NORMAL SUBGROUPS OF LOW INDEX IN THE MODULAR GROUP AND OTHER HECKE GROUPS

Marston Conder and *Peter Dobcsányi*

Department of Mathematics
University of Auckland
Private Bag 92019 Auckland
NEW ZEALAND

Abstract

An account is given of the determination of all normal subgroups of index up to 1500 in the modular group $\mathrm{PSL}(2,\mathbb{Z}) \cong C_2 * C_3$, and all normal subgroups of index up to 500 in the Hecke groups $H_q = C_2 * C_q$ for $4 \le q \le 12$, using an adaptation of the low index subgroups algorithm.

1991 Mathematics Subject Classification: 20F05

1. Introduction

The *modular group* is the group $\mathrm{PSL}(2,\mathbb{Z})$, consisting of all Möbius transformations $z \mapsto (az+b)/(cz+d)$ with $a,b,c,d \in \mathbb{Z}$ and $ad-bc = 1$. This group has a defining presentation $\langle\, x, y \mid x^2 = y^3 = 1 \,\rangle$ in terms of the transformations $x : z \mapsto -1/z$ and $y : z \mapsto (z-1)/z$, and is thus isomorphic to a free product $C_2 * C_3$ of cyclic groups of orders 2 and 3. Note that the product of these two transformations is the translation $z \mapsto z+1$. More generally, for any integer $q \ge 3$ the *Hecke group* H_q is the discrete group generated by the two Möbius transformations $z \mapsto -1/z$ and $z \mapsto z+\lambda_q$, where $\lambda_q = 2\cos(\pi/q)$, and is isomorphic to $C_2 * C_q$ (with the product of the two generators having order q).

Both the modular group and the more general family of Hecke groups (to which it belongs) play a significant role in many branches of mathematics, not only due to their action on the upper-half complex plane, but also because of their actions on other discrete structures such as infinite regular trees. For similar reasons, finite homomorphic images of these groups arise as groups of automorphisms of regular combinatorial and geometric structures such as graphs and maps.

The connection with regular maps was pursued recently by Cangül and Singerman in [**2**], extending previous work by Newman in [**11**] on normal subgroups of the modular group. In particular, this connection was exploited in [**2**] in order to find all normal subgroups of index up to 60 in Hecke groups.

In fact each H_q has a very rich subgroup and quotient structure. For example, in the modular group the index 6 subgroup generated by $xy^{-1}xy$ and $xyxy^{-1}$ is free of rank 2, and thus has every 2-generator finite group as a quotient. Also by the main result of [**7**] and similar arguments, each H_q has all but finitely alternating and symmetric groups of finite degree among its quotients.

In this paper we considerably extend the results of [**2**] and [**11**], producing a list of all normal subgroups of index up to 1500 in the modular group, and a summary of the normal subgroups of index up to 500 in each of the Hecke groups H_q for $4 \leq q \leq 12$.

2. Method: the low index normal subgroups algorithm

The results outlined in the next two Sections were obtained with the help of an adaptation of the *low index subgroups procedure*, which is an algorithm due to Charles Sims and others for determining all subgroups of up to a given index n in a finitely-presented group $G = \langle X \mid R \rangle$, and described in [**8**, **12**].

To find a representative of each such class of subgroups, the standard algorithm uses a back-track search through a tree, with nodes at level k corresponding to certain subgroups generated by k elements. The search begins (at level 0) with the identity subgroup, generated by the empty set ϕ, and uses coset enumeration at each node to determine how to proceed. In a systematic fashion, certain elements are successively adjoined and removed to and from the generating set for the subgroup, on a last-in first-out basis.

The adaptation we have made finds only normal subgroups, by treating these elements as relators rather than just subgroup generators, and is explained in detail in [**3**]. This is appreciably faster than finding all subgroups (or representatives of conjugacy classes of subgroups) of index up to n and eliminating those which are not normal. In turn this enables a search up to much higher index within given computing resources, and has numerous applications, some of which are outlined also in [**3**]. One such application is the determination of all orientable regular maps of genus 2 to 15 and all non-orientable regular maps of genus 3 to 30 (see [**4**]), and another to finding all finite connected trivalent symmetric graphs on up to 768 vertices (see [**5**]).

A parallel implementation of this low index normal subgroups algorithm has been developed by the second author, for use on either a multi-threaded computer or via a distributed processing system. Using a one of the latter, in a platform known as Kaláka (and described in his PhD thesis [**9**]), all normal subgroups of index up to 1500 in the modular group were found in 129 hours 32 minutes. The results are given in Section 3. The same approach was taken for subgroups of index up to 500 in other Hecke groups, and a summary list of the numbers of subgroups of each index is given in Section 4.

For illustrative purposes, we point out that a serial implementation of our adaptation on a moderately-resourced laptop computer takes less than one second to find all 13 normal subgroups of index up to 48 in the modular group, while the standard low index subgroups algorithm takes hours to find representatives of conjugacy classes of all subgroups of index only up to 30 — not surprising as there are millions of such classes.

3. Normal subgroups of index up to 1500 in the modular group

The following table gives information about normal subgroups of the modular group $G = \langle\, x, y \mid x^2 = y^3 = 1 \,\rangle$, of increasing index from 1 to 1500.

Each row of the table describes one normal subgroup K, and the first entry is the index $|G : K|$ (or equivalently, the order of the factor group G/K). The second entry is the *level* of the subgroup K, defined as the smallest positive integer m for which $(xy)^m \in K$ (or equivalently, the order of the image of xy in G/K), and the third entry is its genus g, given by the Riemann-Hurwitz formula $2g-2+N/m = N/6$ where $N = |G:K|$ (as explained in [**2**]). These are followed by representatives of conjugacy classes of elements of G which generate K (or equivalently, additional relators which produce G/K when added to the presentation for G), and for notational convenience, we use u to denote xy and v to denote xy^{-1}. Finally the fifth column contains composition factors for the quotient group $\overline{G} = G/K$, from a composition series $\{1\} = \overline{G}_0 \lhd \overline{G}_1 \lhd \ldots \lhd \overline{G}_{s-1} \lhd \overline{G}_s = \overline{G}$ (which begins with the identity subgroup on the left and ends with $\overline{G} = G/K$ on the right). These were found with the help of the MAGMA system [**1**].

The table contains a large number of members of one particular family of normal subgroups, namely those containing the element $v^6 = (xy^1)^6$. Such normal subgroups contain also $xv^{-6}x^{-1} = x(yx)^6x^{-1} = (xy)^6 = u^6$, and therefore produce factor groups of the $(2,3,6)$ triangle group $\langle\, x, y \,|\, x^2 = y^3 = (xy)^6 = 1 \,\rangle$. In this group the elements $xyxy^{-1}$ and $xy^{-1}xy$ generate a free abelian normal subgroup of rank 2 (and index 6 with quotient C_6), since $[xy^{-1}xy, xyxy^{-1}] = (y^{-1}xyx)(yxy^{-1}x)(xy^{-1}xy)(xyxy^{-1}) = y^{-1}(xy)^6y = 1$.

Accordingly quotients of the modular group via normal subgroups containing the element v^6 are extensions of cyclic or 2-generator abelian groups by a cyclic group of order dividing 6. For each prime $p \equiv 1$ modulo 6 there are two such normal subgroups of index $6p$ giving quotients of the form $C_p : C_6$, and intersections of these provide 2^s normal subgroups of index $6n$ (with quotient $C_n : C_6$) whenever $n = p_1p_2\ldots p_s$ is the product of s distinct primes $p_1, p_2, \ldots, p_s$ congruent to 1 modulo 6. Similarly for every positive integer m there exists a normal subgroup of index $6m^2$ with quotient $(C_m \times C_m) : C_6$. These and other such quotients have special significance in the study of regular maps on the torus (see [**6**; Section 8.4]) and trivalent symmetric graphs of girth 6 (see [**10**]).

Index	Level	Genus	Additional Relators	Composition Factors
1	1	0	x, y	—
2	2	0	y	C_2
3	3	0	x	C_3
6	6	1	uv	C_2, C_3
6	2	0	v^2	C_3, C_2
12	3	0	v^3	C_2, C_2, C_3
18	6	1	u^2v^2	C_3, C_3, C_2

Index	Level	Genus	Additional Relators	Composition Factors
24	6	1	$(uv)^2$	C_2, C_2, C_2, C_3
24	4	0	v^4	C_2, C_2, C_3, C_2
42	6	1	u^2vuv^2, v^6	C_7, C_2, C_3
42	6	1	uvu^2v^2, v^6	C_7, C_2, C_3
48	12	3	u^3v^3	C_2, C_2, C_2, C_2, C_3
48	8	2	$(uv^2)^2$	C_2, C_2, C_2, C_3, C_2
54	6	1	$(uv)^3, v^6$	C_3, C_3, C_3, C_2
60	5	0	v^5	A_5
72	12	4	$(uv)^3, u^4v^4$	C_2, C_3, C_2, C_3, C_2
72	6	1	v^6, uvu^2vuv^2	C_2, C_2, C_3, C_3, C_2
78	6	1	v^6, u^2vuvuv^2	C_{13}, C_2, C_3
78	6	1	$v^6, (uv)^2u^2v^2$	C_{13}, C_2, C_3
96	8	3	$(uv)^3, v^8$	$C_2, C_2, C_2, C_2, C_3, C_2$
96	6	1	$v^6, (uv)^4$	$C_2, C_2, C_2, C_2, C_2, C_3$
114	6	1	$v^6, u^2v^2u^2vuv^2$	C_{19}, C_2, C_3
114	6	1	$v^6, (uv)^2u^2vuv^2$	C_{19}, C_2, C_3
120	10	5	$(uv^3)^2$	C_2, A_5
126	6	1	$v^6, u^2v(uv)^2uv^2$	C_7, C_3, C_3, C_2
126	6	1	$v^6, (uv)^3u^2v^2$	C_7, C_3, C_3, C_2
144	24	10	u^4v^4	$C_2, C_2, C_3, C_2, C_3, C_2$
144	12	7	uvu^2vuv^2	$C_2, C_2, C_2, C_3, C_3, C_2$
150	10	6	$(uv)^3, v^{10}$	C_5, C_5, C_3, C_2
150	6	1	$v^6, (uv)^5$	C_5, C_5, C_2, C_3
162	18	10	$(uv)^3, u^6v^6$	C_3, C_3, C_3, C_3, C_2
162	6	1	$v^6, (uv)^2u^2vuvuv^2$	C_3, C_3, C_3, C_3, C_2
168	6	1	$v^6, u^2v^2u^2vuvuv^2$	C_7, C_2, C_2, C_2, C_3
168	6	1	$v^6, (uv)^3u^2vuv^2$	C_7, C_2, C_2, C_2, C_3
168	7	3	$v^7, (uv)^4$	$L_2(7)$
180	15	10	$(uv^2)^3, u^5v^5$	A_5, C_3
186	6	1	$v^6, u^2v(uv)^3uv^2$	C_{31}, C_2, C_3
186	6	1	$v^6, (uv)^4u^2v^2$	C_{31}, C_2, C_3
192	12	9	$(uv)^4, (uv^4)^2$	$C_2, C_2, C_2, C_2, C_2, C_2, C_3$
192	8	5	v^8, uvu^3vuv^3	$C_2, C_2, C_2, C_2, C_2, C_3, C_2$
216	12	10	$(uv)^3, v^{12}$	$C_2, C_3, C_3, C_2, C_3, C_2$
216	6	1	$v^6, (uv)^6$	$C_2, C_2, C_3, C_3, C_3, C_2$
222	6	1	$v^6, u^2v^2uvu^2vuvuv^2$	C_{37}, C_2, C_3
222	6	1	$v^6, (uv)^3u^2vuvuv^2$	C_{37}, C_2, C_3
234	6	1	$v^6, u^2vuv^2u^2vuvuv^2$	C_{13}, C_3, C_3, C_2
234	6	1	$v^6, (uv)^4u^2vuv^2$	C_{13}, C_3, C_3, C_2
240	20	15	$u^5v^5, (uvuv^2)^2$	C_2, C_2, A_5
258	6	1	$v^6, u^2v(uv)^4uv^2$	C_{43}, C_2, C_3
258	6	1	$v^6, (uv)^5u^2v^2$	C_{43}, C_2, C_3

Index	Level	Genus	Additional Relators	Composition Factors
288	12	13	$(u^2v^2)^2, v^{12}$	$C_2, C_2, C_2, C_3, C_2, C_3, C_2$
288	24	19	$(uv)^3, u^8v^8$	$C_2, C_2, C_3, C_2, C_2, C_3, C_2$
288	6	1	$v^6, (uv)^3u^2v(uv)^2uv^2$	$C_2, C_2, C_2, C_2, C_3, C_3, C_2$
294	14	15	$(uv)^3, v^{14}$	C_7, C_7, C_3, C_2
294	6	1	$v^6, u^2v^2uvu^2v(uv)^2uv^2$	C_7, C_7, C_2, C_3
294	6	1	$v^6, (uv)^4u^2vuvuv^2$	C_7, C_7, C_2, C_3
294	6	1	$v^6, (uv)^7$	C_7, C_7, C_2, C_3
312	6	1	$v^6, u^2vuv^2u^2v(uv)^2uv^2$	$C_{13}, C_2, C_2, C_2, C_3$
312	6	1	$v^6, (uv)^5u^2vuv^2$	$C_{13}, C_2, C_2, C_2, C_3$
324	9	10	$v^9, u^2vu^2v^2uv^2$	$C_3, C_3, C_2, C_3, C_2, C_3$
336	14	17	$(uv)^4, (uv^5)^2$	$C_2, L_2(7)$
336	8	8	$(uv)^4, v^8$	$L_2(7), C_2$
336	12	15	$(uv^4)^2, u^2v^2u^2vuvuv^2$	$C_7, C_2, C_2, C_2, C_2, C_3$
336	12	15	$(uv^4)^2, (uv)^2u^2v^2u^2v^2$	$C_7, C_2, C_2, C_2, C_2, C_3$
336	8	8	$v^8, uv^2u^3v^2uv^3$	$L_2(7), C_2$
342	6	1	$v^6, u^2v(uv)^5uv^2$	C_{19}, C_3, C_3, C_2
342	6	1	$v^6, (uv)^6u^2v^2$	C_{19}, C_3, C_3, C_2
360	30	25	$u^5v^5, (uv)^5$	C_2, A_5, C_3
360	10	13	$(uvuv^2)^2, v^{10}$	A_5, C_3, C_2
366	6	1	$v^6, u^2v^2(uv)^2u^2v(uv)^2uv^2$	C_{61}, C_2, C_3
366	6	1	$v^6, (uv)^4u^2v(uv)^2uv^2$	C_{61}, C_2, C_3
378	6	1	$v^6, u^2vuv^2uvu^2v(uv)^2uv^2$	C_7, C_3, C_3, C_3, C_2
378	6	1	$v^6, (uv)^5u^2vuvuv^2$	C_7, C_3, C_3, C_3, C_2
384	24	25	$u^2vu^2v^2uv^2, u^6v^6$	$C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_3$
384	16	21	$(u^2v^3)^2, (uv^2)^4$	$C_2, C_2, C_2, C_2, C_2, C_2, C_3, C_2$
384	16	21	$(uv)^3, v^{16}$	$C_2, C_2, C_2, C_2, C_2, C_2, C_3, C_2$
384	6	1	$v^6, (uv)^8$	$C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_3$
402	6	1	$v^6, u^2vuvuv^2u^2v(uv)^2uv^2$	C_{67}, C_2, C_3
402	6	1	$v^6, (uv)^6u^2vuv^2$	C_{67}, C_2, C_3
432	24	28	$uvu^3vuv^3, u^2vu^2v^9$	$C_2, C_2, C_3, C_3, C_2, C_3, C_2$
432	12	19	$(uv^4)^2, u^2v(uv)^3u^2v^3$	$C_2, C_2, C_2, C_3, C_3, C_3, C_2$
432	8	10	$v^8, u^2vu^2vuv^2uv^2$	$C_3, C_3, C_2, C_2, C_2, C_3, C_2$
432	8	10	$v^8, uvu^2vu^2v^2uv^2$	$C_3, C_3, C_2, C_2, C_2, C_3, C_2$
438	6	1	$v^6, u^2v(uv)^6uv^2$	C_{73}, C_2, C_3
438	6	1	$v^6, (uv)^7u^2v^2$	C_{73}, C_2, C_3
450	30	31	$(uv)^3, u^{10}v^{10}$	C_5, C_3, C_5, C_3, C_2
450	6	1	$v^6, (uv)^4u^2v(uv)^3uv^2$	C_5, C_5, C_3, C_3, C_2
456	6	1	$v^6, u^2v^2(uv)^2u^2v(uv)^3uv^2$	$C_{19}, C_2, C_2, C_2, C_3$
456	6	1	$v^6, (uv)^5u^2v(uv)^2uv^2$	$C_{19}, C_2, C_2, C_2, C_3$
474	6	1	$v^6, u^2vuv^2uvu^2v(uv)^3uv^2$	C_{79}, C_2, C_3
474	6	1	$v^6, (uv)^6u^2vuvuv^2$	C_{79}, C_2, C_3

Index	Level	Genus	Additional Relators	Composition Factors
486	18	28	$u^4v^2u^2v^4$	$C_3, C_3, C_3, C_3, C_3, C_2$
486	18	28	$u^2v^2u^4v^4$	$C_3, C_3, C_3, C_3, C_3, C_2$
486	18	28	$(uv)^3, v^{18}$	$C_3, C_3, C_3, C_3, C_3, C_2$
486	6	1	$v^6, (uv)^9$	$C_3, C_3, C_3, C_3, C_3, C_2$
486	18	28	$u^6v^6, (uv)^2u^2vuvuv^2, (uv^3)^3$	$C_3, C_3, C_3, C_3, C_3, C_2$
504	6	1	$v^6, u^2vuvuv^2u^2v(uv)^3uv^2$	$C_7, C_2, C_2, C_3, C_3, C_2$
504	6	1	$v^6, (uv)^7u^2vuv^2$	$C_7, C_2, C_2, C_3, C_3, C_2$
504	7	7	$v^7, uvu^2vu^2vuv^2uv^2$	$L_2(8)$
504	9	15	$v^9, (uvuv^3)^2$	$L_2(8)$
504	12	22	$u^3v^2uvu^2v^3, (uv^3)^3, v^{12}$	$C_7, C_2, C_3, C_2, C_3, C_2$
504	12	22	$u^2vuv^2u^3v^3, (uv^3)^3, v^{12}$	$C_7, C_2, C_3, C_2, C_3, C_2$
504	21	31	$(uv)^4, (u^2v^2)^3, u^3v^3uv^2uv^3$	$L_2(7), C_3$
546	6	1	$v^6, u^2v(uv)^7uv^2$	C_{13}, C_7, C_2, C_3
546	6	1	$v^6, u^2v^2(uv)^3u^2v(uv)^3uv^2$	C_{13}, C_7, C_2, C_3
546	6	1	$v^6, (uv)^5u^2v(uv)^3uv^2$	C_{13}, C_7, C_2, C_3
546	6	1	$v^6, (uv)^8u^2v^2$	C_{13}, C_7, C_2, C_3
558	6	1	$v^6, u^2vuv^2(uv)^2u^2v(uv)^3uv^2$	C_{31}, C_3, C_3, C_2
558	6	1	$v^6, (uv)^6u^2v(uv)^2uv^2$	C_{31}, C_3, C_3, C_2
576	24	37	$(u^2v^2)^2$	$C_2, C_2, C_2, C_2, C_3, C_2, C_3, C_2$
576	24	37	$u^2vu^3v^2uv^3, (uv)^6$	$C_2, C_2, C_2, C_2, C_3, C_2, C_3, C_2$
576	12	25	$u^2vu^3v^2uv^3, v^{12}$	$C_2, C_2, C_2, C_2, C_3, C_2, C_3, C_2$
576	24	37	uvu^3vuv^3, u^8v^8	$C_2, C_2, C_3, C_2, C_2, C_2, C_3, C_2$
576	12	25	$(uv^4)^2, (uv)^3u^2v(uv)^2uv^2$	$C_2, C_2, C_2, C_2, C_2, C_3, C_3, C_2$
582	6	1	$v^6, u^2vuvuv^2uvu^2v(uv)^3uv^2$	C_{97}, C_2, C_3
582	6	1	$v^6, (uv)^7u^2vuvuv^2$	C_{97}, C_2, C_3
600	20	36	$(uv)^3, v^{20}$	$C_2, C_5, C_5, C_2, C_3, C_2$
600	6	1	$v^6, (uv)^{10}$	$C_5, C_2, C_5, C_2, C_2, C_3$
618	6	1	$v^6, u^2v(uv)^2uv^2u^2v(uv)^3uv^2$	C_{103}, C_2, C_3
618	6	1	$v^6, (uv)^8u^2vuv^2$	C_{103}, C_2, C_3
624	12	27	$(uv^4)^2, u^2vuv^2u^2v(uv)^2uv^2$	$C_{13}, C_2, C_2, C_2, C_2, C_3$
624	12	27	$(uv^4)^2, (uv)^5u^2vuv^2$	$C_{13}, C_2, C_2, C_2, C_2, C_3$
648	18	37	$u^2vu^2v^2uv^2, (uv)^6$	$C_3, C_3, C_2, C_3, C_2, C_2, C_3$
648	12	28	$(u^2v^3)^2, v^{12}$	$C_3, C_3, C_3, C_2, C_2, C_3, C_2$
648	36	46	$(uv)^3, u^{12}v^{12}$	$C_2, C_3, C_3, C_3, C_2, C_3, C_2$
648	6	1	$v^6, (uv)^5u^2v(uv)^4uv^2$	$C_2, C_2, C_3, C_3, C_3, C_3, C_2$
648	18	37	$u^6v^6, (uv)^6, (uv^3)^3$	$C_3, C_2, C_2, C_3, C_3, C_3, C_2$
648	12	28	$(uv)^2u^2vuvuv^2, (uv^3)^3, v^{12}$	$C_3, C_2, C_3, C_3, C_2, C_3, C_2$
654	6	1	$v^6, u^2v^2(uv)^3u^2v(uv)^4uv^2$	C_{109}, C_2, C_3
654	6	1	$v^6, (uv)^6u^2v(uv)^3uv^2$	C_{109}, C_2, C_3
660	11	26	$(uv)^5, v^{11}, (u^3v^3)^2$	$L_2(11)$
666	6	1	$v^6, u^2v(uv)^8uv^2$	C_{37}, C_3, C_3, C_2
666	6	1	$v^6, (uv)^9u^2v^2$	C_{37}, C_3, C_3, C_2

Index	Level	Genus	Additional Relators	Composition Factors
672	28	45	$u^2vu^2v^3uv^3$	$C_2, C_2, L_2(7)$
672	16	36	$u^2vuvu^2v^5$	$C_2, L_2(7), C_2$
672	16	36	$uv^2u^3v^2uv^3$	$C_2, L_2(7), C_2$
672	6	1	$v^6, u^2vuv^2(uv)^2u^2v(uv)^4uv^2$	$C_2, C_2, C_7, C_2, C_2, C_2, C_3$
672	6	1	$v^6, (uv)^7u^2v(uv)^2uv^2$	$C_7, C_2, C_2, C_2, C_2, C_2, C_3$
702	6	1	$v^6, u^2vuvuv^2uvu^2v(uv)^4uv^2$	$C_{13}, C_3, C_3, C_3, C_2$
702	6	1	$v^6, (uv)^8u^2vuvuv^2$	$C_{13}, C_3, C_3, C_3, C_2$
720	60	55	u^5v^5	C_2, C_2, A_5, C_3
720	20	43	$(uvuv^2)^2$	C_2, A_5, C_3, C_2
720	15	37	$(uv^2)^3, v^{15}$	C_2, C_2, A_5, C_3
720	10	25	$(uv)^4, v^{10}, uv^3u^4v^3uv^4$	A_6, C_2
720	8	16	$v^8, (uv)^5, u^2v^3u^3vuv^2uv^3$	A_6, C_2
726	22	45	$(uv)^3, v^{22}$	C_{11}, C_{11}, C_3, C_2
726	6	1	$v^6, (uv)^{11}$	C_{11}, C_{11}, C_2, C_3
744	6	1	$v^6, u^2v(uv)^2uv^2u^2v(uv)^4uv^2$	$C_{31}, C_2, C_2, C_2, C_3$
744	6	1	$v^6, (uv)^9u^2vuv^2$	$C_{31}, C_2, C_2, C_2, C_3$
750	30	51	$(uv)^5, u^6v^6$	C_5, C_5, C_5, C_2, C_3
750	10	26	$v^{10}, (uv)^2u^2vuvuv^2$	C_5, C_5, C_5, C_3, C_2
762	6	1	$v^6, u^2v^2(uv)^4u^2v(uv)^4uv^2$	C_{127}, C_2, C_3
762	6	1	$v^6, (uv)^6u^2v(uv)^4uv^2$	C_{127}, C_2, C_3
768	12	33	$u^2vu^2v^2uv^2, v^{12}$	$C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_3$
768	12	33	$u^2v^2u^3vuv^3, v^{12}$	$C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_3$
768	16	41	$(u^2v^3)^2, (uv)^6$	$C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_3, C_2$
768	16	41	uvu^3vuv^3, v^{16}	$C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_3, C_2$
768	12	33	$uvu^3v^2u^2v^3, v^{12}$	$C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_3$
768	12	33	$(uv^4)^2, (uv)^8$	$C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_3$
768	8	17	$v^8, (uv^2)^4$	$C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_3, C_2$
768	12	33	$(uv)^4, (u^3v^3)^2, v^{12}$	$C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_2, C_3$
774	6	1	$v^6, u^2vuv^2(uv)^3u^2v(uv)^4uv^2$	C_{43}, C_3, C_3, C_2
774	6	1	$v^6, (uv)^7u^2v(uv)^3uv^2$	C_{43}, C_3, C_3, C_2
798	6	1	$v^6, u^2v(uv)^9uv^2$	C_{19}, C_7, C_2, C_3
798	6	1	$v^6, u^2vuvuv^2(uv)^2u^2v(uv)^4uv^2$	C_{19}, C_7, C_2, C_3
798	6	1	$v^6, (uv)^8u^2v(uv)^2uv^2$	C_{19}, C_7, C_2, C_3
798	6	1	$v^6, (uv)^{10}u^2v^2$	C_{19}, C_7, C_2, C_3
834	6	1	$v^6, u^2v(uv)^2uv^2uvu^2v(uv)^4uv^2$	C_{139}, C_2, C_3
834	6	1	$v^6, (uv)^9u^2vuvuv^2$	C_{139}, C_2, C_3
864	24	55	$(uv)^3, v^{24}$	$C_2, C_3, C_2, C_3, C_2, C_2, C_3, C_2$
864	6	1	$v^6, (uv)^{12}$	$C_2, C_2, C_2, C_2, C_3, C_3, C_3, C_2$
864	12	37	$(uv)^6, (uv^3)^3, v^{12}$	$C_3, C_2, C_2, C_2, C_3, C_2, C_3, C_2$
882	42	64	$(uv)^3, u^{14}v^{14}$	C_7, C_3, C_7, C_3, C_2
882	6	1	$v^6, u^2v(uv)^3uv^2u^2v(uv)^4uv^2$	C_7, C_7, C_3, C_3, C_2
882	6	1	$v^6, (uv)^6u^2v(uv)^5uv^2$	C_7, C_7, C_3, C_3, C_2
882	6	1	$v^6, (uv)^{10}u^2vuv^2$	C_7, C_7, C_3, C_3, C_2

Index	Level	Genus	Additional Relators	Composition Factors
888	6	1	$v^6, u^2v^2(uv)^4u^2v(uv)^5uv^2$	$C_{37}, C_2, C_2, C_2, C_3$
888	6	1	$v^6, (uv)^7u^2v(uv)^4uv^2$	$C_{37}, C_2, C_2, C_2, C_3$
906	6	1	$v^6, u^2vuv^2(uv)^3u^2v(uv)^5uv^2$	C_{151}, C_2, C_3
906	6	1	$v^6, (uv)^8u^2v(uv)^3uv^2$	C_{151}, C_2, C_3
912	12	39	$(uv^4)^2, u^2v^2(uv)^2u^2v(uv)^3uv^2$	$C_{19}, C_2, C_2, C_2, C_2, C_3$
912	12	39	$(uv^4)^2, (uv)^4u^2v^2u^2vuvuv^2$	$C_{19}, C_2, C_2, C_2, C_2, C_3$
936	6	1	$v^6, u^2vuvuv^2(uv)^2u^2v(uv)^5uv^2$	$C_{13}, C_2, C_2, C_3, C_3, C_2$
936	6	1	$v^6, (uv)^9u^2v(uv)^2uv^2$	$C_{13}, C_2, C_2, C_3, C_3, C_2$
936	12	40	$(uv^3)^3, v^{12}, u^2v^2u^2v(uv)^2uv^2$	$C_{13}, C_2, C_3, C_2, C_3, C_2$
936	12	40	$(uv^3)^3, v^{12}, (uv)^3u^2v^2u^2v^2$	$C_{13}, C_2, C_3, C_2, C_3, C_2$
942	6	1	$v^6, u^2v(uv)^{10}uv^2$	C_{157}, C_2, C_3
942	6	1	$v^6, (uv)^{11}u^2v^2$	C_{157}, C_2, C_3
978	6	1	$v^6, u^2v(uv)^2uv^2uvu^2v(uv)^5uv^2$	C_{163}, C_2, C_3
978	6	1	$v^6, (uv)^{10}u^2vuvuv^2$	C_{163}, C_2, C_3
1008	24	64	$(u^3v^3)^2, (u^2v^2)^3$	$L_2(7), C_2, C_3$
1008	24	64	$u^2vu^2vuv^2uv^2, (u^2v^5)^2$	$C_7, C_2, C_2, C_3, C_2, C_3, C_2$
1008	14	49	$(u^2v^2)^3, (uv^2)^4$	$L_2(7), C_3, C_2$
1008	24	64	$uvu^2vu^2v^2uv^2, (u^2v^5)^2$	$C_7, C_2, C_2, C_3, C_2, C_3, C_2$
1008	24	64	$(uv)^4, u^2v^2u^3v^2u^2v^3$	$L_2(7), C_2, C_3$
1008	42	73	$(uv)^4, (u^2v^2)^3$	$C_2, L_2(7), C_3$
1008	18	57	$(uvuv^3)^2, (u^3v^4)^2$	$C_2, L_2(8)$
1008	12	43	$(uv^4)^2, u^2vuvuv^2u^2v(uv)^3uv^2$	$C_7, C_2, C_2, C_2, C_3, C_3, C_2$
1008	12	43	$(uv^4)^2, (uv)^6u^2v^2u^2v^2$	$C_7, C_2, C_2, C_2, C_3, C_3, C_2$
1008	14	49	$(uv^5)^2, uvu^2vu^2vuv^2uv^2$	$C_2, L_2(8)$
1008	8	22	$v^8, (u^2v^2)^3$	$L_2(7), C_3, C_2$
1008	8	22	$v^8, (u^2v^2uv^2)^2$	$L_2(7), C_3, C_2$
1014	26	66	$(uv)^3, v^{26}$	C_{13}, C_{13}, C_3, C_2
1014	6	1	$v^6, u^2v^2(uv)^5u^2v(uv)^5uv^2$	C_{13}, C_{13}, C_2, C_3
1014	6	1	$v^6, (uv)^7u^2v(uv)^5uv^2$	C_{13}, C_{13}, C_2, C_3
1014	6	1	$v^6, (uv)^{13}$	C_{13}, C_{13}, C_2, C_3
1026	6	1	$v^6, u^2vuv^2(uv)^4u^2v(uv)^5uv^2$	$C_{19}, C_3, C_3, C_3, C_2$
1026	6	1	$v^6, (uv)^8u^2v(uv)^4uv^2$	$C_{19}, C_3, C_3, C_3, C_2$
1032	6	1	$v^6, u^2v(uv)^3uv^2u^2v(uv)^5uv^2$	$C_{43}, C_2, C_2, C_2, C_3$
1032	6	1	$v^6, (uv)^{11}u^2vuv^2$	$C_{43}, C_2, C_2, C_2, C_3$
1050	6	1	$v^6, u^2vuvuv^2(uv)^3u^2v(uv)^5uv^2$	C_7, C_5, C_5, C_2, C_3
1050	6	1	$v^6, (uv)^9u^2v(uv)^3uv^2$	C_5, C_7, C_5, C_2, C_3
1080	30	73	$(u^2v^2)^3, uvu^4vuv^4$	C_3, A_5, C_3, C_2
1086	6	1	$v^6, u^2v(uv)^2uv^2(uv)^2u^2v(uv)^5uv^2$	C_{181}, C_2, C_3
1086	6	1	$v^6, (uv)^{10}u^2v(uv)^2uv^2$	C_{181}, C_2, C_3
1092	7	14	$v^7, (uv)^6$	$L_2(13)$
1092	7	14	$v^7, (uv)^7$	$L_2(13)$
1092	7	14	$v^7, (uvuv^2)^3$	$L_2(13)$
1092	13	50	$(uvuv^3)^2, v^{13}, u^2v^2u^2v^4uv^4$	$L_2(13)$

Index	Level	Genus	Additional Relators	Composition Factors
1098	6	1	$v^6, u^2v(uv)^{11}uv^2$	C_{61}, C_3, C_3, C_2
1098	6	1	$v^6, (uv)^{12}u^2v^2$	C_{61}, C_3, C_3, C_2
1134	6	1	$v^6, u^2v(uv)^3uv^2uvu^2v(uv)^5uv^2$	$C_7, C_3, C_3, C_3, C_3, C_2$
1134	6	1	$v^6, (uv)^{11}u^2vuvuv^2$	$C_7, C_3, C_3, C_3, C_3, C_2$
1134	18	64	$u^6v^6, (uv^3)^3, u^2v^2u^2v^2u^2v(uv)^2uv^2$	$C_3, C_7, C_3, C_3, C_3, C_2$
1134	18	64	$u^6v^6, (uv^3)^3, (uv)^5u^2vuvuv^2$	$C_3, C_7, C_3, C_3, C_3, C_2$
1152	24	73	$u^2vu^3v^2uv^3$	$C_2, C_2, C_2, C_2, C_2, C_3, C_2, C_3, C_2$
1152	48	85	$(uv)^3, u^{16}v^{16}$	$C_2, C_2, C_2, C_3, C_2, C_2, C_2, C_3, C_2$
1152	6	1	$v^6, (uv)^7u^2v(uv)^6uv^2$	$C_2, C_2, C_2, C_2, C_2, C_2, C_3, C_3, C_2$
1152	24	73	$u^6v^6, (uv^2uv^3)^2, (uv)^3u^2v(uv)^2uv^2$	$C_2, C_2, C_2, C_2, C_2, C_2, C_3, C_3, C_2$
1152	48	85	$(uv)^2u^2vuvuv^2, u^3vu^3v^3uv^3, u^8v^8$	$C_2, C_2, C_2, C_3, C_2, C_2, C_2, C_3, C_2$
1152	24	73	$(uv)^6, (uv^3)^3, u^2v^2u^4v^2u^2v^4$	$C_2, C_2, C_2, C_2, C_3, C_2, C_2, C_3, C_2$
1152	12	49	$(uv^3)^3, v^{12}, u^2v^2uvu^2v^2u^2vuv^2$	$C_2, C_2, C_2, C_2, C_2, C_3, C_2, C_3, C_2$
1158	6	1	$v^6, u^2v^2(uv)^5u^2v(uv)^6uv^2$	C_{193}, C_2, C_3
1158	6	1	$v^6, (uv)^8u^2v(uv)^5uv^2$	C_{193}, C_2, C_3
1176	28	78	$(uv)^3, v^{28}$	$C_2, C_7, C_7, C_2, C_3, C_2$
1176	6	1	$v^6, u^2vuv^2(uv)^4u^2v(uv)^6uv^2$	$C_7, C_7, C_2, C_2, C_2, C_3$
1176	6	1	$v^6, (uv)^9u^2v(uv)^4uv^2$	$C_7, C_7, C_2, C_2, C_2, C_3$
1176	6	1	$v^6, (uv)^{14}$	$C_7, C_2, C_7, C_2, C_2, C_3$
1194	6	1	$v^6, u^2v(uv)^4uv^2u^2v(uv)^5uv^2$	C_{199}, C_2, C_3
1194	6	1	$v^6, (uv)^{12}u^2vuv^2$	C_{199}, C_2, C_3
1200	40	86	$uvu^3vuv^3, u^2vu^2v^{17}$	$C_2, C_2, C_5, C_5, C_2, C_3, C_2$
1200	12	51	$(uv^4)^2, u^2v(uv)^7u^2v^3$	$C_2, C_5, C_2, C_5, C_2, C_2, C_3$
1200	12	51	$(uv)^4, v^{12}, u^5vu^2v^2uv^5$	$C_5, C_5, C_2, C_2, C_2, C_2, C_3$
1200	12	51	$(uv)^4, v^{12}, uv^2u^2vu^5v^5$	$C_5, C_5, C_2, C_2, C_2, C_2, C_3$
1206	6	1	$v^6, u^2vuvuv^2(uv)^3u^2v(uv)^6uv^2$	C_{67}, C_3, C_3, C_2
1206	6	1	$v^6, (uv)^{10}u^2v(uv)^3uv^2$	C_{67}, C_3, C_3, C_2
1248	6	1	$v^6, u^2v(uv)^2uv^2(uv)^2u^2v(uv)^6uv^2$	$C_{13}, C_2, C_2, C_2, C_2, C_2, C_3$
1248	6	1	$v^6, (uv)^{11}u^2v(uv)^2uv^2$	$C_{13}, C_2, C_2, C_2, C_2, C_2, C_3$
1266	6	1	$v^6, u^2v(uv)^{12}uv^2$	C_{211}, C_2, C_3
1266	6	1	$v^6, (uv)^{13}u^2v^2$	C_{211}, C_2, C_3
1296	36	91	$u^6v^6, u^2vuvu^2v^2uvuv^2$	$C_3, C_2, C_2, C_2, C_3, C_3, C_3, C_2$
1296	24	82	$u^2vu^2vuv^2uv^2, (uv)^6$	$C_3, C_3, C_2, C_2, C_3, C_2, C_3, C_2$
1296	36	91	$u^2vu^2v^2uv^2, u^9v^9$	$C_2, C_3, C_3, C_2, C_3, C_2, C_2, C_3$
1296	24	82	$(u^2v^3)^2, (uv)^2u^2vuvuv^2uv^2uv^2$	$C_2, C_3, C_3, C_3, C_2, C_2, C_3, C_2$
1296	72	100	$uvu^3vuv^3, u^{12}v^{12}$	$C_2, C_2, C_3, C_3, C_3, C_2, C_3, C_2$
1296	24	82	$uvu^2vu^2v^2uv^2, (uv)^6$	$C_3, C_3, C_2, C_2, C_3, C_2, C_3, C_2$
1296	24	82	$(uv)^2u^2vuvuv^2, (u^2v^5)^2$	$C_3, C_2, C_2, C_3, C_3, C_2, C_3, C_2$
1296	12	55	$(uv^4)^2, (uv)^5u^2v(uv)^4uv^2$	$C_2, C_2, C_2, C_3, C_3, C_3, C_3, C_2$
1296	8	28	$v^8, u^2v^2u^3vuvuv^3$	$C_3, C_3, C_3, C_2, C_2, C_2, C_3, C_2$
1296	8	28	$v^8, (uv)^2u^3v^2u^2v^3$	$C_3, C_3, C_3, C_2, C_2, C_2, C_3, C_2$

Index	Level	Genus	Additional Relators	Composition Factors
1302	6	1	$v^6, u^2v(uv)^3uv^2uvu^2v(uv)^6uv^2$	C_{31}, C_7, C_2, C_3
1302	6	1	$v^6, u^2v^2(uv)^6u^2v(uv)^6uv^2$	C_{31}, C_7, C_2, C_3
1302	6	1	$v^6, (uv)^8u^2v(uv)^6uv^2$	C_{31}, C_7, C_2, C_3
1302	6	1	$v^6, (uv)^{12}u^2vuvuv^2$	C_{31}, C_7, C_2, C_3
1314	6	1	$v^6, u^2vuv^2(uv)^5u^2v(uv)^6uv^2$	C_{73}, C_3, C_3, C_2
1314	6	1	$v^6, (uv)^9u^2v(uv)^5uv^2$	C_{73}, C_3, C_3, C_2
1320	10	45	$v^{10}, (u^3v^3)^2$	$L_2(11), C_2$
1320	22	81	$(uv)^5, (u^3v^3)^2, (u^2v^2uv^2)^2$	$C_2, L_2(11)$
1320	12	56	$(uv)^5, (uv^2)^4, v^{12}$	$L_2(11), C_2$
1320	10	45	$v^{10}, (uv^2)^4, u^2v^2u^3v^2u^2v^3$	$L_2(11), C_2$
1320	12	56	$(uv)^6, v^{12}, (u^3v^4)^2, uvu^2vu^2vuv^2uv^2$	$L_2(11), C_2$
1338	6	1	$v^6, u^2vuvuv^2(uv)^4u^2v(uv)^6uv^2$	C_{223}, C_2, C_3
1338	6	1	$v^6, (uv)^{10}u^2v(uv)^4uv^2$	C_{223}, C_2, C_3
1344	12	57	$(uv^4)^2, u^2vuv^2(uv)^2u^2v(uv)^4uv^2$	$C_7, C_2, C_2, C_2, C_2, C_2, C_2, C_3$
1344	12	57	$(uv^4)^2, (uv)^7u^2v(uv)^2uv^2$	$C_7, C_2, C_2, C_2, C_2, C_2, C_2, C_3$
1344	7	17	$v^7, (uv)^8, u^2v^2u^2vu^2vuvuv^2uv^2$	$C_2, C_2, C_2, L_2(7)$
1344	7	17	$v^7, (uv)^8, (uv)^2u^2vu^2v^2u^2v^2uv^2$	$C_2, C_2, C_2, L_2(7)$
1350	30	91	$(uv)^3, v^{30}$	$C_5, C_3, C_5, C_3, C_3, C_2$
1350	6	1	$v^6, (uv)^{15}$	$C_5, C_5, C_3, C_3, C_3, C_2$
1368	6	1	$v^6, u^2v(uv)^4uv^2u^2v(uv)^6uv^2$	$C_{19}, C_2, C_2, C_3, C_3, C_2$
1368	6	1	$v^6, (uv)^{13}u^2vuv^2$	$C_{19}, C_2, C_2, C_3, C_3, C_2$
1368	12	58	$(uv^3)^3, v^{12}, u^2v(uv)^2u^2v^2uvu^2v^3$	$C_{19}, C_2, C_3, C_2, C_3, C_2$
1368	12	58	$(uv^3)^3, v^{12}, u^2vuv^2u^2v(uv)^2u^2v^3$	$C_{19}, C_2, C_3, C_2, C_3, C_2$
1374	6	1	$v^6, u^2v(uv)^2uv^2(uv)^3u^2v(uv)^6uv^2$	C_{229}, C_2, C_3
1374	6	1	$v^6, (uv)^{11}u^2v(uv)^3uv^2$	C_{229}, C_2, C_3
1422	6	1	$v^6, u^2v(uv)^3uv^2(uv)^2u^2v(uv)^6uv^2$	C_{79}, C_3, C_3, C_2
1422	6	1	$v^6, (uv)^{12}u^2v(uv)^2uv^2$	C_{79}, C_3, C_3, C_2
1440	30	97	$(uv^2)^3$	C_2, C_2, C_2, A_5, C_3
1440	20	85	$(u^2v^4)^2, uv^2(uv)^4uv^2uv^3$	C_2, A_5, C_2, C_3, C_2
1440	30	97	$(u^2vuv^2)^2, (uvu^2v^2)^2, u^3vu^3v^{11}$	C_2, C_2, A_5, C_2, C_3
1440	15	73	$(u^2vuv^2)^2, (uvu^2v^2)^2, v^{15}$	C_2, C_2, C_2, A_5, C_3
1440	16	76	$(uv)^5, (uv^6)^2, u^2v^3u^3vuv^2uv^3$	C_2, A_6, C_2
1440	10	49	$v^{10}, (uv)^2u^3vuvuv^3, u^3vu^2v^2uv^2uv^4$	C_2, A_6, C_2
1446	6	1	$v^6, u^2v(uv)^{13}uv^2$	C_{241}, C_2, C_3
1446	6	1	$v^6, (uv)^{14}u^2v^2$	C_{241}, C_2, C_3
1458	54	109	$(uv)^3, u^{18}v^{18}$	$C_3, C_3, C_3, C_3, C_3, C_3, C_2$
1458	6	1	$v^6, (uv)^8u^2v(uv)^7uv^2$	$C_3, C_3, C_3, C_3, C_3, C_3, C_2$
1458	18	82	$u^6v^6, (uv^3)^3, (uv)^9$	$C_3, C_3, C_3, C_3, C_3, C_3, C_2$
1458	18	82	$(u^2v^2)^3, (uv^3)^3, uvu^5vuv^5$	$C_3, C_3, C_3, C_3, C_3, C_3, C_2$
1458	18	82	$(uv)^2u^2vuvuv^2, (uv^3)^3, v^{18}$	$C_3, C_3, C_3, C_3, C_3, C_3, C_2$
1458	18	82	$(uv^3)^3, uvu^5vuv^5, u^3v^2uvu^2vu^2v^4$	$C_3, C_3, C_3, C_3, C_3, C_3, C_2$

Index	Level	Genus	Additional Relators	Composition Factors
1464	6	1	$v^6, u^2v^2(uv)^6u^2v(uv)^7uv^2$	$C_{61}, C_2, C_2, C_2, C_3$
1464	6	1	$v^6, (uv)^9u^2v(uv)^6uv^2$	$C_{61}, C_2, C_2, C_2, C_3$
1482	6	1	$v^6, u^2v(uv)^4uv^2uvu^2v(uv)^6uv^2$	C_{19}, C_{13}, C_2, C_3
1482	6	1	$v^6, u^2vuv^2(uv)^5u^2v(uv)^7uv^2$	C_{19}, C_{13}, C_2, C_3
1482	6	1	$v^6, (uv)^{10}u^2v(uv)^5uv^2$	C_{19}, C_{13}, C_2, C_3
1482	6	1	$v^6, (uv)^{13}u^2vuvuv^2$	C_{19}, C_{13}, C_2, C_3
1488	12	63	$(uv^4)^2, u^2v(uv)^2uv^2u^2v(uv)^4uv^2$	$C_{31}, C_2, C_2, C_2, C_2, C_3$
1488	12	63	$(uv^4)^2, (uv)^9u^2vuv^2$	$C_{31}, C_2, C_2, C_2, C_2, C_3$
1500	15	76	$u^2vu^2v^2uv^2, v^{15}$	$C_5, C_5, C_2, C_5, C_2, C_3$

4. Normal subgroups of Hecke groups

The tables below give the numbers of normal subgroups of index up to 1500 in the modular group, and of normal subgroups of index up to 500 in each of the Hecke groups $H_q = C_2 * C_q$ for $4 \le q \le 12$. The subgroups themselves have been found but would take too much space to present in this paper. Further details are available upon request from the second author.

Case $q = 3$: 309 normal subgroups of index up to 1500 in the group $C_2 * C_3$:

Index	1	2	3	6	12	18	24	42	48	54	60	72
Number	1	1	1	2	1	1	2	2	2	1	1	2

Index	78	96	114	120	126	144	150	162	168	180
Number	2	2	2	1	2	2	2	2	3	1

Index	186	192	216	222	234	240	258	288	294	312
Number	2	2	2	2	2	1	2	3	4	2

Index	324	336	342	360	366	378	384	402	432	438
Number	1	5	2	2	2	2	4	2	4	2

Index	450	456	474	486	504	546	558	576	582	600
Number	2	2	2	5	7	4	2	5	2	2

Index	618	624	648	654	660	666	672	702	720	726
Number	2	2	6	2	1	2	5	2	5	2

Index	744	750	762	768	774	798	834	864	882	888
Number	2	2	2	8	2	4	2	3	4	2

Index	906	912	936	942	978	1008	1014	1026	1032
Number	2	2	4	2	2	12	4	2	2

Index	1050	1080	1086	1092	1098	1134	1152	1158
Number	2	1	2	4	2	4	7	2

Index	1176	1194	1200	1206	1248	1266	1296	1302
Number	4	2	4	2	2	2	10	4

Index	1314	1320	1338	1344	1350	1368	1374	1422
Number	2	5	2	4	2	4	2	2

Index	1440	1446	1458	1464	1482	1488	1500
Number	6	2	6	2	4	2	1

Case $q = 4$: 688 normal subgroups of index up to 500 in the group $C_2 * C_4$:

Index	1	2	4	6	8	10	12	14	16	18	20	22
Number	1	3	3	1	3	1	1	1	3	1	3	1

Index	24	26	28	30	32	34	36	38	40	42	44
Number	4	1	1	1	5	1	2	1	5	1	1

Index	46	48	50	52	54	56	58	60	62	64	66
Number	1	4	1	3	1	3	1	1	1	7	1

Index	68	70	72	74	76	78	80	82	84	86	88
Number	3	1	6	1	1	1	5	1	1	1	3

Index	90	92	94	96	98	100	102	104	106	108
Number	1	1	1	8	1	4	1	5	1	2

Index	110	112	114	116	118	120	122	124	126	128
Number	1	3	1	3	1	8	1	1	1	11

Index	130	132	134	136	138	140	142	144	146	148
Number	1	1	1	5	1	1	1	8	1	3

Index	150	152	154	156	158	160	162	164	166	168
Number	1	3	1	1	1	12	1	3	1	5

Index	170	172	174	176	178	180	182	184	186	188
Number	1	1	1	3	1	3	1	3	1	1

Index	190	192	194	196	198	200	202	204	206	208
Number	1	15	1	2	1	9	1	1	1	5

Index	210	212	214	216	218	220	222	224	226	228
Number	1	3	1	8	1	1	1	5	1	1

Index	230	232	234	236	238	240	242	244	246	248
Number	1	5	1	1	1	10	1	3	1	3

Index	250	252	254	256	258	260	262	264	266	268
Number	1	1	1	19	1	5	1	4	1	1

Index	270	272	274	276	278	280	282	284	286	288
Number	1	5	1	1	1	5	1	1	1	16

Index	290	292	294	296	298	300	302	304	306	308
Number	1	3	1	5	1	1	1	3	1	1

Index	310	312	314	316	318	320	322	324	326	328
Number	1	6	1	1	1	22	1	2	1	5

Index	330	332	334	336	338	340	342	344	346	348
Number	1	1	1	10	1	5	1	3	1	1

Index	350	352	354	356	358	360	362	364	366	368
Number	1	5	1	3	1	12	1	1	1	3

Index	370	372	374	376	378	380	382	384	386	388
Number	1	1	1	3	1	1	1	29	1	3

Index	390	392	394	396	398	400	402	404	406	408
Number	1	5	1	1	1	9	1	3	1	6

Index	410	412	414	416	418	420	422	424	426	428
Number	1	1	1	11	1	1	1	5	1	1

Index	430	432	434	436	438	440	442	444	446	448
Number	1	12	1	3	1	5	1	1	1	7

Index	450	452	454	456	458	460	462	464	466	468
Number	1	3	1	4	1	1	1	5	1	3

Index	470	472	474	476	478	480	482	484	486	488
Number	1	3	1	1	1	25	1	2	1	5

Index	490	492	494	496	498	500
Number	1	1	1	3	1	6

Case $q = 5$: 44 normal subgroups of index up to 500 in the group $C_2 * C_5$:

Index	1	2	5	10	50	60	80	110	120	160
Number	1	1	1	2	1	2	1	4	4	4

Index	240	250	300	310	320	360	410
Number	4	1	2	4	6	2	4

Case $q = 6$: 862 normal subgroups of index up to 500 in the group $C_2 * C_6$:

Index	1	2	3	4	6	8	10	12	14	16	18	20
Number	1	3	1	1	5	1	1	4	1	1	3	1

Index	22	24	26	28	30	32	34	36	38	40	42
Number	1	7	1	1	2	1	1	4	1	1	6

Index	44	46	48	50	52	54	56	58	60	62	64
Number	1	1	9	1	1	6	1	1	4	1	1

Index	66	68	70	72	74	76	78	80	82	84	86
Number	2	1	1	8	1	1	6	1	1	7	1

Index	88	90	92	94	96	98	100	102	104	106
Number	1	2	1	1	13	1	1	2	1	1

Index	108	110	112	114	116	118	120	122	124	126
Number	9	1	1	6	1	1	10	1	1	6

Index	128	130	132	134	136	138	140	142	144	146
Number	1	1	3	1	1	2	1	1	13	1

Index	148	150	152	154	156	158	160	162	164	166
Number	1	5	1	1	7	1	1	11	1	1

Index	168	170	172	174	176	178	180	182	184	186
Number	13	1	1	2	1	1	5	1	1	6

Index	188	190	192	194	196	198	200	202	204	206
Number	1	1	23	1	1	2	1	1	3	1

Index	208	210	212	214	216	218	220	222	224	226
Number	1	4	1	1	21	1	1	6	1	1

Index	228	230	232	234	236	238	240	242	244	246
Number	7	1	1	6	1	1	13	1	1	2

Index	248	250	252	254	256	258	260	262	264	266
Number	1	1	12	1	1	6	1	1	4	1

Index	268	270	272	274	276	278	280	282	284	286
Number	1	5	1	1	3	1	1	2	1	1

Index	288	290	292	294	296	298	300	302	304	306
Number	23	1	1	11	1	1	7	1	1	2

Index	308	310	312	314	316	318	320	322	324	326
Number	1	1	12	1	1	2	1	1	24	1

Index	328	330	332	334	336	338	340	342	344	346
Number	1	2	1	1	27	1	1	6	1	1

Index	348	350	352	354	356	358	360	362	364	366
Number	3	1	1	2	1	1	15	1	1	6

Index	368	370	372	374	376	378	380	382	384	386
Number	1	1	7	1	1	15	1	1	57	1

Index	388	390	392	394	396	398	400	402	404	406
Number	1	4	1	1	4	1	1	6	1	1

Index	408	410	412	414	416	418	420	422	424	426
Number	4	1	1	2	1	1	7	1	1	2

Index	428	430	432	434	436	438	440	442	444	446
Number	1	1	33	1	1	6	1	1	7	1

Index	448	450	452	454	456	458	460	462	464	466
Number	1	5	1	1	12	1	1	4	1	1

Index	468	470	472	474	476	478	480	482	484	486
Number	12	1	1	6	1	1	24	1	1	20

Index	488	490	492	494	496	498	500
Number	1	1	3	1	1	2	1

Case $q = 7$: 26 normal subgroups of index up to 500 in the group $C_2 * C_7$:

Index	1	2	7	14	56	98	112	168
Number	1	1	1	2	2	1	2	3

Index	336	406	448
Number	6	6	1

Case $q = 8$: 1187 normal subgroups of index up to 500 in the group $C_2 * C_8$:

Index	1	2	4	6	8	10	12	14	16	18	20	22
Number	1	3	3	1	5	1	1	1	7	1	3	1

Index	24	26	28	30	32	34	36	38	40	42	44
Number	4	1	1	1	9	1	2	1	5	1	1

Index	46	48	50	52	54	56	58	60	62	64	66
Number	1	10	1	3	1	3	1	1	1	17	1

Index	68	70	72	74	76	78	80	82	84	86	88
Number	3	1	8	1	1	1	13	1	1	1	3

Index	90	92	94	96	98	100	102	104	106	108
Number	1	1	1	17	1	4	1	5	1	2

Index	110	112	114	116	118	120	122	124	126	128
Number	1	7	1	3	1	8	1	1	1	35

Index	130	132	134	136	138	140	142	144	146	148
Number	1	1	1	9	1	1	1	22	1	3

Index	150	152	154	156	158	160	162	164	166	168
Number	1	3	1	1	1	24	1	3	1	5

Index	170	172	174	176	178	180	182	184	186	188
Number	1	1	1	7	1	3	1	3	1	1

Index	190	192	194	196	198	200	202	204	206	208
Number	1	41	1	2	1	11	1	1	1	13

Index	210	212	214	216	218	220	222	224	226	228
Number	1	3	1	10	1	1	1	9	1	1

Index	230	232	234	236	238	240	242	244	246	248
Number	1	5	1	1	1	22	1	3	1	3

Index	250	252	254	256	258	260	262	264	266	268
Number	1	1	1	71	1	5	1	4	1	1

Index	270	272	274	276	278	280	282	284	286	288
Number	1	17	1	1	1	5	1	1	1	37

Index	290	292	294	296	298	300	302	304	306	308
Number	1	3	1	5	1	1	1	7	1	1

Index	310	312	314	316	318	320	322	324	326	328
Number	1	6	1	1	1	60	1	2	1	9

Index	330	332	334	336	338	340	342	344	346	348
Number	1	1	1	26	1	5	1	3	1	1

Index	350	352	354	356	358	360	362	364	366	368
Number	1	9	1	3	1	12	1	1	1	7

Index	370	372	374	376	378	380	382	384	386	388
Number	1	1	1	3	1	1	1	99	1	3

Index	390	392	394	396	398	400	402	404	406	408
Number	1	7	1	1	1	29	1	3	1	6

Index	410	412	414	416	418	420	422	424	426	428
Number	1	1	1	23	1	1	1	5	1	1

Index	430	432	434	436	438	440	442	444	446	448
Number	1	42	1	3	1	5	1	1	1	17

Index	450	452	454	456	458	460	462	464	466	468
Number	1	3	1	4	1	1	1	13	1	3

Index	470	472	474	476	478	480	482	484	486	488
Number	1	3	1	1	1	52	1	2	1	5

Index	490	492	494	496	498	500
Number	1	1	1	7	1	6

Case $q = 9$: 184 normal subgroups of index up to 500 in the group $C_2 * C_9$:

Index	1	2	3	6	9	12	18	24	36	42	48	54
Number	1	1	1	2	1	1	3	2	1	2	2	5

Index	60	72	78	96	114	120	126	144	150	162
Number	1	4	2	2	2	1	4	4	2	11

Index	168	180	186	192	216	222	234	240	258	288
Number	3	1	2	2	10	2	4	1	2	5

Index	294	312	324	336	342	360	366	378	384	402
Number	4	2	4	5	10	2	2	10	4	2

Index	432	438	450	456	474	486
Number	12	2	4	2	2	30

Case $q = 10$: 469 normal subgroups of index up to 500 in the group $C_2 * C_{10}$:

Index	1	2	4	5	6	8	10	12	14	16	18	20
Number	1	3	1	1	1	1	5	1	1	1	1	3

Index	22	24	26	28	30	32	34	36	38	40	42
Number	1	1	1	1	2	1	1	1	1	3	1

Index	44	46	48	50	52	54	56	58	60	62	64
Number	1	1	1	3	1	1	1	1	5	1	1

Index	66	68	70	72	74	76	78	80	82	84	86
Number	1	1	2	1	1	1	1	4	1	1	1

Index	88	90	92	94	96	98	100	102	104	106
Number	1	2	1	1	1	1	4	1	1	1

Index	108	110	112	114	116	118	120	122	124	126
Number	1	10	1	1	1	1	11	1	1	1

Index	128	130	132	134	136	138	140	142	144	146
Number	1	2	1	1	1	1	3	1	1	1

Index	148	150	152	154	156	158	160	162	164	166
Number	1	3	1	1	1	1	11	1	1	1

Index	168	170	172	174	176	178	180	182	184	186
Number	1	2	1	1	1	1	3	1	1	1

Index	188	190	192	194	196	198	200	202	204	206
Number	1	2	1	1	1	1	5	1	1	1

Index	208	210	212	214	216	218	220	222	224	226
Number	1	2	1	1	1	1	11	1	1	1

Index	228	230	232	234	236	238	240	242	244	246
Number	1	2	1	1	1	1	15	1	1	1

Index	248	250	252	254	256	258	260	262	264	266
Number	1	8	1	1	1	1	3	1	1	1

Index	268	270	272	274	276	278	280	282	284	286
Number	1	2	1	1	1	1	3	1	1	1

Index	288	290	292	294	296	298	300	302	304	306
Number	1	2	1	1	1	1	8	1	1	1

Index	308	310	312	314	316	318	320	322	324	326
Number	1	10	1	1	1	1	21	1	1	1

Index	328	330	332	334	336	338	340	342	344	346
Number	1	6	1	1	1	1	3	1	1	1

Index	348	350	352	354	356	358	360	362	364	366
Number	1	2	1	1	1	1	7	1	1	1

Index	368	370	372	374	376	378	380	382	384	386
Number	1	2	1	1	1	1	3	1	1	1

Index	388	390	392	394	396	398	400	402	404	406
Number	1	2	1	1	1	1	5	1	1	1

Index	408	410	412	414	416	418	420	422	424	426
Number	1	10	1	1	1	1	3	1	1	1

Index	428	430	432	434	436	438	440	442	444	446
Number	1	2	1	1	1	1	11	1	1	1

Index	448	450	452	454	456	458	460	462	464	466
Number	1	3	1	1	1	1	3	1	1	1

Index	468	470	472	474	476	478	480	482	484	486
Number	1	2	1	1	1	1	16	1	1	1

Index	488	490	492	494	496	498	500
Number	1	2	1	1	1	1	11

Case $q = 11$: 6 normal subgroups of index up to 500 in the group $C_2 * C_{11}$:

Index	1	2	11	22	242
Number	1	1	1	2	1

Case $q = 12$: 1798 normal subgroups of index up to 500 in the group $C_2 * C_{12}$:

Index	1	2	3	4	6	8	10	12	14	16	18	20
Number	1	3	1	3	5	3	1	6	1	3	3	3

Index	22	24	26	28	30	32	34	36	38	40	42
Number	1	14	1	1	2	5	1	5	1	5	6

Index	44	46	48	50	52	54	56	58	60	62	64
Number	1	1	20	1	3	6	3	1	6	1	7

Index	66	68	70	72	74	76	78	80	82	84	86
Number	2	3	1	20	1	1	6	5	1	7	1

Index	88	90	92	94	96	98	100	102	104	106
Number	3	2	1	1	36	1	4	2	5	1

Index	108	110	112	114	116	118	120	122	124	126
Number	13	1	3	6	3	1	25	1	1	6

Index	128	130	132	134	136	138	140	142	144	146
Number	11	1	3	1	5	2	1	1	32	1

Index	148	150	152	154	156	158	160	162	164	166
Number	3	5	3	1	13	1	12	11	3	1

Index	168	170	172	174	176	178	180	182	184	186
Number	29	1	1	2	3	1	7	1	3	6

Index	188	190	192	194	196	198	200	202	204	206
Number	1	1	71	1	2	2	9	1	5	1

Index	208	210	212	214	216	218	220	222	224	226
Number	5	4	3	1	56	1	1	6	5	1

Index	228	230	232	234	236	238	240	242	244	246
Number	7	1	5	6	1	1	42	1	3	2

Index	248	250	252	254	256	258	260	262	264	266
Number	3	1	12	1	19	6	5	1	11	1

Index	268	270	272	274	276	278	280	282	284	286
Number	1	5	5	1	3	1	5	2	1	1

Index	288	290	292	294	296	298	300	302	304	306
Number	77	1	3	11	5	1	12	1	3	2

Index	308	310	312	314	316	318	320	322	324	326
Number	1	1	37	1	1	2	22	1	26	1

Index	328	330	332	334	336	338	340	342	344	346
Number	5	2	1	1	54	1	5	6	3	1

Index	348	350	352	354	356	358	360	362	364	366
Number	5	1	5	2	3	1	41	1	1	6

Index	368	370	372	374	376	378	380	382	384	386
Number	3	1	7	1	3	15	1	1	181	1

Index	388	390	392	394	396	398	400	402	404	406
Number	3	4	5	1	4	1	9	6	3	1

Index	408	410	412	414	416	418	420	422	424	426
Number	17	1	1	2	11	1	7	1	5	2

Index	428	430	432	434	436	438	440	442	444	446
Number	1	1	104	1	3	6	5	1	13	1

Index	448	450	452	454	456	458	460	462	464	466
Number	7	5	3	1	27	1	1	4	5	1

Index	468	470	472	474	476	478	480	482	484	486
Number	14	1	3	6	1	1	97	1	2	20

Index	488	490	492	494	496	498	500
Number	5	1	5	1	3	2	6

Acknowledgments The authors are grateful to the N.Z. Marsden Fund and the University of Auckland Research Committee for its support.

References

[**1**] W. Bosma, J. Cannon & C. Playoust, The Magma Algebra System I: The User Language, *J. Symbolic Comput.* **24** (1997), 235–265.

[**2**] I.N. Cangül & D.Singerman, Normal subgroups of Hecke groups and regular maps, *Math. Proc. Camb. Phil. Soc.* **123** (1998), 59–74.

[**3**] M.D.E. Conder & P. Dobcsányi, Applications and adaptations of the low index subgroups procedure, *preprint*, currently available on the world-wide web at the URL `http://www.math.auckland.ac.nz/~conder/preprints/lowindex.ps`.

[**4**] M.D.E. Conder & P. Dobcsányi, Determination of all regular maps of small genus, *J. Combin. Theory Ser. B* **81** (2001), 224–242.

[**5**] M.D.E. Conder & P. Dobcsányi, Trivalent symmetric graphs on up to 768 vertices, *J. Combin. Math. Combin. Comput.* **40** (2002), 41–63.

[**6**] H.S.M. Coxeter & W.O.J. Moser, *Generators and Relations for Discrete Groups*, 4th ed. Springer-Verlag (Berlin), 1980.

[**7**] I.M.S. Dey & J. Wiegold, Generators for alternating and symmetric groups, *J. Austral. Math. Soc.* **12** (1971), 63–68.

[**8**] A. Dietze & M. Schaps, Determining subgroups of a given finite index in a finitely presented group, *Canadian J. Math.* **26** (1974), 769–782.

[**9**] P. Dobcsányi, *Applications, Adaptations and Parallelisation of the Low Index Subgroups Algorithm*, PhD thesis, University of Auckland, 1999.

[**10**] R.C. Miller, The trivalent symmetric graphs of girth at most six, *J. Combin. Theory Ser. B* **10** (1971), 163–182.

[**11**] M. Newman, Classification of normal subgroups of the modular group, *Trans. Amer. Math. Soc.* **126** (1967), 267–277.

[**12**] C.C. Sims, *Computation with Finitely-Presented Groups* (Cambridge University Press, 1994).

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand

Marston Conder: `conder@math.auckland.ac.nz`
`http://www.math.auckland.ac.nz/~conder`

Peter Dobcsányi: `peter@math.auckland.ac.nz`
`http://www.math.auckland.ac.nz/~peter`