SURFACE AREA OF ELLIPSOID SEGMENT

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Abstract

The surface area of a general segment of a 3–dimensional ellipsoid is computed.

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1 Surface Area of Ellipsoid

Consider the area of the surface (or part of it) of an ellipsoid centred at the coordinate origin, with rectangular Cartesian coordinate axes along the semi–axes α, β, γ :

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1 . \tag{1}$$

In applications, the semi–axes could be of any magnitude from atomic nuclei to cosmological structures. For numerical computation, it is advantageous to consider a non–dimensionalized form. Take some representative length K, e.g. K could be the average of $\alpha,\ \beta,\ \gamma$, or it could be their maximum. Then, consider an ellipsoid $\mathcal{E}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 , \tag{2}$$

which is a scaled version of (1), with semi–axes $a = \alpha/K,\ b = \beta/K,\ c = \gamma/K$ which are of the order of 1. Computation involving $\mathcal{E}$ is numerically more convenient than with (1); and the surface area of any part of the original ellipsoid (1) is K^2 times the corresponding area on $\mathcal{E}$. And any other point (x, y, z), to be considered in relation to (1), could be compatibly scaled to $(x/K, y/K, z/K)$ and then considered in relation to the scaled ellipsoid $\mathcal{E}$.

Hereafter, we shall consider the ellipsoid $\mathcal{E}$, with semi–axes a, b, c.

A point $P = (\xi, \eta, \zeta)$ is outside $\mathcal{E}$, if and only if

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} > 1 , \tag{3}$$

and P is inside $\mathcal{E}$, if and only if

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} < 1 . \tag{4}$$

1.1 Ellipsoids of revolution

An ellipsoid of revolution with semi–axes $a,\ b,\ b$ is called a spheroid, prolate if $(a > b)$, and oblate if $(a < b)$.

In 1714, Roger Cotes found the surface area of ellipsoids of revolution [Cotes]. Denote $q = 1 - b^2/a^2$. The surface areas for prolate spheroids $(a > b)$, spheres $(a = b)$ and oblate spheroids $(a < b)$ are:

$$A \;=\; \begin{cases} 2\pi b\left(a \times \dfrac{\arcsin\sqrt{q}}{\sqrt{q}} + b\right) & \text{(prolate)}, \\ 2\pi b(a+b) \;=\; 4\pi a^2 & \text{(sphere)}, \\ 2\pi b\left(a \times \dfrac{\operatorname{arcsinh}\sqrt{-q}}{\sqrt{-q}} + b\right) & \text{(oblate)}\ . \end{cases} \tag{5}$$

These are now standard problems in first–year calculus.

Neither the hyperbolic functions nor their inverses had been invented by 1714, and Cotes gave a logarithmic formula for the oblate spheroid [Cotes, pp. 169-171]. In modern notation [Cotes, p.50],

$$A \;=\; \pi\left[2a^2 + b^2\frac{1}{\sqrt{-q}}\log\left(\frac{1+\sqrt{-q}}{1-\sqrt{-q}}\right)\right]. \tag{6}$$

For $|q| \ll 1$, use the power series for $(\arcsin x)/x$ to get the rapidly convergent series

$$A \;=\; 2\pi b\left(a\left[1 + \tfrac{1}{6}q + \tfrac{3}{40}q^2 + \tfrac{5}{112}q^3 + \cdots\right] + b\right). \tag{7}$$

1.2 Surface area of general ellipsoid

For a surface defined by $z \;=\; z(x,y)$ in rectangular Cartesian coordinates xyz, the standard formula for surface area is:

$$\text{Area} \;=\; \int\!\!\int\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ \mathrm{d}x\ \mathrm{d}y. \tag{8}$$

On the ellipsoid $\mathcal{E}$,

$$\frac{\partial z}{\partial x} \;=\; \frac{-c^2x}{a^2z}\ , \qquad \frac{\partial z}{\partial y} \;=\; \frac{-c^2y}{b^2z}\ . \tag{9}$$

Consider the octant for which $x,\ y,\ z$ are all non–negative. Then the surface area for that octant is

$$\begin{aligned} S \;&=\; \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1 + \frac{c^4x^2}{a^4z^2} + \frac{c^4y^2}{b^4z^2}}\ \mathrm{d}y\ \mathrm{d}x \\ &=\; \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{\dfrac{z^2}{c^2} + \dfrac{c^2}{a^2}\dfrac{x^2}{a^2} + \dfrac{c^2}{b^2}\dfrac{y^2}{b^2}}{z^2/c^2}}\ \mathrm{d}y\ \mathrm{d}x \\ &=\; \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{c^2}{a^2}\dfrac{x^2}{a^2} + \dfrac{c^2}{b^2}\dfrac{y^2}{b^2}}{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}}\ \mathrm{d}y\ \mathrm{d}x \end{aligned}$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{1 - \left(1 - \frac{c^2}{a^2}\right)\frac{x^2}{a^2} - \left(1 - \frac{c^2}{b^2}\right)\frac{y^2}{b^2}}{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\, \mathrm{d}y\ \mathrm{d}x\ . \qquad (10)$$

Hence, if two semi-axes (a and b) are fixed and the other semi-axis c increases, then the surface area increases.

Denote

$$\delta \ = \ 1 - \frac{c^2}{a^2}, \qquad \epsilon = 1 - \frac{c^2}{b^2}\ , \qquad (11)$$

and then (10) becomes

$$S \ = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{1 - \delta\frac{x^2}{a^2} - \epsilon\frac{y^2}{b^2}}{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\, \mathrm{d}y\ \mathrm{d}x\ . \qquad (12)$$

If $c > a$ then $\delta < 0$, and if $c > b$ then $\epsilon < 0$. For a general ellipsoid, the coordinate axes can be named so that $a \geq b \geq c$, and then $1 > \delta \geq \epsilon > 0$.

1.2.1 Legendre's series expansion for ellipsoid area

In 1788, Adrien-Marie Legendre (1752-1833) converted this double integral to a convergent series for the surface area A [Legendre 1788] [Legendre 1825, pp. 350–351] [Tee pp. 5–7]:

$$A \ = \ 4\pi ab\big(1 - \tfrac{1}{1\cdot 3}P_1 - \tfrac{1}{3\cdot 5}P_2 - \tfrac{1}{5\cdot 7}P_3 - \tfrac{1}{7\cdot 9}P_4 - \cdots\big), \qquad (13)$$

where

$$\begin{aligned} P_1 &= \tfrac{1}{2}\delta + \tfrac{1}{2}\epsilon, \\ P_2 &= \tfrac{1\cdot 3}{2\cdot 4}\delta^2 + \tfrac{1\cdot 1}{2\cdot 2}\delta\epsilon + \tfrac{1\cdot 3}{2\cdot 4}\epsilon^2, \\ P_3 &= \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\delta^3 + \tfrac{1\cdot 3\cdot 1}{2\cdot 4\cdot 2}\delta^2\epsilon + \tfrac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4}\delta\epsilon^2 + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\epsilon^3, \quad \textit{et cetera.} \end{aligned} \qquad (14)$$

Legendre's series converges rapidly for near-spheres (with $\|\delta\| \ll 1$ and $\|\epsilon\| \ll 1$), but it converges slowly for ellipsoids which are far from spherical.

1.3 Legendre on elliptic integrals

Legendre worked on elliptic integrals for over 40 years, and he summarized his work in [Legendre 1825]. He investigated systematically the integrals of the form $\int R(t, y)\,\mathrm{d}t$, where R is a general rational function and $y^2 = P(t)$, where P is a general polynomial of degree 3 or 4. Legendre called them "fonctions élliptique", because the formula (23) for arclength of ellipse is of that form — now they are called elliptic integrals. He shewed how to express any such integral in terms of elementary functions, supplemented by 3 standard types of elliptic integral[1].

[1] But for a general elliptic integral, reduction to the standard elliptic integrals is an extremely complicated operation [Milne-Thomson §**17.1**]. It is usually simplest to evaluate elliptic integrals directly by Romberg integration, possibly after performing some substitution (e.g. $t = \sin\theta$) to make the integrand smooth.

Each of Legendre's standard integrals has 2 (or 3) parameters, including $x = \sin\phi$. Notation for those integrals varies considerably between various authors. Milne-Thomson's notation for Legendre's elliptic integrals [Milne-Thomson, §**17.2**] uses the parameter m, where Legendre (and many other authors) had used k^2.

Each of the three kinds is given as two integrals. In each case, the second form is obtained from the first by the substitutions $t = \sin\theta$ and $x = \sin\phi$.

The Incomplete Elliptic Integral of the First Kind is:

$$F(\phi|m) \stackrel{\text{def}}{=} \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} = \int_0^x \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-mt^2)}} . \tag{15}$$

The Incomplete Elliptic Integral of the Second Kind is:

$$E(\phi|m) \stackrel{\text{def}}{=} \int_0^{\phi} \sqrt{1 - m\sin^2\theta}\ \mathrm{d}\theta = \int_0^x \sqrt{\frac{1-mt^2}{1-t^2}}\ \mathrm{d}t . \tag{16}$$

That can be rewritten as

$$\int_0^x \frac{1-mt^2}{\sqrt{(1-t^2)(1-mt^2)}}\ \mathrm{d}t , \tag{17}$$

which is of the form $\int R(t, y)\ \mathrm{d}t$, where $y^2 = (1-t^2)(1-mt^2)$.

The Incomplete Elliptic Integral of the Third Kind is:

$$\begin{aligned} \Pi(n;\phi|m) &\stackrel{\text{def}}{=} \int_0^{\phi} \frac{\mathrm{d}\theta}{(1 - n\sin^2\theta)\sqrt{1 - m\sin^2\theta}} \\ &= \int_0^x \frac{\mathrm{d}t}{(1-nt^2)\sqrt{(1-t^2)(1-mt^2)}} . \end{aligned} \tag{18}$$

The special cases for which $\phi = \frac{1}{2}\pi$ (and $x = 1$) are found to be particularly important, and they are called the Complete Elliptic Integrals [Milne-Thomson, §**17.3**].

The Complete Elliptic Integral of the First Kind is:

$$\begin{aligned} K(m) \stackrel{\text{def}}{=} F\left(\tfrac{1}{2}\pi|m\right) &\stackrel{\text{def}}{=} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} \\ &= \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-mt^2)}} . \end{aligned} \tag{19}$$

The Complete Elliptic Integral of the Second Kind is:

$$E(m) \stackrel{\text{def}}{=} E\left(\tfrac{1}{2}\pi|m\right) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\ \mathrm{d}\theta = \int_0^1 \sqrt{\frac{1-mt^2}{1-t^2}}\ \mathrm{d}t . \tag{20}$$

The complete elliptic integrals $K(m)$ and $E(m)$ can efficiently be computed to high precision, by constructing arithmetic-geometric means [Milne-Thomson. §**17.6.3** & §**17.6.4**].

1.3.1 Circumference of ellipse

In 1742, Colin MacLaurin constructed a definite integral for the circumference of an ellipse [MacLaurin]. Consider an ellipse with semi-axes a and b, with Cartesian coordinates along the axes:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \;=\; 1 \;, \tag{21}$$

Denote the circumference by $C(a, b)$.

On that ellipse, $2x\,\mathrm{d}x/a^2 + 2y\,\mathrm{d}y/b^2 = 0$, and hence $\mathrm{d}y/\mathrm{d}x = -b^2x/(a^2y)$, and $C(a, b)$ is 4 times the ellipse quadrant with $x \geq 0$ and $y \geq 0$. That quadrant has arclength

$$\begin{aligned}\frac{C(a,b)}{4} \;&=\; \int_0^a \sqrt{1 + \frac{b^4x^2}{a^4y^2}}\;\mathrm{d}x \;=\; \int_0^a \sqrt{1 + \frac{b^2(x/a)^2}{a^2(y/b)^2}}\;\mathrm{d}x \\ &=\; \int_0^a \sqrt{1 + \frac{(b/a)^2(x/a)^2}{1-(x/a)^2}}\;\mathrm{d}x \;.\end{aligned} \tag{22}$$

Substitute $t = x/a$, and the circumference becomes

$$C(a,b) \;=\; 4a\int_0^1 \sqrt{1 + \frac{(b/a)^2t^2}{1-t^2}}\;\mathrm{d}t \;=\; 4a\int_0^1 \sqrt{\frac{1-mt^2}{1-t^2}}\;\mathrm{d}t \;, \tag{23}$$

where

$$m \;=\; 1 - \frac{b^2}{a^2} \;. \tag{24}$$

With $a \geq b$ this gives $0 \leq m < 1$.

That integral could not (in 1742) be expressed finitely in terms of standard functions. But, in terms of Legendre's Complete Elliptic Integral of the Second Kind, the circumference of the ellipse is

$$C(a,b) \;=\; 4aE\left(1 - \frac{b^2}{a^2}\right) \;. \tag{25}$$

1.4 Legendre's explicit formula for ellipsoid area

In 1825, Legendre constructed an explicit expression for the area of a general ellipsoid, in terms of Incomplete Elliptic Integrals of the First and Second Kinds [Legendre 1825, pp.352–359]. In Milne-Thomson's notation, with δ and ϵ as in (11),

$$A \;=\; 2\pi c^2 + \frac{2\pi ab}{\sqrt{\delta}}\left[(1-\delta)F(\sqrt{\delta}|\epsilon/\delta) + \delta\; E(\sqrt{\delta}|\epsilon/\delta)\right] \;. \tag{26}$$

Note that Legendre's formula does not hold for a sphere, and for a near-sphere Legendre's rapidly–convergent series (13) should be used for the area. Otherwise, A can conveniently be evaluated by using Romberg integration to compute F from the integral (15) and E from the integral (16). Both integrands with t are singular at $t = 1$, and so the integrands with θ should be used for evaluation by Romberg integration.

Legendre's proof of his elliptic integral formula for area is long and complicated, and that formula has been very little known. For example, in 1979

Stuart P. Keller asserted that "Except for the special cases of the sphere, the prolate spheroid and the oblate spheroid, no closed form expression exists for the surface area of the ellipsoid. This situation arises because of the fact that it is impossible to carry out the integration in the expression for the surface area in closed form for the most general case of three unequal axes"[Keller, p.310].

In 1897, Ernesto Cesàro computed the surface area of the ellipsoid with axes $\sqrt{3},\ \sqrt{2},\ 1$ as $3\pi \times 2{\cdot}52620923$ [Cesàro, p.338]. In 1953, Frank Bowman published [Bowman] an obscure derivation of the formula (26) for the area (without mentioning Legendre), in 1958 Albert Eagle proved Legendre's formula [Eagle, p.284 (12)], and in 1994 Leo Maas proved Legendre's formula [Maas]. The clearest proof was given by Derek Lawden in 1989 [Lawden, pp.100–102] (without mentioning Legendre). In 1999 an incorrect version of the formula was published [Wolfram, p.976], without proof[2] or references.

1.5 Area by double integration

The double integral (12) for the area S of an octant can be computed by integrating the integrand in the y direction, and then integrating that in the x direction. Putting

$$x \;=\; as, \quad y = bt\sqrt{1-s^2}, \quad m \;=\; \frac{\epsilon\left(1-s^2\right)}{1-\delta s^2}\ , \tag{27}$$

we get the integral with respect to y as

$$\int\limits_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{1-\delta\dfrac{x^2}{a^2}-\epsilon\dfrac{y^2}{b^2}}{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}}}\,\mathrm{d}y \;=\; b\sqrt{1-s^2}\int_0^1\sqrt{\frac{1-\delta s^2-\epsilon(1-s^2)t^2}{1-s^2-t^2(1-s^2)}}\,\mathrm{d}t$$

$$= b\int_0^1\sqrt{\frac{1-\delta s^2-\epsilon(1-s^2)t^2}{1-t^2}}\,\mathrm{d}t = b\sqrt{1-\delta s^2}\int_0^1\sqrt{\frac{1-\dfrac{\epsilon(1-s^2)}{1-\delta s^2}\,t^2}{1-t^2}}\,\mathrm{d}t$$

$$= \; b\sqrt{1-\delta s^2}\int_0^1\sqrt{\frac{1-mt^2}{1-t^2}}\,\mathrm{d}t \;=\; b\sqrt{1-\delta s^2}\;E(m)\ . \tag{28}$$

Here E is the Complete Elliptic Integral of the Second Kind (20), and m is a function (27) of s. And $E(m)$ can efficiently be computed to high precision, by using arithmetic-geometric means [Milne-Thomson. §**17.6.3** §**17.6.4**].

Then the area of the ellipsoid is

$$A \;=\; 8b\int_0^a\sqrt{1-\frac{\delta x^2}{a^2}}\;E(m)\;\mathrm{d}x \;=\; 8ab\int_0^1\sqrt{1-\delta s^2}\;E(m(s))\;\mathrm{d}s\ \ , \tag{29}$$

and that integral can efficiently be evaluated by Romberg integration [Klette & Rosenfeld, pp. 287–290].

However, evaluation of the integrand in (29) requires the computation of an arithmetic-geometric mean. Legendre's formula (13) involves computation of 2

[2] An obscure statement of that incorrect formula is also given in the website <http://documents.wolfram.com/v4/MainBook/G.1.7.html>. At least 3 people have reported that error to the administrator of that website, but without effect.

Incomplete Elliptic Integrals F and E, both of which have integrands which are simple to compute. Accordingly, computation of the surface area by (29) takes somewhat more time than does Legendre's formula.

1.5.1 Ellipsoids in n dimensions

For n–dimensional ellipsoids with $n \geq 3$, the surface area and electrostatic capacity have been found [Tee].

2 Intersection Between Plane and Ellipsoid

For a surface with equation $F(x, y, z) = 0$, the plane tangential at a point $(\bar{x}, \bar{y}, \bar{z})$ on the surface has the equation [Helmut & Kopka, p.146]

$$\frac{\partial F}{\partial x}(x - \bar{x}) + \frac{\partial F}{\partial y}(y - \bar{y}) + \frac{\partial F}{\partial z}(z - \bar{z}) \;=\; 0, \tag{30}$$

and the normal at that point has the equation

$$\frac{x - \bar{x}}{\dfrac{\partial F}{\partial x}} \;=\; \frac{y - \bar{y}}{\dfrac{\partial F}{\partial y}} \;=\; \frac{z - \bar{z}}{\dfrac{\partial F}{\partial z}}\;. \tag{31}$$

Each of the partial derivatives of F is evaluated at $(\bar{x}, \bar{y}, \bar{z})$.

For the ellipsoid $\mathcal{E}$, $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$, and so

$$\frac{\partial F}{\partial x} = \frac{2\bar{x}}{a^2}, \quad \frac{\partial F}{\partial y} = \frac{2\bar{y}}{b^2}, \quad \frac{\partial F}{\partial z} = \frac{2\bar{z}}{c^2}\;. \tag{32}$$

Accordingly, the plane tangent to $\mathcal{E}$ at $(\bar{x}, \bar{y}, \bar{z})$ has the equation

$$\frac{2\bar{x}(x - \bar{x})}{a^2} + \frac{2\bar{y}(y - \bar{y})}{b^2} + \frac{2\bar{z}(z - \bar{z})}{c^2} \;=\; 0\;, \tag{33}$$

which reduces (in view of (2)) to

$$\frac{\bar{x}x}{a^2} + \frac{\bar{y}y}{b^2} + \frac{\bar{z}z}{c^2} \;=\; 1\;. \tag{34}$$

And the normal to $\mathcal{E}$ at $(\bar{x}, \bar{y}, \bar{z})$ has the equation (with parameter $w/2$)

$$x = \bar{x}\left(1 + \frac{w}{a^2}\right),\; y = \bar{y}\left(1 + \frac{w}{b^2}\right),\; z = \bar{z}\left(1 + \frac{w}{c^2}\right)\;. \tag{35}$$

The unit vector

$$\boldsymbol{\Upsilon} \stackrel{\text{def}}{=} \begin{bmatrix} \dfrac{\bar{x}}{a^2} \\ \dfrac{\bar{y}}{b^2} \\ \dfrac{\bar{z}}{c^2} \end{bmatrix} \left(\left(\frac{\bar{x}}{a^2}\right)^2 + \left(\frac{\bar{y}}{b^2}\right)^2 + \left(\frac{\bar{z}}{c^2}\right)^2\right)^{-1/2} \tag{36}$$

has the direction of the normal to $\mathcal{E}$, at $(\bar{x}, \bar{y}, \bar{z})$.

2.1 Poles and polar planes

Consider a point $P = (\xi, \eta, \zeta)$ which is on the tangent plane (34). Hence P is outside (or on) $\mathcal{E}$, and if P is outside $\mathcal{E}$ then the line from P to $(\bar{x}, \bar{y}, \bar{z})$ is tangential to $\mathcal{E}$ at $(\bar{x}, \bar{y}, \bar{z})$. Then (34) shews that

$$\frac{\bar{x}\xi}{a^2} + \frac{\bar{y}\eta}{b^2} + \frac{\bar{z}\zeta}{c^2} \;=\; 1 \; . \tag{37}$$

Thus, each line through P which is tangential to $\mathcal{E}$ touches it at a point $(\bar{x}, \bar{y}, \bar{z})$, which is on $\mathcal{E}$ and also is on the plane

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} \;=\; 1 \; . \tag{38}$$

That plane intersects $\mathcal{E}$ in an ellipse.

For any point $P = (\xi, \eta, \zeta)$, which can be outside or on or inside $\mathcal{E}$, its *polar plane* for the ellipsoid $\mathcal{E}$ is defined by this equation (38). The point P is called the *pole* of that plane.

If P is on the ellipsoid $\mathcal{E}$, then (34) and (38) shew that the polar plane is tangential to $\mathcal{E}$ at P.

2.2 Criterion for intersection of ellipsoid and plane

A general plane

$$\ell x + my + nz \;=\; f \tag{39}$$

is the polar plane of the point $P = (\xi, \eta, \zeta)$, where equation (39) is f times equation (38). Thus the pole of the plane (39) is the point

$$P \;=\; (\xi, \eta, \zeta) \;=\; \left(\frac{a^2\ell}{f}, \frac{b^2 m}{f}, \frac{c^2 n}{f} \right) \; , \tag{40}$$

unless the plane passes through the centre of $\mathcal{E}$. If the plane passes through the centre of $\mathcal{E}$ then $f = 0$, and the pole is then the point at infinity on the line with direction cosines proportional to $(a^2\ell, b^2m, c^2n)$.

The point P is outside $\mathcal{E}$, if and only if its polar plane intersects $\mathcal{E}$. Hence, the plane (39) intersects $\mathcal{E}$, if and only if

$$1 \;<\; \left(\frac{\xi}{a}\right)^2 + \left(\frac{\eta}{b}\right)^2 + \left(\frac{\zeta}{c}\right)^2 \;=\; \left(\frac{a\ell}{f}\right)^2 + \left(\frac{bm}{f}\right)^2 + \left(\frac{cn}{f}\right)^2 \; . \tag{41}$$

Thus, the plane (39) intersects $\mathcal{E}$, if and only if

$$a^2\ell^2 + b^2m^2 + c^2n^2 \;>\; f^2 \; ; \tag{42}$$

and the plane (39) touches $\mathcal{E}$ (at (40)), if and only if

$$a^2\ell^2 + b^2m^2 + c^2n^2 \;=\; f^2 \; . \tag{43}$$

Divide the equation (39) by $\kappa = \sqrt{\ell^2 + m^2 + n^2}$ if $f \geq 0$, and otherwise divide (39) by $-\kappa$. Then the plane (39) is equally specified by the scaled equation

$$\lambda x + \mu y + \nu z \;=\; h \; , \tag{44}$$

where

$$\lambda = \frac{\ell}{\kappa}, \qquad \mu = \frac{m}{\kappa}, \qquad n = \frac{\nu}{\kappa}, \qquad h = \frac{f}{\kappa} \ , \tag{45}$$

and $h \geq 0$. Here, h is the length of the normal to the plane from the origin, and that normal has direction cosines (λ, μ, ν), with $\lambda^2 + \mu^2 + \nu^2 = 1$. Thus the vector

$$\mathbf{\Gamma} \stackrel{\text{def}}{=} \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} \tag{46}$$

is a unit vector, normal to the plane (44). The pole P (40) can equally be represented as

$$P \ = \ (\xi, \eta, \zeta) \ = \ \left(\frac{a^2\lambda}{h}, \frac{b^2\mu}{h}, \frac{c^2\nu}{h} \right) \ , \tag{47}$$

and the plane (44) intersects $\mathcal{E}$, if and only if

$$h^2 < a^2\lambda^2 + b^2\mu^2 + c^2\nu^2 \ . \tag{48}$$

For planes not through the centre of $\mathcal{E}$, h can be indefinitely small, and division by h is numerically unstable for h close to 0. Hence, division by h should be avoided wherever that is feasible. For instance, the pole P can be specified by $(h\xi, h\eta, h\zeta) = (a^2\lambda, b^2\mu, c^2\nu)$.

Hereafter we shall consider only planes which do intersect $\mathcal{E}$.

3 Area of Segment

The intersection between the ellipsoid (2) and the plane (44) is an ellipse, which is called here the *ellipse of intersection*. The ellipse of intersection plus its interior is called here the *elliptical disk*. The parallel projection, normal to the xy–plane, of the ellipse of intersection onto the xy–plane is called here the *projected ellipse*, and that projected ellipse (with its interior) is called here the *projected elliptical disk*.

For a general ellipsoid cut by a plane, consider the areas of the two segments of that ellipsoid.

If the plane is normal to an axis of the ellipsoid, then the area of the segment can readily be computed by integration. For example, with the plane $x = g$, the area of the segment from $x = g$ to $x = a$ is given (cf. (29)) as

$$4ab \int_{g/a}^{1} \sqrt{1 - \delta s^2} \ E(m(s)) \ \mathrm{d}s \ \ , \tag{49}$$

which can readily be evaluated by Romberg integration.

If the plane passes through the centre of the ellipsoid then each segment has half the area of the full ellipsoid.

Otherwise, the segment on the side of the plane away from the centre has surface area smaller than the other segment, on the side of the plane with the centre. Compute the area of the smaller segment — if the area of the larger segment is required then compute the full area by Legendre's formula (or Legendre's series, for a near–sphere) and subtract from it the area of the smaller segment.

In §**1.5** the ellipsoid $\mathcal{E}$ is intersected by the xy–plane, and the surface area is found (10) by parallel projection of elements of area in the xy–plane normal to that plane, onto the surface of the ellipsoid.

For an ellipsoid segment, if the ellipse of intersection does not pass through the xy–plane (i.e. on the ellipse of intersection either $z \geq 0$ everywhere, or $z \leq 0$ everywhere), then the area of the ellipsoid segment could be computed by normal projection from the projected elliptical disk onto the entire segment. But otherwise, some part of the xy–plane outside the projected ellipse would have to be projected onto the segment, both for $z > 0$ and for $z < 0$. Several cases would have to be treated separately, which would complicate greatly the computation.

A simpler approach is to use normal projection from the projected elliptical disk onto the elliptical disk, and then to project that elliptical disk onto the segment by projection from the pole (47) of the plane (44).

3.1 Ellipse of intersection

Choose the axis making the smallest angle with the normal to the intersecting plane as the z–axis, so that ν is the largest coefficient (in modulus) in equation (44). Then,

$$z \;=\; \frac{h - \lambda x - \mu y}{\nu} \tag{50}$$

on the ellipse of intersection. Since $\lambda^2 + \mu^2 + \nu^2 = 1$, this choice of axis gives $|\nu| \geq 1/\sqrt{3}$.

Substitute (50) for z in (2) (times $\nu^2 c^2$), and we get the x–y equation of a conic section (in the plane (44)) as

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + G = 0 \;, \tag{51}$$

where

$$\begin{aligned} A = \frac{\nu^2 c^2}{a^2} + \lambda^2 \;, \quad & B = 2\lambda\mu \;, \quad C = \frac{\nu^2 c^2}{b^2} + \mu^2 \;, \\ D = -2h\lambda \;, \quad & E = -2h\mu \;, \quad G = h^2 - \nu^2 c^2 \;, \end{aligned} \tag{52}$$

so that $A > 0$ and $C > 0$. Each point on the conic section is on the ellipsoid $\mathcal{E}$ and hence x and y are bounded[3], and so that conic section must be an ellipse.

The area and the circumference of the ellipse of intersection are constructed in the Appendix.

On that ellipse of intersection, z is given as a function of x and y by (50). The equation of the projected ellipse (in the xy–plane) is also (51).

That projected ellipse is centred (116) at $U = (v, w)$, where

$$\begin{aligned} 2Av + Bw &= -D, \\ Bv + 2Cw &= -E, \end{aligned} \tag{53}$$

and hence

$$v \;=\; \frac{BE - 2CD}{4AC - B^2} \;, \qquad w \;=\; \frac{BD - 2AE}{4AC - B^2} \;, \tag{54}$$

and the centre of the ellipse of intersection is at $(v,\ w,\ [h - \lambda v - \mu w]/\nu)$.

[3] In fact, $|x| \leq a$ and $|y| \leq b$.

The cosine of the angle between the plane (44) and the xy–plane is ν, and hence normal projection of any region of area S from the xy–plane onto the plane (44) gives a region with enlarged area S/ν. In particular, the area of the elliptical disk is the area of the projected elliptical disk, divided by ν. Note that we can take $|\nu| \geq 1/\sqrt{3}$, and so the modulus of the area is enlarged by a factor no more than $\sqrt{3}$.

Since P is outside $\mathcal{E}$ then its polar plane intersects $\mathcal{E}$ in an ellipse of intersection, and the lines from P to points on that ellipse of intersection are tangents to $\mathcal{E}$ on that ellipse of intersection. Those tangents from P generate a circular cone which is generally oblique; but if $\mathcal{E}$ is an ellipsoid of revolution and P is on an axis of revolution, then those tangents form a right circular cone.

Thus the ellipsoid $\mathcal{E}$ is contained wholly within that cone, except for tangential contact at each point on the ellipse of intersection. Each line from P through any point inside that ellipse of intersection will intersect $\mathcal{E}$ at one point of the larger segment (containing the centre of $\mathcal{E}$), and at one point of the smaller segment.

The area of the (smaller) segment of the ellipsoid will be found by projection, from the pole P, of the elliptical disk onto that segment, between the plane (44) and its pole P.

4 Formula for Area of Segment

Consider planes H, I and J, with angle α between H and J and angle β between I and J. Thus, each line normal to J makes angle α with each line normal to H, and angle β with each line normal to I. If a region R (with area s) on plane H projects (normal to plane J) onto a region T in J, then the area of T is $s\ \cos\alpha$. If T projects (normal to plane J) onto a region U in the plane I, then the area of U is $s\ \cos\alpha/\cos\beta$.

If the planes H and I are replaced by smooth surfaces, then we can consider parallel projection of points on surface H onto surface I, by lines normal to plane J. Let point U on H project to point V on surface I, with the line UV (normal to plane J) making angle α with the normal to surface H at U, and making angle β with the normal to surface I at V. Then an infinitesimal area ds on surface H at U projects to an infinitesimal area $\mathrm{d}s\ \cos\alpha/\cos\beta$ on surface I at V.

Now consider projection from a point W of an infinitesimal area ds at U on surface H to V on surface I, with the line WUV making angle α with the normal to surface H at U, and making angle β with the normal to surface I at V. At U on H, the infinitesimal area ds projects (from W) onto the plane through U normal to WUV as $\mathrm{d}s\ \cos\alpha$. From W, that projects onto the plane through V normal to WUV as $(WV/WU)^2\mathrm{d}s\ \cos\alpha$. Hence, the projection from W of ds on surface H at U to surface I at V has area

$$\left(\frac{WV}{WU}\right)^2 \frac{\cos\alpha}{\cos\beta}\,\mathrm{d}s\ .$$

Each point $L = (x, y)$ inside the projected ellipse (51) is projected (normal to the xy–plane) to a point $M = (x, y, z)$ in the elliptical disk on the plane (44), where $z = (h - \lambda x - \mu y)/\nu$ (cf. (50)). The element of area $\mathrm{d}x\,\mathrm{d}y$ at L is projected to $\mathrm{d}x\,\mathrm{d}y/\nu$ at M. On the line from the pole P (47) to M, each point $(\bar{x}, \bar{y}, \bar{z})$ has the equation

$$\bar{x} = x + (\xi - x)t, \;\; \bar{y} = y + (\eta - y)t, \;\; \bar{z} = z + (\zeta - z)t, \tag{55}$$

with parameter t. As t increases from 0 to 1, the point $(\bar{x}, \bar{y}, \bar{z})$ moves from M (in the elliptical disk) to the pole P. In terms of $\varpi = t/h$, this becomes

$$\bar{x} = x + (a^2\lambda - hx)\varpi, \;\; \bar{y} = y + (b^2\mu - hy)\varpi, \;\; \bar{z} = z + (c^2\nu - hz)\varpi, \tag{56}$$

with $t = \varpi h$.

If the point $(\bar{x}, \bar{y}, \bar{z})$ on the interval PM is also on $\mathcal{E}$, then

$$\frac{\bar{x}^2}{a^2} + \frac{\bar{y}^2}{b^2} + \frac{\bar{z}^2}{c^2} \;=\; 1\;. \tag{57}$$

Substitute (56) in (57), and we get a quadratic equation in the parameter ϖ:

$$\left(\frac{x + (a^2\lambda - hx)\varpi}{a}\right)^2 + \left(\frac{y + (b^2\mu - hy)\varpi}{b}\right)^2 + \left(\frac{z + (c^2\nu - hz)\varpi}{c}\right)^2 = 1\;. \tag{58}$$

This reduces to

$$\phi\varpi^2 + \chi\varpi + \psi \;=\; 0\;, \tag{59}$$

where

$$\begin{aligned}
\phi &= \left(a\lambda - \frac{hx}{a}\right)^2 + \left(b\mu - \frac{hy}{b}\right)^2 + \left(c\nu - \frac{hz}{c}\right)^2, \\
\chi &= \frac{2x(a^2\lambda - hx)}{a^2} + \frac{2y(b^2\mu - hy)}{b^2} + \frac{2z(c^2\nu - hz)}{c^2}, \\
\psi &= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\;,
\end{aligned} \tag{60}$$

and hence $\phi > 0$. In view of (44), χ simplifies to $\chi = -2h\psi$.

The point $M = (x, y, z)$ is on the ellipse of intersection, which is inside $\mathcal{E}$, and hence $(x/a)^2 + (y/b)^2 + (z/c)^2 < 1$. Therefore $\psi < 0$, and hence the quadratic equation (59) has a positive root and a negative root (of larger modulus).

For numerical stability, compute the negative root of that quadratic equation $\phi\varpi^2 - 2h\psi\varpi + \psi = 0$ as

$$u \;=\; \frac{h\psi - \sqrt{h^2\psi^2 - \phi\psi}}{\phi}\;, \tag{61}$$

and then compute the root with smaller modulus as $\varpi = \phi/(u\psi)$. Use that positive root ϖ in (56) to get the intersection $N = (\bar{x}, \bar{y}, \bar{z})$ of PM with the smaller segment of $\mathcal{E}$, between the plane (44) and its pole P. (The other root gives the intersection of PM with the larger segment of $\mathcal{E}$). Then compute $t = \varpi h$, which is non–negative.

This holds for all h including $h = 0$, in which case $\chi = 0$ and

$$\varpi \;=\; \sqrt{\frac{-\psi}{\phi}} \;=\; \sqrt{\frac{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}}{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2}}\;, \tag{62}$$

and $t = 0$. And the negative root of (59) is[4] $-\varpi$.

Thus, the element of area $\mathrm{d}x\,\mathrm{d}y$ at L in the projected ellipse is transformed to the element of area $(1-t)^2 \cos\alpha\ \mathrm{d}x\,\mathrm{d}y/(\nu\ \cos\beta)$ at N on the segment, where the line PNM makes angle α with the normal to the plane (44), and it makes angle β with the normal to $\mathcal{E}$ at N.

The unit vector $\mathbf{\Gamma}$ in (46) is normal to the plane (44), and the unit vector $\mathbf{\Upsilon}$ in (36) is normal to $\mathcal{E}$ at N. The vector

$$\mathbf{\Delta} \stackrel{\text{def}}{=} \begin{bmatrix} x-\xi \\ y-\eta \\ z-\zeta \end{bmatrix} h \ = \ \begin{bmatrix} hx - a^2\lambda \\ hy - b^2\mu \\ hz - c^2\nu \end{bmatrix} \tag{63}$$

has the direction from the pole P to M, and it is numerically stable for all h (including $h = 0$). Accordingly, in terms of the Euclidean vector norm,

$$\begin{aligned} \mathbf{\Delta}^T\mathbf{\Gamma} &= \|\mathbf{\Delta}\|\ \|\mathbf{\Gamma}\| \cos\alpha \ = \ \|\mathbf{\Delta}\| \cos\alpha, \\ \mathbf{\Delta}^T\mathbf{\Upsilon} &= \|\mathbf{\Delta}\|\ \|\mathbf{\Upsilon}\| \cos\beta \ = \ \|\mathbf{\Delta}\| \cos\beta\ , \end{aligned} \tag{64}$$

and so

$$\frac{\cos\alpha}{\cos\beta} \ = \ \frac{\mathbf{\Delta}^T\mathbf{\Gamma}}{\mathbf{\Delta}^T\mathbf{\Upsilon}}\ . \tag{65}$$

Furthermore, $M = (x, y, z)$ is on the plane (44), and so

$$\begin{aligned} \mathbf{\Delta}^T\mathbf{\Gamma} \ &= \ [(hx - a^2\lambda),\ (hy - b^2\mu),\ (hz - c^2\nu)] \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} \\ &= \ h(\lambda x + \mu y + \nu z) - a^2\lambda^2 - b^2\mu^2 - c^2\nu^2 \\ &= \ h^2 - a^2\lambda^2 - b^2\mu^2 - c^2\nu^2, \end{aligned} \tag{66}$$

which is a negative constant, since the plane (44) does intersect $\mathcal{E}$ (cf. (48)). Indeed, $\|\mathbf{\Gamma}\| \cos\alpha$ (in (64)) is the length (times h) of the normal from P onto its polar plane (47), for all $M = (x, y, z)$ on that plane.

Hence, the area Ξ of the (smaller) segment of $\mathcal{E}$ is

$$\Xi \ = \ \frac{h^2 - a^2\lambda^2 - b^2\mu^2 - c^2\nu^2}{\nu} \int\!\!\int \frac{(1-t)^2}{\mathbf{\Delta}^T\mathbf{\Upsilon}}\ \mathrm{d}y\,\mathrm{d}x\ , \tag{67}$$

where the double integration is done over the projected elliptical disk in the xy–plane. In that integrand $\mathbf{\Delta}, \mathbf{\Upsilon}$ and t are functions of x and y.

That expression computes a signed area, and so its modulus should be used for the unsigned area.

With

$$\omega \stackrel{\text{def}}{=} \frac{1}{\sqrt{\left(\dfrac{\bar{x}}{a^2}\right)^2 + \left(\dfrac{\bar{y}}{b^2}\right)^2 + \left(\dfrac{\bar{z}}{c^2}\right)^2}}\ , \tag{68}$$

[4] In (44), the equation of the plane was scaled to give $h \geq 0$. If that had not been done, then for $h = 0$ (or very close to it), roundoff could make the root ϖ with smaller modulus flip sign from + to -, so that the point N on the ellipsoid would then flip from one side of the plane (44) to the other, and that would make the computation fail. But with $h \geq 0$ we do get $\varpi > 0$ inside the ellipse of intersection.

we get from (63) and (36) that

$$
\begin{aligned}
\boldsymbol{\Delta}^T \boldsymbol{\Upsilon} &= [(hx - a^2\lambda),\ (hy - b^2\mu),\ (hz - c^2\nu)] \begin{bmatrix} \dfrac{\bar{x}}{a^2} \\ \dfrac{\bar{y}}{b^2} \\ \dfrac{\bar{z}}{c^2} \end{bmatrix} \omega \\
&= \omega \left(h \left(\frac{x\bar{x}}{a^2} + \frac{y\bar{y}}{b^2} + \frac{z\bar{z}}{c^2} \right) - \lambda\bar{x} - \mu\bar{y} - \nu\bar{z} \right) . \qquad (69)
\end{aligned}
$$

For $L = (x, y)$ on the projected ellipse, $M = N$ and $(x, y, z) = (\bar{x}, \bar{y}, \bar{z})$, so that $t = 0$. Then, (x, y, z) satisfies both equation (2) for $\mathcal{E}$ and equation (44) for the plane, so that $\boldsymbol{\Delta}^T \boldsymbol{\Upsilon} = \omega(h - h) = 0$ and $\cos\beta = 0$. Thus, the integrand in (67) is singular on the boundary (51) of the domain of double integration.

4.1 Polar coordinates

Integration of (67) over x then over y (or conversely) leads to some awkward singularities. Accordingly, we shall perform the double integration over the projected elliptic disk in polar coordinates, centred at the centre $U = (v, w)$ of that ellipse (51), with

$$
x = v + r\ \cos\vartheta, \qquad y = w + r\ \sin\vartheta\ , \qquad (70)
$$

where ϑ is the angle from the x–axis. The element of area $\mathrm{d}x\,\mathrm{d}y$ is replaced by $r\,\mathrm{d}\vartheta\ \mathrm{d}r$.

The line through the ellipse centre U with angle ϑ intersects the projected ellipse at $r = \pm\rho(\vartheta)$, where ρ is the positive root of the quadratic equation got by substituting (70) in (51):

$$
\begin{aligned}
0 &= A(v + r\ \cos\vartheta)^2 + B(v + r\ \cos\vartheta)(w + r\ \sin\vartheta) + C(w + r\ \sin\vartheta)^2 \\
&\qquad + D(v + r\ \cos\vartheta) + E(w + r\ \sin\vartheta) + r\ \sin\vartheta + G \\
&= \left[A\cos^2\vartheta + B\cos\vartheta\ \sin\vartheta + C\sin^2\vartheta\right] r^2 \\
&\qquad + \left[(2Av + Bw + D)\cos\vartheta + (Bv + 2Cw + E)\sin\vartheta\right] r \\
&\qquad + Av^2 + Bvw + Cw^2 + Dv + Ew + G\ . \qquad (71)
\end{aligned}
$$

Since (v, w) is the centre of the ellipse then $2Av + Bw + D = 0$ and $Bv + 2Cw + E = 0$ from (53), and so the coefficient of r is 0, as we expected. Therefore,

$$
\rho(\vartheta) = \sqrt{\frac{-(Av^2 + Bvw + Cw^2 + Dv + Ew + G)}{A\ \cos^2\vartheta + B\ \cos\vartheta\ \sin\vartheta + C\sin^2\vartheta}}\ . \qquad (72)
$$

Thus, (67) becomes

$$
\Xi = \frac{h^2 - a^2\lambda^2 - b^2\mu^2 - c^2\nu^2}{\nu} \int_0^{2\pi} J(\vartheta)\ \mathrm{d}\,\vartheta\ , \qquad (73)
$$

where

$$
J(\vartheta) \stackrel{\text{def}}{=} \int_0^{\rho(\vartheta)} \frac{(1-t)^2}{\boldsymbol{\Delta}^T\boldsymbol{\Upsilon}}\ r\ \mathrm{d}r\ . \qquad (74)
$$

Here, t, $\mathbf{\Delta}$ and $\mathbf{\Upsilon}$ are functions of x and y, and hence they are functions of the polar coordinates r and ϑ.

For the integration over ϑ, the integrand $J(\vartheta)$ is a smooth function with period 2π. Accordingly, that integral is best evaluated by Composite Rectangular Rule [Davis & Rabinowitz, pp.134–146], which is equivalent to Composite Trapezoidal Rule for a periodic function integrated over its period.

But for the integration over r, the integrand in (74) is singular at $r = \rho(\vartheta)$.

When the intersecting plane (44) is the xy–plane, the integrand (12) has the factor $1/\sqrt{1 - (x/a)^2 - (y/b)^2}$, which is singular on the boundary of the domain of integration.

This suggests that the singularity of the integrand in (74) at $r = \rho$ is of the form $1/\sqrt{\rho - r}$, so that (74) can be rewritten as

$$J(\vartheta) \;=\; \int_0^{\rho} (\rho - r)^{-1/2}\; F(r)\; \mathrm{d}r, \tag{75}$$

where $F(r)$ is a smooth function of r. Substitute $u = \sqrt{\rho - r}$ so that $r = \rho - u^2$ and $\mathrm{d}r = -2u\; \mathrm{d}u$, and (75) becomes

$$J(\vartheta) \;=\; 2\int_0^{\sqrt{\rho}} F(\rho - u^2)\; \mathrm{d}u\ , \tag{76}$$

which can readily be evaluated efficiently if $F(r)$ is smooth. Thus, (74) can be converted to

$$J(\vartheta) \;=\; 2\int_0^{\sqrt{\rho}} (1-t)^2(\rho - u^2)\frac{u}{\mathbf{\Delta}^T\mathbf{\Upsilon}}\; \mathrm{d}u\ , \tag{77}$$

where t, $\mathbf{\Delta}$ and $\mathbf{\Upsilon}$ are evaluated for $r = \rho - u^2$.

Note that the integrand in (77) has the indeterminate form $0/0$ at $u = 0$, on the projected ellipse.

More detailed analysis (too lengthy to include here) shews that the integrand in (77) is a smooth function of u, which converges to a limit as $u \searrow 0$ on the projected ellipse. But that limit is very complicated to compute, for each ϑ.

4.2 Romberg integration

The integral of a smooth function f over a finite interval $L \stackrel{\text{def}}{=} \int_a^b f(x)\;\mathrm{d}x$ is usually evaluated by Romberg integration [Davis & Rabinowitz, pp.434–446]. That is a systematic extrapolation of Composite Trapezoidal Rule, based on the correction terms in Euler–MacLaurin quadrature. Romberg's algorithm generates a triangular array of estimates $T_{i,j}$, with $i \geq 0$ and $0 \leq j \leq i$. The initial estimate is computed by the simple Trapezoidal Rule $T_{0,0} = (b-a)(f(a) + f(b))/2$, and then successive estimates $T_{i,0}$ are computed by Composite Trapezoidal Rule, with the interval $[a, b]$ being divided into 2^i equal sub–intervals and with simple Trapezoidal Rule being applied over each sub–interval. Each estimate $T_{i,0}$ is then subjected to an iterated extrapolation process, generating successively $T_{i,1},\ T_{i,2},\ \cdots,\ T_{i,i}$. For a sufficiently smooth function f, the diagonal estimates $T_{i,i}$ converge very rapidly to L.

Romberg integration of $f(x)$ over $[a, b]$ is a closed quadrature formula, which uses the values of $f(a)$ and of $f(b)$. Hence, if either $f(a)$ or $f(b)$ has indefinite form, then $\int_a^b f(x)\;\mathrm{d}x$ cannot be computed by standard Romberg integration — but it can be computed by a modified version.

4.2.1 Midpoint Romberg integration

In Romberg's algorithm, replace the simple Trapezoidal Rule (which is a closed formula) by the simple Midpoint Rule $T_{0,0} = (b-a)f\left(\frac{1}{2}(a+b)\right)$, which is an open quadrature formula. (For integrating any quadratic polynomial f, the truncation error of simple Midpoint Rule equals -2 times the truncation error of simple Trapezoidal Rule.) Then neither $f(a)$ nor $f(b)$ is required, and so this Midpoint Romberg integration is an open quadrature formula, which can be applied when $f(x)$ is indefinite at either $x = a$ or $x = b$.

In particular, the integral (77) for $J(\vartheta)$ can readily be computed by Midpoint Romberg integration over u.

4.3 Computed examples

A program to compute segment area by Midpoint Romberg integration over u and then by Composite Rectangular Rule over ϑ, has been written in THINK Pascal with **extended** variables (equivalent to about 19 or 20 significant decimal figures), and used on a Macintosh computer. That program gives rapid convergence for the double integration, as was expected from the detailed analysis of the integrand of (77).

4.3.1 Test cases

Archimedes proved that, for a sphere of radius a intersected by a plane at distance g from the centre, the surface area of the sphere segment is $2\pi a(a-g)$ [Archimedes, Prop. 42 & 43]. For a unit sphere, the THINK Pascal program computes the area of the segment from $z = g$ to $z = 1$ for $g = 0,\ 0{\cdot}5,\ 0{\cdot}9$ and $0{\cdot}95$, as $6{\cdot}283185307179586$, $3{\cdot}141592653589794$, $0{\cdot}628318530717959$ and $0{\cdot}314159265358979$ respectively. Those computed areas do equal $2\pi,\ \pi,\ \pi/5$ and $\pi/10$, within 10^{-15}.

If the intersecting plane (44) is normal to an axis (which we call the x–axis) of the ellipsoid $\mathcal{E}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \;=\; 1\;, \tag{78}$$

then the area of the segment from $x = g$ to $x = a$ is given by the definite integral (49). Table 1 gives various examples of such segment areas computed by our algorithm of double integration, and the difference from the definite integral (49).

Table 1

a	b	c	g	**Double integration**	**Difference**
1	1	1	0·5	3·141592653589793	$-2{\cdot}70\times10^{-16}$
3	1	2	1	14·898564166191096	$-2{\cdot}32\times10^{-16}$
1·1	0·7	1·3	0·4	4·102165227994899	$-1{\cdot}65\times10^{-16}$
2	0·8	1·5	1·5	2·419521133392435	$-1{\cdot}18\times10^{-16}$
1·6	1·1	0·9	1·3	1·320531492584690	$2{\cdot}74\times10^{-17}$

Table 2 gives various computed examples of half the area of the ellipsoid (78) (computed by Legendre's formula), and the area of the (smaller) segment intersected by the plane $\ell x + my + nz = f$.

Each plane with $f = 0$ passes through the centre of the ellipsoid, and its segment area computed by our algorithm agrees closely with half the ellipsoid area computed from Legendre's formula.

Table 2

a	b	c	**Half Ellipsoid**	ℓ m n f	**Segment**
2	0·7	1·2	10·21169524241981	4 -3 1 0	10·21169524241981
				3 -2 4 7	0·805727202906178
				1 -7 4 1	8·88982342451370
1·9	0·6	1	8·060185336491328	-3 2 1 0	8·060185336491328
				-3 2 1 -4	2·020205465454151
				-3 2 1 5·8	0·086701019126932
				1 3 -2 -3	0·719344732924807
1·5	0·8	0·9	6·978898751442496	1 -1 7 0	6·978898751442496
				-2 1 -4 4	1·050044459459405
				-2 1 4 1	5·458346340746586
0·7	1·3	1·4	8·015485343804755	1 -1 0 0	8·015485343804754
				2 2 -1 -1	5·452684579507425
				-7 5 3 9	0·138647695911539
				-7 5 3 1	7·124341012035889

5 Intersecting Ellipsoids

When software is developed for computing area of surfaces from digitized pictures, then it can be tested on simple surfaces of known area, e.g. spheres and ellipsoids [Klette & Rosenfeld, Chapter 8]. More general test surfaces could be constructed from intersections of such surfaces, e.g. 2 intersecting spheres.

In particular, a segment of an ellipsoid together with its elliptical disk forms a convex closed surface, whose area can be computed. And further plane segments could be sliced off, giving a convex faceted ellipsoid whose area can be computed.

If 2 ellipsoids intersect, then their intersection is in general a complicated space curve (or pair of curves), and the computation of the surface area of a segment of an ellipsoid intersected by another ellipsoid would be a very difficult problem. Accordingly, we shall consider an ellipsoid intersected by a plane in an ellipse of intersection, and another ellipsoid with the same ellipse of intersection.

5.1 Intersection of similar ellipsoids

For example, if $\mathcal{E}$ does intersect the plane (44), then their intersection is the plane ellipse of intersection (51) and (50). For any real parameter ς, the quadric surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \varsigma(\lambda x + \mu y + \nu z) \;=\; 1 + \varsigma h \tag{79}$$

intersects $\mathcal{E}$ and the plane (44) in that ellipse of intersection. Equation (79) may be rewritten as

$$\frac{1}{a^2}\left(x^2 + \varsigma\lambda a^2 x + \tfrac{1}{4}\varsigma^2\lambda^2 a^4\right) + \frac{1}{b^2}\left(y^2 + \varsigma\mu b^2 y + \tfrac{1}{4}\varsigma^2\mu^2 b^4\right)$$
$$+\frac{1}{c^2}\left(z^2 + \varsigma\nu c^2 x + \tfrac{1}{4}\varsigma^2\nu^2 c^4\right) = 1 + \varsigma h + \tfrac{1}{4}\varsigma^2(\lambda^2 a^2 + \mu^2 b^2 + \nu^2 c^2). \tag{80}$$

Denote

$$\kappa \stackrel{\text{def}}{=} 1 + \varsigma h + \tfrac{1}{4}\varsigma^2(\lambda^2 a^2 + \mu^2 b^2 + \nu^2 c^2) \ , \tag{81}$$

and this reduces to

$$\frac{(x+\frac{1}{2}\varsigma\lambda a^2)^2}{a^2\kappa} + \frac{(y+\frac{1}{2}\varsigma\mu b^2)^2}{b^2\kappa} + \frac{(z+\frac{1}{2}\varsigma\nu c^2)^2}{c^2\kappa} \ = \ 1 \ . \tag{82}$$

From (81), it follows that $\kappa = 0$ if and only if

$$1 + h\varsigma h + \tfrac{1}{4}\varsigma^2(\lambda^2 a^2 + \mu^2 b^2 + \nu^2 c^2) \ = \ 0. \tag{83}$$

This quadratic equation in ς has the discriminant $h^2 - \lambda^2 a^2 - \mu^2 b^2 - \nu^2 c^2$, and that is negative (48) since the plane (44) intersects $\mathcal{E}$. Consequently, $\kappa > 0$ for all real ς.

Hence, (82) is the equation of an ellipsoid similar to $\mathcal{E}$ centred at the point $(-\frac{1}{2}\varsigma\lambda a^2, -\frac{1}{2}\varsigma\mu b^2, -\frac{1}{2}\varsigma\nu c^2)$, with semi–axes $a\sqrt{\kappa},\ b\sqrt{\kappa},\ c\sqrt{\kappa}$, which are parallel to the $x,\ y,\ z$ axes. This ellipsoid intersects $\mathcal{E}$ in the ellipse of intersection with the plane (44), and the areas of both segments of the ellipsoid (82) can be computed as for $\mathcal{E}$ (after a parallel shift of origin to $(-\frac{1}{2}\varsigma\lambda a^2, -\frac{1}{2}\varsigma\mu b^2, -\frac{1}{2}\varsigma\nu c^2)$). Either of the segments of $\mathcal{E}$ could be joined to either of the segments of the other ellipsoid (82) to give 4 figures, whose surface area can be computed as above. The 2 surfaces which are composed of segments of both ellipsoids on the same side of the plane (44) are concave — the other 2 surfaces, which are composed of segments of both ellipsoids on opposite sides of the plane (44), can be convex or concave.

5.2 Intersection of general ellipsoids

More generally, for real parameters $\rho,\ \varsigma,\ \tau,\ k$, the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + (\rho x + \varsigma y + \tau z - k)(\lambda x + \mu y + \nu z - h) \ = \ 1 \tag{84}$$

is a quadric surface, which intersects $\mathcal{E}$ at the ellipse of intersection with the plane (44). This equation can be expanded as

$$\begin{aligned} &\left(\frac{1}{a^2} + \rho\lambda\right)x^2 + \left(\frac{1}{b^2} + \varsigma\mu\right)y^2 + \left(\frac{1}{c^2} + \tau\nu\right)z^2 \\ &\quad +(\rho\mu + \varsigma\lambda)xy + (\rho\nu + \tau\lambda)xz + (\varsigma\nu + \tau\mu)yz \\ &\quad -(h\rho + k\lambda)x - (h\varsigma + k\mu)y - (h\tau + k\nu)z \ = \ 1 - hk. \end{aligned} \tag{85}$$

That is the equation of an ellipsoid, if and only if the Hessian matrix $\mathbf{M}$ of the function of $x,\ y$ and z on the left–hand side

$$\mathbf{M} \ = \ \begin{bmatrix} 2\left(\dfrac{1}{a^2} + \rho\lambda\right) & (\rho\mu + \varsigma\lambda) & (\rho\nu + \tau\lambda) \\ (\rho\mu + \varsigma\lambda) & 2\left(\dfrac{1}{b^2} + \varsigma\mu\right) & (\varsigma\nu + \tau\mu) \\ (\rho\nu + \tau\lambda) & (\varsigma\nu + \tau\mu) & 2\left(\dfrac{1}{c^2} + \tau\nu\right) \end{bmatrix} \tag{86}$$

is positive–definite, and $1 - hk$ is sufficiently large[5] (or -$\mathbf{M}$ is positive–definite, and $hk-1$ is sufficiently large). Both conditions hold if the parameters $\rho,\ \varsigma,\ \tau,\ k$ have sufficiently small moduli, and also for some sets of parameters with larger moduli.

The centre and the axes of that ellipsoid (85) can be found by the n–dimensional analysis in the Appendix, and those axes can be taken as orthogonal Cartesian axes. The ellipsoid (84) then has an equation (in the new coordinates) of the standard form (2), and the areas of its segments by the plane (44) (represented in the new coordinates) can be found as for $\mathcal{E}$. Either of the segments of $\mathcal{E}$ could be joined to either of the segments of the other ellipsoid (84), to give 4 figures whose surface area has been computed. The 2 surfaces which are composed of segments of both ellipsoids on the same side of the plane (44) are concave — the other 2 surfaces, which are composed of segments of both ellipsoids on opposite sides of the plane (44), can be convex or concave.

5.3 Tangential ellipsoids

If $k^2 < a^2\rho^2 + b^2\varsigma^2 + c^2\tau^2$, then (42) the plane

$$\rho x + \varsigma y + \tau z \ = \ k \tag{87}$$

intersects $\mathcal{E}$ in an ellipse of intersection. In that case, equation (84) holds on that ellipse of intersection with (87). Thus, the quadric surface (84) then intersects $\mathcal{E}$ in 2 ellipses of intersection, on the planes (44) and (87). If the plane (87) approaches the plane (44), then the ellipse of intersection in the plane (87) approachesthe ellipse of intersection in the plane (44). Hence, for any real parameter υ the quadric surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \upsilon(\lambda x + \mu y + \nu z - h)^2 \ = \ 1 \tag{88}$$

is <u>tangential</u> to the ellipsoid $\mathcal{E}$ on the ellipse of intersection in the plane (44).

Indeed, if $\upsilon > 0$ then on that quadric surface (88),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \ = \ 1 - \upsilon(\lambda x + \mu y + \nu z - h)^2 \ \le \ 1, \tag{89}$$

with equality only on the ellipse of intersection. Hence, the quadric surface (88) is contained in the ellipsoid $\mathcal{E}$, and therefore it is an ellipsoid if $\upsilon > 0$. And similarly if $\upsilon < 0$ then the quadric surface (88) is wholly outside $\mathcal{E}$, except on the ellipse of intersection.

Equation (88) may be rewritten as

$$\begin{aligned} &\left(\frac{1}{a^2} + \upsilon\lambda^2\right)x^2 + \left(\frac{1}{b^2} + \upsilon\mu^2\right)y^2 + \left(\frac{1}{c^2} + \upsilon\nu^2\right)z^2 \\ &\qquad +2\upsilon(\lambda\mu xy + \lambda\nu xz + \mu\nu yz) - 2\upsilon h(\lambda x + \mu y + \nu z) \ = \ 1 - \upsilon h^2 \ . \end{aligned} \tag{90}$$

Even for negative υ this represents an ellipsoid (at any rate, for sufficiently small $-\upsilon$), and the areas of its segments by the plane (44) can be computed, as for the ellipsoid (84).

[5]Details are given in the Appendix.

Either of the segments of $\mathcal{E}$ could be joined to either of the segments of the other ellipsoid (90) to give 4 figures, whose surface area has been computed. Those ellipsoids are tangential on the ellipse of intersection. Accordingly, if a segment of $\mathcal{E}$ on one side of the plane (44) is joined to the segment of the ellipsoid (90) on the other side of the plane then the compound figure is convex and smooth, with the direction normal to the surface varying continuously over the surface. But, if a segment of $\mathcal{E}$ on one side of the plane (44) is joined to the segment of the ellipsoid (90) on the same side of the plane then the compound figure is concave, with cusps everywhere on the ellipse of intersection.

5.4 Sets of intersecting ellipsoids

This process could be repeated, to build up the union of a set of ellipsoids, each of which has plane intersection with at least one other ellipsoid. Also, no 3 ellipsoids have any interior point in common. (This latter condition could be relaxed somewhat.) The area of each segment can be computed as above, and added to give the area of the surface of the union of those ellipsoids.

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Appendix: Ellipsoid Axes

Consider an ellipse in rectangular Cartesian coordinates

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = w. \tag{91}$$

In particular, the projected ellipse (51) has $w = -G = \nu^2 c^2 - h^2$.

If $B = 0$ (which happens for the projected ellipse if and only if either $\lambda = 0$ or $\mu = 0$, or both), then this becomes

$$\begin{aligned} w &= A\left[x^2 + \frac{Dx}{A}\right] + C\left[y^2 + \frac{Ey}{C}\right] \\ &= A\left[\left(x + \frac{D}{2A}\right)^2 - \left(\frac{D}{2A}\right)^2\right] + C\left[\left(y + \frac{E}{2C}\right)^2 - \left(\frac{E}{2C}\right)^2\right] . \end{aligned} \tag{92}$$

Define the new coordinates $z_1 = x + D/(2A),\ z_2 = y + E/(2C)$, with a parallel shift of axes to $(u_1, u_2) = (-D/(2A), -E/(2C))$, and this becomes

$$Az_1^2 + Cz_2^2 \ = \ f, \tag{93}$$

where

$$f \stackrel{\text{def}}{=} w + \frac{D^2}{4A} + \frac{E^2}{4C} \ . \tag{94}$$

That can be rewritten as

$$\frac{z_1^2}{d_1^2} + \frac{z_2^2}{d_2^2} \ = \ 1 \ , \tag{95}$$

where

$$d_1 \stackrel{\text{def}}{=} \sqrt{f/A}, \qquad d_2 \stackrel{\text{def}}{=} \sqrt{f/C} \ . \tag{96}$$

This is the equation (in the shifted rectangular Cartesian coordinates $z_1,\ z_2$), of an ellipse with axes parallel to the xy axes, and its semi–axes are d_1 and d_2.

Therefore, if $B = 0$ then the ellipse (91) has the area

$$K \ = \ \pi d_1 d_2 \ = \ \frac{\pi f}{\sqrt{AC}} \ = \ \frac{\pi\left(w + \frac{D^2}{4A} + \frac{E^2}{4C}\right)}{\sqrt{AC}} \ . \tag{97}$$

But if $B \neq 0$, then it is convenient to apply the theory for general ellipsoids in n dimensions, with $n > 1$.

Axes Of Ellipsoids In n Dimensions

With rectangular Cartesian coordinates $x_1,\ x_2, \ldots,\ x_n$, the equation of a general ellipsoid is

$$\mathbf{x}^T\mathbf{M}\mathbf{x} + \mathbf{q}^T\mathbf{x} \ = \ w, \tag{98}$$

where $\mathbf{M}$ is a real symmetric positive–definite matrix of order n, the vector $\mathbf{x} = [x_1\ x_2\ \ldots\ x_n]^T$, $\mathbf{q}$ is a real n–vector and w is a real scalar (which could be 0).

Substitute $\mathbf{x} = \mathbf{z} + \mathbf{u}$, and this becomes

$$\begin{aligned} w \ &= (\mathbf{z}+\mathbf{u})^T\mathbf{M}(\mathbf{z}+\mathbf{u}) + \mathbf{q}^T(\mathbf{z}+\mathbf{u}) \\ &= \ \mathbf{z}^T\mathbf{M}\mathbf{z} + \mathbf{z}^T\mathbf{M}\mathbf{u} + \mathbf{u}^T\mathbf{M}\mathbf{z} + \mathbf{u}^T\mathbf{M}\mathbf{u} + \mathbf{q}^T\mathbf{z} + \mathbf{q}^T\mathbf{u} \ . \end{aligned} \tag{99}$$

Now,

$$\mathbf{q}^T\mathbf{z} \ = \ \left(\mathbf{q}^T\mathbf{z}\right)^T \ = \ \mathbf{z}^T\mathbf{q}, \tag{100}$$

and

$$\mathbf{u}^T\mathbf{M}\mathbf{z} \ = \ \left(\mathbf{u}^T\mathbf{M}\mathbf{z}\right)^T \ = \ \mathbf{z}^T\mathbf{M}^T\mathbf{u} \ = \ \mathbf{z}^T\mathbf{M}\mathbf{u} \tag{101}$$

since $\mathbf{M}$ is symmetric. Therefore

$$\mathbf{z}^T\mathbf{M}\mathbf{u} + \mathbf{u}^T\mathbf{M}\mathbf{z} + \mathbf{q}^T\mathbf{z} \ = \ \mathbf{z}^T(2\mathbf{M}\mathbf{u} + \mathbf{q}) \ . \tag{102}$$

Solve the equation $2\mathbf{M}\mathbf{u} + \mathbf{q} = \mathbf{0}$ for $\mathbf{u}$ by Cholesky factorization of $\mathbf{M}$, so that $\mathbf{z}^T\mathbf{M}\mathbf{u} + \mathbf{u}^T\mathbf{M}\mathbf{z} + \mathbf{q}^T\mathbf{z} = 0$. Also,

$$\mathbf{q}^T\mathbf{u} \ = \ -2\mathbf{u}^T\mathbf{M}^T\mathbf{u} \ = \ -2\mathbf{u}^T\mathbf{M}\mathbf{u} \ , \tag{103}$$

and so (99) reduces to the equation

$$\mathbf{z}^T\mathbf{M}\mathbf{z} \ = \ f \ , \tag{104}$$

where

$$f \ = \ w + \mathbf{u}^T\mathbf{M}\mathbf{u} \ . \tag{105}$$

Make a parallel shift of the axes from $\mathbf{x} = \mathbf{0}$ to $\mathbf{u}$, with the coordinate vector $\mathbf{z} = \mathbf{x}$ - $\mathbf{u}$. The ellipsoid is centred at that shifted origin, since if $\mathbf{z}$ satisfies (104) then so does $-\mathbf{z}$.

Since $\mathbf{M}$ is real–symmetric then $\mathbf{MV} = \mathbf{V\Lambda}$, where $\mathbf{\Lambda} = \lceil \lambda_1,\ \lambda_2, \ldots,\ \lambda_n \rfloor$ is a diagonal matrix of the (positive) eigenvalues, and $\mathbf{V}$ is a real orthonormal matrix of eigenvectors of $\mathbf{M}$. Therefore $\mathbf{V}^T\mathbf{V} = \mathbf{I} = \mathbf{VV}^T$, $\mathbf{M} = \mathbf{V\Lambda V}^T$, and in the shifted coordinates

$$f \;=\; \mathbf{z}^T\mathbf{Mz} \;=\; \mathbf{z}^T\mathbf{V\Lambda V}^T\mathbf{z} \;=\; \mathbf{y}^T\mathbf{\Lambda y}, \tag{106}$$

where $\mathbf{y} \stackrel{\text{def}}{=} \mathbf{V}^T\mathbf{z}$, and hence $\mathbf{x} = \mathbf{Vy}$. The vector $\mathbf{y}$ can be used for coordinates, as an alternative to the vector $\mathbf{z}$, with the transformation $\mathbf{z} = \mathbf{Vy}$. This is an orthogonal transformation which preserves lengths, since for all $\mathbf{w}$ and $\mathbf{z}$,

$$\begin{aligned} \left\|\mathbf{V}^T\mathbf{w} - \mathbf{V}^T\mathbf{z}\right\|^2 \;&=\; \left\|\mathbf{V}^T(\mathbf{w}\text{-}\mathbf{z})\right\|^2 \;=\; (\mathbf{V}^T(\mathbf{w}\text{-}\mathbf{z}))^T\mathbf{V}^T(\mathbf{w}\text{-}\mathbf{z}) \\ &= (\mathbf{w}-\mathbf{z})^T\mathbf{VV}^T(\mathbf{w}-\mathbf{z}) \;=\; (\mathbf{w}-\mathbf{z})^T(\mathbf{w}-\mathbf{z}) \;=\; \|\mathbf{w}-\mathbf{z}\|^2 . \end{aligned} \tag{107}$$

Hence, angles also are preserved. Then

$$f \;=\; \mathbf{y}^T\mathbf{\Lambda y} \;=\; \sum_{i=1}^{n} \lambda_i y_i^2. \tag{108}$$

Thus the ellipsoid has the equation in the $\mathbf{y}$ coordinates

$$\sum_{i=1}^{n} \frac{y_i^2}{d_i^2} \;=\; 1, \tag{109}$$

where $d_i = \sqrt{f/\lambda_i}$. Hence the d_i are the semi–axes of the ellipsoid.

The unit ball in n dimensions has volume [Smith & Vamanamurthy]

$$\Omega_n \;=\; \frac{\pi^{n/2}}{\Gamma\left(1 + \frac{1}{2}n\right)} . \tag{110}$$

Thus,

$$\Omega_2 \;=\; \pi,\ \Omega_3 \;=\; \tfrac{4}{3}\pi,\ \Omega_4 \;=\; \tfrac{1}{2}\pi^2,\ \Omega_5 \;=\; \tfrac{8}{15}\pi^2,\ \Omega_6 \;=\; \tfrac{1}{6}\pi^3 \ \textit{et cetera}. \tag{111}$$

For each $i = 1,\ 2,\ \cdots,n$, scaling the i-th semi-axis from 1 to d_i multiplies the volume by d_i, and hence the ellipsoid (109) has the n–dimensional volume

$$K \;=\; \Omega_n\, d_1 d_2 d_3 \cdots d_n \;=\; \Omega_n \sqrt{\prod_{i=1}^{n} \frac{f}{\lambda_i}} . \tag{112}$$

But $\lambda_1\lambda_2 \cdots \lambda_n = \det \mathbf{M}$, and hence the n–dimensional volume is

$$K \;=\; \Omega_n \sqrt{\frac{f^n}{\det \mathbf{M}}} . \tag{113}$$

Note that the volume K can be computed without evaluating any eigenvalues. And if $\mathbf{M}$ is factorized by the rational version of Cholesky factorization to compute $\mathbf{u}$ and $\det \mathbf{M}$, then only one square root needs to be extracted.

Area of Ellipse

We now consider the equation (91) of an ellipse with $B \neq 0$.

That is a 2-dimensional ellipsoid, and so its area K can be found from the general result for n dimensions. Rewrite x and y as x_1 and x_2.

In matrix form, (91) becomes $\mathbf{x}^T\mathbf{M}\mathbf{x} + \mathbf{q}^T\mathbf{x} \;=\; w$, where

$$\mathbf{M} = \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}, \; \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; \mathbf{q} = \begin{bmatrix} D \\ E \end{bmatrix}, \; w = -G, \tag{114}$$

and $\det \mathbf{M} = AC - B^2/4$.

For the projected ellipse, in view of (52) this reduces to $w = \nu^2c^2 - h^2$, and

$$\det \mathbf{M} \;=\; AC - \tfrac{1}{4}B^2 \;=\; \big(\lambda^2a^2 + \mu^2b^2 + \nu^2c^2\big)\frac{\nu^2c^2}{a^2b^2} \; . \tag{115}$$

The ellipse is centred at $\mathbf{u}$, where

$$2\mathbf{M}\mathbf{u} = -\mathbf{q} \; , \tag{116}$$

and hence

$$\mathbf{u} \;=\; \begin{bmatrix} \left(\dfrac{BE - 2CD}{4AC - B^2}\right) \\ \left(\dfrac{BD - 2AE}{4AC - B^2}\right) \end{bmatrix} \; . \tag{117}$$

Hence,

$$\mathbf{u}^T\mathbf{M}\mathbf{u} \;=\; Au_1^2 + Bu_1u_2 + Cu_2^2$$

$$= \frac{A(BE - 2CD)^2 + B(BE - 2CD)(BD - 2AE) + C(BD - 2AE)^2}{(4AC - B^2)^2} \; , \tag{118}$$

and the constant term is

$$f \;=\; w + \; \mathbf{u}^T\mathbf{M}\mathbf{u} \; . \tag{119}$$

The eigenvalues of $\mathbf{M}$ are

$$\lambda_1 = \frac{1}{2}\big(A{+}C{+}\sqrt{(A - C)^2 + B^2}\big), \quad \lambda_2 = \frac{1}{2}\big(A{+}C{-}\sqrt{(A - C)^2 + B^2}\big). \tag{120}$$

The discriminant is positive, and hence $\lambda_1 > \lambda_2$.

Each eigenvalue λ_j has eigenvector

$$\begin{bmatrix} B \\ 2(A - \lambda_j) \end{bmatrix} \;=\; \begin{bmatrix} B \\ A - C \mp \sqrt{(A - C)^2 + B^2} \end{bmatrix} \; ,$$

with normalizing factor $\omega_j = \big(B^2 + 4(A - \lambda_j)^2\big)^{-1/2}$, or

$$\omega_1 \;=\; \frac{1}{\sqrt{2\Big((A - C)^2 + B^2 - (A - C)\sqrt{(A - C)^2 + B^2}\,\Big)}} \; ,$$

$$\omega_2 \;=\; \frac{1}{\sqrt{2\Big((A - C)^2 + B^2 + (A - C)\sqrt{(A - C)^2 + B^2}\,\Big)}} \; . \tag{121}$$

Thus, the orthonormalized matrix of eigenvectors is

$$\mathbf{V} \;=\; \left[\begin{array}{cc} B\omega_1 & B\omega_2 \\ 2\omega_1(A-\lambda_1) & 2\omega_2(A-\lambda_2) \end{array} \right] , \tag{122}$$

and the coordinate systems are related by $\mathbf{x} = \mathbf{V}\mathbf{y} + \mathbf{u}$.

The semi-axes of the ellipse are

$$d_1 \;=\; \sqrt{\frac{f}{\lambda_1}}, \qquad d_2 \;=\; \sqrt{\frac{f}{\lambda_2}} , \tag{123}$$

with $d_1 < d_2$. The area of the ellipse (91) is

$$K(h) \;=\; \Omega_2\sqrt{\frac{f^2}{\det \mathbf{M}}} \;=\; \frac{\pi f}{\sqrt{\det \mathbf{M}}} \;=\; \frac{\pi f}{\sqrt{AC - B^2/4}} . \tag{124}$$

And if (91) is a projected ellipse, then the area of the ellipse of intersection is $L(h) \;=\; K(h)/\nu$.

Circumference of Ellipse of Intersection

In the plane (44), construct orthogonal Cartesian axes centred at $(x, y, z) = (0, 0, h/\nu)$, with η–axis parallel to the x–axis, and with ζ–axis in the plane (44) orthogonal to the η–axis. Thus, each point (η, ζ) in the ellipse of intersection projects to $(x, y) = (\eta, \nu\zeta)$ in the projected ellipse (91).

Thus, in the plane (44), the ellipse of intersection has the equation

$$A\eta^2 + (B\nu)\eta\zeta + (C\nu^2)\zeta^2 + D\eta + (E\nu)\zeta \;=\; w, \tag{125}$$

which can be rewritten as

$$\hat{A}\eta^2 + \hat{B}\eta\zeta + \hat{C}\zeta^2 + \hat{D}\eta + \hat{E}\zeta \;=\; \hat{w}. \tag{126}$$

For this ellipse of intersection the eigenvalues $\hat{\lambda}_1 > \hat{\lambda}_2$ and the semi–axes $\hat{d}_1 < \hat{d}_2$ ¿can be computed as above, for the projected ellipse (51). Then, equation (25) gives the circumference of the ellipse of intersection as

$$C(\hat{d}_2, \hat{d}_1) \;=\; 4\hat{d}_2 E\left(1 - \frac{\hat{d}_1^2}{\hat{d}_2^2}\right) \;=\; 4\hat{d}_2 E\left(1 - \frac{\hat{\lambda}_2}{\hat{\lambda}_1}\right) . \tag{127}$$

Plane Through Centre of Ellipsoid

If the plane (44) passes through the centre of $\mathcal{E}$ then $h = 0$, and accordingly $D = E = 0$ in (52). Therefore $\mathbf{u} = \mathbf{0}$ and $f = w = \nu^2 c^2$, and the area of the projected ellipse is

$$K(0) \;=\; \frac{\pi\nu^2 c^2}{\sqrt{AC - B^2/4}} . \tag{128}$$

And the area of the ellipse of intersection is

$$L(0) \;=\; \frac{K(0)}{\nu} \;=\; \frac{\pi\nu c^2}{\sqrt{AC - B^2/4}} \;=\; \frac{\pi abc}{\sqrt{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2}} . \tag{129}$$

Note that the expression (129) is unchanged by any permutation of (a with λ), (b with μ) and (c with ν).