Symplectic operator-extension techniques and zero-range quantum models

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Abstract

F. Berezin and L. Faddeev interpreted Fermi zero-range model as a self-adjoint extension of the Laplacian. Various modifications of this model in conventional Hilbert space possess rich spectral properties, but unavoidably have the negative effective radius and contain numerous parameters which do not have a direct physical meaning. We suggest, for spherically-symmetric scattering, a generalization of the Fermi zero-range model supplied with an indefinite metric in the inner space and a Hamiltonian of the inner degrees of freedom. Effective radius of this model may be both positive or negative. We propose also a general *principle of analyticity* formulated in terms of $k\cot\delta(k)$ as a function of the scattering phase shift $\delta(k)$ depending on the wave-number k. This principle allows us to evaluate all parameters of the model, including the indefinite metric tensor of the inner space, once the basic parameters of the model: the spectrum of the inner Hamiltonian, the scattering length and the effective radius, are fixed, such that the sign of the effective radius is connected with the spectrum of the inner Hamiltonian by the appropriate consistency condition.

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I. INTRODUCTION

In [1] a realistic description of quantum scattering by a point-wise object was interpreted, based on [2], in terms of J. von Neumann operator extensions. From the point of view developed in [1, 3], the solvable zero-range model suggested by Fermi in [2], is actually a self-adjoint extension, see [4], of the Laplace operator defined on smooth functions which have a singularity at the origin

$$\Psi(r) = \frac{\mathcal{A}^{\psi}}{4\pi r} + \mathcal{B}^{\psi} + o(1), \tag{1}$$

with the asymptotic boundary values $\mathcal{A}$ and $\mathcal{B}$ submitted to the real *boundary condition*, $\gamma = \bar{\gamma}$

$$\gamma\mathcal{A} + \mathcal{B} = 0. \tag{2}$$

This operator has only trivial spectral structure (one negative eigenvalue $-\kappa^2$, for $\kappa = 4\pi\gamma > 0$ or one resonance $\kappa' = i4\pi\gamma$ for $\gamma < 0$). In attempt to extend the construction to solvable models with rich spectrum, Wigner suggested in [5] to calculate scattered waves for the Schrödinger operator with compactly supported potentials $V(\mathbf{r})$, ($V(\mathbf{r}) = 0$ for $r = |\mathbf{r}| > r_0$) imposing an appropriate energy-dependent boundary condition at $r = r_0$. From the modern point of view, the suggestion of Wigner is equivalent to matching the scattering Ansatz in the outer space $r > r_0$ with solutions of the Schrödinger equation on the inner space $r < r_0$ based on Dirichlet-to-Neumann map (DN-map) Λ of the Schrödinger operator on the inner space, see [6]. In particular, for the spherically-symmetric potential V, when using the spherically symmetric Ansatz for the scattered waves (in "s-channel"):

$$\psi(r,k) = \frac{C}{r}\sin(kr + \delta(k)) := \frac{C}{r}\chi(r,k),\ r > r_0 \tag{3}$$

in the outer space, one can present the matching condition in form

$$\left(\frac{\partial}{\partial r}\chi(r,k) + \alpha(k)\chi(r,k)\right)\Big|_{r=r_0} = 0. \tag{4}$$

Here α is the corresponding Weyl-Titchmarsh function, see [7] - the component of the DN-map in s-channel. Unfortunately, this condition contains the full spectral data (encoded in $\alpha(k)$) of the inner problem, thus reducing the scattering problem to the inner spectral problem. Nevertheless the suggestion of Wigner inspired numerous attempts to construct energy-dependent potentials characterized by the energy-dependent boundary conditions. In [8] the "zero-range potential with inner structure" was suggested based on replacement, for spherically-symmetric scattering, the Schrödinger operator in the inner space by a finite matrix which plays a role of the Hamiltonian of the inner degrees of freedom. Then the role of the component α of the DN-map is played by the analog $F(k)$ of the corresponding Krein-function, see [9]. The operator extension procedure suggested in [8] is equivalent to imposing of the "asymptotic boundary condition" at the origin :

$$\left(\frac{1}{\chi(\mathbf{r},k)}\frac{\partial}{\partial r}\chi(\mathbf{r},k)\right)\Big|_{r=0} = k\cot\delta(k) := F(k). \tag{5}$$

Here the function $F(k)$ is defined by the spectral structure of the Hamiltonian of the inner degrees of freedom and by the boundary parameters of the model, see [8]. The operator extensions approach allows to construct simple solvable models for few - channels and few - body scattering systems with interesting resonance properties, without solving sophisticate boundary problems, see [10–13] and more references in [14]. The behavior of the function

$F(k)$ which corresponds to the zero-range model [8] looks similar to the low-energy behavior of the corresponding function in s-channel for conventional rapidly decreasing potentials, see [15]:

$$F(k) = k\cot\delta(k) = -\frac{1}{a} + \frac{r_0}{2}k^2 \dots . \tag{6}$$

The parameters r_0, a in (6) are called, respectively, the *effective radius* of the scatterer and the scattering length, see [15]. They are measured in experiment. Unfortunately the zero-range model [8] with inner structure and *positive metric in the inner space*

a. unavoidably has a negative effective radius, [16], because in that case $-F$ is a Nevanlinna class function of $\lambda = k^2$, $\Im F\, \Im\lambda < 0$ and

b. It contains numerous "extra parameters" which *have no straightforward physical interpretation* and hence are not *fitting parameters*. These are: the deficiency vector in the inner space and the boundary parameters.

We propose a procedure to repair both defects a,b for a "special zero-range model", with an *indefinite metric on the inner space*, see section 4, and the scattering phase shift $\delta(k)$ (we call it further just "the scattering phase") *if the function F is selected as an entire function of the parameter $\lambda = k^2$*:

$$F(k) = \sum_{l=0}^{\infty} g_l k^{2l}. \tag{7}$$

In case of finite-dimensional "inner space" $\dim E = n$ the function $F(k)$ submitted to this *principle of analyticity* is just a polynomial degree $2n$. The condition (7) enables us to evaluate all "extra parameters" of the special zero-range model with inner structure, if the *fitting data* : the scattering length, the effective radius and the spectrum of the Hamiltonian of the inner degrees of freedom are connected by the appropriate consistency condition. The standard self-adjoint quantum Hamiltonian of the model is obtained via restriction of the model onto the absolutely continuous subspace which has the positive metric. More physical discussion and fitting of the model to experimental data can be found in [17].

II. PRELIMINARIES: SYMPLECTIC EXTENSION PROCEDURE

Zero-range model with inner structure can be constructed via operator extension procedure applied to an orthogonal sum of a restricted bounded inner Hamiltonian and the restricted Laplacian in $L_2(R_3)$. But the classical version of the operator-extension procedure, [4], is inconvenient for differential operators. The extension procedure for them is usually reduced to choosing of a Lagrangian plane of the corresponding symplectic *boundary form* e.g $\frac{\partial u}{\partial r}(r_0)\bar{v}(r_0) - u(r_0)\frac{\partial \bar{v}}{\partial r}(r_0)$,for Laplacian in the outer space $r > r_0$. I.M. Gelfand in the 1960s, [18], attracted attention of specialists to necessity of developing a simplectic version of the operator extension procedure for abstract operators. It was done in [8] for solvable quantum models with bounded inner Hamiltonian and positive metric in the inner spaces. In this preliminary section we describe the symplectic version of the extension procedure for operators in Pontryagin space, based on an abstract analog of the boundary form.

Consider the scattering problem for two quantum particles with masses m_1, m_2 and the reduced mass $\mu = m_1 m_2 (m_1 + m_2)^{-1}$. Choosing some characteristic wave number $k_0 > 0$ and the corresponding characteristic energy $E_0 = (2\mu)^{-1}\hbar^2 k_0^2$, we introduce the dimensionless coordinates and the corresponding spectral parameter, connected with the standard Euclidean coordinates $\mathbf{r}$ in R_3 and the conventional energy $E = (2\mu)^{-1}\hbar^2 k^2$ as

$$x = k_0\mathbf{r},\ \lambda = k^2 k_0^{-2} = E(E_0)^{-1}. \tag{8}$$

Then we re-write the conventional Schrödinger equation as

$$\left(\triangle + \lambda - \tilde{V}(x)\right) u(x) = 0. \tag{9}$$

Here $\tilde{V}(x) = E_0^{-1} V(\mathbf{r})$ is the dimensionless interaction potential. The wave functions $u(x)$ are dimensionless. We will replace the equation (9) by the zero-range model, assuming that the component of the model in the outer space is defined by the Schrödinger equation with zero potential. The "free" Green's function which corresponds to the equation with zero potential on the whole space R^3, satisfies the equation

$$-(\triangle + \lambda) G_\lambda(x, x') = \delta^{(3)}(x - x') \tag{10}$$

and has a form of an outgoing wave generated at the pole x':

$$G_\lambda(x, x') = -(\triangle + \lambda)^{-1} \delta^{(3)}(x - x') = \frac{e^{i\sqrt{\lambda}|x-x'|}}{4\pi|x - x'|}. \tag{11}$$

We assume that the outer space is supplied with the standard $L_2(R^3)$ - dot-product $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle = \langle \bar{\alpha} u, v\rangle$ and the non-perturbed two-body Hamiltonian is defined by the three-dimensional self-adjoint Laplacian with respect to dimensionless coordinates

$$lu = -\triangle u, \tag{12}$$

on the Sobolev class $W_2^2(R^3)$. Variables $\tilde{x}, \tilde{y}, \ldots$ are used for the inner space E of the model.

Following [1] we restrict the Laplacian l onto the domain consisting of all smooth functions vanishing near the point x_0. The closure l_0 of the restricted operator is defined on the domain D_0 which consists of all W_2^2-functions vanishing just at x_0. This operator is symmetric and has deficiency indices $(1, 1)$:

$$\overline{(l_0 - iI)D_0} = L_2(R_3) \ominus N_i, \ \overline{(l_0 + iI)D_0} = L_2(R_3) \ominus N_{-i},$$

dim N_i = dim $N_{-i} = 1$. The adjoint 0perator $l_0^+ = (-\triangle)^+$ is defined, due to the von Neumann theorem, see [4, 19], by the same differential expression (12) on the domain

$$D_0^+ = D_0 + N_i + N_{-i}. \tag{13}$$

The one-dimensional deficiency subspaces N_i, N_{-i} are spanned by the corresponding non-perturbed free Green functions

$$N_i = \{G_{-i}(*, x_0)\}, \ N_{-i} = \{G_i(*, x_0)\},$$

$$G_{-i}(x, x_0) = \frac{e^{i\sqrt{-i}|x-x_0|}}{4\pi|x - x_0|}, \ G_i(x, x_0) = \frac{e^{i\sqrt{i}|x-x_0|}}{4\pi|x - x_0|} = \frac{l + iI}{l - iI} G_{-i}(x, x_0), \tag{14}$$

with branches of square roots defined by the condition $\mathrm{Im}\sqrt{\lambda} > 0$. We assume further that the restriction point is $x_0 = 0$, if another point is not selected .

The above representation (13) of the domain of the adjoint operator was used in [1, 19]. Another representation of the domain of the adjoint Laplacian as a set of singular elements described above (1) was suggested by Fermi [2] and is now commonly used in physical literature, see for instance [3, 20]. It characterizes each element u from the domain of the adjoint operator by *dimensionless asymptotic boundary values* A, B for $|x| \to 0$:

$$u(x) = \frac{A^u}{4\pi|x|} + B^u + o(1). \tag{15}$$

The boundary form of l_0^+ on elements from the domain of the adjoint Laplacian is calculated via integration by parts as

$$\mathcal{J}(u,v) = \langle (l_0)^+ u,\, v\rangle - \langle u,\, (l_0)^+ v\rangle = \bar{B}^u A^v - \bar{A}^u B^v. \tag{16}$$

Due to the connection (8) between dimensional and dimensionless coordinates the corresponding asymptotic values are connected by the dimensional factor k_0 : $A = \mathcal{A}k_0$, $B = \mathcal{B}$.

It is well known that two descriptions (13,15) of the domain of the adjoint Laplacian are equivalent, [1]. They lie as foundations, respectively, for the classical von-Neumann and for the symplectic version of the extension procedure for Laplacian. Planning to develop the symplectic version for abstract operators we will derive the equivalence of both above representations based on decomposition of elements of the defect $N = N_i + N_{-i}$ with respect to the *symplectic basis*

$$W_+(x) = \frac{1}{2}\left[G_{-i}(x,0) + G_i(x,0)\right] = \frac{l}{l - iI} G_{-i}(x,0),$$

$$W_-(x) = \frac{1}{2i}\left[G_{-i}(x,0) - G_i(x,0)\right] = -\ \frac{I}{l - iI} G_{-i}(x,0). \tag{17}$$

We will present the von-Neumann formula for elements of the domain of the adjoint operator in terms of the decomposition with respect to the basis $W_\pm$:

$$u = u_0 + \eta_+^u W_+ + \eta_-^u W_-, \tag{18}$$

where u_0 is an element from the domain of the closure l_0 of the restricted operator and $\eta_\pm^u$ are complex coefficients. Deficiency elements $G_{\pm i}(*,0)$ are eigenvectors of the adjoint operator with eigenvalues $\mp i$ respectively. Then the elements $W_\pm$ are transformed by the adjoint operator as:

$$l_0^+ W_+ = W_-,\ l_0^+ W_- = -W_+. \tag{19}$$

Lemma II.1 *The boundary form (16) of the adjoint operator l_0^+ depends only on components $u_d = \eta_+^u W_+ + \eta_-^u W_-$, $v_d = \eta_+^v W_+ + \eta_-^v W_-$ of the elements u, v in the defect. It is an Hermitian symplectic form of the variables $\eta_\pm$ and can be presented alternatively as*

$$\mathcal{J}(u,v) = \mathcal{J}_l(u_d, v_d) = \langle l_0^+ u_d,\ v_d\rangle - \langle u_d,\ l_0^+ v_d\rangle = \ \frac{1}{4\pi\sqrt{2}}\left(\bar{\eta}_+^u \eta_-^v - \bar{\eta}_-^u \eta_+^v\right). \tag{20}$$

The above variables $\eta_\pm$ are connected with the asymptotic boundary values A, B via the transformation:

$$\begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -4\pi\sqrt{2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \tag{21}$$

Remark Both pairs of variables $\eta_\pm$ and A, B are *symplectic coordinates* of the element u in representations (15,18). The representation (i7) of the boundary form in terms of the symplectic variables A, B is usually obtained, see for instance [1–3, 20] via straightforward integration by parts. An alternative calculation based on (19) goes in line with an abstract analog (Lemma II.2 below) of the above lemma.

Proof of the formula (20) is obtained via the direct application of the adjoint operator l_0^+ to the above version (18) of the von-Neumann decomposition of elements from D_0^+ in the boundary form of the adjoint operator: $\langle l_0^+ u,\ v\rangle - \langle u,\ l_0^+ v\rangle := \mathcal{J}_l(u,v)$. One can easily see that the boundary form depends only on the parts of the elements $u_d = \eta_+^u W_+ + \eta_-^u W_-$, $v_d =$

$\eta^v_+ W_+ + \eta^v_- W_-$ in the defect, and can be calculated using (19), since $l_0^+ u_0 = l_0 u_0$. Then due to $\int_{R_3} |G_i(\xi,0)|^2 d^3\xi = 1/(4\pi\sqrt{2})$ we obtain:

$$\mathcal{J}(u,v) = \langle l_0^+ u_d,\ v_d\rangle - \langle u_d,\ l_0^+ v_d\rangle = \left(\bar\eta^u_+\eta^v_- - \bar\eta^u_-\eta^v_+\right)\ 1/(4\pi\sqrt{2}). \tag{22}$$

The second formula (21) is based on the asymptotic behavior of the non-perturbed Green function for $x \to 0$:$W_+(x) = \frac{1}{4\pi|x|} - \frac{1}{4\pi\sqrt{2}} + o(1),\ W_-(x) = -\frac{1}{4\pi\sqrt{2}} + o(1)$. End of the proof.

Self-adjoint extensions of l_0 were obtained in [1] via submitting A, B to the Fermi boundary conditions $B = \gamma A$. For general construction of extensions of densely defined operators in terms of boundary forms see also [23]. Though the boundary forms (16) and (20) are equivalent, our version (20) can be used for the whole class of all differential operators with square-integrable Green-function, in particular for the Schrödinger operators in a bounded domain with Robin boundary conditions, even restricted at the point $x)_0$ on the boundary boundary.

We choose the non-perturbed Hamiltonian of inner degrees of freedom (we call it further just inner Hamiltonian) as a J-self-adjoint operator in the inner Pontryagin space supplied with an indefinite metric. After introducing the dimensionless spectral parameter λ the zero-range model of two particles Hamiltonian will be obtained as an extension of the orthogonal sum $l_0 \oplus \mathcal{H}_0$ of the restricted Laplacian and the restricted inner Hamiltonian $\mathcal{H}$ in the Pontryagin space $L_2(R^3) \oplus E_J$. The inner component E_J is obtained via equipping a finite-dimensional Hilbert space E, $\dim E = N$, with an indefinite metric. Consider two complementary orthogonal projections $P_\pm$ with respect to the conventional dot-product $\langle *, *\rangle$ in E. Then the indefinite dot product in E (J-dot product) is defined as :

$$\langle J\tilde x,\ \tilde y\rangle = [\tilde x, \tilde y] = \langle P_+\tilde x, P_+\tilde y\rangle - \langle P_-\tilde x, P_-\tilde y\rangle.$$

Obviously $[\bar\alpha\tilde x, \tilde y] = \alpha[\tilde x, \tilde y] = [\tilde x, \alpha\tilde y]$.The space E supplied with the above J-dot product is denoted by E_J. For general properties of Pontryagin spaces see [21]. We sketch below a symplectic version of the operator-extension technique in the inner component E_J.

A bounded operator $\mathcal{H}$ is called J-symmetric (symmetric with respect to the above J-dot product) if $[\mathcal{H}\tilde x, \tilde y] = [\tilde x, \mathcal{H}\tilde y]$. In finite-dimensional sace each J-symmetric operator is J-self-adjoint. We will apply the symplectic scheme to the *bounded* operator which is also *symmetric with respect to the conventional dot-product in E and commutes with J*. We assume that the spectrum of the J- symmetric operator $\mathcal{H}$ consists of a finite number of simple *positive* eigenvalues $\lambda_s = k_s^2\, k_0^{-2}$, $(\mathcal{H} - \lambda_s I)\, e_s = 0$ and $\mathcal{H} - iI$ is invertible. Choose a normalized *generating* vector $e \in E_J$, $[e, e] = 1$ (which is non-orthogonal to each eigenvector e_s of the operator $\mathcal{H}$, $[e, e_s] := e^s \neq 0,\ s = 1, 2, \dots N$). For selected vector e define the domain $D_{\mathcal{H}_0}$ of the restricted operator as $D_{\mathcal{H}_0} = [\mathcal{H} - iI]^{-1}\ \ (E_J \ominus_J \{e\})$ and set $\mathcal{H}_0 = \mathcal{H}|_{D_{\mathcal{H}_0}}$, where the J-orthogonal difference is denoted by $\ominus_J$. The vectors e, e' play roles of deficiency vectors of the restricted operator at the spectral points $\pm i$: $[(\mathcal{H} - iI)D_{\mathcal{H}_0}, e] = 0,\ [(\mathcal{H} + iI)D_{\mathcal{H}_0}, e'] = 0$. Then the vectors e and $e' = \frac{\mathcal{H}+iI}{\mathcal{H}-iI}e$ form a linearly-independent pair, with a positive angle between them. The subspaces $M_i := \bigvee_\alpha \{\alpha e\}$, $M_{-i} := \bigvee_\alpha \{\alpha e'\}$ will play roles of the deficiency subspaces.

At any regular points λ, $\bar\lambda$ of the operator $\mathcal{H}$ the deficiency subspaces M_λ, $M_{\bar\lambda}$ are also one-dimensional, the defect is two-dimensional and the deficiency vectors are calculated as

$$e_\lambda = \frac{\mathcal{H} + iI}{\mathcal{H} - \bar\lambda I}e,\ \ e_{\bar\lambda} = \frac{\mathcal{H} + iI}{\mathcal{H} - \lambda I}e.$$

Generally the J-adjoint operator $\mathcal{H}_0^+$ is defined by the formula $[\mathcal{H}_0\tilde x, \tilde y] = [\tilde x, \mathcal{H}_0^+\tilde y]$ on elements $\tilde y$ for which the J-dot product in the left hand side may be continued onto the whole

space E as an "anti-linear" functional of $\tilde{x}$. For densely-defined operators this condition implies $(\mathcal{H}_0^+ + iI)e = 0,\ (\mathcal{H}_0^+ - iI)e' = 0$. For a bounded operator $\mathcal{H}$ *we just define* the formal adjoint operator $\mathcal{H}_{_0}^{^+}$ on the defect $M = M_{_i} + M_{_{-i}}$ by the above formulae. Then, for complex values of λ, the deficiency vectors are eigenvectors of the corresponding formal adjoint operator:

$$\left(\mathcal{H}_0^+ - \bar{\lambda}I\right) e_\lambda = 0,\ \left(\mathcal{H}_0^+ - \lambda I\right) e_{\bar{\lambda}} = 0.$$

An analog of the von-Neumann representation of the domain of the adjoint operator remains true in Ponryagin space, see [21]. For elements from the domain of the formal adjoint operator we have the representation $\tilde{x} = \alpha e + \beta e',\ \tilde{x}_0 \in D_0$ with

$$\mathcal{H}_0^+(+\alpha e + \beta e') = -i\alpha e + i\beta e'.$$

M. Krasnosel'skii noticed, see [22], that the extension construction for non-densely defined closed operators is actually developed in the *defect* $M = M_{_i} + M_{_{-i}}$, and can be accomplished similarly to von-Neumann construction, if the deficiency subspaces do not overlap. We obtain a symmetric extension of the restricted operator $\mathcal{H}_{_0}$ on the defect via restriction of the formal adjoint $\mathcal{H}_{_0}^{^+}|_{_\mathcal{L}}$ onto some Lagrangian plane $\mathcal{L} \subset M$ where the corresponding boundary form (25) vanishes. Thereafter an extension in the whole space is obtained as the a direct sum $\mathcal{H}_{_0} + \mathcal{H}_{_0}^{^+}|_{_\mathcal{L}}$. We develop here the operator-extension construction with a pair of deficiency subspaces assuming that dim $E_{_J} \geq 2$. In case dim $E_{_J} = 1$ the resulting formulae describing the extended operator also remain true and may be verified via direct calculation.

Consider a new basis in the defect $M = M_i + M_{-i}$ which is similar to the above basis (17) combined of the Greens-functions:

$$w_+ = \frac{e + e'}{2} = \frac{\mathcal{H}}{\mathcal{H} - iI}e,\ w_- = \frac{e - e'}{2i} = \frac{-I}{\mathcal{H} - iI}e. \tag{23}$$

Due to above definition of the formal adjoint operator we have:

$$\mathcal{H}_0^+ w_+ = w_-,\ \mathcal{H}_0^+ w_- = -w_+. \tag{24}$$

We use the new basis $\{w_+,\, w_-\}$ to represent elements from the domain of the formal adjoint operator via new dimensionless symplectic variables $\xi_\pm$ which play a role similar to the above pair $\eta_\pm$:

$$\tilde{x} = \xi_+^x w_+ + \xi_-^x w_-.$$

Lemma II.2 *The boundary form of the (formal) adjoint operator in terms of the variables* $\xi_\pm$ *is given by an Hermitian symplectic form :*

$$\mathcal{K}(\tilde{x}, \tilde{y}) = \left[\mathcal{H}_0^+\tilde{x},\, \tilde{y}\right] - \left[\tilde{x},\, \mathcal{H}_0^+\tilde{y}\right] = \bar{\xi}_+^x \xi_-^y - \bar{\xi}_-^x \xi_+^y. \tag{25}$$

Proof exactly follows the pattern of the previous lemma (II.1). The vector e is normalized in $E_{_J}$, hence the normalization constant similar to $(4\pi\sqrt{2})^{-1}$ in (20) is equal to 1.

Note that the formula (25) can be interpreted as an "abstract formula of integration by parts " for the operator $\mathcal{H}_{_0}^{^+}$. The symplectic version of the extension procedure allows to obtain the J-self-adjoint extension of the formal adjoint operator via restriction of it onto the Lagrangian plane $\mathcal{L}_{_\gamma}$ in the defect defined by the boundary conditions as $\xi_- = \gamma\xi_+,\ \gamma = \bar{\gamma}$. Then the construction of the J-self-adjoint extension in $E_{_J}$ is accomplished via forming of the direct sum $\mathcal{H}_{_0} + \mathcal{H}_{_0}^{^+}|_{_{\mathcal{L}_\gamma}}$.

Our aim is the construction of the joint extension $\mathcal{A}_\Gamma$ of $l_{_0} \oplus \mathcal{H}_{_0}$. Consider the orthogonal sum of operators $l \oplus \mathcal{H}$ in the Pontryagin space $L_2(R^3) \oplus E_{_J}$ with elements $U = (u,\, \tilde{x})$, $u \in$

$L_2(R^3)$, $\tilde{x} \in E_J$. Restricting both operators as above we obtain the symmetric operator $l_0 \oplus \mathcal{H}_0$ with the defect $N \oplus M$ and the deficiency index $(2,2)$. We define the adjoint operator as an orthogonal sum of "adjoint" operators $l_0^+ \oplus \mathcal{H}_0^+$ and calculate the boundary form of it as a sum of boundary forms of the outer and inner components:

$$\mathbf{J}(U,V) = \mathcal{J}(u,v) + \mathcal{K}(\tilde{x},\tilde{y}) = (\bar{B}^u A^v - \bar{A}^u B^v) + (\bar{\xi}_+^x \xi_-^y - \bar{\xi}_-^x \xi_+^y),$$

where we choose the dimensionless symplectic variables A, B defined by (15) for the "outer" boundary form. The boundary form $\mathbf{J}(U,V)$ of the orthogonal sum of operators $l_0^+ \oplus \mathcal{H}_0^+$ in the Pontryagin space $L_2(R^3) \oplus_J E_J$ is a symplectic Hermitian form. Hermitian extensions of the operator $\mathcal{A}_0 = l_0 \oplus \mathcal{H}_0$ (or Hermitian restrictions of the adjoint operator $\mathcal{A}_0 = l_0^+ \oplus \mathcal{H}_0^+$ is constructed via imposing proper boundary conditions on symplectic variables with an hermitian matrix $\mathbf{\Gamma}$:

$$\begin{pmatrix} B \\ -\xi_- \end{pmatrix} = \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix} \begin{pmatrix} A \\ \xi_+ \end{pmatrix}. \tag{26}$$

The joint boundary form $\mathbf{J}(U,V)$ vanishes on the corresponding Lagrangian plane $\mathcal{L}_\Gamma \subset M+N$ defined by the above boundary condition. All Lagrangian planes of the joint boundary form may be constructed either with a help of various Hermitian matrices Γ, or obtained from already constructed planes by proper J-unitary transformation. As previously, the extended operator $\mathcal{A}_\Gamma$ is obtained as the direct sum of the constructed restriction of the adjoint $\mathcal{A}_\gamma^+ = l_0^+ \oplus \mathcal{H}_0^+ \mid_{\mathcal{L}_\Gamma}$ onto the Lagrangian plane $\mathcal{L}_\Gamma$ in the defect and the closure of the restricted operator , $\mathcal{A}_\Gamma = (l_0 \oplus \mathcal{H}_0) + \mathcal{A}_\gamma^+$.

III. THE SPECTRAL STRUCTURE OF $\mathcal{A}_\Gamma$ AND THE POSITIVE SUBSPACE

One can show, combining [13, 22], that the constructed operator $\mathcal{A}_\Gamma$ is a J-self-adjoint operator in the Pontryagin space $L_2(R^3) \oplus E_J$. According to general description of the spectral structure of self-adjoint operators in Pontryagin space, see [21], the operator $\mathcal{A}_\Gamma$ has a *positive* invariant subspace $H \subset L_2(R^3) \oplus E_J$ of a finite co-dimension in $L_2(R^3) \oplus E_J$, where the part of the operator is a conventional self-adjoint operator with respect to the positive metric $P_H[I \oplus J]P_H$. This self-adjoint operator may play a role of a conventional quantum mechanical Hamiltonian of the constructed model. Normally, description of the positive subspace is complicated task since the indefinite metric can be degenerated. In our case we can provide an alternative description of the subspace by showing that it coincides with the subspace of the absolutely-continuous spectrum of $\mathcal{A}_\Gamma$.

Assume that the extension procedure is developed as described in the previous section based on a generating vector $e = e_i$ with use of the boundary conditions (26). The spectrum of the constructed operator $\mathcal{A}_\Gamma$ is defined by the classical Krein formula, see [9]. In our case it coincides, see for instance [13], with singularities (the cut and the poles) of the scattering matrix (33) and consists of a finite number of eigenvalues and Lebesgue absolutely continuous spectrum filling the positive semi-axis $0 < \lambda < \infty$ with infinite multiplicity. The generalized eigenfunctions of the absolutely continuous spectrum of $\mathcal{A}_\Gamma$ are obtained via fitting to the above boundary conditions (26) the Ansatz for scattered waves :

$$\Psi_\lambda = \begin{pmatrix} \psi_\lambda(x,\nu) \\ \tilde{\psi}_\lambda \end{pmatrix} = \begin{pmatrix} e^{i\sqrt{\lambda}(\nu,x)} + k_0\, f(\sqrt{\lambda}) \frac{e^{i\sqrt{\lambda}|x|}}{|x|} \\ \xi_+ \frac{\mathcal{H}+iI}{\mathcal{H}-\lambda I} e \end{pmatrix}, \tag{27}$$

with components $\psi_\lambda, \tilde{\psi}_{\bar{\lambda}}$ in the outer and inner spaces. The symplectic variables $\xi_\pm$ of the inner component of the solution of the adjoint homogeneous equation are connected $\xi_- = -Q(\lambda)\xi_+$ via the Krein Q-function of the inner Hamiltonian - an abstract analog of the Weyl-Titchmarsh m-function, see [7]:

Lemma III.1 *The Q-function for the operator $\mathcal{H}_0^+$ has the form:*

$$Q(\lambda) = \left[e, \frac{I + \lambda\mathcal{H}}{\mathcal{H} - \lambda I} e\right].$$

Proof of the statement, similarly to the proof of the corresponding statement in case of positive metric in [10], is reduced to the calculation of symplectic coordinates of the deficiency element $e_{\bar{\lambda}}$. Consider the J-orthogonal projections $P_e = e][e$ and $I - P_e$ onto the one-dimensional deficiency subspace N_i and onto the complementary subspace $E \ominus_J N_i$, respectively. Then the solution of the adjoint homogeneous equation $e_{\bar{\lambda}}$ can be be presented as:

$$e_{\bar{\lambda}} = \frac{\mathcal{H} + iI}{\mathcal{H} - \lambda I} e = \frac{\mathcal{H}}{\mathcal{H} - iI} e + \frac{I}{\mathcal{H} - iI} e \left[e, \frac{I + \lambda\mathcal{H}}{\mathcal{H} - \lambda I}\, e\right] + \frac{I}{\mathcal{H} - iI}(I - P_e)\frac{I + \lambda\mathcal{H}}{\mathcal{H} - \lambda I} e. \quad (28)$$

The last term in the right-hand side is an element u_0 from the domain of the restricted operator $\mathcal{H}_0$, but the first two terms belong to the defect M so the whole linear combination with use of the notation $Q(\lambda) \equiv [e, \frac{I+\lambda\mathcal{H}}{\mathcal{H}-\lambda I} e]$ can be presented as

$$\frac{\mathcal{H} + iI}{\mathcal{H} - \lambda I} e = w_+ - Q(\lambda) w_- + u_0.$$

Hence the symplectic coordinates $\xi_\pm$ of $e_{\bar{\lambda}}$ are $\xi_+ = 1$, $\xi_- = -Q(\lambda)$. End of the proof.

We consider an indefinite metric tensor J defined by a diagonal matrix, for instance $\{J_{ss}\} = \pm 1$, $s = 1, 2, \ldots N$, commuting with a positive diagonal matrix $\mathcal{H} = \{\lambda_s\}$, $s = 1, 2, \ldots \dim E$. Recall that J-normalized deficiency vector e ($[e, e] = 1$) has non-zero components e^s with respect to the standard basis in E. Then the Q-function has the form

$$Q(\lambda) = \sum_{s=1}^{N} \frac{1 + \lambda\lambda_s}{\lambda_s - \lambda} J_{ss} |e^s|^2 = \sum_{s=1}^{N} \frac{1 + \lambda\lambda_s}{\lambda_s - \lambda} P_s, \quad (29)$$

with *real* weights $P_s = J_{ss}|e^s|^2$ and poles of first order at the eigenvalues λ_s of $\mathcal{H}$. In case of positive metric $J_{ss} > 0$ the Q- function belongs to Nevanlinna class.

The previous lemma allows us to solve the adjoint non-homogeneous equation and obtain the Krein formula [9] for the resolvent of the J-self-adjoint extension of the symmetric operator $l_0 \oplus \mathcal{H}_0$ with the boundary condition (26). The scattered waves in s-channel may be derived from it in a rather standard way, see [13]. One may also show, see for instance [3, 14], that the boundary conditions (26) formulated for elements of the domain of the extension $\mathcal{A}_\Gamma$ are fulfilled for the corresponding scattered waves. We focus now on the straightforward derivation of the expressions for the scattered waves $\{\Psi\}$ based on the above Ansatz (27). The symplectic variables in the ansatz for components of the scattered wave in the outer and the inner spaces are:

$$B = \left(1 + i\sqrt{\lambda} k_0 f\right), \; A = 4\pi k_0 f, \; \xi_- = -Q\,\xi_+. \quad (30)$$

This gives, due to (26), the following equation for the amplitudes f, ξ_+ :

$$\begin{pmatrix} 1 + i\sqrt{\lambda} k_0 f \\ Q\xi_+ \end{pmatrix} = \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix} \begin{pmatrix} 4\pi k_0 f \\ \xi_+ \end{pmatrix}. \quad (31)$$

Solving this equation we obtain the expression for the amplitude f:

Theorem III.1 *The amplitude f, for real positive $\lambda = k^2\ k_0^{-2}$, is equal to*

$$f(k) = \left(4\pi k_0 \gamma_{00} - \frac{4\pi k_0 |\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)} - ik\right)^{-1}. \tag{32}$$

Proof follows from the previous discussion if the dimension of the inner space is ≥ 2,and $\xi_+ = \gamma_{01}\frac{4\pi k_0 f}{Q-\gamma_{11}}$. In case of the one-dimensional inner space the formulae (32) are verified by the direct calculation.

Corollary The scattering matrix in s-channel connected to the amplitude $f(\sqrt{\lambda})$ via the formula, see [14]:

$$S(\sqrt{\lambda}) = 1 + 2ik\, f(\sqrt{\lambda}) = \frac{\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11}-Q} + i\frac{\sqrt{\lambda}}{4\pi}}{\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11}-Q} - i\frac{\sqrt{\lambda}}{4\pi}}, \tag{33}$$

with the branch of $\sqrt{\lambda}$ defined by the condition $\sqrt{\lambda} > 0$ for $\lambda > 0$. The dimensionless eigenvalues λ_s of $\mathcal{A}_\gamma$ are found as poles of the scattering matrix on the spectral sheet $\Im\sqrt{\lambda} > 0$:

$$\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)} - i\frac{\sqrt{\lambda}}{4\pi} = 0.$$

When considering the non-stationary scattering problem, see for instance [24], it is convenient to use the complex conjugate scattering matrix, instead of (33). Non-pure-imaginary zeroes of our scattering matrix $S(\sqrt{\lambda})$ sit in the upper half-plane $\sqrt{\lambda} > 0$, and the corresponding complex-conjugate poles - in the lower half-plane. Vice versa, the non-pure-imaginary zeroes of the complex conjugate matrix sit in the lower half-plane $\sqrt{\lambda} < 0$, and the corresponding poles sit in complex-conjugate points in the upper half-plane.

In the next section we consider zero-range models with scattering matrix approaching 1 at infinity. For these models the scattering matrix is presented as a ratio of two finite Blaschke products $S(k) = S_+(k)\ S_-(k)$ with zeroes in upper and lower half-planes correspondingly: $S_+(k) = \prod_l \frac{k-k_l}{k-\bar{k}_l}$. $\Im k_l > 0$, $S_-(k) = \prod_n \frac{k-\kappa_n}{k-\bar{\kappa}_n}$. $\Im\kappa_n < 0$. We proceed assuming that the Scattering matrix has this form.

In the remaining part of this section we describe the minimal positive subspace of the operator $\mathcal{A}_\Gamma$. In case of s-scattering, only eigenfunctions with spherically - symmetric outer part differ from the non-perturbed exponentials. The whole absolutely-continuous subspace is split orthogonally into the trivial part constituted by eigenfunctions with trivial inner components and the complementary part H_a characterized by spherically-symmetric outer components of the corresponding (generalized) eigenfunctions. Consider the restriction of $\mathcal{A}_\Gamma$ onto subspace H_a. The scattered waves from H_a are obtained via fitting an appropriate ansatz to (26). This gives:

$$\Psi_k(r) = \begin{pmatrix} \frac{1}{4\pi r}\left[e^{-ikr} - S(k)\, e^{ikr}\right] \\ \xi(k)\, \frac{\mathcal{H}+iI}{\mathcal{H}-\lambda I} e \end{pmatrix}, \tag{34}$$

$S(k)$ coincides with (33) and $\xi(k) = \gamma_{01}\, \frac{2i\sqrt{\lambda}}{Q-\gamma_{11}}\, f(k)$. Further construction of the positive subspace is developed in H_a.

Theorem III.2 *The sperically - symmetric subspace H_a of the absolutely- continuous spectrum of $\mathcal{A}_\Gamma$ is positive, and the restriction of the operator $\mathcal{A}_\Gamma$ onto that subspace is a conventional self-adjoint operator which is equivalent to the multiplication by k^2 in $L_2(R)$.*

Proof We suggest an indirect proof of this statement based on analysis of the energy norm associated with the wave equation $u_{tt} + \mathcal{A}_\Gamma u = 0$. Introduce the corresponding energy norm on a dense linear set of all Cauchy data $\mathbf{U} = \{u,\, u_t\}$ with both components from H_a:

$$\mathbf{U}(t) = \begin{pmatrix} u \\ u_t \end{pmatrix} = \int_{-\infty}^{\infty} \begin{pmatrix} \frac{1}{ik}\Psi \\ \Psi \end{pmatrix} e^{ikt} h(k)dk, \;\; -\infty < t < \infty.$$

We can choose $h = h_- \in S_+^{-1} m_- H_-^2$, with a rapidly decreasing at infinity outer function $m_-(k)$ in lower half-plane. Then both components of $\mathbf{U}(t)$ belong to the domain of $\mathcal{A}_\Gamma$. The closure of $\bigvee_{t>0} e^{ikt} S_+^{-1} m_- H_-^2$ in L_2-norm coincides with $L_2(R)$ and gives the spectral representation of the absolutely-continuous subspace. As usual, see [24], the evolution of the above wave equation on the space of Cauchy data with the energy-norm $||\mathbf{U}||^2 = \frac{1}{2}\{[Au, u] + [u_t, u_t]\}$ is energy-preserving. It can be presented as $\exp(i\mathcal{L})$ with the corresponding matrix generator:

$$\mathcal{L} = i \begin{pmatrix} 0 & -1 \\ \mathcal{A}_\Gamma & 0 \end{pmatrix}.$$

The absolutely continuous spectrum of the operator $\mathcal{L}$ is simple on the whole real axis k, and corresponding generalized eigenfunctions are two-component vectors $\left(\frac{1}{ik}\Psi, \Psi\right)$. To prove that the energy norm is positive, it suffice, due to the energy conservation, to verify the positivity on elements $\mathbf{U}(t)$ for $t \to -\infty$. Note that, due to the special choice of h_-,the parts of each component u, u_t in the inner subspace vanish for large negative t, so that the total energy norm coincides with one of the component of $\mathbf{U}$ in the outer space and is equal to the standard energy norm for the wave equation. Hence it is positive for large negative t. Then due to the energy conservation, the energy norm is positive on elements $\mathbf{U}(t)$ for each t. Hence the generator $\mathcal{L}$ of the evolution is a self-adjoint operator in the space of Cauchy data $\mathbf{U}$ with the positive energy. The square of it is defined on smooth elements by the diagonal matrix in the space of energy-normed Cauchy data

$$\mathcal{L}^2 = \begin{pmatrix} \mathcal{A}_\Gamma & 0 \\ 0 & \mathcal{A}_\Gamma \end{pmatrix}.$$

It can be extended by Friedrichs procedure onto the maximal domain in the energy normed space. The square norm of the first component of the corresponding decomposition of the Cauchy data coincides with the quadratic form of $\mathcal{A}_\Gamma$. The restriction of the energy norm onto the second component u_t of Cauchy data is still positive and equivalent to the L_2 - norm on kh_-. Closure of the subspace of the second components of Cauchy data gives the subspace of the absolutely-continuous spectrum of $\mathcal{A}_\Gamma$. Hence the corresponding absolutely-continuous subspace is positive, and the part of $\mathcal{A}_\Gamma$ in it is a conventional self-adjoint operator.

IV. FITTING OF PARAMETERS FOR ZERO-RANGE MODEL

In this section we define the metric of the inner space, and evaluate the boundary parameters and the moduli of components of the deficiency vector in the finite-dimensional case $\dim E = N$, assuming that the spectrum of the inner Hamiltonian and the real values of the scattering length and of the effective radius are given.

The scattering matrix $S(k) = \exp 2i\delta(k)$ can be presented as the Caley-transform of the function cot $\delta(k)$:

$$S(k) = \frac{\cot\ \delta(k) + i}{\cot\ \delta(k) - i} = 1 + 2ikf(k). \tag{35}$$

Hence the scattering amplitude $f(k)$ is presented as a function of the scattering phase $\delta(k)$:

$$f(k) = \left[k\cot\ \delta(k) - ik\right]^{-1}. \tag{36}$$

Combining Eqs.(32), (35) and (36) one can find the S-scattering matrix in the form:

$$S(k) = 1 + \frac{2ik}{4\pi k_0[\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)}] - ik} = \frac{F(k) + ik}{F(k) - ik}, \tag{37}$$

with

$$F(k) = 4\pi k_0 \left[\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)}\right] = k\cot\delta(k). \tag{38}$$

Here we used the above notation (29) and the dimensionless energy $\lambda = (k/k_0)^2$

In this section we consider a special class of zero-range models for which the scattering matrix tends to 1 at infinity. This implies

$$\gamma_{11} + \sum_{s=1}^{N} \lambda_s P_s = 0, \tag{39}$$

and the representation of the denominator in (38) as a ratio of two polynomials $\mathcal{P}_m, \mathcal{P}_n$ with real coefficients , $M < N$:

$$D(\lambda) = \gamma_{11} - Q(\lambda) = -\sum_{s=1}^{N} \frac{1 + \lambda_s^2}{\lambda_s - \lambda} P_s = \Lambda_{NM} \frac{\prod_{r=1}^{M} (\lambda - h_r)}{\prod_{s=1}^{N} (\lambda - \lambda_s)} := \Lambda_{NM} \frac{\mathcal{P}_M}{\mathcal{P}_M}. \tag{40}$$

Points $\{h_r\}$ sit in complex plane λ symmetrically with respect to complex conjugation. The real coefficient Λ_{MN} will be found later. Consider the Laurent series

$$D(\lambda) = \Lambda_{NM} \left[\frac{1}{\lambda^{(N-M)}} + \frac{d_{N-M+1}}{\lambda^{(N-M+1)}} + \ldots \frac{d_N}{\lambda^N} + \ldots\right], \tag{41}$$

with $d_l = 0$ for $l = 1, 2, \ldots, N - M - 1,\ d_{N-M} = 1$, and

$$D^{-1}(\lambda) = \frac{1}{\Lambda_{NM}} \frac{\prod_{s=1}^{N} (\lambda - \lambda_s)}{\prod_{r=1}^{M} (\lambda - h_r)} = \frac{1}{\Lambda_{NM}} \left[\lambda^{(N-M)} + \ldots + q_1\lambda + q_0 + \frac{q_{-1}}{\lambda} \ldots\right]. \tag{42}$$

The real coefficients d_l, q_l are uniquely defined by the poles λ_s and zeroes h_s of D. For instance, integrating D^{-1} on a large circle with proper weights z^{-1}, z^{-2} we obtain

$$q_0 = \frac{\mathcal{P}_N(0)}{\mathcal{P}_M(0)} + \sum_{r=1}^{M} \frac{\prod_{s=1}^{N} (h_r - \lambda_s)}{h_r \prod_{t \neq r} (h_r - h_t)},\ q_1 = \frac{d}{d\lambda} \left(\frac{\mathcal{P}_N}{\mathcal{P}_M}\right)(0) + \sum_{r=1}^{M} \frac{\prod_{s=1}^{N} (h_r - \lambda_s)}{h_r^2 \prod_{t \neq r} (h_r - h_t)}, \tag{43}$$

Comparing coefficients in front of powers of λ in (40, 41) we obtain a linear system for the variables $p_s = \left(1+\lambda_s^2\right) P_s$:

$$\sum_1^N \left(1+\lambda_s^2\right) \lambda_s^{l-1} P_s = \Lambda_{NM} d_l,\ l = 0, 1, \ldots, N-1. \tag{44}$$

. This system has Vandermond determinant, which is positive if the eigenvalues are enumerated in increasing order $W(\lambda_1, \lambda_2, \ldots \lambda_N) = \prod_{s>t} |\lambda_s - \lambda_t| := W$. The solution $\{p_s\}$ is unique and is presented by the Cramer formula. Denote by W^s_{NM} the determinant with the column $\left(1, \lambda_s, \lambda_s^2, \lambda_s^3, \ldots \lambda_s^{N-1}\right)$ replaced by the column $\left(1, d_1, \ldots d_{N-1}\right)$. Then the parameter Λ_{NM} is defined from the normalization condition $[e,e] = \sum_{s=1}^N P_s = 1$, see lemma II.2 :

$$\Lambda_{NM} = \left(\sum_{s=1}^N (1+\lambda_s^2)^{-1} W^{-1} W^s_{NM} \right)^{-1} \tag{45}$$

The role of fitting parameters will play :

1.The coefficients $\frac{r_0}{2}$, $\frac{1}{a}$ in front of the powers k^2, k^0 in the Laurent expansion

$$F(k) = 4\pi k_0 \left\{ \gamma_0 - \frac{|\gamma_{01}|^2}{D(\lambda)} \right\} = 4\pi k_0 \left\{ \gamma_0 - \frac{|\gamma_{01}|^2}{\Lambda_{NM}} \left[\lambda^{(N-M)} + \ldots + q_1 \lambda + q_0 + \frac{q_{-1}}{\lambda} \ldots \right] . \right\} \tag{46}$$

of the function $F(k)$ at infinity, with $\lambda = k^2 k_0^{-2}$.

2. The eigenvalues $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_N$ of the inner Hamiltonian.

3. The zeroes $h_1, h_2, \ldots h_M$ of $D(\lambda)$

We assume that the above data are *consistent* in the following sense:

$$\text{sign } r_0 = -\text{sign } \Lambda_{NM}\ q_1 \tag{47}$$

Lemma IV.1 *If the consistency condition (47) is fulfilled, then the boundary parameters γ_{st} and weights $P_s = J_{ss}|e^s|^2$ of the model $\mathcal{A}_\Gamma$ are defined in several steps:*

1. Calculate the weights $P_s = J_{ss}|e^s|^2 = \Lambda_{NM} W^s W^{-1} (1+\lambda_s^2)^{-1}$ via solving the system (44).

2. Define the boundary parameters γ_{00}, $|\gamma_{01}|^2$, for given r_0, a from the equations:

$$\frac{r_0}{2} = -4\pi k_0^{-1} |\gamma_{01}|^2 \Lambda_{NM}^{-1} q_1,\ \ -\frac{1}{a} = 4\pi k_0 \left[\gamma_{00} - |\gamma_{01}|^2 \Lambda_{NM}^{-1} q_0 \right]. \tag{48}$$

3. Set $\gamma_{11} = -\Lambda_{NM} \sum_s \lambda_s W^s_{NM} W^{-1} (1+\lambda_s^2)^{-1}$.

Note that the poles h_r of F are the points where the scattering matrix is equal to 1. In real physical problems in 3-d space this is never observed for finite real energy. In [17] we suggested extending of this observation to complex values of energy in form of *the principle of analyticity* , assuming that F is analytic on the whole plane λ. In finite-dimensional case this means that F is just a polynomial degree N, $M = 0$. The corresponding operator $\mathcal{A}_\Gamma$ is

called "the special zero-range model". Arranging the eigenvalues of the inner Hamiltonian in increasing order and denoting by W_s the Vandermond determinant with λ_s just omitted, we obtain via direct calculation the following consistency condition:

$$\text{sign } r_0 = \text{sign } \Lambda, \text{ where } \Lambda = (-1)^n \Lambda_{N0} = \sum_{s=1}^{N} \frac{W_s}{W} (-1)^s \frac{1}{1+\lambda_s^2}, \tag{49}$$

and fit the model based on equations:

$$\Lambda = \sum_{s=1}^{n} \frac{W_s}{W} (-1)^s \frac{1}{1+\lambda_s^2}, \;\; P_s = \frac{W_s}{W} (-1)^s \frac{\Lambda}{1+\lambda_s^2}, \;\; |\gamma_{01}|^2 = \frac{r_0 \, k_0 \Lambda}{8\pi k_0 \prod_{s=1}^{N} \lambda_s^N \sum_s^N \lambda_s^{-1}},$$

$$\gamma_{11} + \sum_{s=1}^{N} \lambda_s \; P_s, \;\; -\frac{1}{a} = 4\pi k_0 \left[\gamma_{00} - |\gamma_{01}|^2 \sum_{s=1}^{N} \frac{W_s}{W} (-1)^s \frac{1}{1+\lambda_s^2} \prod_{s=1}^{N} \lambda_s^n \sum_s^n \lambda_s^{-1} \right]. \tag{50}$$

Physical meaning of the above formulae becomes more transparent if we return to dimensional wave number k and spectrum $k_s^2 = k_0^2 \lambda_s$. In particular, the consistency condition (49) can be presented as sign r_0 =sign $\sum_{s=1}^{N} \frac{(-1)^s}{\prod_{l \neq s} |k_s^2 - k_l^2|} \frac{k_0^{2N}}{k_0^4 + k_s^4}$. This gives the following statement:

Theorem IV.1 *For scattering systems which can be modelled by special zero-range potentials the only fitting parameters are: the scattering length, the effective radius, and the spectrum of the inner Hamiltonian. The* $\text{sign}\, r_0$ *of the effective radius is necessarily connected to the spectrum of the inner Hamiltonian by the consistency condition. The components of the metric tensor for those models are defined as* $J_{ss} = (-1)^s \text{sign}\, r_0$. *Other essential parameters of the model like: boundary parameters* $\gamma_{00}, \gamma_{11}, |\gamma_{01}|$, *and components* $|e^s|^2$ *of the deficiency vector are defined as functions of the scattering length* a, *effective radius* r_0, *and* $\lambda_s \;\; (s = 1, 2, \ldots, n)$

$$a = \left(-4\pi\gamma_{00} k_0 + \frac{4\pi |\gamma_{01}|^2}{k_0^{2N-1} \Lambda} \prod_{s=1}^{N} k_s^2 \right)^{-1}, \;\; r_0 = \frac{8\pi |\gamma_{01}|^2}{\Lambda \, k_0^{2N-1}} \left(\prod_{t=1}^{N} k_t^2 \right) \sum_{s=1}^{N} k_s^{-2}. \tag{51}$$

In particular, if the effective radius is positive, $r_0 > 0$, *then* $\Lambda > 0$ *and* $J_{ss} = (-1)^s$.

Introducing the notations $\varepsilon = 4\pi\gamma_{00} k_0, \;\; \gamma = 4\pi\gamma_{01}|^2 \; k_0^{-2N+1} \; \Lambda^{-1}$ we obtain for the function $F(k)$ the following representation:

$$F(k) = k \text{cot} \delta(k) = \varepsilon - \gamma \prod_{s=1}^{N} (k_s^2 - k^2). \tag{52}$$

Sign of the effective radius is the same as the sign γ :

$$a = \left(-\varepsilon + \gamma \prod_{s=1}^{N} k_s^2 \right)^{-1}, \;\; r_0 = 2\gamma \sum_{s=1}^{N} \prod_{t(t \neq s)} k_t^2. \tag{53}$$

Corollary 2 Using (37,52) and above parameters ε, γ, we may obtain a convenient representation for the scattering matrix S in terms of the resonance parameters k_s:

$$S(k) = 1 + \frac{2ik}{\varepsilon - ik - \gamma \prod_{s=1}^{N} (k_s^2 - k^2)}. \tag{54}$$

The total scattering cross-section is $\sigma(k) = 4\pi|f(k)|^2$ or can be written in explicit form (see next section) using equations (8) and (52). This expression for the scattering matrix in s-channel describes the resonance scattering of particles with resonances defined by the spectral properties of the Hamiltonian $\mathcal{H}$ of the inner degrees of freedom.

Re-normalization. We already proved that our resonance scattering model depends only the scattering length a, effective radius r_0, the spectrum k_s ($s = 1, 2, \ldots, N$) of the inner Hamiltonian and the typical wave number. Now we introduce instead of the scattering length and effective radius a new parameter a_0 with dimension ($[a_0] = cm$) and a dimensionless parameter α by the formulae:

$$k_0 := -\frac{1}{4\pi\gamma_{00}a_0}, \quad \alpha := -\frac{4\pi|\gamma_{01}|^2}{\Lambda}\left(4\pi\gamma_{00}\right)^{2N-1}, \tag{55}$$

Then the function $F(k) = k\cot\delta(k)$ for our zero-range potential is:

$$F(k) = -\frac{1}{a_0} - \frac{\alpha}{a_0}\prod_{s=1}^{N}\left(a_0^2k_s^2 - a_0^2k^2\right). \tag{56}$$

The effective radius and the scattering length can be found from (56) as first coefficients of the polynomial $F(k) = -a^{-1} + \frac{r_0}{2}k^2$:

$$\frac{r_0}{2} = \alpha a_0\sum_{n=1}^{N}\prod_{s(s\neq n)}^{N}a_0^2k_s^2, \;\; a = \frac{a_0}{1+\alpha\prod_{s=1}^{N}a_0^2k_s^2}. \tag{57}$$

Then the parameter a_0 can be interpreted as a *non- re-normalized scattering length* for the zero-range potential without inner structure, which corresponds to $\alpha = 0$. The above equation connects the re-normalized scattering length a to the *non-re-normalized scattering length* a_0, taking into account resonance scattering.

Thus the function $F(k)$ given by Eq. (56) depends on the typical wave-number and $N+2$ parameters: a_0, α and k_s, $s = 1, 2, \ldots N$. Consequently, the non-re-normalized scattering length a_0, the dimensionless parameter α and the spectrum k_s ($s = 1, 2, \ldots, N$) of the Hamiltonian define the scattering length a and effective radius r_0 of the model.

Resonance cross-section. The "total" cross-section for spherically-symmetric scattering is generally calculated as

$$\sigma(k) = 4\pi|f(k)|^2 = \frac{4\pi}{|F(k) - ik|^2}.$$

This gives due to (56) an explicit formula for total cross-section of the special zero-range model:

$$\sigma(k) = \frac{4\pi a_0^2}{1 + a_0^2k^2 + 2\alpha\prod_{s=1}^{N}\left(a_0^2k_s^2 - a_0^2k^2\right) + \alpha^2\prod_{s=1}^{N}\left(a_0^2k_s^2 - a_0^2k^2\right)^2}. \tag{58}$$

One can see from (58,56) that the maxima of the total cross-section $\sigma(k)$ are shifted from the eigenvalues of the inner Hamiltonian. They can be interpreted again as *re-normalized* eigenvalues of the inner Hamiltonian. The re-normalization is caused by the interaction introduced via the boundary condition (26). Note that the final formula obtained via substitution of (56) into (58) is not a phenomenological formula, but an exact formula derived for certain Hamiltonian. The corresponding solvable model has equal rights with other quantum solvable models, but unlike them it may have resonances at positive energy, the sign $\pm$ of the effective radius defined by the spectrum of the inner Hamiltonian via (49) and

allows complete fitting of all parameters from the experimental data, thus prescribing them a certain physical meaning.

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