# Discrete wave scattering on a star-graph

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**Abstract**

In this paper we consider the spectral problem for the adjacency matrix of a graph composed of a compact part with a few semi-infinite periodic leads attached. Based on the spectral properties of the adjacency matrix we develop Lax-Phillips scattering theory for the corresponding discrete wave equation.

## 1 Introduction

The spectral method is widely used for the study of transport phenomena on a compact graph, see [1]. The spectrum of the adjacency matrix of a compact graph is discrete and consists of a finite number of eigenvalues. The simplest non-compact star-graphs are obtained by attaching semi-infinite periodic leads to a compact graph. The corresponding adjacency matrices have an absolutely-continuous component in the spectrum. One can study the spectral and transport properties of non-compact graphs using scattering theory. One of the most interesting questions concerns the connection between the characteristics of the discrete spectrum of the compact part of a graph and the resonances and the resonance states of the scattering problem. The resonances and the shapes of the resonance states control the transmission of signals from one lead to another. In this paper we derive the dispersion equation for resonances in terms of the corresponding Dirichlet-to-Neumann map of the compact sub-graph and the quasi-momentum on the leads. We also consider the discrete wave equation on the graph by reducing it to a discrete version of the Lax-Phillips scattering problem.

The paper has the following structure. In the second section we introduce an analog of the Dirichlet-to-Neumann map of the compact part of the graph and construct the scattered waves and the scattering matrix of the non-compact graph with semi-infinite periodic leads attached. In the third section we consider the discrete wave-equation on the non-compact graph. We introduce the energy norm and reveal the geometry of the energy-normed space of Cauchy data. In particular we describe the incoming and outgoing subspaces and observe that they are orthogonal. In the fourth section we briefly discuss the connection between the scattering problem on the star-graph and Lax-Phillips theory. In particular we establish a connection between the discrete spectrum of the adjacency matrix of the compact subgraph and resonances and establish completeness of the resonance states. We postpone to forthcoming publications analysis of the scattering problem with leads having non-trivial periods. This analysis requires spectral theory of functions on multiply-connected domains.

## 2 The discrete Schrödinger equation on the non-compact graph

Consider a non-compact graph $\Omega$ consisting of a compact part $\Omega_{in}$ and a few simplest semi-infinite periodic leads $\omega = \left\{\omega^l\right\}_{l=1}^N$ attached to some vertices $a_r \in \Omega_{in}$, $r = 1, 2, \dots N < \infty$. The simplest lead $\omega^l$ is a periodic lattice $\left\{b_0^l, b_1^l, b_2^l, b_3^l, \dots\right\}$, where the node $b_s^l$ has two nearest neighbors $b_{s-1}^l$, $b_{s-1}^l$. Following [1] we consider the adjacency matrix $L$ of the graph $\Omega$ in the the space of square summable sequences $\mathbf{U} = \{\mathbf{u}_{in}, \vec{\mathbf{u}}\}$. Here $\mathbf{u}_{in} = (u_1, u_2, u_3 \dots u_M)$— the complex coordinates of the inner component $\mathbf{u}_{in}$ of $\mathbf{U}$, defined at the vertices $a_s$, $s = 1, 2, 3 \dots M$, $M \geq N$. Furthermore, $\vec{\mathbf{u}} = \left(\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3, \dots \dots \mathbf{u}^N\right)$—the set of $l_2$-vectors $\mathbf{u}^l = \left(u_1^l, u_1^l, u_3^l, \dots\right)$ on the leads $\omega^l$, $l = 1, 2, \dots N$. The first component of $\mathbf{U}$ in the decomposition $L_2(\Omega) = l_2(\Omega_{in}) \oplus l_2(\omega)$ is finite-dimensional, dim $l_2(\Omega_{in}) = M$, the second component is of course infinite-dimensional. If the lead $\omega^k$ is attached to the node $a_k \in \Omega_{in}$ we impose on vectors $\mathbf{U}$ from the domain of the operator $L$ the boundary conditions $u_{in}(a_k) = u_0^k$, thus assuming that $b_0^k \equiv a_k$, $k = 1, 2, \dots N$. We introduce also the contact space $E_{cont} = E = C_N$ as a space of vectors constituted by the values of the components of $\mathbf{u}_{in}$ at the contact points $a_s$, $s = 1, 2, \dots N$.

The operator $L$ can be interpreted as a self-adjoint extension, see [2], of the properly restricted, orthogonal sum $L_{in} \oplus \sum_{k=1}^N \mathbf{l}^k$. Here $\mathbf{l}^k := l$ is the non-perturbed adjacency matrix on the lead $\omega^k$.

$$\mathbf{l}\,\mathbf{u} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ \dots \end{pmatrix}.$$

Note that the restricted operator is not densely defined. Nevertheless, the corresponding self-adjoint extension can be constructed after [2].

We revisit first the spectral properties of the non-perturbed operator $\mathbf{l}$. It is self-adjoint in $l_2$ and has a simple, absolutely continuous spectrum. The spectrum consists of a single spectral band $[-2, 2]$ with eigenfunctions parametrized by the quasi-momentum exponential $\Theta = e^{ip}$ with real quasi-momentum $p$ on the interval $0 \leq p < 2\pi$. The eigenfunctions $\Psi_\lambda$ are obtained as linear combinations $\Psi_\lambda = \left\{1 + S, \Theta + S\bar{\Theta}, \Theta^2 + S\bar{\Theta}^2 \dots\right\}$ of Bloch-solutions $\chi_\pm$

$$\chi_+ = \left(1, \Theta, \Theta^2, \Theta^3, \dots\right), \ \chi_- = \left(1, \bar{\Theta}, \bar{\Theta}^2, \bar{\Theta}^3, \dots\right)$$

of the homogeneous equation $\mathbf{l}\,\chi_\pm = \lambda\chi_\pm$, $\lambda = \Theta + \bar{\Theta}$. Substitution of the ansatz $\Psi_\lambda$ into the homogeneous equation $\mathbf{l}\Psi_\lambda - \lambda\Psi_\lambda = 0$ gives $S = -\bar{\Theta}^2$. It is convenient to use the quasi-momentum exponential as a spectral parameter instead of $\lambda$. Then we may write $\Psi_\lambda = \Psi_\Theta$

where $\lambda = \Theta + \bar{\Theta}$ is a point on the spectrum and $|\Theta| = 1$. The spectral decomposition of $\mathbf{l}$ is given by integration in the quasi-momentum exponential $\Theta$ in the positive direction over the unit circle $\Sigma$

$$\frac{1}{2\pi i}\int_{\Sigma}\langle \mathbf{u},\, \Psi_{\theta}\rangle \Psi_{\theta}\frac{d\Theta}{\Theta} = \mathbf{u}. \tag{1}$$

The formula (1) can be easily verified on the dense set of finite elements $\mathbf{u}$ and extended via closure to $l_2(\Sigma)$. The system of all eigenfunctions $\Psi_{\Theta}$, $0 \leq p < 2\pi$ is over-complete. The corresponding system on the interval $0 \leq p < \pi$ is complete and orthogonal. Hence the spectral integral can be reduced to the integral over the upper semi-circle $\theta = e^{ip}$, $0 \leq p < \pi$, which corresponds to the upper shore of the spectral band $[-2, 2]$ (since $\lambda = \Theta + \bar{\Theta}$)

$$\frac{1}{\pi}\int_{0}^{\pi}\langle \mathbf{u},\, \Psi_{\Theta}\rangle \Psi_{\Theta}\, dp = \mathbf{u}, \tag{2}$$

or to the integral over the spectral parameter $\lambda$

$$\frac{1}{2\pi}\int_{-2}^{2}\langle \mathbf{u},\, \Psi_{\Theta}\rangle \Psi_{\Theta}\frac{d\lambda}{\sin p} = \mathbf{u}, \tag{3}$$

with $\lambda = 2\cos p$, $\sin p = \sqrt{1 - \lambda^2/4}$. More about spectral properties of discrete and continuous periodic operators can be found in [3], see also references therein. The operator $L_{out} = \oplus \sum_{k=1}^{N} \mathbf{l}^k$ is defined in the space $l_2(E)$ of vectors $\vec{\mathbf{u}} = (\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3, \ldots \mathbf{u}^N)$ with coordinates $\left\{u_s^l\right\} = \vec{u}_s \in E = C_N$, $s = 0, 1, 2, \ldots$. Hereafter we call the space $E$ a *contact space*. The expansion over the system of eigenvectors $\vec{\Psi} = \left\{\Psi^k\right\}_{k=1}^{N}$ is obtained as an orthogonal sum

$$\frac{1}{2\pi i}\int_{\Sigma}\langle \mathbf{U},\, \vec{\Psi}_{\Theta}\rangle \vec{\Psi}_{\Theta}\frac{d\Theta}{\Theta} = \mathbf{U} \tag{4}$$

where the summation is over the standard basis $\nu_s \in E$, $\nu_s = \delta_{st}$, $1 \leq s, t \leq N$. We have also spectral expansions for $L_{out}$ similar to (2, 3):

$$\frac{1}{\pi}\int_{0}^{\pi}\sum_{\nu}\langle \mathbf{U},\, \vec{\Psi}_{\Theta}(\nu)\rangle \vec{\Psi}_{\Theta}(\nu)\, dp = \mathbf{U} \tag{5}$$

and

$$\frac{1}{2\pi}\int_{-2}^{2}\sum_{\nu}\langle \mathbf{U},\, \vec{\Psi}_{\Theta}(\nu)\rangle \vec{\Psi}_{\Theta}(\nu)\frac{d\lambda}{\sin p} = \mathbf{U}. \tag{6}$$

The resolvent and the scattered waves of the perturbed operator $L$ can be constructed by matching a linear combination of Bloch solutions of the homogeneous equation on the leads with an appropriate solution of the homogeneous equation on $\Omega_{in}$. If $\lambda$ does not belong to the spectrum of $L_{in}$, then the inner component $\Psi_{in}$ of the scattered wave on $\Omega_{in}$ is constructed as a linear combination of resolvent kernels $G_{in}(t, s, \lambda) := G_s(t)$, $t \in \Omega_{in}$ with poles at the contact points $a_s$ where the wires $\Omega_s$ are attached

$$\Psi_{in} = \sum_{s=1}^{N}\alpha_s G_s, \tag{7}$$

see a similar construction of the scattering ansatz for quantum graphs in [4]. The Greens function $G_s(t)$ satisfies the non-homogeneous equation on the compact part $\Omega_{in}$ of the graph

$$L_{in}G_s(\lambda) - \lambda G_{in}(\lambda) = \sum_{t\in U_s} G_{in}(s,t,\lambda) - \lambda G_{in}(s,t,\lambda) = \delta_{s,t}.$$

Here the summation is over the 'star' $U_s$ of nearest neighbors to the node $a_s$. The Kronecker symbol is defined in the obvious way: $\delta_{s,t} = 0$ on all nodes $t \neq s$ in $\Gamma_{in}$ and it is equal to 1 at the node $a_s$. On the complement of the (discrete) spectrum $\sigma_d^{in}$ of $L_{in}$ the matrix $G_{in}(s,t,\lambda)$ coincides with inverse matrix $(L_{in} - \lambda I)^{-1}$. In this paper we construct the scattered wave $\Psi_s$ of the perturbed operator $L$ initiated by an incoming wave on the semi-infinite wire $\omega^s$ attached to the vertex $a_s$. The components $\psi_s^t$ of the scattered wave $\Psi_s$ on the leads $\omega^t$, $t \neq s$ are proportional to the Bloch-solution

$$\psi_s^t = S_s^t \left(1,\, \bar\Theta,\, \bar\Theta^2,\, \bar\Theta^3,\, \dots,\right), \tag{8}$$

such that the complex conjugate $\bar\psi$ admits an analytic continuation on the spectral sheet $\Theta$, $|\Theta| < 1$, of the spectral parameter as square-summable sequences $\bar\psi_s^t = \bar S_s^t \left(1, \Theta,\, \Theta^2, \dots\right)$. The component of the scattered wave on $\omega^s$ is constructed of two Bloch solutions

$$\psi_s^s = \left(1, \Theta,\, \Theta^2, \dots\right) + S_s^s \left(1, \bar\Theta,\, \bar\Theta^2, \dots\right). \tag{9}$$

We introduce the matrix $\{G(t,s)\} := G$ consisting of the values of the Greens functions of the inner operator $L_{in}$. It coincides with the restriction of the inverse matrix $(L_{in} - \lambda I)^{-1}$ onto the contact space $E$

$$G = P_E\, (L_{in} - \lambda I)^{-1} \Big|_E .$$

Assume that all the scattered waves initiated by incoming waves from all leads are constructed. We combine the coefficients $\left\{S_s^t\right\} := S$ to form the scattering matrix.

**Lemma 2.1**

$$S = -\frac{I + \Theta G}{I + \bar\Theta G} \tag{10}$$

*Proof* Consider the matching conditions for the components of the scattered wave $\Psi_s$ initiated from the lead $\omega^s$:

$$\sum_{r=1}^{N} \alpha_r G(t,r) = S_s^t,\ t \neq s\,; \sum_{r=1}^{N} \alpha_r G(s,r) = 1 + S_s^s,\ t = s.$$

The equation $L\Psi_s - \lambda\Psi_s = 0$ can be written as

$$\alpha_t + S_s^t \bar\Theta = 0,\ t \neq s;\ \alpha_s + \Theta + S_s^s \bar\Theta = 0,\ t = s.$$

Eliminating $\alpha$ using the second pair of equations and the Kronecker symbol we can re-write the linear system for $S$ as

$$I + S + \Theta G + \bar\Theta S G = 0$$

and $S := \left\{S^t_s\right\}_{s,t=1}^N$. Then we have for the matrix of coefficients $S^t_s$ of the scattered waves $\Psi_s$, $s = 1, 2, \dots N$:

$$S = -\frac{I + \Theta G}{I + \bar{\Theta} G}$$

*End of the proof*

**Remark** The matrix $G$ plays the role of the inverse Dirichlet-to-Neumann map, [5, 6]—the multi-dimensional version of the Weyl-Titchmarsh function which attracts much attention from the specialists, see for instance the recent publications [7, 8, 9, 10]. The matrix $S$ is the scattering matrix of the adjacency matrix $L$ of the non-compact graph $\Omega$ with respect to the non-perturbed operator $L_{out} = \oplus \sum_{k=1}^N \mathbf{l}^k$. The formula (10) is the analog of the formula expressing the scattering matrix in terms of the Dirichlet-to-Neumann map, see [6]. For the one-dimensional analog of the formula see also [11, 12, 13].

The scattering matrix is defined on the continuous spectrum $[-2, 2]$ of the operator $L$ which coincides with the continuous spectrum of the non-perturbed operator $L_{out}$. This interval corresponds to the unit circle $|\Theta| = 1$ in terms of the quasi-momentum exponential $\Theta = e^{ip}$, $0 \leq p \leq 2\pi$. Using the connection between the quasi-momentum and the spectral parameter we conclude that the scattering matrix admits an analytic continuation by symmetry $S^+(\bar{\Theta}^{-1}) = S^{-1}(\Theta)$ from the unit circle onto the complex plane of $\Theta$, with real zeros at the points $\Theta_s$ in the unit disc which correspond to the eigenvalues $\lambda_s = \Theta_s + (\Theta_s)^{-1}$ of $L$. Due to symmetry the scattering matrix has also complex poles situated symmetrically to the zeros with respect to the unit circle.

The spectral expansion of the operator $L$ includes generally a finite sum over the eigenvalues and an integral over the continuous spectrum. We omit the standard derivation of the spectral expansion which follows from the compression of the Riesz integral around the spectrum of $L$ to the real interval $[-2, 2]$ followed by the use of the Hilbert identity for the jump of the resolvent across the continuous spectrum. Here is the final formula:

$$\mathbf{U} = \sum_\nu \Psi_m \langle \mathbf{U}, \Psi_m \rangle + \frac{1}{2\pi} \int_{-2}^{2} \sum_{s=1}^N \Psi^s_\Theta \langle \mathbf{U}, \Psi^s_\Theta \rangle \frac{d\lambda}{\sin p}.$$

The scattered waves $\Psi^s_\Theta$, constructed above by matching linear combination of Bloch waves to linear combinations of Greens functions of $L_{in}$ with poles at the contact points, can also be obtained from the asymptotic of the resolvent kernel $G(t, s)$ of the operator $L$ when $t \to \infty$ along the lead attached to the corresponding contact point $a_s$.

$$G(\tau, t) \approx \Psi^s_\tau G^s(t), \text{ when } t \to \infty,\ t \in \omega^s.$$

Here $G^s(s) = G^s(\tau_0, s)$ is the Green function of the component of $L_{out}$ on the lead $\omega^s$, $\tau_0 \in \omega^s$.

We consider two simple examples:

**Example 1** Consider a non-compact graph consisting of three leads attached to the nodes $a_1$, $a_2$, $a_3$, of an equilateral triangle $\Omega_{in}$. The adjacency matrix $L_{in}$ of the triangle is

$$L_{in} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues are $-1$, multiplicity 2, and 2, multiplicity 1. The corresponding normalized eigenvectors are, respectively

$$\phi_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ \text{and} \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since $\Omega_{in}$ coincides with the set of contact points, we see that the equation $I + \bar{\Theta}G$ splits into two equations in complex plane $\Theta$, $\bar{\Theta} = \Theta^{-1}$, which correspond to the eigenvalues $-1$, 2

$$\Theta + \frac{1}{-1 - \Theta - \Theta^{-1}} = 0,\ \Theta + \frac{1}{2 - \Theta - \Theta^{-1}} = 0$$

The first equation gives $\Theta = -1$. The singularity $\Theta = -1$ corresponds to the eigenvalue $\lambda_1 = -2$. The second equation defines the resonance $\Theta = 2$. The zeros at the origin $\Theta = 0$ in the numerator and in the denominator of the formula (10) for the scattering matrix cancel each other and thus give no contribution to the resulting spectrum and resonances.

**Example 2** Consider a ring $\Omega_{in}$ with $N$ equidistant nodes $\left\{e^{2\pi l/M}\right\}_{l=0}^{l=M-1}$. The eigenvalues of the adjacency matrix $L_{in}$ are $2\cos 2\pi m/N$, and the corresponding eigenvectors are $\Phi_m = \left\{1,\, e^{2i\pi m/N},\, e^{2i\pi 2m/N},\, e^{2i\pi 3m/N},\, \dots\right\}$, $m = 1, 2, 3 \dots N-1$. Assume that only one lead is attached to the ring at $a_0 = 1$. Then there is only one contact point, and the corresponding scattering matrix is

$$S(p) = -\frac{1 + \bar{\Theta}\sum_{m=0}^{N-1}\frac{1}{N}[2\cos 2\pi m/N - \Theta - \bar{\Theta}]^{-1}}{1 + \Theta\sum_{m=0}^{N-1}\frac{1}{N}[2\cos 2\pi m/N - \Theta - \bar{\Theta}]^{-1}} =$$

$$-\frac{N - \sum_{m=0}^{N-1}\left[\Theta - e^{2\pi i m/N}\right]^{-1}\left[\Theta - e^{-2\pi i m/N}\right]^{-1}}{N - \sum_{m=0}^{N-1}\left[\bar{\Theta} - e^{2\pi i m/N}\right]^{-1}\left[\bar{\Theta} - e^{-2\pi i m/N}\right]^{-1}}$$

Zeros of the numerator of the scattering matrix can be found numerically.

# 3 The discrete wave equation and Lax-Phillips scattering

Once the spectral analysis of the adjacency matrix is completed one can solve various dynamical problems on the graph $\Omega$. Denote by $\vec{\mathbf{u}}(t)$ a function ( a sequence) depending on the discrete time variable $t = 0, \pm 1, \pm 2 \dots$ and taking complex values at the nodes on the compact subgraph $\Omega_{in}$ and on the leads $\omega^s$, $s = 1, 2, \dots N$

$$\vec{\mathbf{u}}(t) = \left\{\mathbf{u}_{in}(t),\, \mathbf{u}^1(t),\, \mathbf{u}^2(t),\, \mathbf{u}^3(t), \dots \mathbf{u}^N(t)\right\}.$$

Consider the discrete wave equation on the graph $\Omega$

$$\mathbf{u}(t+1) + \mathbf{u}(t-1) = L\mathbf{u}(t),$$

with Cauchy data $\mathbf{U}(0) = (U_0(0), U_1(0))$ fixed at the initial moment of time. Generally we consider the Cauchy data at the moment $t$:

$$U_0(t) = \mathbf{u}(t), \qquad U_1(t) = \mathbf{u}(t+1) - \mathbf{u}(t-1).$$

One can see that the compactly supported functions $\vec{\mathbf{u}}(t \pm s)$ on the leads represent incoming and outgoing waves. The energy dot-product associated with the adjacency matrix $L_{out}$ on the leads $\omega$

$$[\mathbf{U},\, \mathbf{V}]_{\mathcal{E}_{out}} = \frac{1}{2}\langle (4 - L^2_{out}) U_0,\, V_0 \rangle_{L_2(\omega)} + \frac{1}{2}\langle U_1,\, V_1 \rangle_{L_2(\omega)}$$

vanishes if $\mathbf{U}$ and $\mathbf{V}$ are Cauchy data of incoming and outgoing waves respectively and is positive if $\mathbf{U} = \mathbf{V}$. Thus the restriction of the evolution defined by the discrete wave equation onto the outer space (supported by the wires) has the typical properties of the Lax-Phillips unitary group, see [14]. In particular it has an orthogonal pair of incoming and outgoing subspaces constituted of Cauchy data of incoming and outgoing waves obtained via closure in the energy-normed space of the subspaces of all compactly supported incoming and outgoing data.

This structure is inherited also by the wave evolution on the whole graph. To see this let us introduce the dot-product associated with the adjacency matrix $L$ on $\Omega$:

$$[\mathbf{U},\, \mathbf{V}]_{\mathcal{E}} = \frac{1}{2}\langle (4 - L^2) U_0,\, V_0 \rangle_{L_2(\Omega)} + \frac{1}{2}\langle U_1,,\, V_1 \rangle_{L_2(\Omega)}. \tag{11}$$

Generally the energy dot-product defined by the formula (11) may be indefinite. However, for numerous interesting non-compact graphs it is positive. *Hereafter we proceed under the assumption that* $[\mathbf{U},\, \mathbf{U}]_{\mathcal{E}} > 0$, *if* $\mathbf{U} \neq 0$. Consider the energy-normed space $\mathcal{E}$ of Cauchy data on $\Omega$. We represent it as an orthogonal sum of incoming and outgoing subspaces $\mathcal{D}_{in}$, $\mathcal{D}_{out}$ of Cauchy data supported on the leads and the co-invariant subspace $\mathcal{K}$

$$\mathcal{K} := \mathcal{E} \ominus [\mathcal{D}_{in} \oplus \mathcal{D}_{out}]\,.$$

**Theorem 3.1** *The discrete wave equation on the space of energy-normed Cauchy data is equivalent to the unitary group in $\mathcal{E}$ defined by the appropriate Dirac operator:*

$$\mathcal{U} := \frac{1}{2}\begin{pmatrix} L & 1 \\ L^2 - 4 & L \end{pmatrix},\ \mathcal{U} + \mathcal{U}^{-1} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}.$$

*The eigenfunctions of the absolutely-continuous spectrum of the generator $\mathcal{U}$ are represented as*

$$\mathbf{\Phi}_\Theta = \begin{pmatrix} \frac{1}{\Theta - \Theta^{-1}}\, \mathbf{\Psi}_\Theta \\ \mathbf{\Psi}_\Theta \end{pmatrix}. \tag{12}$$

*They correspond to the spectral points $\Theta = e^{ip}$, $0 \leq p \leq 2\pi$ : $\mathcal{U}\mathbf{\Phi}_\Theta = \Theta \mathbf{\Phi}_\Theta$. The spectral representation of the transformation $\mathcal{U}$ is given by the formula:*

$$\mathbf{U} \xrightarrow{\mathcal{J}} [\mathbf{U},\, \mathbf{\Phi}_\Theta]_{\mathcal{E}} := (\mathcal{J}\mathbf{U})\,(\Theta),$$

$$(\mathcal{J}\mathbf{U})\,(\Theta) \xrightarrow{\mathcal{J}^{-1}} \frac{1}{2\pi\, i}\int_{\Sigma_1} \mathbf{\Phi}_\Theta\ (\mathcal{J}\mathbf{U})\,(\Theta) \frac{d\Theta}{\Theta} = \mathbf{U}.$$

*Proof* The identity

$$\mathcal{U}+\mathcal{U}^{-1} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$$

can be obtained by direct calculation. This means that all spectral objects for the operator $\mathcal{U}$, including the resolvent and the spectral expansion, can be constructed from the corresponding details of the operator $L$—see the similar calculation for the standard Lax-Phillips generator in [15]. In particular, the eigenfunctions of the absolutely-continuous spectrum of the generator $\mathcal{U}$ can be obtained from the columns (12) of the scattered waves $\Psi_{\Theta}$ of $L$ in the course of the solution of the wave equation by the Fourier method. It is sufficient to verify that $\mathbf{\Phi}_{\Theta}$ satisfies the homogeneous equation $\mathcal{U}\mathbf{\Phi}_{\Theta} = \Theta\mathbf{\Phi}_{\Theta}$ in the weak sense. We use the fact that $L\mathbf{\Psi}_{\Theta} = (\Theta+\Theta^{-1})\mathbf{\Psi}_{\Theta}$, in the weak sense, on a dense domain in $L_2(\Omega)$. Then, using the facts $(L^2-4)\mathbf{\Psi}_{\Theta} = (\Theta-\Theta^{-1})^2\,\mathbf{\Psi}_{\Theta}$ and $L\mathbf{\Psi}_{\Theta} = (\Theta+\Theta^{-1})\mathbf{\Psi}_{\Theta}$, we obtain the desired statement. To prove the equivalence of the unitary group to the original wave equation we substitute $\vec{\mathbf{u}}(t+1)+\vec{\mathbf{u}}(t-1) = L\vec{\mathbf{u}}(t)$ into $\mathcal{U}$. We obtain:

$$2\vec{\mathbf{u}}(t+1) = \vec{\mathbf{u}}(t+1)+\vec{\mathbf{u}}(t-1)+\vec{\mathbf{u}}(t+1)-\vec{\mathbf{u}}(t-1),$$

and

$$2\left(\vec{\mathbf{u}}(t+2)-\vec{\mathbf{u}}(t)\right) = L\left(\vec{\mathbf{u}}(t+1)+\vec{\mathbf{u}}(t-1)\right)-4\vec{\mathbf{u}}(t)+L\vec{\mathbf{u}}(t+1)-L\vec{\mathbf{u}}(t-1).$$

Using $\vec{\mathbf{u}}(t+1)+\vec{\mathbf{u}}(t-1) = L\vec{\mathbf{u}}(t)$ again we obtain the announced statement. The unitarity of the transformation $\mathcal{U}$ follows from the spectral representation.

*End of the proof.*

# 4 Connection to Lax-Phillips scattering

If the spectrum of the operator $\mathcal{U}$ is purely continuous then the corresponding unitary group $\mathcal{U}^l$ has all the typical properties of the Lax-Phillips unitary group. In particular, it possesses an orthogonal pair of incoming and outgoing subspaces of the Cauchy data, supporte3d by the leads, see our remark in the beginning of previous section. The Lax-Phillips scattering matrix is obtained from the stationary scattering matrix S, see (10), via complex conjugation on the spectrum

$$\mathbf{S}_{LPh}(\Theta) = -\frac{1+\bar{\Theta}G}{1+\Theta G}$$

and can be continued onto the whole complex plane of the quasi-momentum exponential $\Theta$ by the formula

$$\mathbf{S}_{LPh}(\Theta) = -\frac{1+\Theta^{-1}G}{1+\Theta G} = -\Theta\,\frac{\Theta+G}{1+\theta G},\ \text{with } G = G(\lambda) = G(\Theta+\Theta^{-1}). \tag{13}$$

The Lax-Phillips scattering matrix is analytic in the unit disc because the pair of incoming and outgoing subspaces supported by the leads is orthogonal, see [14]. Using the spectral representation for the Greens function of $L_{in}$ in terms of eigenvectors $\varphi_s$ and eigenvalues $\Lambda_s$,

$$G(\lambda) = \sum_{s=1}^{M}\frac{P_E\varphi_s\rangle\,\langle P_E\varphi_s}{\lambda_s-\lambda}, \tag{14}$$

we obtain an equation for the resonances—vector zeros $\Theta_r,\ \nu_r\ :\ \mathbf{S}_{LPh}(\Theta_r)\ \nu_r\ = 0$ of the Lax-Philips scattering matrix in the unit disc $|\Theta| < 1$—in the form

$$\Theta\nu + \sum_{s=1}^{M} \frac{P_E\varphi_s\rangle\,\langle P_E\varphi_s,\,\nu\rangle}{\lambda_s - \lambda} = 0, \text{ with } \lambda = \Theta + \Theta^{-1}.$$

From the solution of this equation we can observe the dependence of the resonances on the eigenvalues of $L_{in}$ and the on the projection $P_E\Psi_s$ of the eigenvectors onto the contact subspace $E$. This way the shape of eigenvectors defines the the transmission from one lead to another.

The matrix-function $\mathbf{S}_{LPh}(\Theta)$ contains all the spectral information on the dynamical properties of the evolution defined by the wave-equation. In particular, the incoming and outgoing subspaces of the corresponding unitary evolution group $\mathcal{U}^t$ can be shown, in the spectral representation $\mathcal{J}$, to be $H^2_-$, $\mathbf{S}_{LPh}\,H^2_+$ respectively while the co-invariant subspace is $H^2_+ \ominus \mathbf{S}_{LPh}\,H^2_+ := \mathcal{K}$. The eigenvalues of the generator $\mathcal{T}$ of the Lax-Phillips semigroup [14]:

$$\mathcal{P}_{\mathcal{K}}\mathcal{U}^t\Big|_{\mathcal{K}} := \mathcal{T}^t,\ t = 0, 1, 2\ldots$$

coincide with the zeros $\Theta_s$ of the Lax-Phillips scattering matrix and the eigenvectors—the resonance states—in the spectral representation $\mathcal{J}$ are simply $\mathbf{S}_{LPh}\nu_r(\Theta_r - \Theta)^{-1}$. The bi-orthogonal system of eigenvectors of $\mathcal{T}$ consists of reproducing kernels $\left(1 - \bar{\Theta}_r\,\Theta\right)^{-1}$, see [16]. Completeness of the system of resonance states is equivalent to the absence of the singular factor in $\mathbf{S}_{LPh}(\Theta)$ in the unit disc.

**Corollary** The scattering matrix $\mathbf{S}_{LPh}$ of the wave-evolution with positive adjacency matrix is a Blaschke product. This is simply because the above formula (13) represents it as a ratio of two polynomials of $\Theta$. Hence the system of eigenvectors of the discrete spectrum of the semigroup $\mathcal{T}^t$, $t = 0, 1, 2\ldots$ is complete.

One can also study, based on [17], the joint completeness of eigenvectors of both semi-groups $\mathcal{T}^t$, $\left[\mathcal{T}^+\right]^t$ $t = 0, 1, 2\ldots$.

In the case when the leads have a richer period, the spectrum of the adjacency matrix of the non-compact graph may have a more sophisticated structure. In particular, it may have several spectral bands. Then the corresponding spectral theory of functions should be developed not on the complex plane, but on the relevant Riemann surface. Nevertheless, a modified version of Lax-Phillips theory can be applied to the problem [18] using a properly re-defined Dirac operator $\mathcal{U}$. This approach is based on recent developments in the spectral theory of functions on Riemann surfaces, see [19] and references therein. We postpone discussion of these interesting questions to a forthcoming publication.

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