Probabilistic Solutions to Merchant Problems: Locating of the False Stack

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Abstract

In [1] a probabilistic solution to the Infinite Merchant's Problem, an undecidable problem equivalent to the Halting Problem, was proposed. The solution uses a real Hilbert space and is based on the estimation of the exponential growth of an unbounded semigroup. In [2] was offered an alternative solution in terms of scattering processes on quantum dots. The authors reduced the problem to a special scattering problem and testify the "halting phenomenon" based on the quantum measurement of results of scattering with random input data. The a-posteriori probability of halting, subject to the negative results of multiple independent tests, was estimated. The possibility of location the number of the bag with false coins in finite-dimensional case was noticed in [1] and proved in [2]. The aim of this paper is to offer a solution of the latter problem in the infinite case.

Keywords: Merchant problem, average on Weiner measure, Hilbert space

1 Introduction:

In [1] a probabilistic solution to the Infinite Merchant's Problem, an undecidable problem equivalent to the Halting Problem, was proposed. The solution uses a real Hilbert space and is based on the estimation of the exponential growth of an unbounded semigroup. In [2] was offered an alternative solution in terms of scattering processes on quantum dots. The authors reduced the problem to a special scattering problem and testify the "halting phenomenon" based on the quantum measurement of results of scattering with random input data. The a-posteriori probability of halting, subject to the negative results of multiple independent tests, was estimated. The possibility of location the number of the bag with false coins in finite-dimensional case was noticed in [1] and proved in [2]. The aim of this paper is to offer a solution of the latter problem in the infinite case.

2 Locating of the False Stack

Following [1], [2] we will consider the real Hilbert space

$$h := l^2(\mathbf{N}), \;\; x := \{x_n\}_{n=1}^{\infty} \in h, \;\; x_n \in \mathbf{R}, \;\; ||x|| := \left(\sum_n |x_n|^2 \right)^{1/2},$$

as the quantum space. We suppose that for a fixed computable positive number γ, $\gamma > 0$, the stack with a number $\beta \in \mathbf{N}$ contains false coins of the weight $q_\beta = 1 + \gamma$, while each of the rest consists of true coins of the weight $q_n = 1$, $n \neq \beta$.

To localized the false stack, that is, to calculate the number β, we will use two diagonal operators Q and L on h, which are defined as follows:

$$Qx := (1+\gamma)P_\beta + P_\perp x$$

and

$$Lx := \left\{ \frac{x_n}{n^2} \right\}_{n=1}^{\infty},$$

where P_β is the ortho-projector on the β-th element $e_\beta \in h$ of the standard basis in h, e.g. $P_\beta x = x_\beta e_\beta$, and $P_\perp := I - P_\beta$.

Then we construct two quadratic forms induced by the T-th iteration of the operator Q, namely,

$$\langle Q^T x, x\rangle := (1+\gamma)^T \langle P_\beta x, x\rangle + \langle P_\perp x, x\rangle,$$
$$\langle Q^T Lx, x\rangle := \beta^{-2}(1+\gamma)^T \langle P_\beta x, x\rangle + \langle P_\perp Lx, x\rangle.$$

After that, similar to [1], [2], for a random vector x one may consider the Rayleigh ratio

$$\mathcal{J}_T := \frac{\langle Q^T Lx,\, x\rangle}{\langle Q^T x,\, x\rangle}. \tag{1}$$

Then the difference:

$$\mathcal{J}_T - \beta^{-2} = \frac{\beta^{-2}(1+\gamma)^T \langle P_\beta x,\, x\rangle + \sum_{n\neq\beta} |x_n/n|^2}{(1+\gamma)^T \langle P_\beta x,\, x\rangle + \sum_{n\neq\beta} |x_n|^2} - \beta^{-2}$$

$$= \frac{\sum_n |x_n/n|^2}{(1+\gamma)^T |x_\beta|^2 + \sum_{n\neq\beta} |x_n|^2} - \frac{\beta^{-2}\sum_n |x_n|^2}{(1+\gamma)^T |x_\beta|^2 + \sum_{n\neq\beta} |x_n|^2}.$$

can be rewritten as:

$$\mathcal{J}_T - \beta^{-2}\left[1 - \frac{\sum_n |x_n|^2}{(1+\gamma)^T |x_\beta|^2 + \sum_{n\neq\beta} |x_n|^2}\right] = \frac{\sum_n |x_n/n|^2}{(1+\gamma)^T |x_\beta|^2 + \sum_{n\neq\beta} |x_n|^2} \tag{2}$$

or in a short form:

$$\mathcal{J}_T - \beta^{-2} I_1 = I_2 \tag{3}$$

with

$$I_1 := 1 - \frac{\sum_n |x_n|^2}{(1+\gamma)^T |x_\beta|^2 + \sum_{n\neq\beta} |x_n|^2} = \frac{[(1+\gamma)^T - 1]|x_\beta|^2}{(1+\gamma)^T |x_\beta|^2 + \sum_{n\neq\beta} |x_n|^2}$$
$$= \frac{[(1+\gamma)^T - 1](|x_\beta|/||x_\perp||)^2}{(1+\gamma)^T (|x_\beta|/||x_\perp||)^2 + 1}, \tag{4}$$

where $x_\perp = P_\perp x$.

Because the vector x is random we need to imply some average procedure to the equation (3). A suitable average approach in a general way is considered in the following section.

2.1 Average Process

In this section A is a positive self-adjoint (symmetric) operator on a real separable Hilbert space H with a orthonormal basis $f := \{f_n\}_{n=1}^{\infty}$.

Let us choose and fix some natural number β, then for each vector $u := \sum_{n=1}^{\infty} x_n f_n \in H$ we have the following decomposition: $u = u_\beta + u_\perp$, with $u_\beta := x_\beta f_\beta$ and $u_\perp := \sum_{n\neq\beta}^{\infty} x_n f_n$.

For a fixed natural number N we denote by H_N a finite-dimensional subspace, which is a linear span of the first N elements of the basis f. Following [1], [2] we are interested in the behaviour of the ratio:

$$\mathcal{R}_\beta := \frac{\int_{|x_\beta|<\varepsilon||u_\perp||} e^{-\langle Au,\, u\rangle} d\mu_u}{\int e^{-\langle Au,\, u\rangle} d\mu_u}. \tag{5}$$

as N goes to infinity and then $0 < \varepsilon \to 0+$. Here u is an element of H_N and $d\mu_u := dx_1 ... dx_N$ is the Lebesgue measure on $\mathbf{R}^N$, which is isomorphic to H_N.

The quadratic form of the operator A may be rewritten by the following way ($u \in H$):

$$\langle Au,\, u\rangle = a_\beta x_\beta^2 + 2x_\beta \langle u_\perp,\, A_{\perp\beta} f_\beta\rangle + \langle A_{\perp\perp} u_\perp,\, u_\perp\rangle$$

$$= a_\beta \left[x_\beta + \frac{\langle u_\perp,\, A_{\perp\beta} f_\beta\rangle}{a_\beta} \right]^2 - a_\beta^{-1} \langle u_\perp,\, A_{\perp\beta} f_\beta\rangle \langle A_{\perp\beta} f_\beta,\, u_\perp\rangle + \langle A_{\perp\perp} u_\perp,\, u_\perp\rangle$$

$$=: S_1 - S_2 + S_3, \tag{6}$$

where $a_\beta := \langle Af_\beta,\, f_\beta\rangle$ and the second addend S_2 is the quadratic form of some one dimensional operator. Then the denominator and nominator in (5) may be simplified as:

$$\int e^{-\langle Au,\, u\rangle} d\mu_u = \int \left(\int e^{-S_1} dx_\beta \right) e^{S_2 - S_3} d\mu_\perp = \sqrt{\frac{\pi}{a_\beta}} \int e^{S_2 - S_3} d\mu_\perp \tag{7}$$

and

$$\int_{|x_\beta|<\varepsilon||u_\perp||} e^{-\langle Au,\, u\rangle} d\mu_u = \int_{|x_\beta|<\varepsilon||u_\perp||} e^{-S_1 + S_2 - S_3} d\mu_u. \tag{8}$$

Thus

$$\mathcal{R}_\beta := \frac{\int \sqrt{\frac{a_\beta}{\pi}} \int_{|x_\beta|<\varepsilon||u_\perp||} e^{-S_1 + S_2 - S_3} d\mu_u}{\int_{\mathbf{R}^{N-1}} e^{S_2 - S_3} d\mu_\perp}. \tag{9}$$

It has been noticed above that S_2 is one-dimension operator, therefore, it can be represented by an ortho-projector P on an element

$$p := \frac{A_{\perp\beta} f_\beta}{||A_{\perp\beta} f_\beta||},$$

So,

$$S_2 = r\langle Pu_\perp, u_\perp\rangle$$

and putting $B := A_{\perp\perp}$ one get

$$S_3 - S_2 = \langle (B - rP) u_\perp, u_\perp\rangle.$$

Finally, for the denominator (7) we have

$$\int e^{-\langle Au,\, u\rangle} d\mu_u = \sqrt{\frac{\pi}{a_\beta}}\sqrt{\frac{\pi}{\det(B - rP)}}$$

with

$$r := a_\beta^{-1}||A_{\perp\beta} f_\beta||^2.$$

The determinant can be calculated as the product of the eigenvalues α^r, which subject to the equation:

$$(B - rP)e = \alpha^r e$$

or

$$e - r(B - \alpha^r)^{-1} Pe = 0.$$

Because

$$Pu = \langle p,\, u\rangle p,$$

then

$$1 - r\langle (B - \alpha^r)^{-1} p,\, p\rangle = 0.$$

Finally, one have an equation on the eigenvalues of the operator $B - rP$:

$$1 - r\sum \frac{\langle p,\, p_s\rangle^2}{\alpha_s^0 - \alpha^r} = 0,$$

where p_s - eigenfunctions and α_s^0 - eigenvalues of the operator B.

Expression for the numerator:

$$\int \sqrt{\frac{a_\beta}{\pi}} \int_{|u_\beta| < \varepsilon ||u_\perp||} e^{-S_1 + S_2 - S_3} d\mu_u = \int \left[\sqrt{\frac{a_\beta}{\pi}} \int_{|x_\beta| < \varepsilon ||u_\perp||} e^{-I_1} dx_\beta\right] e^{-\langle (B - rP)u_\perp,\, u_\perp\rangle} d\mu_\perp. \quad (10)$$

It has been shown above that

$$\det(B - rP) = \prod_s \sqrt{\frac{\pi}{\alpha_s^0}} \sqrt{\prod \frac{\alpha_s^0}{\alpha_s^r}}. \quad (11)$$

1) What may appear from

$$\sqrt{\frac{a_\beta}{\pi}} \int_{|u_\beta| < \varepsilon ||u_\perp||} e^{-a_\beta\left[x_\beta + \frac{\langle u_\perp,\, A_{\perp\beta} f_\beta\rangle}{a_\beta}\right]^2} dx_\beta \ ?$$

2) Is it possible under some conditions on A and β to obtain uniform convergence of the ratio $\mathcal{R}_\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$?
If this is done, than one may operate on the indistinguishable set defined as

$$\mathcal{F}_\varepsilon := \{u \in H : |x_\beta| < \varepsilon ||u_\perp||\} \quad (12)$$

with a non-trivial operator A. For instance (see [1], [2]), choosing

$$\langle Au,\, u\rangle := \sum_{s=1}^{\infty} |x_s - x_{s-1}|^2, \ \ x_0 = 0, \quad (13)$$

we have got the Wiener measure in the limit $N \to \infty$.

2.2 Estimates of Averages

The averages on Wiener measure W of the expressions for I_l in (3) can be estimated as follows

$$\langle I_l \rangle_W = \langle I_l \rangle_{\mathcal{F}_\varepsilon} + \langle I_l \rangle_{\mathbf{R}^N \setminus \mathcal{F}_\varepsilon}, \;\; l = 1,\, 2. \tag{14}$$

The first addend on the right side in (14) doesn't exceed the W-measure $m_\varepsilon := W(\mathcal{F}_\varepsilon)/W(\mathbf{R}^N)$ of the set $\mathcal{F}_\varepsilon$ because $I_l \leq 1$ there, $l = 1,\, 2$. Furtheremore $m_\varepsilon \to 0$ as $\varepsilon \to 0$ (see [1], [2]). The second term in (14) with $l = 1$ is estimated from below as:

$$\langle I_1 \rangle_{\mathbf{R}^N \setminus \mathcal{F}_\varepsilon} \geq \frac{(1+\gamma)^T \varepsilon^2}{(1+\gamma)^T \varepsilon^2 + 1} = 1 - \frac{1}{(1+\gamma)^T \varepsilon^2 + 1} \approx I, \;\; T \to \infty. \tag{15}$$

It remains to estimate the corresponding average for I_2 in (3) on the domain $\mathbf{R}^N \setminus \mathcal{F}_\varepsilon$. Firstly, notice that

$$\sum_n (|x_n|/n)^2 = \langle Lx,\, x \rangle.$$

Then

$$\langle I_2 \rangle_{\mathbf{R}^N \setminus \mathcal{F}_\varepsilon} \leq \frac{1}{(1+\gamma)^T \varepsilon^2 + 1} \langle \frac{\langle Lx,\, x \rangle}{\langle x_\perp,\, x_\perp \rangle} \rangle_W. \tag{16}$$

Because $\langle x,\, x \rangle^{-1} \in L^1(\mathbf{R}^N)$ with $N \geq 3$, the costante

$$C_\beta := \langle \frac{\langle Lx,\, x \rangle}{\langle x_\perp,\, x_\perp \rangle} \rangle_W$$

is finite. Thus,

$$\langle I_2 \rangle_{\mathbf{R}^N \setminus \mathcal{F}_\varepsilon} \to 0, \;\; T \to \infty. \tag{17}$$

To sum up, we have proved that if the system with high probability contains a false coin under the same general condition of the paper [2], i.e. we do not know in advance whether there is a false coin and we want to find out whether there is one (and we know how to do it with a high probability [2]) and, then, if this is the case, we can locate the false coin, because by (3), (14), (15) and (17) the average of the ratio $\mathcal{J}_\beta$ tends to some number β^{-2}, where β is the number of the stack with false coins.

3 Appendix: Difference Hardy Inequality

In this section we will prove that the operator L, which has been used in the previous section, is subordinated by the average operator A from (13). That is,

$$\langle Lx,\, x \rangle = \sum_{n=1}^{\infty} \frac{x_n^2}{n^2} \leq 16 \sum_{n=1}^{\infty} |x_n - x_{n-1}|^2, \;\; x_0 = 0. \tag{18}$$

for any $x \in h$ such that the right part of the inequality is finite.

The proof easy follows from the Cauchy inequality applied to the following estimation ($x_0 = 0$):

$$\sum_{n=1}^{\infty} \frac{x_n^2}{n^2} \leq 2 \sum_{n=1}^{\infty} \frac{x_n^2}{n(n+1)} = 2 \sum_{n=1}^{\infty} (x_n - x_{n-1}) \frac{x_n + x_{n-1}}{n}$$

$$\leq 2\sqrt{\sum_{n=1}^{\infty}(x_n - x_{n-1})^2}\sqrt{\sum_{n=1}^{\infty}\frac{(x_n + x_{n-1})^2}{n^2}}.$$

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