On the stability of swirling flows in a finite pipe

S. Wang*

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Abstract

We study the stability mechanism of the swirling flow in a finite pipe. We first revisited the Rayleigh's linear stability theory, and build up the nonlinear theory in the framework of Hamiltonian system. We then consider the Lamb-Oseen vortex in a finite pipe with fixed flowrate condition at the boundaries. By using recently developed perturbation method of the linear operators, we analyzed the global stability equation and found the disturbance flow fields. We then conducted a study of the kinetic energy transfer mechanism between the disturbance and the base flow by using the Reynolds-Orr equation. We found that the energy transfer takes place actively at the boundaries as well as inside the flow. This is contrast to the solid body rotation flow. We further investigated Lamb-Oseen vortex in a slightly divergent pipe and showed that the internal flow has a leading role in the energy transfer mechanism. This study clarifies the relation of the Rayleigh stability and the global stability found by Wang and Rusak, and provide a basic understanding of the stability mechanism of swirling flows in a finite pipe.

*University of Auckland, Department of mathematics, 38 Princes Street, Auckland, New Zealand

Contents

1 Introduction

The study of the stability of axisymmetric swirling flows in a pipe has a long history. There are excellent review papers on this topic, see for example Leibovich [8]. The development of the concept and method of global stability is relatively new, see for example Huerre and Rossi [6], for a review on this subject. In the following we will briefly state some of the previous results that are directly related to this article. Rayleigh [10] established a fundamental criterion for swirling flows in an infinitely long straight pipe in 1916. His criterion states that a columnar swirling flow with a swirl velocity component $V(r)$ and uniform axial velocity components is stable to infinitesimal axially symmetric disturbances only if the square of the circulation function, $K = rV$, decreases nowhere as r increases from the center of the pipe to the pipe wall, i.e.:

$$\Gamma \equiv \frac{1}{r^3}\frac{d}{dr}(K^2) > 0. \tag{1}$$

The Rayleigh criterion was later strengthened by Synge to be also sufficient to linear stability.

Howard and Gupta[5] considered a similar flow but with non-uniform axial flow $W(r)$, and showed that if

$$J \equiv \frac{\Gamma}{(dW/dr)^2} > \frac{1}{4}, \tag{2}$$

then the flow is linearly stable. There is a counter example showing that (2) is not necessary for linear stability. In finite pipe, one shall assume the periodic boundary conditions imposed at the inlet and outlet. The Rayleigh and Howard and Gupta criteria are all valid for such flows.

The nonlinear stability of swirling flow was studied by Szeri and Holmes by using the Arnold's energy-Casimir method. However, the derived quadratic form suffers the lack of definiteness, a necessity for obtaining the nonlinear stability. This is essentially due to the vortex stretch mechanism. A high wavenumber cut-off method was thereby proposed to overcome this difficulty, and many interesting stability results are obtained under the assumption that the disturbance is limited to the wave frequency below certain cut-off value. One may justify the assumption such that the high frequency disturbance may eventually be damped by the viscous effect.

In the application of these criteria to real flows in a finite pipe, caution must be taken as the periodic boundary conditions can be severely violated. In a long pipe, where the boundary condition's influence is relatively weak, and the evolution of the disturbance sufficiently away from the pipe ends may well be predicted by these criteria. However, there are important cases, for example, the vortex breakdown phenomenon, wherever in a test rig or in a open flow field, one finds significant upstream and down stream influences. It is questionable whether or not the Rayleigh criterion can be applied in these situations. This type of problem was first studied by Wang and Rusak [16] [17]. They considered certain boundary conditions imposed at the pipe inlet and outlet, and studied the stability of the swirling flow under such boundary conditions. It was found that the boundary conditions imposed dramatically alter the stability nature of the swirling flow. In particular, an instability related to the swirl strength was found that can not be explained by the Rayleigh's stability theory. This type of stability has been aptly interpreted by Gallaire and Chomaz [3] as being *global* in nature.

The global instability is found to be in good correlation with the experimental observations. Numerical computations can accurately reproduce the initial evolution of the bubble type vortex breakdown at the swirl predicted by the global instability onset. However, The physical mechanism of this new stability has remained largely unexplained. The original analysis of Wang and Rusak was mainly relied on a bifurcation argument, which is mathematically sound, but lack of a clear physical insight.

Recently Gallaire and Chomaz [3] considered this problem by revisiting the case of the solid body rotation flow in a finite pipe. They revealed that the unstable mode found in the solid body rotation flow at a sufficiently high swirl is actually originated by the gain of energy at the boundaries of the pipe. The vortex core serves only as a neutral waveguide. This identifies the physical mechanism of the instability in the solid body rotation flow, and raise a question whether or not this is still true for general swirling flows in a finite pipe.

In this article, we concentrate on revealing the physical mechanism of the stability of the swirling flow in a finite pipe and filling the major gaps in the research. More specifically, we consider the following questions:

1. *Is the Rayleigh's criterion a sufficient condition for the nonlinear stability? If it is, what is the physical mechanism?*

2. *What is the mechanism of the global stability? How is it related to the Rayleigh's stability theory?*

3. *What is the distinguishing roles of the boundary and the internal flow in the energy*

transfer of general swirling flows in a finite pipe? Is it a necessary condition for the global instability onset that the disturbance gains energy at the boundaries?

We give definite answers to all these questions in this study in a unified approach. The stability mechanism is mainly examined from a viewpoint of the energy transfer between the disturbance and the base flow. The Hamiltonian system theory has a crucial role in this study.

We first revisited the Rayleigh's stability theory. Thus, we considered swirling flows in a finite pipe with periodic conditions being imposed at the inlet and outlet. Such a flow admits a Lie-Poisson bracket derived from the relabellings symmetry. As shown in Szeri and Holmes, Arnold's energy-Casimir method can be extended to study the nonlinear stability of the swirling flows. *For columnar swirling flows in a finite pipe with uniform axial velocity (the case considered by Rayleigh), a crucial observation is that there is no active energy transfer between the azimuthal and axial velocity components, and thus, a simplified Arnold function can be found which enable us to avoid the high wave cut-off procedure. We are then able to prove the formal stability. For the Lamb-Oseen vortex, we show further the nonlinear stability and derive a global, a prior bound for the disturbance's kinetic energy.*

We then consider the Lamb-Oseen vortex in a finite pipe with fixed flowrate at the outlet. This flow does not hold the Hamiltonian structure. The stability nature is expected to be altered. By using recently developed perturbation method of the linear operators, we were able to find the approximated growth rate function and the corresponding eigenmodes. We then conducted a study of the kinetic energy transfer mechanism between the disturbance and the base flow by using the Reynold-Orr equation. *We found that the energy transfer takes place actively at the boundaries as well as in the internal flow. This is sharply contrast to the solid body rotation flow. We further investigated Lamb-Oseen vortex in a slightly divergent pipe. It has been found that even though overall the boundaries have a damping mechanism, the flow can nevertheless become neutral or even unstable. In this case, the internal flow has been shown to be a main agency to transfer energy from the base flow to the disturbance.*

This study clarifies the relation of the Rayleigh stability and the global stability, and provides the understanding of the basic stability mechanism for swirling flows in a finite pipe with various boundary conditions imposed. The new findings and their physical implications are summarized and discussed in depth at the concluding section.

The article is organized in such a way as to focus on revealing the physics, and necessary mathematical analysis has been postponed to the appendixes. One shall notice that the mathematical methods developed in this article are novel and of importance in future study of vortex dynamics.

2 Mathematical model and stability equation

We consider axisymmetric, incompressible and inviscid flow in a finite length pipe. We use cylindrical coordinates (r, θ, x), and the velocity components (u, v, w) corresponding to the radial, azimuthal, and axial velocity, respectively . In the dimensionless form, the pipe radius is set as a unit and the pipe length as L, rescaled with respect to the pipe radius. By virtue of the axisymmetry, the stream function $\psi(x, r, t)$ can be defined such that $u = -\psi_x/r$, and $w = \psi_r/r$. Let $y = r^2/2$, in terms of this new variable, $w = \psi_y$, $u = -\frac{\psi_x}{\sqrt{2y}}$ and the reduced form of azimuthal vorticity $\chi = -(\psi_{yy} + \psi_{xx}/2y)$ (the azimuthal vorticity $\eta = \frac{\chi}{r}$).

The Euler equation in terms of ψ, χ and the circulation function K, defined as $K = rv$, can be written in a compact form (see for example Szeri and Holmes [15]):

$$\begin{aligned} K_t + \{\psi, K\} &= 0, \\ \chi_t + \{\psi, \chi\} &= \frac{1}{4y^2}(K^2)_x, \end{aligned} \tag{3}$$

where the brackets $\{f, g\}$ is the canonical Poisson bracket or Jacobian defined as:

$$\{f, g\} = f_y g_x - f_x g_y. \tag{4}$$

The first equation in (3) is a transport equation describing simply that K is conserved along the stream line and the second describes the interaction of K and χ or the vortex stretch mechanism.

Let us now consider a steady, columnar swirling base flow with all velocity components depending only on the radius:

$$(U(r), V(r), W(r)) = (0, \omega v_0(r), w_0(r)), \tag{5}$$

where $\omega > 0$ is the swirl parameter, from which, one may find $\psi = \psi_0(y) = \int_0^y w_0(y)dy$, $K = \omega K_0(y)$ with $K_0(y) = \sqrt{y}v_0(y)$. This base flow is a steady state solution of the Euler equation (3). $\psi_0(y)$ satisfies the well-known steady Squire-Long equation, see Squire [13] and Long [9]:

$$\Delta_{SL}\psi = H'(\psi) - \frac{I'(\psi)}{2y}, \tag{6}$$

where Δ_{SL} denotes the operator

$$\Delta_{SL}\psi = \psi_{yy} + \frac{\psi_{xx}}{2y}, \tag{7}$$

and $H = p/\rho + (u^2 + v^2 + w^2)/2$ is the total head function (p is pressure and ρ density), $I = K^2/2$ is the extended circulation, both of which are functions of $\psi_0(y)$ only. We may write $I = \omega^2 I_0$ with $I_0 = K_0^2/2$.

In the study of linear stability disturbances of stream function ψ_1 and circulation K_1 are superimposed to the base flow:

$$\begin{aligned} \psi(x, y, t) &= \psi_0(y) + \epsilon\psi_1(x, y, t)..., \\ K(x, y, t) &= \omega K_0(y) + \epsilon K_1(x, y, t)..., \end{aligned} \tag{8}$$

with $\epsilon << 1$. On substituting these expressions into the Squire-Long equation (3) and neglecting the second order perturbation terms, one obtains the linearized equations of motion of the swirling flow

$$\begin{aligned} K_{1t} + w_0 K_{1x} - \omega K_{0y}\psi_{1x} &= 0, \\ \chi_{1t} + w_0\chi_{1x} - \chi_{0y}\psi_{1x} &= \frac{(\omega K_0 K_1)_x}{2y^2}, \end{aligned} \tag{9}$$

where $\chi_1 = -\Delta_{SL}\psi_1$ is the disturbance of the azimuthal vorticity. This equation governs the evolution of the small disturbance.

With introducing mode analysis:

$$\begin{aligned} \psi_1(x, y, t) &= \phi(x, y)e^{\sigma t}, \\ K_1(x, y, t) &= k(x, y)e^{\sigma t}. \end{aligned} \tag{10}$$

where σ is the growth rate, one may derive from (9):

$$\begin{aligned} &\left(\phi_{yy} + \frac{\phi_{xx}}{2y} - (H''(\psi_0) - \frac{\Omega I_0''(\psi_0)}{2y})\phi\right)_{xx} \\ &+\frac{\sigma\chi_{0y}}{w_0^2}\phi_x + \frac{2\sigma}{w_0}\left(\phi_{yy} + \frac{\phi_{xx}}{2y}\right)_x + \frac{\sigma^2}{w_0^2}\left(\phi_{yy} + \frac{\phi_{xx}}{2y}\right) = 0. \end{aligned} \tag{11}$$

where $\Omega = \omega^2$, a rescaled swirl parameter. For the detailed derivation of (11), see Wang and Rusak [17].

In this article, we focus on a particular vortex flow: the Lamb-Oseen vortex with uniform advection (hither, it will be addressed as Lamb-Oseen vortex.). It contains a vortex core at the center in which the flow is similar to the solid body rotation flow whereas outside this vortex core the flow is close to irrotational flow. The axial velocity of the Lamb-Oseen vortex is uniform and the movement is specified by its axial velocity and its circumferential velocity,

$$\begin{aligned} w_0(r) &= W_0, \\ \omega v_0(r) &= \omega \frac{(1 - e^{-r^2/r_c^2})}{r}, \end{aligned} \tag{12}$$

in which r_c is the vortex core. In the study of the stability, the axial velocity can be always rescaled as an unit. For convenience, let $\beta = 1/r_c^2$. It can be shown that (see detail in S.Wang [18]) the stability equation reads as:

$$\left(\phi_{yy} + \frac{\phi_{xx}}{2y} + \frac{\Omega}{m(y)}\phi\right)_{xx} + 2\sigma\left(\phi_{yy} + \frac{\phi_{xx}}{2y}\right)_x + \sigma^2\left(\phi_{yy} + \frac{\phi_{xx}}{2y}\right) = 0. \tag{13}$$

where,

$$m(y) = \frac{y^2}{\beta(1 - e^{-2\beta y})e^{-2\beta y}}. \tag{14}$$

To study the swirling flow in a finite pipe certain conditions are imposed on the boundaries to reflect the physical setting. We consider mainly the following boundary conditions:

1. We assume that for any time t, $\psi(0, y, t) = \psi_0(y)$ and $K(0, y, t) = \omega K_0(y)$ are given at the inlet.
2. We set $\psi_{xx}(0, y, t)$ to fix the azimuthal vorticity χ along the inlet for all time t, $\chi(0, y, t) = -\psi_{0yy}$.
3. At the outlet the flowrate is assumed unchanged at all time t, $\psi(L, y, t) = 0$.
4. As for a axisymmetric flow, we shall impose $\psi(x, 0, t) = 0$ for all time t and $\psi(x, 1/2, t) = \int_0^{0.5} w_0(y)dy$ to describe the total mass flux across the pipe.

From this, we may derive the corresponding boundary conditions for ϕ:

$$\begin{aligned} \phi(x, 0) &= 0, \ \phi(x, 1/2) = 0, \ \text{for} \ \ 0 \le x \le L, \\ \phi(0, y) &= 0, \ \phi_{xx}(0, y) = 0, \ k(0, y) = 0, \ \text{for} \ \ 0 \le y \le 1/2, \\ \phi(L, y) &= 0, \ \text{for} \ \ 0 \le y \le 1/2. \end{aligned} \tag{15}$$

Further more, $k(0, y) = 0$ can be replaced by:

$$\phi_{yyx}(0, y) + \frac{\phi_{xxx}(0, y)}{2y} - \left(H''(\psi_0) - \frac{I''(\psi_0)}{2y}\right)\phi_x(0, y) = 0. \tag{16}$$

We also consider the periodic boundary conditions imposed at the inlet and outlet.

3 Rayleigh stability theory and the nonlinear theory

3.1 Rayleigh stability theory

We will first reexamine the basic idea of Rayleigh in his deduction of his criterion, and then show how by extending his idea to obtain the nonlinear stability results. We consider a finite straight pipe with periodic boundary conditions specified at the inlet and outlet throughout this section. This is the finite pipe version of the Rayleigh's theory.

In his original approach, Rayleigh observed that by Kelvin's circulation theorem, K is conserved along the stream line, and that the work done by the centrifugal force acting in the radial direction actually can be associated with a potential energy density related to the circulation function as $\frac{\rho K^2}{2r^2}$, which is precisely the kinetic energy of the azimuthal motion $\frac{\rho v^2}{2}$. By further observing that if in the base flow, $K_0(r)$ decreases somewhere, the aforementioned potential energy is not at the minimum state, and the motion in the radial direction will be liberated with the release of the potential energy. This is Rayleigh's physical argument for demonstrating the necessity of stability: Flow can not be stable if $K_0(r)$ decreases somewhere. It is remakable that Rayleigh's insight of the physics truly contains some deep mathematics. Indeed, his using potential energy related to the conservation of circulation could be considered as a pioneer work of Arnold's energy-Casimir method.

It was shown by Synge[14]that the Rayleigh criterion is necessary and sufficient condition for linear stability. However, it is not known yet whether or not the Rayleigh criterion is a sufficient condition for nonlinear stability. We will give a definite answer to this long standing problem in the following.

3.2 The nonlinear theory

Consider swirling flows in a finite pipe with periodic conditions being imposed at the inlet and outlet. Such flows admit a Lie-Poisson bracket derived from the relabellings symmetry. As shown in Szeri and Holmes, Arnold's energy-Casimir method can be extended to study the nonlinear stability of the swirling flow.

The general Arnold function for columnar swirling flow can be written as, see the details in Szeri and Holmes,

$$A(\chi, K) \quad = \quad H(\chi, K) + C(\chi, K) \tag{17}$$

$$= \overbrace{2\pi \int_D \frac{1}{2}\left(\psi\chi + \frac{K^2}{2y}\right) dydx + \pi \int_0^L w(x,0.5)dx}^{H(\chi,K)}$$

$$\underbrace{-\pi \int_0^L w(x,0.5)dx + 2\pi \int_D \chi f(k)dydx + 2\pi \int_D j(K)dydx + 2\pi \int_D cy\chi dydx}_{C(\chi,K)}$$

where, $H(\chi, K)$ is the kinetic energy, and $C(\chi, K)$ is the Casimir function; j and f are arbitrary real-valued function, c an arbitrary constant. The fourth to sixth terms are constants of motion, known as generalized swirl, generalized helicity and generalized impulse, respectively. The term $\pi \int_0^L w(x, 0.5)dx$ is also a constant of motion. The Arnold function is thus an invariant quantity of the flow.

For columnar swirling flows in a finite pipe with uniform axial velocity $W_0 = 1$, i.e $\psi_0(y) = y$, a key observation of this article is that there is no active energy transfer between the azimuthal and axial velocity components, and thus, the general helicity can be outed from the Arnold function:

$$A(\chi, K) = 2\pi \int_D \frac{1}{2}\left(\psi\chi + \frac{K^2}{2y}\right) dydx + 2\pi \int_D j(K)dydx + 2\pi \int_D cy\chi dydx. \tag{18}$$

We assume that the circulation function $K_0(y)$ is monotonic increasing function of y, and the Rayleigh's criterion is thereby satisfied for this flow. We now define the inverse function $K_0(y)$ as Y, namely,

$$y = Y(K_0). \tag{19}$$

One finds the first variation of (24) as

$$\delta A = 2\pi \int_D \left[(\frac{\psi_0}{2} + cy)\delta\chi + \frac{\chi_0 \delta\psi}{2} \right] dydx + 2\pi \int_D \left(\frac{K_0}{2y} + j'(K_0)\right) \delta K dydx. \tag{20}$$

To make it vanishing at the equilibrium, noticing that $\chi_0 \equiv 0$ and thus the term $\frac{\chi_0\delta\psi}{2}$ vanishes, one needs only to choose

$$c = -\frac{1}{2}, \quad j'(K_0) = -\frac{K_0}{2Y(K_0)}. \tag{21}$$

The second variation is

$$\delta^2 A = 2\pi \int_D \left(\delta\psi\delta\chi + \frac{K_0 Y'(K_0)}{2y^2}\delta K \delta K\right) \, dydx. \tag{22}$$

We may write it, in terms of the velocities of the disturbance, as

$$\delta^2 A = 2\pi \int_D \left(u_1^2 + w_1^2 + \frac{K_0 Y'(K_0)}{y} v_1^2\right) \, dydx. \tag{23}$$

Therefore, if $Y'(K_0) \geq 0$, which is exactly the Rayleigh criterion, flow is formally stable. It is a well known fact that the formal stability implies the linear stability. In finite dimensions, formal stability also implies the nonlinear stability. However, our system is of infinite dimensions, and the nonlinear stability must be studied case by case. This is basically because of the non-compactness of the infinite dimensional space.

We now consider the nonlinear stability. By using the remainder of the second order Taylor's expansion, one obtains the following estimate

$$\begin{aligned} A(\chi_0+\delta\chi, K_0+\delta K) - A(\chi_0, K_0) &= 2\pi \int_D \frac{1}{2}\left(u_1^2+w_1^2\right) dydx \\ &\quad +2\pi \int_D \left\{\frac{K}{4}\left(\frac{1}{y}-\frac{1}{Y(K)}\right)\right\}'_{K=K_m} dydx, \end{aligned} \tag{24}$$

where, $K_m(x,y) = \theta(K(x,y,t) - K_0(x,y)) + K_0(x,y)$ with $0 \leq \theta \leq 1$ is an intermediate value between $K(x,y,t)$ and $K_0(x,y)$. For the solid body rotation flow, one finds $Y(K) = \frac{K}{2\omega}$ and

$$\left\{\frac{K}{4}\left(\frac{1}{y}-\frac{1}{Y(K)}\right)\right\}'_{K=K_m} = \frac{1}{4y}. \tag{25}$$

And one obtains

$$\begin{aligned} A(\chi_0+\delta\chi, K_0+\delta K) - A(\chi_0, K_0) &= 2\pi \int_D \frac{1}{2}\left(u_1^2+w_1^2\right) + 2\pi \int_D \frac{1}{4y}\delta K \delta K \\ &= 2\pi \int_D \frac{1}{2}\left(u_1^2+w_1^2+v_1^2\right)\, dydx. \end{aligned} \tag{26}$$

This is exactly the kinetic energy of the disturbance (after multiplying the density). This is a result as anticipated. In fact, consider a reference frame attached to the solid body of rotation. The flow is seen static in this reference frame. There are two non-inertial forces exerted on the flow, the centrifugal and Coriolis forces. The kinetic energy of the disturbance is conserved because neither can perform work to the flow. Actually, the solid body rotation flow does strongly indicate that the Rayleigh criterion is essentially a criterion for the nonlinear stability. It also shows the correctness of our method and the derivation. Applying the same method to other swirling flows will lead to non-trivial nonlinear stability result and obtain the global estimate of the disturbance's energy. We conduct such analysis for our base line flow: the Lamb-Oseen vortex.

For the Lamb-Oseen vortex, one has $K_0(y) = \omega(1-e^{-2\beta y})$ and thus

$$Y'(K_0) = \frac{1}{K_0'(Y)} = \frac{1}{2\omega\beta e^{-2\beta Y}}. \tag{27}$$

Using this, one obtains

$$\left\{\frac{K}{4}\left(\frac{1}{y}-\frac{1}{Y(K)}\right)\right\}' = \frac{1}{4}\left(\frac{1}{y}-\frac{Y-(e^{2\beta Y}-1)/2\beta}{Y^2}\right). \tag{28}$$

We may estimate the Arnold function as

$$\begin{aligned} &A(\chi_0+\delta\chi, K_0+\delta K) - A(\chi_0, K_0) \qquad (29)\\ &= 2\pi\int_D \frac{1}{2}\left(u_1^2+w_1^2\right)dydx + 2\pi\int_D \frac{1}{4}\left(\frac{1}{y}-\frac{Y-(e^{2\beta Y}-1)/2\beta}{Y^2}\right)_{Y=Y(K_m)}\delta K\delta K\ dydx, \end{aligned}$$

or

$$\begin{aligned} &A(\chi_0+\delta\chi, K_0+\delta K) - A(\chi_0, K_0) \qquad (30)\\ &= 2\pi\int_D \frac{1}{2}\left(u_1^2+v_1^2+w_1^2\right)dydx + 2\pi\int_D \frac{1}{4}\left(-\frac{Y-(e^{2\beta Y}-1)/2\beta}{Y^2}\right)_{Y=Y(K_m)}\delta K\delta K\ dydx. \end{aligned}$$

It is easy to verify that the function in the bracket is an increasing function for $Y \in (0, 0.5)$ and

$$0.5 = \lim_{Y\to 0^+} -\frac{Y-(e^{2\beta Y}-1)/2\beta}{Y^2} \le -\frac{Y-(e^{2\beta Y}-1)/2\beta}{Y^2} \le -\frac{0.5-(e^{\beta}-1)/2\beta}{0.5^2} \approx 24.4. \tag{31}$$

One obtains thus

$$\begin{aligned} &2\pi\int_D \frac{1}{2}\left(u_1^2+v_1^2+w_1^2\right)dydx + 2\pi\int_D \frac{24.4y}{4y}\delta K\delta K\ dydx \qquad (32)\\ &\ge A(\chi_0+\delta\chi, K_0+\delta K) - A(\chi_0, K_0). \end{aligned}$$

Considering $y \in (0, 0.5)$, one has

$$2\pi\int_D \frac{1}{2}\left(u_1^2+w_1^2+12.2v_1^2\right)dydx \ge A(\chi_0+\delta\chi, K_0+\delta K) - A(\chi_0, K_0). \tag{33}$$

One has from (30) obviously

$$A(\chi_0+\delta\chi, K_0+\delta K) - A(\chi_0, K_0) \ge 2\pi\int_D \frac{1}{2}\left(u_1^2+w_1^2+v_1^2\right)dydx. \tag{34}$$

One therefore obtains the following global estimate of the disturbance's kinetic energy related to the initial disturbance

$$2\pi\int_D \frac{1}{2}\left(u_1^2+w_1^2+12.2v_1^2\right)|_{t=0}dydx \ge 2\pi\int_D \frac{1}{2}\left(u_1^2+w_1^2+v_1^2\right)dydx. \tag{35}$$

It is interesting to observe that in the solid body rotation case, the kinetic energy of the disturbance is conserved whereas in Lamb-Oseen vortex case, the kinetic energy of the disturbance is bounded by the kinetic energy of the initial disturbance. The estimate of the bound is global and a priori in nature.

The stability theory of inviscid swirling flows in a finite straight pipe with periodic boundary conditions (or equivalent infinite straight pipe) can be considered as completed. The Rayleigh criterion is essentially a necessary and sufficient conditions for the nonlinear stability.

4 Energy transfer mechanism of the swirling flows in a finite pipe

4.1 The global stability as a consequence of the breakup of the Hamiltonian structure

It is clearly shown in the previous section that the stability nature ties strongly to the Hamiltonian, namely, the conservation of the kinetic energy, circulation and impulse. In particular, the conservation of circulation leads to a potential energy-like Casimir function which modulates the energy exchange between the azimuthal component and the other components. However, if the boundary conditions are not assumed periodic, the system is not Hamiltonian, and the conservation of kinetic energy, impulse and circulation does not hold any more. One shall notice that in practical problem the periodic conditions often do not reflect the real physical situation. The question is therefore raised as in which degree the more realistic boundary conditions can change the stability and the dynamics of the flow.

Wang and Rusak introduced the analysis of the linear stability of the swirling flow in a finite pipe with a set of boundary conditions which models the flow inlet and outlet physical conditions. They found that the stability nature is very different from the Rayleigh's stability. Original neutral mode becomes asymptotic stable at low swirl whereas it becomes unstable at sufficient high swirl. Comparing this to the Rayleigh stability theory, it is evident that the boundary conditions used in Wang and Rusak break up the Hamiltonian nature of the system, and new linearly stable and unstable modes thereby emerge into the picture. It is clear that such instability can not be understood in the framework of the Rayleigh stability. In our view, the Rayleigh's stability and the global stability found by Wang and Rusak are naturally complement each other, and the combination of the two gives a rather complete picture of the stability of swirling flow in a finite pipe.

A subtle issue about this new stability is that at low swirling level, the modes are all asymptotic stable, and do not change the base flow state. The instability only occurs at sufficiently high swirl when the standing wave is at the first time sustainable in the flow field. This coincidence of the wave characteristics has shaded the global instability mechanism for decades. In real flows such as pipe swirling flows, leading edge vortex over a Delta wing or many other real situations, such instability often becomes predominant and induces the drastic change of the flow field.

4.2 Reynolds-Orr equation in the finite pipe

We will concentrate on the study of the energy transfer mechanism for swirling flows in a finite pipe. We rely on more on the linear stability analysis. Recent progress Wang [18] based on the perturbation method of linear operators enables us to conduct a semi-analytic analysis of the stability problem for any columnar swirling flow. In particular, we are now able to find an approximated analytic expression of the eigenmode. This is very useful in the study of the energy transfer mechanism. In fact, with such explicit expression, The energy transfer between the disturbance and the base flow can be readily analyzed by the Reynolds-Orr equation.

We introduce the inviscid Reynolds-Orr equations for swirling flow in a finite pipe. The total kinetic energy of the disturbance contained in a finite pipe is (with multiplying density ρ)

$$E(t) = \pi \int_D \left(u_1^2 + v_1^2 + w_1^2 \right) dydx. \tag{36}$$

For an axisymmetric base flow with velocity components (U, V, W), The Reynolds-Orr equation (see for example, Schmid and Henningson[12] and Wu, Ma and Zhou[19]) is written as

$$\frac{dE(t)}{dt} = \int_D (u_1, v_1, w_1)\mathbf{D}(u_1, v_1, w_1)^T dxdy - \int_0^{0.5} [u_1 p_1]_{x=0}^{x=L} dy - \int_0^{0.5} \left[U(u_1^2 + v_1^2 + w_1^2) \right]_{x=0}^{x=L} dy \tag{37}$$

where p_1 is the disturbance of the pressure, and $\mathbf{D}$ is the symmetric strain rate of the base flow,

$$\mathbf{D} = \frac{1}{2} \begin{bmatrix} 2U_r & r(\frac{V}{r})_r & W_r + U_z \\ r(\frac{V}{r})_r & \frac{2U}{r} & V_z \\ W_r + U_z & V_z & 2W_r \end{bmatrix}. \tag{38}$$

For the columnar swirling flow with uniform axial flow $W \equiv 1$,

$$(U(r), V(r), W(r)) = (0, \omega v_0(r), 1), \tag{39}$$

it takes particularly simple form:

$$\mathbf{D} = \frac{1}{2}\begin{bmatrix} 0 & \omega(\frac{dv_0}{dr} - \frac{v_0}{r}) & 0 \\ \omega(\frac{dv_0}{dr} - \frac{v_0}{r}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{40}$$

Inserting this into (37), one obtains

$$\begin{aligned} \frac{dE(t)}{dt} &= -2\pi(\int_D \omega\left(\frac{dv_0}{dr} - \frac{v_0}{r}\right) v_1 u_1 dy dx \\ &- \int_0^{0.5} [w_1 p_1]_{x=0}^{x=L} dy - \frac{1}{2}\int_0^{0.5} \left[u_1^2 + v_1^2 + w_1^2\right]_{x=0}^{x=L} dy). \end{aligned} \tag{41}$$

The first term in RHS represents the contribution to the kinetic energy of the disturbance from the internal flow. The second term in RHS represents the work done to the disturbance due to the pressure disturbance which is exerted on the fluid at the inlet and outlet. The third term in RHS represents the disturbance's kinetic energy flux at the inlet and outlet.

With periodic boundary conditions imposed at inlet and outlet, the boundary terms vanish, and one has

$$\frac{dE(t)}{dt} = -2\pi \int_D \omega\left(\frac{dv_0}{dr} - \frac{v_0}{r}\right) v_1 u_1 dy dx \tag{42}$$

One may easily verify that for the solid body rotation flow the integral at RHS vanishes (the solid body rotation flow is apparently strain free). We thus recover the conservation of the kinetic energy of the disturbance. For the Lamb-Oseen vortex, the same term does not vanishes and the kinetic energy transfer between the base flow and the disturbance takes place inside the flow.

In the next section, we consider the Lamb-Oseen vortex with boundary conditions (15). The fixed flowrate boundary conditions have an advantage such that the second term in (41) becomes inactive. For Lamb-Oseen vortex, one has

$$\frac{dv_0}{dr} - \frac{v_0}{r} = 2\beta e^{-2\beta y} - \frac{(1 - e^{-2\beta y})}{y}. \tag{43}$$

And the Reynolds-Orr equation reads as

$$\frac{dE(t)}{dt} = 2\pi\left(\int_0^{0.5}\int_0^L \omega\left(2\beta e^{-2\beta y} - \frac{(1 - e^{-2\beta y})}{y}\right) u_1 v_1 dx dy + \frac{1}{2}\int_0^{0.5} [u_1^2 + v_1^2]_{x=L}^{x=0} dy\right). \tag{44}$$

This equation clearly distinguishes the contributions to the kinetic energy of the disturbance from the internal flow and the boundaries.

One shall notice that the Reynolds-Orr equation alone does not lead to any sharp estimate of the energy transfer for general swirling flows. However, when the disturbance flow field has been found, a precise estimate of the energy transfer can be obtained by applying the Reynolds-Orr equation.

4.3 The energy transfer mechanism of Lamb-Oseen vortex in a straight pipe

The stability of the solid body rotation flow in a finite pipe has been thoroughly studied by Wang and Rusak [17], Gallaire and Chomaz [3] and Gallaire, Chomaz and Huerre,[4]. It has been shown in Gallaire and Chomaz [3]that the stability of the solid body rotation flow is solely dependent on the net gain of the energy at the inlet and outlet. This is natural from the Reynolds-Orr equation as has been shown that there is no kinetic energy transfer takes place inside the flow. Actually, a special form of Reynolds-Orr equation is derived in Gallaire and Chomaz [3].

The subtle point is that the solid body rotation is an exceptional case, and in general case, the internal flow is active in energy transfer. Flow being neutral under the Rayleigh's sense does not by any means imply its being inactive in energy transfer between the disturbance and base flow. The disturbance may absorb energy from the base flow at one location and gives it back at another location. The discussion of the nonlinear stability of the Lamb-Oseen vortex with the periodic boundary conditions shows solely that such energy transfer does not lead to the net gain of the energy to the disturbance, globally. When boundary conditions other than the periodic condition are imposed, the Hamiltonian breaks up, and thereby an important global constraint on the energy transfer is lifted. The stability nature shall thus be changed. It shall be emphasized that the boundary conditions do not only change the physics at the boundaries but also the entire flow field. This is the main result of this section.

The stability equation of the Lamb-Oseen vortex has been recently studied by Wang [18]by using perturbation theory of linear operators, and its growth rate function has been found approximately. See Appendix A for a brief description of the method and relevant useful terminologies. An extension of the method would allow us to find the velocity field of the disturbance, and enable us to conduct a study of the energy transfer mechanism. In order to focus on the discussion of the physical mechanism we postpone the technical analysis to the Appendix B.

4.3.1 Analysis of the flow field by the perturbation method of the linear operators

The stability equation of columnar flow in a finite pipe has been solved by Wang [18]. For a typical case: $\beta = 4$ and $L = 6$, which will be used as our base flow in this article, the approximated growth rate function has been found, and is shown in Figure 1. We found that

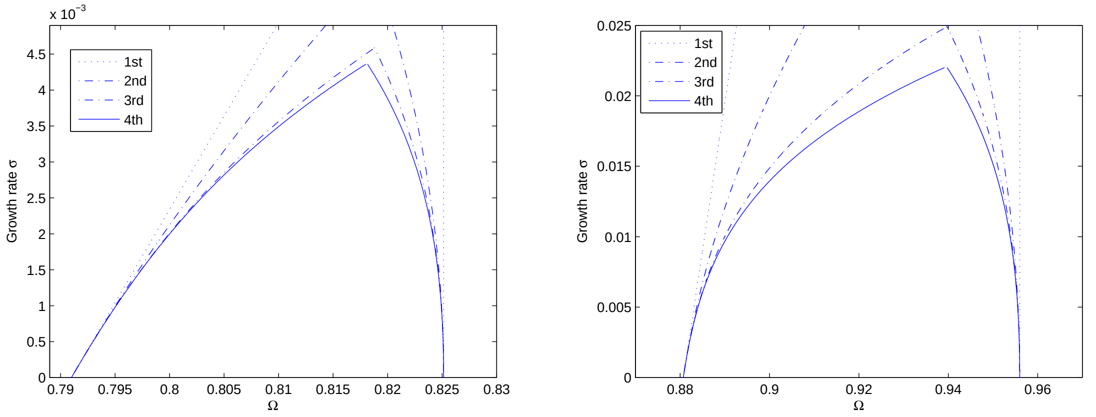

Figure 1: Growth rate σ of the Lamb-Oseen vortex with fixed outlet flow. The first to fourth order approximation. The left plot: First and second branch; The right plot: 3rd and 4th branch

the growth rate function resembles to the solid body rotation flow case. The approximated growth rate curve of the Lamb-Oseen vortex shows that at the swirl $\Omega_{1,1} \approx 0.7910$, flow becomes neutral at the first time, which we define as the first critical swirl. Flow is asymptotic stable when swirl below the critical swirl and becomes unstable when swirl above it. With fourth order approximation, one found the relation between σ and Ω at the neighborhood of the first critical swirl, see [18],

$$\Omega \approx 0.7910 + 3.847\sigma + 1.868 \times 10^2\sigma^2 + 6.474 \times 10^4\sigma^3 + 3.654 \times 10^6\sigma^4. \tag{45}$$

In order to use the basic energy transfer equation (44), the flow field of the disturbance, u_1 and v_1 must be found. This task will be carried out by using the perturbation method. In Appendix B we find the eigenfunction $\phi(x, y; \sigma)$ for sufficient small σ with second order accuracy, see the explicit formula (75) for $\phi(x, y; \sigma)$ and the lengthy derivation there. Figure 2 shows $\phi(x, y; \sigma)$ for $\sigma = 0, 0.0007, 0.0014, 0.0021$ A noticeable feature of the eigenfunctions is that they become more asymmetric as σ increases. In the case where periodic boundary conditions are imposed, the eigenfunctions are all sine function along the axial direction. The

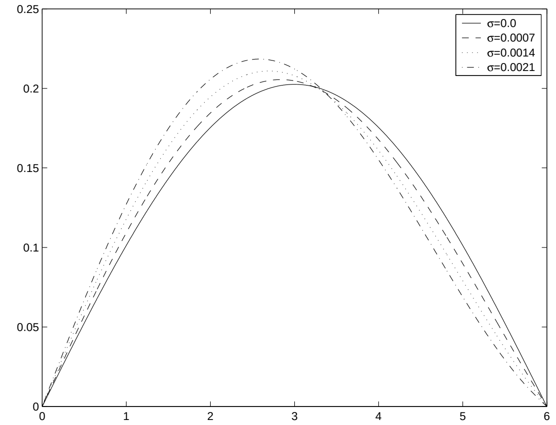

Figure 2: The eigenfunctions $\phi(x, 0.1; \sigma)$ for $\sigma = 0, 0.0007, 0.0014, 0.0021$ and $y = 0.1$.

asymmetry found here reflects the influence of the boundary.

4.3.2 The energy budget for the Lamb-Oseen vortex in a straight finite pipe

After the finding of the stream function of the disturbance, one may readily proceed to find u_1 and v_1 from $\phi(x, y; \sigma)$, (See (78) and (82) in Appendix B for the explicit expressions of u_1 and v_1) and substitute them into the Reynolds-Orr equation. This can be done by a direct numerical computation. However, the mathematical expression can be significantly simplified. And we conduct the analytic work in Appendix C. Based on the analysis we may write the Reynolds-Orr equation of the Lamb-Oseen vortex (44) in the form

$$\frac{dE(t)}{dt} = 2\pi \int_0^{0.5} \left(E_{in}(y;\sigma) + E_b(y;\sigma)\right) dy, \tag{46}$$

where,

$$E_{in}(y;\sigma) = \omega \left(2\beta e^{-2\beta y} - \frac{(1 - e^{-2\beta y})}{y}\right) \int_0^L u_1 v_1 dx, \tag{47}$$

and

$$E_b(y;\sigma) = \frac{1}{2}\left[u_1^2 + v_1^2\right]_{x=L}^{x=0}. \tag{48}$$

$E_b(y;\sigma)$ represents the density of the boundary energy transfer rate at the radial location y, and $E_{in}(y,\sigma)$, the density of the internal energy transfer rate at y which takes into account of the axially accumulated effect. A neat close form (89) has been derived for $E_{in}(y;\sigma)$ in Appendix C. See also (91) for $E_b(y;\sigma)$. Based on these formulas we calculate $E_{in}(y;\sigma)$ and

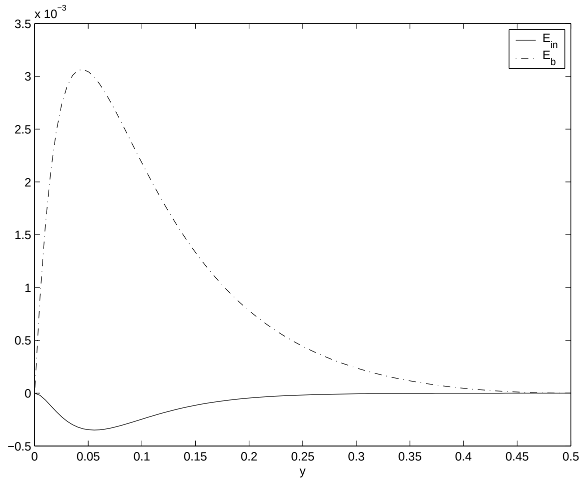

Figure 3: The energy transfer density functions: solid line: $E_{in}(y;\sigma)$ and dashed line: $E_b(y;\sigma)$ with $\sigma = 0.0001$.

$E_b(y;\sigma)$, and plot the results in Figure 3. It is clear from this plot that the energy transfer between the base flow and the disturbance does take place in the internal flow. One shall notice that the Lamb-Oseen vortex is neutrally stable according to the Rayleigh criterion. In the analysis of Rayleigh theory, Fourier modes are introduced in axial direction. This is a valid analysis for swirling flow in a infinitely long pipe, for in the case all disturbance with finite kinetic energy (or mathematically the disturbance is belong to L^2 space) can be decomposed into Fourier modes. One may understand the situation as that in an infinite pipe, all finite energy disturbance appears symmetric from the far field. Applying the Reynolds-Orr equation to this flow, one should find the gain and loss of the disturbance's kinetic energy are perfectly balanced because of the symmetry of the Fourier modes.

In our case, the boundary conditions imposed certainly change the energy transfer mechanism at the boundaries. It is also true that the physical nature of the internal flow is changed too. In fact, we have already found that the eigenmode is not any more sine function along the axial direction, and it becomes more asymmetric when Ω moves further away from the critical swirl. Such an asymmetric mode certainly results from the reaction to the non-symmetric flow conditions at the inlet and outlet. We now demonstrate that the asymmetry directly induces the energy exchange between the disturbance and the base flow. We need thus to examine the integrand in the Reynolds-Orr equation:

$$e_{in}(x,y) = \omega \left(2\beta e^{-2\beta y} - \frac{(1-e^{-2\beta y})}{y} \right) u_1 v_1. \tag{49}$$

We examine the axial variation of this function at a fixed radial position. Figure 4 shows the typical behavior of $e_{in}(x, y)$ by a slice of this function with fixed $y = 0.05$. The computation is based on $\sigma = 0.001$, which gives a swirl level slightly above the critical swirl $\Omega_{1,1}$. The primary flow is slightly unstable. It is found from this plot that $e_{in}(x, 0.05)$ is asymmetric, as expected.

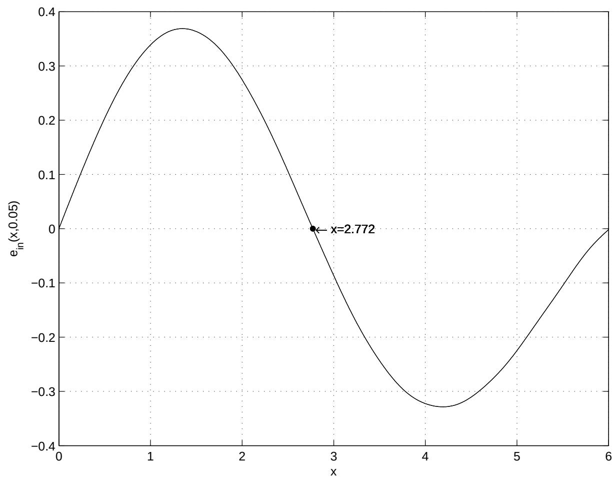

Figure 4: The axial variation of the energy transfer density $e_{in}(x, y_0)$ with $y_0 = 0.05$.

The disturbance absorbs the energy from base flow up to the axial location $x = 2.772$, which is significantly away from the middle point of the pipe $x = 3$. The disturbance loses its energy to the base flow in the rear part of the pipe after $x = 2.772$. From this plot, it can be seen that the area below the x axis is slightly larger than the area above the x axis. Therefore, the axially accumulated disturbance's energy gain is negative.

In the case of the swirling flow in the finite pipe, the first critical swirl, at which the standing wave is sustainable at the first time in the base flow, turns out to be a clear sign for the global instability arising. In fact, as a manifestation of the breakup of the Hamiltonian, the original neutral mode becomes asymptotic stable at low swirl, eventually regains the neutrality at the critical swirl, and becomes unstable when the swirl is above the critical swirl.

For our particular base flow, namely the Lamb-Oseen vortex, it is found that the disturbance actually loses its kinetic energy to the base flow in the internal flow when the flow is in the unstable range whereas it gains the energy otherwise. One shall notice that this is not a general behavior. It is not hard to find examples to show the opposite. The key observation is that the internal flow is generally active in the energy transfer.

One also find that the loss and the gain in this case are seen less than 10% of the total energy exchange. The majority of the energy exchange still occurs at the boundaries. This gives a good explanation why the growth rate curves of Lamb-Oseen vortex resembles the solid body rotation flow.

We also calculate the energy exchange rate based on the second order approximation of the eigenfunction. We found that the contribution from the second order is small, and can be

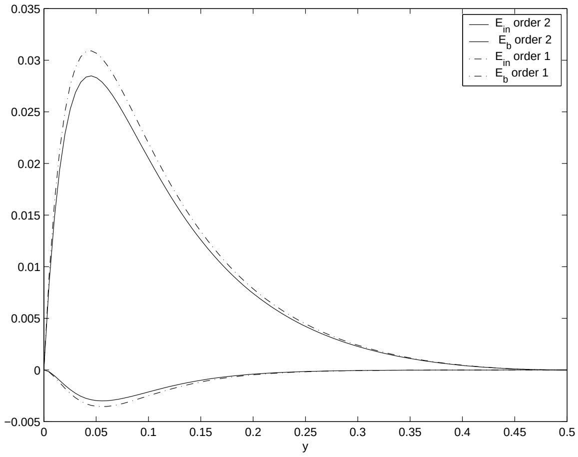

Figure 5: The first and second order of the energy transfer density functions e_{in} and e_b at $\sigma = 0.001$.

ignored. See Figure 5.

4.4 The energy budget of the Lamb-Oseen vortex in a slightly divergent pipe

In the previous paragraph, we found that in the Lamb-Oseen vortex in a straight pipe, the boundaries are still the main agency for energy transfer, and the internal energy transfer does take place but has a minor role in the total energy transfer. It is of great interest, therefore, to find an example such that the internal energy exchange is more significant, and has a leading role in the creation of the flow instability. We found that a divergent pipe can be an appropriate example.

To study the roles of the boundary and the internal flow in the energy transfer mechanism one may concentrate on the neutral state. It has the advantage that the relevant analysis can be greatly simplified, yet the physics can be sufficiently clarified. In the straight pipe case, for example, one finds that the neutral eigenmode, as shown in Figure 2, is a sine-function along the axial direction, and thus the energy gain at the inlet is equal to the loss at the outlet.

Considering this energy balance at the boundaries and the neutrality of the flow state, one may instantly conclude that the net internal energy transfer must be zero.

We will show in the following that the inlet will become less active after introducing a divergent section at the pipe entrance. In another word, the radial velocity at the inlet is less sensitive to the change of the flow in the pipe. Accordingly the energy gain at the inlet is reduced. The basic idea by using a divergent pipe is based on the change of the physical nature of the base flow in such a pipe. In fact, the swirling flow in a long pipe with a divergent section at the pipe entrance reaches the critical state in such a manner that at the divergent section it is locally supercritical and it becomes locally subcritical at the rear part of the pipe. As such, the standing wave consists of two parts of spatial development: an evanescent standing wave at the divergent section and followed by a sinuous bending standing wave. The streamline of the standing wave at the inlet is flatten as its being evanescent, and the amount of energy absorbed at the inlet is thus reduced according to the Reynolds-Orr equation. One finds that the energy gain at the inlet can not supply sufficient energy to offset the loss at the outlet. The total energy gain at the boundaries becomes negative. As the flow considered is in the critical state, the leftover loss must be compensated by the energy gain in the internal flow.

Technically, the critical state in a long divergent pipe can be found by a long wave approach. A similar problem with a different outlet boundary condition, namely, assuming non-radial flow at the outlet, has been studied by Rusak, Judd and Wang [11]. The method used there can be applied to our case almost identically. See Appendix D for the analysis of the Lamb-Oseen vortex in a slightly divergent pipe. The pipe length is chosen as $L = 10$, and the pipe shape is defined by a smooth function, see (92) for the exact definition.

We found the bifurcation diagram of the steady solution with regarding to the swirl change $\Delta\Omega$ for this case, see figure 6. In this figure, the vertical coordinate is the minimum axial velocity of the flow at the centerline of the pipe and the horizontal coordinate refers to the change of the swirl, respectively.

Note that a saddle node bifurcation is formed at the critical state, and the stability is expected to be changed in general. One may conduct a local analysis to confirm it. This can be done by the perturbation method of the linear operators. As this study is concentrated on the basic mechanism, we turn our attention to the critical state and the neutral mode. We found that the neutral mode can be expressed in a form with variables separated, namely, $\varphi_1(y)\Psi(X)$, where $X = \frac{x}{10}$, a rescaled length. $\Psi(X)$ is plotted in Figure 7. It is found that

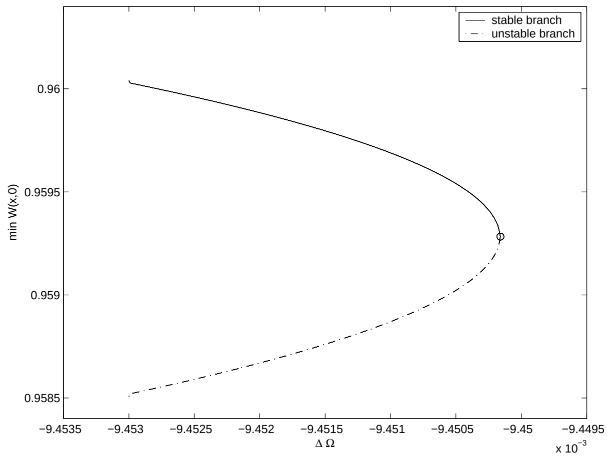

Figure 6: The bifurcation diagram: $\Delta\Omega$ vs. the minimum axial velocity at the centerline of the pipe $\min W(x, 0)$. The circle indicates the bifurcation point.

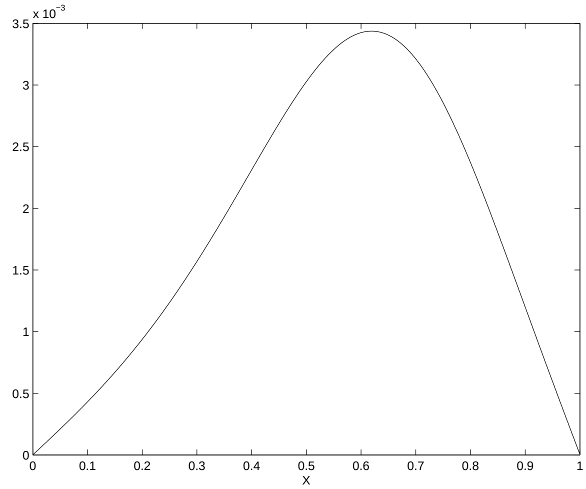

Figure 7: The axial component of the neutral mode, $\Psi(X)$.

$\Psi(X)$ is indeed significantly deviated from the sine-like function along the axial direction. At the first half of the pipe, $\Psi(X)$ appears as an evanescent wave whereas at the second half of the pipe, as a sine-wave. One finds that the magnitudes of the slop at the inlet and outlet are much different. It is clear that the disturbance's energy loss at the outlet is much larger than the gain at the inlet. According to the Reynolds-Orr equation, one finds

$$\frac{\text{Gain at the inlet}}{\text{Loss at the outlet}} = \frac{\Psi_x(0)^2}{\Psi_x(L)^2} \approx 0.12. \tag{50}$$

Therefore, the gain through the internal energy exchange supplies other 88% of the energy loss at the outlet. When the swirl is changed from the critical swirl, an unstable branch is developed as shown in Figure 6. where, the internal gain must surpass the total loss at the boundaries, and sustains the exponential growth of the disturbance.

5 Summary of the findings and the conclusions

We have studied the stability mechanism of swirling flows in a finite pipe. This study has revealed the energy transfer mechanisms in the swirling flows, and provided a basic understanding of the physical nature. We summarize in the following the new findings of this research, discuss the physics in light of the new findings and draw conclusions whenever appropriate.

1. The Rayleigh's criterion is proved to be necessary and sufficient for flow being formally stable. The Lamb-Oseen vortex in a finite pipe with periodic conditions imposed is shown to be nonlinear stable against any axisymmetric disturbance, and a global estimate of the disturbance's kinetic energy is obtained. The Hamiltonian structure and the underline symmetry play crucial role in the proof. The basic physics may be understood as follows. *Considering a pure vortex (with zero axial velocity) as the base flow, one notices that for such flow the velocities of the radial and axial components of the disturbance are coincident with the total flow velocities with the same components. Therefore, if the kinetic energy of the axial and radial disturbances grows, the kinetic energy of the azimuthal component must decay as to balance the total kinetic energy, but this is not possible if the relevant "potential energy" is at a minimum level.* This reveals a subtle energy-locking mechanism existent in the flow being stable in the sense of the Rayleigh's stability. One notes that Rayleigh has given an physical explanation of the necessity of his criterion. The two explanations, as a whole, offer a rather complete picture of the physics.

2. *When other boundary conditions are imposed at the pipe ends, such as those discussed in this article, the flow system changes from a closed system to a open system, and the Hamiltonian structure (Lie-Poisson bracket) does not hold. The aforementioned stability mechanism for the Rayleigh's stability is thus also destroyed.* One finds that typically, the original neutral mode becomes asymptotic stable at the low swirl level, a clear sign that the Hamiltonian structure is destroyed, and that it will becomes neutrally stable at the first critical swirl, and unstable beyond that level. The appearance of the standing wave in the base flow is often an clear indication for the occurrence of flow instability. *One shall, however, notice that even though the occurrence of the instability is in certain way related to the wave propagation nature, the instability itself is certainly more than a wave phenomenon. It involves the energy transfer mechanism between the disturbance and the base flow at the boundaries and inside the flow.*

   For instance, for the Lamb-Oseen vortex one finds such a energy transfer mechanism. Especially, the vortex core is found being active in energy transfer. This is a new mechanism identified in this study. It is a rather surprising result. It is a well known fact that the Lamb-Oseen vortex is stable against the axisymmetric disturbance. It is generally believed that the vortex core in the case serves as solely a neutral waveguide. *One must notice, however, that the boundary conditions have an influence on the whole flow. The non-symmetric flow conditions at the inlet and outlet naturally induce the non-symmetric eigenmode which becomes active in energy transfer.* One also notice that the solid body rotation flow is but an exceptional case where the strain rate of the base flow vanishes everywhere, and thus can not support the internal energy transfer.

3. The boundaries are found still having an important role for the energy transfer mechanism of the Lamb-Oseen vortex in a finite straight pipe. The disturbance's kinetic energy influx and out flux at the pipe ends comprise the principal energy gain and loss in the case. We found that about 90% of the energy transfer occurs at the boundaries in the Lamb-Oseen vortex. The internal energy transfer counts another 10%.

   However, a net energy gain at the boundaries is not a necessary condition for the onset of the global instability. This has been demonstrated by the case of the swirling flow in a divergent pipe. One finds that in this case the disturbance loses its energy at the boundaries, yet the flow can be shown to be neutral or unstable. This clearly demonstrates that the disturbance's energy gain at the boundary alone does not determine the stability of the flow. The internal flow energy transfer is found significantly enhanced in the case.

This seems to suggest that the flow non-uniformity alone can induce the global instability at a sufficient high swirl. This is, however not true. *One may consider an infinitely long, divergent and convergent pipe filled with the solid body rotation flow, assuming there is no axial movement. Such flow is apparently nonlinear stable for any type of disturbance at any swirl level because of the strain-free nature of the base flow.* In fact, Hamiltonian structure is generally preserved for swirling flows in a divergent and convergent pipe with periodic boundary condition specified at the inlet and outlet. The dynamics of such flows must follow the general behavior of the Hamiltonian system. For example, the asymptotic stable mode is not possible, and the stable mode is again always neutral. See, for example, Arnold's classical book [1]. *We thus conclude that it is the breakup of the Hamiltonian not the flow non-uniformity that is directly responsible for the the global stability found in swirling flows.*

4. In real flow situations, the periodic boundary conditions are rarely satisfied. Swirling flows are seen often under strong real flow influence at the boundaries, such as vortex generations at the upstream and viscous dissipations and breakup of a large scale of turbulences at the downstream, which certainly yield the ideal conditions for the global stability onset. *Therefore, the global stability is a real flow phenomenon existent in many flows of importance in real world.* Its effects may not be observed at low swirl level, as flow is asymptotic stable in the case. However, the manifestation of the instability is strong at sufficient high swirl. In fact, the global instability occurs very robustly and often induces drastic change of flow state, as has been observed in the vortex breakdown phenomenon.

The conclusions drawn above shed new light on the physical mechanism of swirling flows in a finite pipe. The stability mechanism has been studied with the Hamiltonian system theory, and the physical nature has been revealed in a rather fundamental level. These results build a clear picture of the stability of swirling flows in a finite pipe. Especially the relation between the Rayleigh's stability and the global stability has been clarified, and the energy transfer mechanism has been identified.

This study has revealed the complexity of the dynamical behaviors of the swirling flows. Swirling flows as comprising high density of kinetic energy are highly sensitive to the flow condition change, and tend to lose the stability whenever circumstances arising that may liberate the confined kinetic energy, which can be caused certainly by a unstable distribution of the centrifugal body force but can also be induced by the imbalance of the flow field at the upstream and the downstream. Therefore, in the study of the swirling flows, the physical problem must be presented precisely. This raises the "need" of the consistency in the future study of the dynamics of the swirling flows. In the context of the vortex breakdown, for example, one uses often different mathematical models for the theoretical studies and for the numerical simulations. This has led to significant discrepancies between the theories and the numerical simulations. The author believes that the consistency in all aspects of the researches would be the key to clarify the fundamental mechanism of this complicated fluid phenomenon.

The physical mechanisms revealed in this article may exist in many other flows. One may notice that in the discussion of the mechanism, we often give a "qualitative" argument at first which is then followed and supported by an accurate mathematical analysis. This gives strong indication that the flow behaviors discussed in this article are of a general nature. Real flows in various similar situations are expected to share some common features found in this study. We believe that the mathematical methods developed in this article can be generally useful for the future extension of the theory to other flow problems, and the physical insight being gained in this study may provide a good guidance for such studies.

A The stability analysis of the Lamb-Oseen vortex

The stability equation of the Lamb-Oseen vortex in a finite pipe has been studied in Wang [18], and the approximated growth rate function has been found. We briefly state the mathematical method used there. This method will be extended in the next section to find the flow field of the disturbance which is needed for applying the Reynolds-Orr equation.

The stability equation (13) can be formulated as the following perturbation problem:

$$\overbrace{-m(y)\left(\phi_{yy}+\frac{\phi_{xx}}{2y}\right)}^{T^{(0)}}+\sigma\underbrace{\int_0^x\left[-2m(y)(\phi_{yy}+\frac{\phi_{xx}}{2y})\right]dx}_{T^{(1)}}$$

$$+\frac{\sigma^2}{2}\underbrace{\int_0^x\int_0^x\left[-2m(y)(\phi_{yy}+\frac{\phi_{xx}}{2y})\right]dxdx}_{T^{(2)}}=\Omega\phi \tag{51}$$

subject to the boundary conditions derived from (15):

$$\begin{aligned}\phi(x,0) &= 0,\ \phi(x,1/2)=0,\ \text{for}\ \ 0\le x\le L,\\ \phi(0,y) &= 0,\ \text{for}\ \ 0\le y\le 1/2,\\ \phi(L,y) &= 0,\ \text{for}\ \ 0\le y\le 1/2.\end{aligned} \tag{52}$$

One may write (51) as:

$$T(\sigma)=T^{(0)}+\sigma T^{(1)}+\sigma^2T^{(2)}, \tag{53}$$

The stability equation can be solved by using the perturbation method of the linear operators.

One first solves the zeroth order eigenvalue problem:

$$-m(y)\left(\phi_{yy}+\frac{\phi_{xx}}{2y}\right)=\Omega\phi \tag{54}$$

with boundary conditions (52), by using the method of separation of variables. The eigenfunctions have been found as:

$$\phi^*_{o,n}(x,y)=\sqrt{\frac{2}{L}}\Phi^*_{o,n}(y)\sin(\frac{n\pi x}{L}) \tag{55}$$

with $\Phi^*_{o,n}(y)$ solving the reduced zeroth order eigenvalue problem

$$\begin{aligned}\Phi_{yy}-\frac{n^2\pi^2\Phi}{2L^2y}+\frac{\Omega_{o,n}}{m(y)}\Phi=0,\\ \Phi(0)=0,\Phi(\frac{1}{2})=0,\end{aligned} \tag{56}$$

and normalized as

$$\left(\int_0^{0.5}\frac{\Phi^{*2}_{o,n}(y)}{m(y)}dy\right)^{\frac{1}{2}}=1, \tag{57}$$

where, $\Omega_{o,n}$ are eigenvalues with $o=1,2,3...$ and $n=1,2,3...$ in the order $\Omega_{o_1,n_1}\le\Omega_{o_2,n_2}$ for $o_1\le o_2$ and $n_1\le n_2$. The eigenvalues $\Omega_{o,n}$ are critical swirls, where neutral mode exist. At each critical swirl $\Omega_{o,m}$, let $\Delta_{o,m}\Omega(\sigma)$ be the function of the swirl increment at the growth rate σ. The actual swirl can be written in terms of $\Delta_{o,m}\Omega(\sigma)$ as

$$\Omega=\Omega_{o,m}+\Delta_{o,m}\Omega(\sigma). \tag{58}$$

The perturbation theory of linear operators claims that the function $\Delta_{o,m}\Omega(\sigma)$ is analytic, see Kato [7] and has power series expansion:

$$\Delta_{o,m}\Omega(\sigma) = \Omega^{(1)}_{o,m}\sigma + \Omega^{(2)}_{o,m}\sigma^2 + \Omega^{(3)}_{o,m}\sigma^3 + ... \tag{59}$$

The coefficients in the power series are explicitly given by

$$\begin{aligned} \Omega^{(1)}_{o,m} &= (T^{(1)}\phi^*_{o,m}, \phi^*_{o,m}) \\ \Omega^{(2)}_{o,m} &= (T^{(2)}\phi^*_{o,m}, \phi^*_{o,m}) - \sum_{o_1,n:(o_1,n)\neq(o,m)} \frac{(T^{(1)}\phi^*_{o,m}, \phi^*_{o_1,n})(T^{(1)}\phi^*_{o_1,n}, \phi^*_{o,m})}{\Omega_{o_1,n} - \Omega_{o,m}} \\ &... \end{aligned} \tag{60}$$

where, (f, g) is a weighted inner product defined as

$$(f, g) = \int_0^{0.5}\int_0^L \frac{fg}{m(y)} dxdy. \tag{61}$$

See Kato [7] for the expansion of $\Omega^{(i)}_{o,m}$ with $i = 3, 4$.

B Find the disturbance flow fields by using the perturbation method

B.1 Find the stream function of the disturbance

The references of this appendix are Kato [7] and Wang [18]. We consider the general linear operator perturbation problem:

$$T(\sigma) = T^{(0)} + \sigma T^{(1)} + \sigma^2 T^{(2)}. \tag{62}$$

The goal is to find the eigenmodes of (62) by using the perturbation method.

We assume that the spectrum of $T^{(0)}$ is discrete and simple, denoted by $\lambda_0, \lambda_1, \lambda_2, ...$, with the corresponding orthonormal eigenvectors $\{e_0, e_1, e_2, ...\}$:

$$\begin{aligned} T^{(0)}e_0 &= \lambda_0 e_0 \\ T^{(0)}e_i &= \lambda_i e_i,\ i = 1, 2, ... \end{aligned} \tag{63}$$

We single out the eigenvalue λ_0 here with which we consider the perturbation problem. We denote the perturbed eigenvalue of λ_0 as $\lambda(\sigma)$,

$$T(\sigma)e = \lambda(\sigma)e. \tag{64}$$

In the perturbation theory, $\lambda(\sigma)$ can be shown being an analytic function of σ in a neighborhood of $\sigma = 0$, and the analytic expansion can be found by using the eigenfunctions of the unperturbed problem. This fact has been used in Appendix A to find the approximations of the growth rate function. In the following, we will show how to find the relevant eigenmodes for the perturbed operator.

The resolvent is defined as

$$R(\zeta, \sigma) = (T(\sigma) - \zeta)^{-1}. \tag{65}$$

The projection $P(\sigma)$ corresponding to eigenvalues $\lambda(\sigma)$ is expressed in terms of the resolvent as

$$P(\sigma) = -\frac{1}{2\pi i} \int_\Gamma R(\zeta, \sigma) d\zeta. \tag{66}$$

where, Γ is a small positively-oriented curve enclosing the eigenvalue of $T(\sigma)$. We denote $P = P(0)$.

It has been shown that $P(\sigma)$ has the following expansion:

$$P(\sigma) = P + \sigma P^{(1)} + \sigma^2 P^{(2)} + ... \tag{67}$$

$P^{(n)}$ can be explicitly expressed as

$$\begin{aligned} P^{(1)} &= -PT^{(1)}S - ST^{(1)}P, \\ P^{(2)} &= -PT^{(2)}S - ST^{(2)}P + PT^{(1)}ST^{(1)}S + ST^{(1)}PT^{(1)}S + ST^{(1)}ST^{(1)}P - PT^{(1)}PT^{(1)}S^2, \\ &- PT^{(1)}S^2T^{(1)}P - S^2T^{(1)}PT^{(1)}P. \end{aligned} \tag{68}$$

in terms of the operators P and S, where S is the reduced resolvent of $T^{(0)}$, see Kato [7] for the precise definition. In terms of the basis formed by the orthonormal eigenvectors $\{e_0, e_1, e_2, ...\}$, P and S can be simply represented as

$$P\varrho = (\varrho, e_0)e_0, \tag{69}$$

and

$$S\varrho = \sum_{i=1}^{\infty} (\lambda_i - \lambda_0)^{-1} (\varrho, e_i) e_i. \tag{70}$$

where, ϱ is any vector in the function space under consideration.

The eigenvector of $T(\sigma)$ can be approximately expressed as:

$$P(\sigma)e_0 = e_0 + \sigma P^{(1)}e_0 + \sigma^2 P^{(2)}e_0, \tag{71}$$

with second order accuracy.

Noticing $Pe_0 = e_0$ and $Se_0 = 0$, one obtains

$$\begin{aligned} P^{(1)}e_0 &= -ST^{(1)}e_0, \\ P^{(2)}e_0 &= -ST^{(2)}e_0 + ST^{(1)}ST^{(1)}e_0 - PT^{(1)}S^2T^{(1)}e_0 - S^2T^{(1)}PT^{(1)}e_0. \end{aligned} \tag{72}$$

By using (69) and (70), one derives from (72)

$$P^{(1)}e_0 = -\sum_i \frac{(T^{(1)}e_0, e_i)e_i}{(\lambda_i - \lambda_0)}, \tag{73}$$

where $\sum_i = \sum_{i=1}^{\infty}{}^1$ and

$$\begin{aligned} P^{(2)} &= \sum_i -\frac{(T^{(1)}e_i, e_0)(T^{(1)}e_0, e_i)}{(\lambda_i - \lambda_0)^2}e_0 + \sum_i [-\frac{(T^{(2)}e_0, e_i)}{(\lambda_i - \lambda_0)} + \sum_j \frac{(T^{(1)}e_0, e_j)(T^{(1)}e_j, e_i)}{(\lambda_j - \lambda_0)(\lambda_i - \lambda_0)} \\ &\quad -\frac{(T^{(1)}e_0, e_0)(T^{(1)}e_0, e_i)}{(\lambda_i - \lambda_0)^2}]e_i, \end{aligned} \tag{74}$$

We now apply (73) and (74) to Lamb-Oseen vortex, and find the eigenmodes near the first critical swirl $\Omega_{1,1} \approx 0.7910$. One obtains the following expansion of $\phi(x, y; \sigma)$ in terms of the eigenfunctions $\phi^*_{1,n}$ for $n = 1, 2, ..., 10$

$$\phi(x, y; \sigma) = \phi^*_{1,1} + \sigma \sum_{n=2}^{10} a_{1,n}\phi^*_{1,n} + \sigma^2 \sum_{n=1}^{10} b_{1,n}\phi^*_{1,n} + o(\sigma^2), \tag{75}$$

where

$$a_{1,n} = -\frac{(T^{(1)}\phi^*_{1,1}, \phi^*_{1,n})}{\Omega_{1,n} - \Omega_{1,1}} \text{ for } n \geq 2. \tag{76}$$

$$\begin{aligned} b_{1,1} &= \sum_{n\neq 1} \frac{(T^{(1)}\phi^*_{1,1}, \phi^*_{1,n})(T^{(1)}\phi^*_{1,n}, \phi^*_{1,1})}{(\Omega_{1,n} - \Omega_{1,1})^2} \\ b_{1,n} &= [-\frac{(T^{(2)}\phi^*_{1,1}, \phi^*_{1,n})}{\Omega_{1,n} - \Omega_{1,1}} + \sum_{l\neq 1} \frac{(T^{(1)}\phi^*_{1.1}, \phi^*_{1,l})(T^{(1)}\phi^*_{1,l}, \phi^*_{1,n})}{(\Omega_{1,n} - \Omega_{1,1})(\Omega_{1,l} - \Omega_{1,1})} \\ &\quad - \frac{(T^{(1)}\phi^*_{1,1}, \phi^*_{1,1})(T^{(1)}\phi^*_{1,1}, \phi^*_{1,n})}{(\Omega_{1,n} - \Omega_{1,1})^2}] \text{ for } n > 1. \end{aligned} \tag{77}$$

These formulas are comprised of the inner products involved with operators $T^{(1)}$ and $T^{(2)}$. In the columnar swirling flow, they can be all analytically evaluated, as shown in [18]. One finds

Table 1: $a_{1,n}$ and $b_{1,n}$ for the Lamb-Oseen vortex with $L = 6$, $\beta = 4$.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$a_{1,n}$	0	-74.5021	14.1724	-6.1605	2.9505	-1.7747	1.1041	-0.7653	0.5391	-0.4042
$b_{1,n}$	5605	136.1	433.7	-80.14	35.31	-23.41	6.745	-10.68	1.506	-6.071

values of $a_{1,n}$ and $b_{1,n}$ in Table 1 for the Lamb-Oseen vortex with $L = 6$, $\beta = 4$. We found from this table that the first few terms are more important in the expansion, and the values of the other terms diminish quickly.

Based on the values shown in Table 1, the modes of the stream function $\phi(x, y; \sigma)$ are calculated for various σ, and the results are plotted in Figure 2.

B.2 The velocity field of the disturbances

One may find the velocity field $u_1(x, y; \sigma)$ from its relation to the stream function.

$$u_1(x, y; \sigma) \quad = \quad -\frac{1}{\sqrt{2y}}\frac{\partial}{\partial x}\left(\phi^*_{1,1} + \sigma\sum_{i=2}^{10} a_{1,n}\phi^*_{1,n} + \sigma^2\sum_{i=2}^{10} b_{1,n}\phi^*_{1,n}\right) \tag{78}$$

We may derive from (9) the following formula of K_1.

$$K_1(x, y) = \frac{2y^2}{K_0(y)}\int_0^x (\sigma\chi_1 + \chi_{1x})dx. \tag{79}$$

Using the relation $\chi_1 = -\Delta_{SL}\phi(x, y; \sigma)$ and the expansion of $\phi(x, y; \sigma)$ (75), one obtains

$$\begin{aligned} K_1(x, y) \quad &= \quad \frac{2y^2}{K_0(y)}(\int_0^x -\sigma\Delta_{SL}(\phi^*_{1,1} + \sigma\sum_{n=2}^{10} a_{1,n}\phi^*_{1,n} + \sigma^2\sum_{n=1}^{10} b_{1,n}\phi^*_{1,n})dx \\ &- \quad \Delta_{SL}(\phi^*_{1,1} + \sigma\sum_{n=2}^{10} a_{1,n}\phi^*_{1,n} + \sigma^2\sum_{n=1}^{10} b_{1,n}\phi^*_{1,n})) \end{aligned} \tag{80}$$

By using the fact: $\phi^*_{o,n}$ are the eigenfunctions of the zeroth order problem, one obtains the following expansion of K_1,

$$\begin{aligned} K_1(x, y) \quad &= \quad \frac{2y^2}{K_0(y)m(y)}(\Omega_{1,1}\phi^*_{1,1} + \sigma[\sum_{i=2}^{10} a_{1,n}\Omega_{1,n}\phi^*_{1,n} + \int_0^x (\Omega_{1,1}\phi^*_{1,1}] \\ &+\sigma^2[\sum_{i=1}^{10} b_{1,n}\Omega_{1,n}\phi^*_{1,n} + \int_0^x \sum_{i=2}^{10} a_{1,n}\Omega_{1,n}\phi^*_{1,n}]). \end{aligned} \tag{81}$$

One finds v_1 from the relation $v_1 = \frac{K_1}{\sqrt{2y}}$,

$$v_1(x, y) \quad = \quad \frac{\sqrt{2}y^{3/2}}{K_0(y)m(y)}(\Omega_{1,1}\phi^*_{1,1} + \sigma[\sum_{i=2}^{10} a_{1,n}\Omega_{1,n}\phi^*_{1,n} + \int_0^x (\Omega_{1,1}\phi^*_{1,1}] \tag{82}$$

[1]This notation will be adopted for the similar terms thither. e.g. $\sum_{i,j} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}$.

$$+\sigma^2[\sum_{i=1}^{10} b_{1,n}\Omega_{1,n}\phi^*_{1,n} + \int_0^x \sum_{i=2}^{10} a_{1,n}\Omega_{1,n}\phi^*_{1,n}]).$$

C Derive energy transfer functions

With the flow field of the disturbance being found, one may insert u_1 and v_1 into the Reynolds-Orr equation for Lamb-Oseen vortex. The equation (44) can be written as

$$\frac{dE(t)}{dt} = 2\pi \int_0^{0.5} (E_{in}(y;\sigma) + E_b(y;\sigma))dy. \tag{83}$$

where,

$$E_{in}(y) = \omega \left(2\beta e^{-2\beta y} - \frac{(1 - e^{-2\beta y})}{y} \right) \int_0^L u_1 v_1 dx \tag{84}$$

and

$$E_b(y;\sigma) = \frac{1}{2}\left[u_1^2 + v_1^2\right]_{x=L}^{x=0}. \tag{85}$$

are the density of the internal energy transfer and the density of the boundary energy transfer, respectively, both being function of y.

We can derive close forms for these density functions. For reason of simple, we only derive the close forms of the first order. First, consider the integral in (84)

$$\begin{aligned} \int_0^L u_1 v_1 dx &= \frac{y}{K_0(y)m(y)} \int_0^L (\Omega_{1,1}\phi^*_{1,1} + \sigma[\sum_{i=2}^{10} a_{1,n}\Omega_{1,n}\phi^*_{1,n} + \int_0^x (\Omega_{1,1}\phi^*_{1,1}] \\ &\quad \times \frac{\partial}{\partial x}(\phi^*_{1,1} + \sigma \sum_{i=2}^{10} a_{1,n}\phi^*_{1,n} + \sigma^2 \sum_{i=2}^{10} b_{1,n}\phi^*_{1,n})dx \end{aligned} \tag{86}$$

By use of integration in parts and the boundary conditions, one finds the first order approximation as

$$\int_0^L u_1 v_1 dx = \frac{\sigma y}{K_0(y)m(y)} \int_0^L \frac{\partial \phi^*_{1,1}}{\partial x}[\sum_{n=2}^{10} (\Omega_{1,n} - \Omega_{1,1})a_{1,n}\phi^*_{1,n} + \int_0^x (\Omega_{1,1}\phi^*_{1,1}]dx \tag{87}$$

Inserting $\phi^*_{1,n} = \Phi^*_{1,n}(y)\sin\frac{n\pi x}{L}$, one may complete the integration and obtain a close form as

$$\int_0^L u_1 v_1 dx = \frac{\sigma y \Phi^*_{1,1}}{K_0(y)m(y)}[\sum_{n=2}^{10} (\frac{a_{1,n}n(1 - (-1)^{n+1})(\Omega_{1,n} - \Omega_{1,1})}{(n+1)(n-1)}\Phi^*_{1,n} - \frac{\Omega_{1,1}L}{2}\Phi^*_{1,1}] \tag{88}$$

In substitution of this into (84), one obtains

$$E_{in} = C(y)[\sum_{n=2}^{10} (\frac{a_{1,n}n(1 - (-1)^{n+1})(\Omega_{1,n} - \Omega_{1,1})}{(n+1)(n-1)}\Phi^*_{1,n} - \frac{\Omega_{1,1}L}{2}\Phi^*_{1,1}] \tag{89}$$

where $C(y)$ denotes the function

$$C(y) = -\omega(2\beta e^{-2\beta y} - \frac{(1-e^{-2\beta y})}{2y})\frac{\sigma y \Phi^*_{1,1}}{K_0(y)m(y)} \tag{90}$$

On the other hand, substituting (78) into (85), one finds:

$$E_b(y) = \frac{\sigma\pi^2}{L^2 2y}\Phi^*_{1,1}(y)(\sum_{n=2}^{10}(1-(-1)^n)na_{1,n}\Phi^*_{1,n}) \tag{91}$$

D Find the steady solution for a long divergent pipe

We will develop the weakly nonlinear solution of the steady state in a long divergent pipe near the critical state. One uses a standard long wave approach. A similar problem with a different outlet boundary condition, namely, assuming non-radial flow at the outlet, has been studied by Rusak, Judd and Wang [11]. The method used there can be applied to our case almost identically. Consider a long pipe with small divergence:

$$Y_0(X) = 0.5 + \delta e^{-(\frac{X-1}{0.5})^6}, \quad x = LX, \tag{92}$$

where, $Y_0(X)$ is the rescaled pipe radius at the pipe location $x = LX$, and the actual radius shall be

$$R_0(x) = \sqrt{2Y_0(x/L)}. \tag{93}$$

Notice that a rescaled length $X = x/L$ is introduced to reflect the influence of the long pipe and $0 \leq X \leq 1$. One chooses $L = 10$ in this study, which is sufficiently long as a long wave approach being valid. The pipe divergence parameter δ has been chosen as $\delta = 0.0001$, which is extremely small, but large enough to ensure the nonlinear effect. The critical state is very sensitive to the pipe shape change.

We seek a steady solution of the Square-Long equation in the form:

$$\psi(x,y) = \psi_0 + \epsilon_1 A(X)\varphi_1(y) + \epsilon_2 B(X)\varphi_2(y) + ... \tag{94}$$

where $\epsilon_1 << 1, \epsilon_1 << \epsilon_2$, the small parameters depending on $\Delta\Omega$ and δ in the expansion. One finds $\epsilon_1 = \frac{1}{L}$. Substituting this expression of $\psi(x,y)$ into the Square-Long equation, after a standard procedure, see Rusak, Judd and Wang [11] for the detailed derivation, one finds that $\varphi_1(y)$ satisfies

$$\varphi_{1yy}(y) - (H''(\psi_0;\Omega_B) - \frac{\Omega_B I_0(\psi_0)}{2y})\varphi_1(y), \quad \varphi_1(0) = \varphi_1(0.5) = 0, \tag{95}$$

where, Ω_B is Benjamin's critical swirl. Let $\Delta\Omega = \Omega - \Omega_B$. One finds $A(x)$ satisfies the following nonlinear equation:

$$A_{XX} - \alpha(A^2) + \beta \frac{\Delta\Omega}{\epsilon_1} A = -\frac{\delta}{\tau\epsilon_1^2} \int_0^X \varphi_{1y}(0.5)\psi_0(0.5)Y_{0X}(X)\ dx. \tag{96}$$

where,

$$\alpha = -\frac{\Omega_B}{\delta} \int_0^{1/2} \frac{1}{\psi_{0y}^{3/2}} \left(\frac{K_0 K_{0y}}{y\psi_{0y}^{3/2}} \right)_y \frac{\varphi_1^3}{2}\ dy, \tag{97}$$
$$\beta = -\frac{1}{\tau} \int_0^{1/2} \frac{I'(\psi_0)}{2y^2\psi_{0y}} \varphi_1^2\ dy, \quad \tau = \int_0^{1/2} \frac{\varphi_1^2}{2y}\ dy.$$

For the Lamb-Oseen vortex with $\beta = 4$,

$$\alpha = 5.597,\ \beta = 23.52,\ \tau = 0.001237,\ \varphi_{1y}(0.5) = -0.1048;\ \Omega_B = 0.7798. \tag{98}$$

One obtains the following concrete equation to determine $A(X)$ for the case $L = 10$ and $\delta = 0.0001$:

$$A_{XX} - 5.597A^2 + 23.52\frac{\Delta\Omega}{100} A = 84.72 e^{-(\frac{X-1}{0.5})^6}, \quad A(0) = 0, \quad A(1) = 0. \tag{99}$$

Notice that the outlet boundary condition is set as the fixed flowrate $A(1) = 0$. The bifurcation diagram of the steady solution vs. the swirl $\Delta\Omega$ in terms of the conventional minimum axial velocity at the centerline of the pipe is obtained by solving (99), and is plotted in figure 6. We are concerned in the flow field of the critical state. $A(X)$ is found for the critical state as shown in Figure 8.

The neutral mode can be approximately expressed in the form of separation of variables $\varphi_1(y)\Psi(X)$, and the axial component is found by the difference of the critical state and a nearby steady state: $\Psi(X) = A(X, \Omega_c) - A(X, \Omega_c - 0.001)$, where Ω_c denotes the critical swirl. $\Psi(X)$ thus found is plotted in Figure 7.

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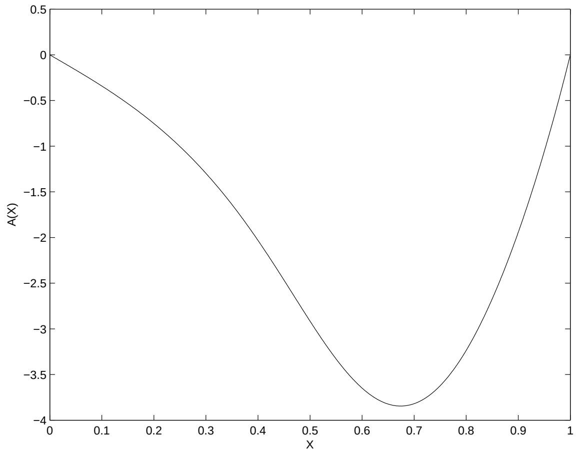

Figure 8: $A(x)$ as found at the critical state.

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