathbf{n}$*, then the projection of* $\mathbf{n}$ *onto L is normal to* $\mathrm{Q} \bigcap \mathrm{L}$.

Exactly the same definition can be used to define projections onto subspaces of $E_{m,k}$. In order for the definition to make sense the matrix $\mathrm{M}^t\mathrm{M}$ must be nonsingular, and, in view of Corollary **??**, this occurs exactly when M is regular.

It is easy to verify that the definition of $\mathrm{P_L}$ depends only on the space L and not on the particular choice of normal vectors, that $\mathrm{P_L}$ maps $E_{m,k}$ onto L and fixes L pointwise, and that ${\mathrm{P_L}}^2 = {\mathrm{P_L}}^t = \mathrm{P_L}$. We also note the following important invariance property of projections. If Λ is an isometry of $E_{m,k}$ and $\mathbf{v}$ and $\mathbf{w}$ are vectors in $E_{m,k}$, then

$$\mathrm{P}_{\Lambda\mathrm{L}}(\Lambda\mathbf{v}) = \Lambda(\mathrm{P}_{\mathrm{L}}\mathbf{v})$$

Definition. If M is a square matrix, M_i^j denotes the matrix obtained from M by deleting the ith row and jth column of M, M_{ik}^{jl} denotes the matrix obtained from M by deleting the ith and kth rows and jth and lth columns of M, etc.

We denote by M_{ij} the signed cofactor corresponding to the matrix M_i^j. That is,

$$M_{ij} = (-1)^{i+j}\mathrm{Det}(\mathrm{M}_i^j)$$

The next result gives the angle between the projections of two hyperplanes onto a given subspace in terms of Gram matrices.

Theorem 13 *Let* P_1 *and* P_2 *be non null oriented hyperplanes in* $S_{n,k}$*, with outward unit normals* $\mathbf{n_1}$ *and* $\mathbf{n_2}$*, respectively. Let* $\mathbf{Q}$ *be a regular sequence of* r *hyperplanes in* $S_{n,k}$*, with linearly independent normals. Let* Q *be the linear subspace (of codimension* r*) obtained by taking the intersection of the hyperplanes in* $\mathbf{Q}$ *and* P_{Q} *the projection onto* Q*. If both of the projections,* $\mathrm{P}_1 \bigcap \mathrm{Q}$*,* $\mathrm{P}_2 \bigcap \mathrm{Q}$ *are nonempty, non null, and have outward unit normals* $\mathbf{n_1'}$ *and* $\mathbf{n_2'}$*, respectively then,*

$$\mathbf{n_1'} \cdot \mathbf{n_2'} = \mathrm{Sign}(\mathrm{Det}(\mathrm{G}(\mathbf{Q}))) \times \frac{N(\mathbf{n_1})N(\mathbf{n_2})\mathrm{Det}(G_2^1)}{\sqrt{|\mathrm{Det}(G_1^1 G_2^2)|}} \tag{5}$$

where $\mathrm{G} = \mathrm{G}(P_1, P_2, \mathbf{Q})$

Proof:- Without loss of generality we may assume that $\mathrm{N} = \mathrm{N}(\mathbf{Q})$ is bitriangular, so that $\mathrm{Q} = \Pi \bigcap S_{n,k}$, where Π is the subset of $E_{n+1,k}$ for which all but the last α real, and the first γ imaginary, coordinates vanish ($\alpha + \gamma = r$). Clearly the map π, which deletes all these vanishing coordinates is an isometry from Q onto $S_{\alpha+\gamma-1,\alpha}$ and $\mathbf{m_i} = \pi(\mathrm{P}_{\mathrm{Q}}(\mathbf{n_i}))$ is normal to $\pi(\mathrm{P}_{\mathrm{Q}}(\mathrm{P}_i))$ (i=1,2). We show

$$\begin{aligned}\mathbf{m_1}\cdot\mathbf{m_2}\times \mathrm{Det}(\mathrm{G}(\mathbf{Q})) &= \mathbf{n_1}^t(\mathrm{I}-\mathrm{N}(\mathrm{N}^t\mathrm{N})^{-1}\mathrm{N}^t)\mathbf{n_2}\times\mathrm{Det}(\mathrm{N}^t\mathrm{N})\\ &= \mathrm{Det}[(\mathbf{n_1};\mathrm{N})^t(\mathbf{n_2};\mathrm{N})]\\ &= \mathrm{Det}(\mathrm{G}_2^1)\times N(\mathbf{n_1})N(\mathbf{n_2}) \qquad (6)\end{aligned}$$

Similarly

$$\mathbf{m_i}\cdot\mathbf{m_i}\times\mathrm{Det}(\mathrm{G}(\mathbf{Q})) = \mathrm{Det}(\mathrm{G}_i^i)\times N(\mathbf{n_i})^2 \qquad (7)$$

for $i = 1, 2$.

Since, by assumption, $\mathbf{m_i}$ is not null, the matrix G_i^i is nonsingular.

The first and last equations of (**??**) are obvious. To prove the second, let X be the column space of N. Since each line of (**??**) is linear in $\mathbf{n_2}$ (in fact, in each of $\mathbf{n_1}$ and $\mathbf{n_2}$), it suffices to consider the two cases $\mathbf{n_2}\in\mathrm{X}$ and $\mathbf{n_2}\in\mathrm{X}^\perp$.

If $\mathbf{n_2}\in\mathrm{X}$, say $\mathbf{n_2}=\mathrm{N}\mathbf{z}$, then $(\mathbf{n_2};\mathrm{M})$ is rank deficient and so the second line of (**??**) is zero. Also,

$$\begin{aligned}\mathrm{N}(\mathrm{N}^t\mathrm{N})^{-1}\mathrm{N}^t\mathbf{n_2} &= \mathrm{N}(\mathrm{N}^t\mathrm{N})^{-1}\mathrm{N}^t\mathrm{N}\mathbf{z}\\ &= \mathrm{N}\mathbf{z} = \mathbf{n_2} \qquad (8)\end{aligned}$$

so the first line of (**??**) vanishes also.

If $\mathbf{n_2}\in\mathrm{X}^\perp$, then $\mathrm{N}^t\mathbf{n_2}=\mathbf{0}$ so that the first line of (**??**) is $\mathbf{n_1}\cdot\mathbf{n_2}\times\mathrm{Det}(\mathrm{N}^t\mathrm{N})$ and,

$$\begin{aligned}\mathrm{Det}[(\mathbf{n_1};\mathrm{N})^t(\mathbf{n_2};\mathrm{N})] &= \mathrm{Det}\begin{pmatrix}\mathbf{n_1}\cdot\mathbf{n_2} & \mathbf{n_1}^t\mathrm{N}\\ \mathrm{N}^t\mathbf{n_2} & \mathrm{N}^t\mathrm{N}\end{pmatrix}\\ &= \mathbf{n_1}\cdot\mathbf{n_2}\,\mathrm{Det}(\mathrm{N}^t\mathrm{N}), \qquad (9)\end{aligned}$$

since $\mathrm{N}^t\mathbf{n_2}=\mathbf{0}$.

Now, since $\mathbf{n_i}'=\mathbf{m_i}/\sqrt{|\mathbf{m_i}\cdot\mathbf{m_i}|}$ (i=1,2) the theorem follows. □

In the hyperbolic and spherical cases (k= 1 and $n+1$ respectively) Theorem **??** simplifies considerably. Using Theorems **??** and **??**, **??** becomes, in these cases,

$$\mathbf{n}_1'\cdot\mathbf{n}_2' = \frac{\pm\mathrm{Det}(G_2^1)}{\sqrt{\mathrm{Det}(G_1^1G_2^2)}} \qquad (10)$$

the sign being positive for spherical space, and negative for hyperbolic.

When $\mathbf{Q}$ comprises r of the hyperplanes that bound a polytope, its intersection, if nonempty, will meet the polytope in a codimension r face. Theorem **??** can thus be used to calculate face angles at all dimensions. In particular, for a simplex, all face angles can be computed from dihedral angles. For this purpose it is convenient to have a more symmetrical from of Theorem **??**. We begin with a simple lemma and a definition.

Lemma 14 *Let* N_1 *be any matrix with* m *columns,* π *a permutation of* $\{1, 2, \ldots, m\}$*, and* N_2 *the matrix obtained by permuting the columns of* N_1 *according to* π*. Let* $\mathrm{G}_i = \mathrm{N}_i^t \mathrm{N}_i$ *(i=1,2), then,*

$$(-1)^{i+j}\mathrm{Det}((\mathrm{G}_1)_i^j) = (-1)^{\pi(i)+\pi(j)}\mathrm{Det}\left((\mathrm{G}_2)_{\pi(i)}^{\pi(j)}\right) \tag{11}$$

Proof:- It suffices to prove the lemma in the case π is a transposition of the form $(k, k+1)$. The matrix G_2 is obtained by applying this transposition to the columns *and* rows of G_1. If, for example, $i = k$, $j \neq k+1$ then $(\mathrm{G}_2)_{\pi(i)}^{\pi(j)}$ is obtained from $(\mathrm{G}_1)_i^j$ by a column transposition so that (**??**) holds in this case. A similar argument applies in all other cases.□

Definition. Let A be an $m \times m$ square matrix and $J \subseteq \{1, 2, \ldots, m\}$. We denote by A_J the principal submatrix of A formed from the rows and columns whose indices belong to J. For $i, j \in J$ we let $\mathrm{A}_J(i, j)$ denote the cofactor of A_J corresponding to the submatrix of A whose rows and columns have indices in $J - \{i\}$ and $J - \{j\}$, respectively.

Theorem 15 *Let* $\mathbf{Q}$ *be a sequence of* $n+1$ *hyperplanes in* $S_{n,k}$ *with Gram matrix G. Suppose the normals of the hyperplanes in* $\mathbf{Q}$ *form a linearly independent set and that for any proper subset* J *of* $\{1, 2, \ldots, n+1\}$*,* $Q = Q_J = \cap_{k \in J} Q_k$ *is a regular subspace.*

For any distinct $i, j \notin J$ *the projections* $Q_i \cap Q$ *and* $Q_j \cap Q$ *are non empty and not null. The dot product of their respective unit normals,* $\mathbf{n_i'}$ *and* $\mathbf{n_j'}$ *is given by*

$$\mathbf{n_i'} \cdot \mathbf{n_j'} = -\mathrm{Sign}(\mathrm{G}_J) \times \frac{N(\mathbf{n_i})N(\mathbf{n_j})\mathrm{G}_{J'}(i,j)}{\sqrt{|\mathrm{G}_{J'}(i,i)\mathrm{G}_{J'}(j,j)|}} \tag{12}$$

where $J' = J \cup \{i, j\}$*.*

Proof:- Since the normals of $\mathbf{Q}$ are linearly independent it follows that the corresponding normal matrices have full rank so that (by Corollary **??**) all the principal submatrices of G are nonsingular (It is easy to show that, conversely, the hypotheses of this theorem follow if G is assumed to have nonsingular principal submatrices). It follows from the assumptions of the theorem that all the projections are non empty, and by (**??**) they are also not null. The theorem now follows from **??** and Lemma **??**.□

For the remainder of this section we suppose that the hyperplanes of $\mathbf{Q}$ form the outward oriented faces of a simplex Δ, and consider only the spherical and hyperbolic cases. As before, (**??**) now simplifies to,

$$\mathbf{n_i'} \cdot \mathbf{n_j'} = \frac{\mp G_{J'}(i,j)}{\sqrt{G_{J'}(i,i)G_{J'}(j,j)}} \tag{13}$$

the sign now being negative in the spherical case and positive in the hyperbolic. It follows that the cosine of the *angle* between $\mathbf{n_i'}$ and $\mathbf{n_j'}$ is obtained by taking the right hand side of **??** with the sign negative in both cases. This angle is the *exterior* angle between the two projected faces $Q_i \cap Q$ and $Q_j \cap Q$. The interior angle θ_{ij} between Q_i and Q_j in face Q being given by,

$$\cos\theta_{ij} = \frac{G_{J'}(i,j)}{\sqrt{G_{J'}(i,i)G_{J'}(j,j)}} \tag{14}$$

in both hyperbolic and spherical space.

When J has cardinality $n-1$ the pair $\{i,j\}$ is uniquely determined and $J' = \{1, 2, \ldots, n+1\}$. The subspace $\mathbf{Q}_J$ is, in this case, isometric to the unit circle or to a pair of hyperbolae, the projections Q_i and Q_j are the vertices opposite faces j and i, respectively, and the "angle" between these two vertices is now an edge length. From Section **??** $-\mathbf{n_i'} \cdot \mathbf{n_j'}$ is $\cos l_{ij}$ where l_{ij} is the edge length of Δ contained in $\mathbf{Q}_J$. In the hyperbolic case l_{ij} is imaginary.

In these cases (**??**) becomes

$$\cos l_{ij} = \frac{(-1)^{i+j}\mathrm{Det}(\mathrm{G}_i^j)}{\sqrt{\mathrm{Det}(\mathrm{G}_i^i \mathrm{G}_j^j)}} \tag{15}$$

Of course we may take edge length in hyperbolic space to be real, in which case the cosine in the above equation would be replaced by a hyperbolic cosine. The advantage of supposing hyperbolic length to be imaginary is that it allows hyperbolic and spherical space to be treated together.

Sylvester's identity (see e.g. [BG] Theorem 1.4.1) gives $\mathrm{Det}(\mathrm{G}_{ij}^{ij})\mathrm{Det}(\mathrm{G}) = \mathrm{Det}(\mathrm{G}_i^i\mathrm{G}_j^j) - \mathrm{Det}(\mathrm{G}_i^j)\mathrm{Det}(\mathrm{G}_j^i))$, whence equation (**??**) can be written in terms of $\sin l_{ij}$ thus,

$$\sin^2 l_{ij} = \frac{\mathrm{Det}(\mathrm{G}_{ij}^{ij})\mathrm{Det}(\mathrm{G})}{\mathrm{Det}(\mathrm{G}_i^i\mathrm{G}_j^j)} \tag{16}$$

Equation (**??**) gives an explicit way of calculating edge lengths from dihedral angles. We can rewrite it in matrix form as follows. Define an $(n+1)\times(n+1)$ matrix Γ by (a_{ij}) where $a_{ij} = \cos l_{ij}$. Let D and D_1 be the diagonal matrices whose diagonal entries coincide with those of the adjoint matrices G^* and Γ^*, respectively. The matrix D is always positive (Corollary **??**) but D_1 is negative in the hyperbolic case when n is even (see (**??**) in the proof below). In this case we define ${\mathrm{D}_1}^{\pm 1/2}$ to have nonnegative multiples of $\pm i$ on the diagonal.

Theorem 16 (n-dimensional Cosine Rule)

$$\Gamma = \mathrm{D}^{-1/2}\mathrm{G}^*\mathrm{D}^{-1/2} \tag{17}$$

and

$$\mathrm{G} = {\mathrm{D}_1}^{-1/2}\Gamma^*{\mathrm{D}_1}^{-1/2} \tag{18}$$

Proof:- Equation (**??**) is simply a restatement, in matrix terms, of (**??**). Taking adjoints gives,

$$\Gamma^* = \mathrm{Det}(\mathrm{G})^{n-1}\mathrm{Det}(\mathrm{D})^{-1}\mathrm{D}^{1/2}\mathrm{G}\mathrm{D}^{1/2} \tag{19}$$

and equating diagonal entries

$$\mathrm{D}_1 = \mathrm{Det}(\mathrm{G})^{n-1}\mathrm{Det}(\mathrm{D})^{-1}\mathrm{D} \tag{20}$$

whence, substituting back into (**??**), we get,

$$\Gamma^* = {\mathrm{D}_1}^{1/2}\mathrm{G}{\mathrm{D}_1}^{1/2}$$

and (**??**) follows.□

We refer to this theorem as the n-dimensional cosine rule because, when $n = 2$, (**??**) and (**??**) are just matrix forms of the familiar cosine rules for spherical and hyperbolic triangles.

The above theorem gives an explicit formula for the dihedral angles of a simplex in terms of its edge lengths. Since the edges of a simplex are also edges of its faces, we may, given all the angles in faces of any given dimension, calculate the edge lengths by (**??**) and so recover the dihedral angles using (**??**). In particular we have shown that the Gram matrix of a simplex is uniquely determined by its face angles at any dimension. As we prove in the next section (Theorem **??**), the Gram matrix determines the simplex itself up to isometry.

6 Further results on Gram Matrices

We have shown in Section **??** that the Gram matrix of a sequence of hyperplanes does not determine the sequence up to isometry, or even to distinguish between the intersecting and parallel cases. We have, however this uniqueness result for *regular* sequences.

Theorem 17 *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are regular sequences of hyperplanes and* $G(\mathbf{P}) = G(\mathbf{Q})$*, then* $\mathbf{P}$ *and* $\mathbf{Q}$ *are isometric.*

In view of Proposition **??** we have

Corollary 18 *A disjoint sequence of hyperplanes is uniquely determined up to isometry by its Gram matrix*

If the zero rows are deleted from a bitriangular matrix M with columns in a given $E_{m,k}$ then the matrix so obtained clearly determines M. Since regular bitriangular matrices, by definition, also have no paired rows, Theorem **??** is a consequence of the following result.

Theorem 19 *If M is a bitriangular matrix without zero or paired rows then the matrix* $\mathrm{G} = \mathrm{M}^t\mathrm{M}$ *uniquely determines M.*

Proof:- In this proof we will suppose that M is given by [M, equation (5)] and refer to the conditions (1)-(7) in the definition following it. Recall that r_j denotes the rank of the matrix comprising the first j columns of M. Suppose $\mathrm{G} = (g_{ij})$ to be $n \times n$.

We prove the theorem by induction on n. The case $n = 1$ is trivial. For the induction step let M be a bitriangular matrix without zero or paired rows such that $\mathrm{M}^t\mathrm{M} = \mathrm{G}$. We must show that M is uniquely determined by G. We let M_1 denote the matrix obtained by deleting the last column of M, and N the matrix obtained by deleting the zero and paired rows from M_1. By the induction hypothesis, G determines N.

Let G_1 be the matrix obtained by deleting the last row and column of G, and let s and s_1 be the ranks of G and G_1, respectively.

Clearly M has n columns and, since M is bitriangular and has no zero or paired rows, it has rank(M) rows. By corollary **??** rank(M) $= s$. Thus G determines the dimensions of M.

Since M has no zero or paired rows, we have either $N = \mathrm{M}_1$, or N is obtained from M_1 by deleting a single zero row or two paired rows. In these three cases we have, respectively, $s = s_1$, $s = s_1 + 1$ and $s = s_1 + 2$ (see remarks preceding Corollary **??**). We consider these cases in turn. Let $\mathbf{v}_j$ denote the jth column of N, N_j the matrix comprising the first j columns of N, and $\mathbf{y} = (y_1, y_2, \ldots, y_s)^t$ the last column of M.

Case 1: $N = \mathrm{M}_1$. By induction on j we see that the values of $g_{1n} = \mathbf{v}_1 \cdot \mathbf{y}, g_{2n} = \mathbf{v}_2 \cdot \mathbf{y} \ldots g_{jn} = \mathbf{v}_j \cdot \mathbf{y}$ determine the values of y_k whenever the kth row of N_j is neither zero nor paired and of $y_k - y_l/i$ whenever the kth and lth rows of N_j are paired. Since M_1 itself has neither zero nor paired rows, this result, when $j = n - 1$, means that $\mathbf{y}$, and hence M, is completely determined.

Case 2: N is M_1 with a zero row deleted. Now M_1 results by inserting a zero row between the real and imaginary rows of N. Now, as in the previous case, the dot products of $\mathbf{y}$ with the columns of M_1 determine all entries of $\mathbf{y}$ except for that corresponding to the zero row of M_1. This entry (which by condition (1) must be nonnegative) is determined by the value of $g_{nn} = \mathbf{y} \cdot \mathbf{y}$.

Case 3: N is M_1 with a pair of rows deleted. Let J be greatest value of j for which, there is a vector $\mathbf{w}$ the same length as the columns of N, for which,

$$\mathbf{w} \cdot \mathbf{v}_i = g_{in} \quad (1 \leq i \leq j)$$

We must have $J < n-1$ or else one of the previous cases would apply. If p and q are the indices of, respectively, the bottom nonzero real, and the top nonzero imaginary, rows of N_J, then M_1 is obtained by inserting a real row $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ below the pth row of N, and the imaginary row $i\mathbf{u}$ above the qth row of N. Let α and $i\beta$ be the entries of $\mathbf{y}$ corresponding to the rows $\mathbf{u}$ and $i\mathbf{u}$ respectively in M_1.

By an inductive argument, similar to that used in the previous cases, we see that the equations,

$$\mathbf{y} \cdot \mathbf{v}_i = g_{in} \quad (1 \leq i \leq n-1)$$

determine all the entries of $\mathbf{y}$ except for α and $i\beta$ and also (from the above equation for $i = J+1$) the value of $\alpha - \beta$. Since the equation $\mathbf{y} \cdot \mathbf{y} = g_{nn}$ then gives the value of $\alpha^2 - \beta^2$, $\mathbf{y}$ is completely determined.

It remains to determine the entries of $\mathbf{u}$. For $j \leq J$, $u_j = 0$. When $j > J$ and $r_j > r_{j-1}$, condition (4) gives $u_j = 1$. The remaining values of u_j can be found from the equations $\mathbf{y} \cdot \mathbf{v}_j = g_{jn}$, again using condition (4) and the now known (and distinct) values of α and β.□

7 Hyperbolic Space

As noted in [M], the hyperboloid model Q_n for hyperbolic space can be obtained by taking the component $x_1 > 0$ of $S_{n,1}$ and changing the sign of the metric. In this section we investigate the particular properties of this space.

We have seen that for pseudospheres any sequence $\mathbf{P}$ of oriented hyperplanes in $S_{n,k}$ can be put into bitriangular form by an isometry or, in algebraic terms, by premultiplying the matrix of normals by a matrix from $\mathrm{Q}(n+1, k)$. In Q_n this result requires slight modification because its isometry group is not $\mathrm{Q}(n+1, 1)$ but the index two subgroup $\mathrm{Q}^+(n+1, 1)$.

The group $\mathrm{Q}(n+1, 1)$ is generated by $\mathrm{Q}^+(n+1, 1)$ and the matrix $-\mathrm{I}_{n+1}$. Multiplying the normal vector to an oriented hyperplane P by $-\mathrm{I}_{n+1}$ of course gives the normal vector to the same hyperplane with its orientation reversed. Thus in Q_n any matrix of normals can be put into bitriangular form by premultiplying it by an isometry and possibly the orientation reversing map.

If $\mathbf{P}$ is a sequence of oriented hyperplanes we denote by $-\mathbf{P}$ the same sequence with the orientation of each hyperplane reversed. The next result characterizes those sequences for which there is an isometry from $\mathbf{P}$ to $-\mathbf{P}$.

Proposition 20 *If $\mathbf{P}$ is a sequence of oriented hyperplanes in Q_n then there is an isometry from $\mathbf{P}$ to $-\mathbf{P}$ if and only if $\mathbf{P}$ is intersecting.*

Proof:- By applying a sequence of isometries we may assume $\pm N(\mathbf{P})$ is bitriangular and so, by interchanging $\mathbf{P}$ and $-\mathbf{P}$ if neccessary, that $\mathrm{N}(\mathbf{P})$ itself is bitriangular. Now suppose that there is a matrix $\mathrm{M} \in \mathrm{Q}^+(n+1,1)$ for which

$$\mathrm{MN}(\mathbf{P}) = \mathrm{N}(-\mathbf{P}) = -\mathrm{N}(\mathbf{P}) \tag{21}$$

If $\mathbf{P}$ is not intersecting let $s \geq 1$ be chosen so that $P_1, P_2 \ldots P_s$ is intersecting and $P_1, P_2 \ldots P_{s+1}$ is not. If the first s columns of $\mathrm{N}(\mathbf{P})$ constitute a matrix of rank r, a simple induction shows that M must take the block form,

$$\mathrm{M} = \begin{pmatrix} \mathrm{A} & 0 \\ 0 & -\mathrm{I}_r \end{pmatrix}$$

where

$$\mathrm{A} \in \mathrm{Q}^+(n+1-r,1) \tag{22}$$

Equating the $(s+1)$th columns in (**??**) now gives an equation of the form $\mathrm{A}\mathbf{v} = -\mathbf{v}$ where $\mathbf{v}$ is of the form either $(1,0\ldots0,i)$ or $(a,0\ldots0,0)$ $(a \neq 0)$, but this is contrary to (**??**). (Recall that the top row of A must begin with a positive entry and be spacelike). Thus (**??**) cannot hold when $\mathbf{P}$ is parallel or disjoint.

On the other hand, when $\mathbf{P}$ is intersecting the top row of $\mathrm{N}(\mathbf{P})$ is zero so the diagonal matrix with an entry of 1 in the top left position and -1 elsewhere on the diagonal is an isometry which changes the orientation of each hyperplane in $\mathbf{P}$.□

We conclude by stating Theorem **??** as it applies to hyperbolic space.

Theorem 21 *A $j \times j$ real symmetric matrix is the Gram matrix of some sequence $\mathbf{P}$ of j oriented hyperplanes in hyperbolic n-space if and only if it has entries of 1 on the leading diagonal, and at most n positive eigenvalues, and at most 1 negative eigenvalue. The sequence $\mathbf{P}$ is disjoint if and only if its Gram matrix has exactly 1 negative eigenvalue.*

Bibliography

[BG] G. A. Baker and P. Graves-Morris, *Padé Approximants, Part I: Basic Theory* Encyclopaedia of Mathematics and its applications Vol. 13, Addison-Wesley 1981

[B] A. F. Beardon, *The Geometry Of Discrete Groups*, Springer-Verlag 1983

[M] T. H. Marshall, *Geometry of Pseudosphers I.* N.Z. J. Math. (previous issue)

[S] G. A. F. Seber, *Linear Regression Analysis*, Wiley, 1977

[V] E. B. Vinberg, *Hyperbolic Reflection Groups*, Russian Math. Surveys **40**, (1985) 31-75

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