# Generalisation of the Danilov-Karzanov-Koshevoy Construction for Peak-Pit Condorcet Domains 

Arkadii Slinko


#### Abstract

Danilov, Karzanov and Koshevoy (2012) geometrically introduced an interesting operation of composition on tiling Condorcet domains and using it they disproved a long-standing problem of Fishburn about the maximal size of connected Condorcet domains. We give an algebraic definition of this operation and investigate its properties. We give a precise formula for the cardinality of composition of two Condorcet domains and improve the Danilov, Karzanov and Koshevoy result showing that Fishburn's alternating scheme does not always define a largest peak-pit Condorcet domain.


Keywords Condorcet domain • Peak-pit domain • Never conditions •
Danilov-Karzanov-Koshevoy construction

## 1 Introduction

The famous Condorcet Paradox shows that if voters' preferences are unrestricted, the majority voting can lead to intransitive collective preference in which case the Condorcet Majority Rule (Condorcet, 1785), despite all its numerous advantages, is unable to determine the best alternative, i.e., it is not always decisive. Domain restrictions is, therefore, an important topic in economics and computer science alike (Elkind, 2018). In particular, for artificial societies of autonomous software agents there is no problem of individual freedom and, hence, for the sake of having transitive collective decisions the designers can restrict choices of those artificial agents in order to make the majority rule work every time.

Condorcet domains represent a solution to this problem, they are sets of linear orders with the property that, whenever the preferences of all voters belong to this set, the majority relation of any profile with an odd number of

[^0]voters is transitive. Maximal Condorcet domains historically have attracted a special attention since they represent a compromise which allows a society to always have transitive collective preferences and, under this constraint, provide voters with as much individual freedom as possible. The question: "How large a Condorcet domain can be?" has attracted even more attention (see the survey of Monjardet (2009) for a fascinating account of historical developments). Kim et al. (1992) identified this problem as a major unsolved problem in the mathematical social sciences. Fishburn (1996) introduced the function
$$
f(n)=\max \{|\mathcal{D}|: \mathcal{D} \text { is a Condorcet domain on the set of } n \text { alternatives. }\}
$$
and put this problem in the mathematical perspective asking for maximal values of this function.

Abello (1991) and Fishburn (1996, 2002) managed to construct some "large" Condorcet domains based on different ideas. Fishburn, in particular, taking a clue from Monjardet example (sent to him in private communication), came up with the so-called alternating scheme domains (that will be defined later in the text), later called Fishburn's domains (Danilov et al. 2012). This scheme produced Condorcet domains with some nice properties, which, in particular, are connected and have maximal width (see the definitions of these concepts later in this paper). Fishburn (1996) conjectured (Conjecture 2) that among Condorcet domains that do not satisfy the so-called never-middle condition (these in Danilov et al. (2012) were later called peak-pit domains), the alternating scheme provides domains of maximum cardinality. Galambos \& Reiner (2008) formulated another similar hypothesis (Conjecture 1) which later appeared to be equivalent to Fishburn's one (Danilov et al., 2012). Monjardet (2006) introduced the function

$$
g(n)=\max \{|\mathcal{D}|: \mathcal{D} \text { is a peak-pit domain on the set of } n \text { alternatives }\}
$$

in terms of which Fishburn's hypothesis becomes $g(n)=\left|F_{n}\right|$, where $F_{n}$ is the $n$th Fishburn domain. Monjardet (2009) also emphasised Fishburn's hypothesis.

It is known that $g(n)=f(n)$ for $n \leq 7$ (Fishburn, 1996, Galambos \& Reiner, 2008) and it is believed that $g(16)<f(16)$ (Monjardet, 2009). This is because Fishburn (1996) showed that $f(16)>\left|F_{16}\right|$. Thus, if Fishburn's hypothesis were true we would get $f(n)>g(n)$ for large $n$. However, this hypothesis is not true.

Danilov et al. (2012) introduced the class of tiling domains which are peakpit domains of maximal width and defined an operation on tiling domains that allowed them to show that $g(42)>\left|F_{42}\right|$. This operation was somewhat informally defined which made investigation of it and application of it in other situations difficult. In the present article we give an algebraic definition and a generalisation of the Danilov-Karzanov-Koshevoy construction and investigate its properties. In our interpretation it involves two peak-pit Condorcet domains $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ on sets of $n$ and $m$ alternatives, respectively, and two linear orders
$u \in \mathcal{D}_{1}$ and $v \in \mathcal{D}_{2}$; the result is denoted as $\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(u, v)$. It is again a peak-pit Condorcet domain on $n+m$ alternatives whose exact cardinality we can calculate. Using this formula we can slightly refine the argument from Danilov et al. (2012) to show that $g(40)>\left|F_{40}\right|$.

## 2 Preliminaries

Let $A$ be a finite set and $\mathcal{L}(A)$ be the set of all (strict) linear orders on $A$. Any subset $\mathcal{D} \subseteq \mathcal{L}(A)$ will be called a domain. Any sequence $P=\left(v_{1}, \ldots, v_{n}\right)$ of linear orders from $\mathcal{D}$ will be called a profile over $\mathcal{L}^{1}$. It usually represents a collective set of opinions of a society about merits of alternatives from $A$. A linear order $a_{1}>a_{2}>\cdots>a_{n}$ on $A$, will be denoted by a string $a_{1} a_{2} \ldots a_{n}$. Let us also introduce notation for reversing orders: if $x=a_{1} a_{2} \ldots a_{n}$, then $\bar{x}=a_{n} a_{n-1} \ldots a_{1}$. If linear order $v_{i}$ ranks $a$ higher than $b$, we denote this as $a \succ_{i} b$.

Definition 1 The majority relation $\succeq_{P}$ of a profile $P$ is defined as

$$
a \succeq_{P} b \Longleftrightarrow\left|\left\{i \mid a \succ_{i} b\right\}\right| \geq\left|\left\{i \mid b \succ_{i} a\right\}\right| .
$$

Verbally, $a \succeq_{P} b$ means that at lest as many voters from a society with profile $P$ prefer $a$ to $b$ as voters who prefer $b$ to $a$. For an odd number of linear orders in the profile $P$ this relation is a tournament, i.e., complete and asymmetric binary relation. In this case we denote it $\succ_{P}$.

Now we can define the main object of this investigation.
Definition 2 A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ over a set of alternatives $A$ is a Condorcet domain if the majority relation $\succ_{P}$ of any profile $P$ over $\mathcal{D}$ with odd number of voters is transitive. A Condorcet domain $\mathcal{D}$ is maximal if for any Condorcet domain $\mathcal{D}^{\prime} \subseteq \mathcal{L}(A)$ the inclusion $\mathcal{D} \subseteq \mathcal{D}^{\prime}$ implies $\mathcal{D}=\mathcal{D}^{\prime}$.

There is a number of alternative definitions of Condorcet domains, see e.g., Monjardet (2009); Puppe \& Slinko (2019).

Up to an isomorphism, there is only one maximal Condorcet domain on the set $\{a, b\}$, namely $C D_{2}=\{a b, b a\}$ and there are only three maximal Condorcet domains on the set of alternatives $\{a, b, c\}$, namely,

$$
\begin{gathered}
C D_{3, t}=\{a b c, a c b, c a b, c b a\}, \quad C D_{3, m}=\{a b c, b c a, a c b, c b a\}, \\
C D_{3, b}=\{a b c, b a c, b c a, c b a\} .
\end{gathered}
$$

The first domain contains all the linear orders on $a, b, c$ where $b$ is never ranked first, second contains all the linear orders on $a, b, c$ where $a$ is never ranked second and the third contains all the linear orders on $a, b, c$ where $b$ is never ranked last. Following Monjardet, we denote these conditions as $b N_{\{a, b, c\}} 1$,

[^1]$a N_{\{a, b, c\}} 2$ and $b N_{\{a, b, c\}} 3$, respectively. We note that these are the only conditions of type $x N_{\{a, b, c\}} i$ with $x \in\{a, b, c\}$ and $i \in\{1,2,3\}$ that these domains satisfy.

A domain that for any triple $a, b, c \in A$ satisfies a condition $x N_{\{a, b, c\}} 1$ with $x \in\{a, b, c\}$ is called never-top domain, a domain that for any triple $a, b, c \in A$ satisfies a condition $x N_{\{a, b, c\}} 2$ with $x \in\{a, b, c\}$ is called never-middle domain, and a domain that for any triple $a, b, c \in A$ satisfies a condition $x N_{\{a, b, c\}} 3$ with $x \in\{a, b, c\}$ is called never-bottom domain.

Definition 3 (Danilov et al. (2012)) A domain that for any triple satisfies either never-top or never-bottom condition is called a peak-pit domain. Both never-top and never-bottom conditions will be called peak-pit conditions.

We note that Danilov et al. (2012), who consider linear orders over $A=$ $\{1,2, \ldots, n\}$, restrict in their investigation the class of peak-pit domains to domains that contain two completely reversed orders (up to an isomorphism they can be taken as $12 \ldots n$ and $\overline{12 \ldots n}=n n-1 \ldots 1$ ) and prove that under this restriction all of them can be embedded into tiling domains (Theorem 2 of Danilov et al. (2012)). We also note that never-bottom domains are also known as Arrow's single-peaked domains and maximal domains among them have all cardinality $2^{n-1}$ (Slinko, 2019).

Given a set of alternatives $A$, we say that

$$
\begin{equation*}
\mathcal{N}=\left\{x N_{\{a, b, c\}} i \mid\{a, b, c\} \subseteq A, x \in\{a, b, c\} \text { and } i \in\{1,2,3\}\right\} \tag{1}
\end{equation*}
$$

is a complete set of never conditions if it contains at least one never condition for every triple $a, b, c$ of distinct elements of $A$. If the set of linear orders that satisfy $\mathcal{N}$ is non-empty, we say that $\mathcal{N}$ is consistent.

Proposition $1 A$ domain of linear orders $\mathcal{D} \subseteq \mathcal{L}(A)$ is a Condorcet domain if and only if it is non-empty and satisfies a complete set of never conditions.

Proof This is well-known characterisation noticed by many researchers. See, for example, Theorem 1(d) in Puppe \& Slinko (2019) and references there.

This proposition, in particular, means that the collection $\mathcal{D}(\mathcal{N})$ of all linear orders that satisfy a certain complete set of never conditions $\mathcal{N}$, if non-empty, is a Condorcet domain. Let us also denote by $\mathcal{N}(\mathcal{D})$ the set of all never conditions that are satisfied by all linear orders from a domain $\mathcal{D}$.

Let $\psi: A \rightarrow A^{\prime}$ be a bijection between two sets of alternatives. It can then be extended to a mapping $\psi: \mathcal{L}(A) \rightarrow \mathcal{L}\left(A^{\prime}\right)$ in two ways: by mapping a linear order $u=a_{1} a_{2} \ldots a_{m}$ onto $\psi(u)=\psi\left(a_{1}\right) \psi\left(a_{2}\right) \ldots \psi\left(a_{m}\right)^{2}$ or to $\overline{\psi(u)}=$ $\psi\left(a_{m}\right) \psi\left(a_{m-1}\right) \ldots \psi\left(a_{1}\right)$.

Definition 4 Let $A$ and $A^{\prime}$ be two sets of alternatives (not necessarily distinct) of equal cardinality. We say that two domains, $\mathcal{D} \subseteq \mathcal{L}(A)$ and $\mathcal{D}^{\prime} \subseteq \mathcal{L}\left(A^{\prime}\right)$ are isomorphic if there is a bijection $\psi: A \rightarrow A^{\prime}$ such that $\mathcal{D}^{\prime}=\{\psi(d) \mid d \in \mathcal{D}\}$ and flip-isomorphic if $\mathcal{D}^{\prime}=\{\overline{\psi(d)} \mid d \in \mathcal{D}\}$.

[^2]Example 1 The single-peaked and single-dipped maximal Condorcet domains on $\{a, b, c\}$ are $C D_{3, b}=\{a b c, b a c, b c a, c b a\}$ and $C D_{3, t}=\{a b c, a c b, c a b, c b a\}$, respectively. They are not isomorphic but flip-isomorphic under the identity mapping of $\{a, b, c\}$ onto itself.

Definition 5 (Puppe (2018)) A Condorcet domain $\mathcal{D}$ is said to have maximal width if it contains two completely reversed orders, i.e., together with some linear order $u$ it also contains its flip $\bar{u}$.

Up to an isomorphism, for any Condorcet domain $\mathcal{D}$ of maximal width we may assume that $A=\{1,2, \ldots, n\}$ and it contains linear orders $e=12 \ldots n$ and $\bar{e}=n \ldots 21$.

The universal domain $\mathcal{L}(A)$ is naturally endowed with the following betweenness structure (as defined by Kemeny (1959)). An order $v$ is between orders $u$ and $w$ if $v \supseteq u \cap w$, i.e., $v$ agrees with all binary comparisons in which $u$ and $w$ agree (see also Kemeny \& Snell (1960)). The set of all orders that are between $u$ and $w$ is called the interval spanned by $u$ and $w$ and is denoted by [ $u, w]$. The domain $\mathcal{L}(A)$ endowed with this betweenness relation is referred to as the permutahedron (Monjardet, 2009).

Given a domain of preferences $\mathcal{D}$, for any $u, w \in \mathcal{D}$ we define the induced interval as $[u, w]_{\mathcal{D}}=[u, w] \cap \mathcal{D}$. Puppe \& Slinko (2019) defined a graph $G_{\mathcal{D}}$ associated with this domain. The set of linear orders from $\mathcal{D}$ are the set of vertices $V_{\mathcal{D}}$ of $G_{\mathcal{D}}$, and for two orders $u, w \in \mathcal{D}$ we draw an edge between them if there is no other vertex between them, i.e., $[u, w]_{\mathcal{D}}=\{u, w\}$. The set of edges is denoted $E_{\mathcal{D}}$ so the graph is $G_{\mathcal{D}}=\left(V_{\mathcal{D}}, E_{\mathcal{D}}\right)$. As established in Puppe \& Slinko (2019), for any Condorcet domain $\mathcal{D}$ the graph $G_{\mathcal{D}}$ is a median graph (Mulder, 1978) and any median graph can be obtained in this way.

A domain $\mathcal{D}$ is called connected if its graph $G_{\mathcal{D}}$ is a subgraph of the permutahedron (Puppe \& Slinko, 2019); we note that domains $C D_{3, t}$ and $C D_{3, b}$ are connected but $C D_{3, m}$ is not. Danilov et al. (2012) called a domain of maximal width semi-connected if the two completely reversed orders can be connected by a path of vertices that is also a path in the permutahedron corresponding to a maximal chain in the Bruhat order. They proved that a maximal Condorcet domain of maximal width is semi-connected if and only if it is a peak-pit domain. Puppe (2017) showed that for a maximal Condorcet domain semi-connectedness implies direct connectedness (Proposition A2) which means that any two linear orders in the domain are connected by a shortest possible (geodesic) path.

Finally, we give two more definitions that express two properties of Condorcet domains. But, firstly, we will introduce the following notation. Suppose $\mathcal{D} \subseteq \mathcal{L}(A)$ be a domain on the set $A$ and let $B \subseteq A$. Suppose also $u \in \mathcal{D}$. Then by $\mathcal{D}_{B}$ and $u_{B}$ we denote the restrictions of $\mathcal{D}$ and $u$ onto $B$, respectively.
Definition 6 We call a Condorcet domain $\mathcal{D}$ ample if for any pair of alternatives $a, b \in A$ the restriction $\mathcal{D}_{\{a, b\}}$ of this domain to $\{a, b\}$ has two distinct orders, that is, $\mathcal{D}_{\{a, b\}}=\{a b, b a\}$.

A rhombus tiling (or simply a tiling) is a subdivision $T$ into rhombic tiles of the zonogon $Z(n ; 2)$ obtained as the Minkowski sum of $n$ segments $s_{i}=\left[0, \psi_{i}\right]$, $i=1, \ldots, n$. This centre-symmetric $2 n$-gon has the bottom vertex $b=(0,0)$ and the top vertex $t=s_{1}+\ldots+s_{n}$. A snake is a path from $b$ to $t$ which, for each $i=1, \ldots, n$ contains a unique segments parallel to $s_{i}$. Each snake corresponds to a linear order on $\{1, \ldots, n\}$ in the following way. If a point traveling from $b$ to $t$ passes segments parallel to $s_{i_{1}}, s_{i_{2}} \ldots, s_{i_{n}}$, then the corresponding linear order will be $i_{1} i_{2} \ldots i_{n}$. The set of snakes of a rhombus tiling, thus, defines a domain which is called tiling domain. Danilov et al. (2012) showed that peakpit domains of maximal width are exactly the tiling domains (see an example on Figure 1).

Definition 7 (Slinko (2019)) A Condorcet domain $\mathcal{D}$ is called copious if for any triple of alternatives $a, b, c \in A$ the restriction $\mathcal{D}_{\{a, b, c\}}$ of this domain to this triple has four distinct orders, that is, $\left|\mathcal{D}_{\{a, b, c\}}\right|=4$.

Of course, any copious Condorcet domain is ample. We note that, if a domain $\mathcal{D}$ is copious, then it satisfies a unique set of never conditions (1).

Definition 8 A complete set of peak-pit conditions (1) is said to satisfy the alternating scheme (Fishburn, 1996), if for all $1 \leq i<j<k \leq n$ it includes

$$
j N_{\{i, j, k\}} 3, \text { if } \mathrm{j} \text { is even, and } j N_{\{i, j, k\}} 1, \text { if } j \text { is odd }
$$

or

$$
j N_{\{i, j, k\}} 1, \text { if } \mathrm{j} \text { is even, and } j N_{\{i, j, k\}} 3, \text { if } j \text { is odd. }
$$

The domains that are determined by these complete sets we define $F_{n}$ and $\overline{F_{n}}$, respectively, and call Fishburn's domains (Danilov et al., 2012). The second domain is flip-isomorphic to the first so we consider only the first one.

In particular, $F_{2}=\{12,21\}, F_{3}=\{123,213,231,321\}$ and

$$
F_{4}=\{1234,1243,2134,2143,2413,2431,4213,4231,4321\}
$$

Figure 1 shows the median graph of $F_{4}$ and its representation as a tiling domain.

Galambos \& Reiner (2008) give the exact formula for the cardinality of $F_{n}$ :

$$
\left|F_{n}\right|=(n+3) 2^{n-3}- \begin{cases}\left(n-\frac{3}{2}\right)\left(\begin{array}{c}
n-2 \\
n-1 \\
n
\end{array}\right) & \text { for even } n  \tag{2}\\
\left(\frac{n-1}{2}\right)\binom{n-1}{\frac{n}{2}} & \text { for odd } n\end{cases}
$$

Given a path in the permutahedron from $e=12 \ldots n$ to $\bar{e}=n n-1 \ldots 1$ where each pair $(i, j)$ with $1 \leq i<j \leq n$ is switched exactly once (which can be associated with a maximal chain in the Bruhat order $\mathbb{B}(n, 1))$ we say that it satisfies the inversion triple $[i, j, k]$ with $i<j<k$ if The pairs in this triple are switched in the order $(j, k),(i, k),(i, j)$. Galambos \& Reiner (2008) showed that if a maximal Condorcet domain $\mathcal{D}$ of maximal width contains one maximal chain in the Bruhat order $\mathbb{B}(n, 1)$ (i.e., is semi-connected), then it is a


1234
1243
2134
2143
2413
4213
2431
4231
4321


Fig. 1 Median graph and the tiling of the hexagon for Fishburn's domain $F_{4}$
union of all equivalent maximal chains, i.e., those chains that satisfy the same set of inversion triples. Thus any maximal semi-connected Condorcet domain $\mathcal{D}$ can be defined by the set of inversion triples. In particular, the domain $F_{4}$ can be defined by the set of inversion triples

$$
\{[1,3,4],[2,3,4]\}
$$

## 3 Main Results

Let us start with an observation.
Proposition 2 Let $\mathcal{D}$ be a semi-connected Condorcet domain of maximal width on the set of alternatives $A$. Then:
(i) For any $a \in A$ its restriction $\mathcal{D}^{\prime}$ on $A^{\prime}=A-\{a\}$ is also a semi-connected domain of maximal width.
(ii) $\mathcal{D}$ is copious peak-pit domain.

Proof (i) If $w$ and $\bar{w}$ are two completely reversed linear orders in $\mathcal{D}$, then after removal of $a$, their images will still be completely reversed. Let $u, v$ be two vertices in $G_{\mathcal{D}}$ which are neighbouring vertices in the permutahedron on the path connecting $w$ and $\bar{w}$. Then $v$ differs from $u$ by a swap of neighbouring alternatives. Let $u^{\prime}, v^{\prime}$ be their images under the natural mapping of $\mathcal{D}$ onto $\mathcal{D}^{\prime}$. If one of these swapped alternatives was $a$, then $u^{\prime}=v^{\prime}$. If not, $u^{\prime}, v^{\prime}$ will still differ by a swap of neighbouring alternatives. Hence $\mathcal{D}^{\prime}$ is semi-connected.
(ii) Let $a, b, c \in A$ and let $\mathcal{D}^{\prime \prime}$ be the restriction of $\mathcal{D}$ onto $\{a, b, c\}$. Since $\mathcal{D}$ is of maximal width, the same can be said about $\mathcal{D}^{\prime \prime}$ and without loss of


Fig. 2 Concatenation of tilings $T$ and $T^{\prime}$
generality we may assume that $\mathcal{D}^{\prime \prime}$ contains $a b c$ and $c b a$. By (i) $\mathcal{D}^{\prime \prime}$ is semiconnected and hence there will be two intermediate orders in $\mathcal{D}^{\prime \prime}$ connecting $a b c$ and $c b a$. These would be either $a c b$ and $c a b$ or $b a c$ and $b c a$. Thus, $\mathcal{D}^{\prime \prime}$ has four linear orders, and, hence, $\mathcal{D}$ is copious domain satisfying $b N_{\{a, b, c\}} 1$ or $b N_{\{a, b, c\}} 3$, respectively. Hence it is a peak-pit domain.

### 3.1 Danilov-Karzanov-Koshevoy construction and its generalisation

Danilov-Karzanov-Koshevoy (Danilov et al. 2012) define the 'concatenation' of two tiling domains by the picture shown in Figure 2 (where one arrow is obviously missing).

Let us now start describing this construction algebraically. In fact, this will be a generalisation of their construction since in our construction two arbitrary linear orders are involved. Firstly, we describe 'pure' concatenation.

Let $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ be two Condorcet domains on disjoint sets of alternatives $A$ and $B$, respectively. We define a concatenation of these domains as the domain

$$
\mathcal{D}_{1} \odot \mathcal{D}_{2}=\left\{x y \mid x \in \mathcal{D}_{1} \text { and } y \in \mathcal{D}_{2}\right\}
$$

on $A \cup B$. It is immediately clear that $\mathcal{D}_{1} \odot \mathcal{D}_{2}$ is also a Condorcet domain of cardinality $\left|\mathcal{D}_{1} \odot \mathcal{D}_{2}\right|=\left|\mathcal{D}_{1}\right|\left|\mathcal{D}_{2}\right|$. We have only to check that one of the never-conditions is satisfied for triples $\left\{a_{1}, a_{2}, b\right\}$ where $a_{1}, a_{2} \in A$ and $b \in B$ (for triples $\left\{a, b_{1}, b_{2}\right\}$ the argument will be similar). The restriction $\left(\mathcal{D}_{1} \odot\right.$ $\left.\mathcal{D}_{2}\right)\left.\right|_{\left\{a_{1}, a_{2}, b\right\}}$ will contain at most two linear orders $a_{1} a_{2} b$ and $a_{2} a_{1} b$, which is consistent with both never-top and never-bottom conditions. This domain corresponds to $T$ and $T^{\prime}$ on Figure 2 ,

Definition 9 Let $A$ and $B$ be two disjoint sets of alternatives, $u \in \mathcal{L}(A)$ and $v \in \mathcal{L}(B)$. An order $w \in \mathcal{L}(A \cup B)$ is said to be a shuffle of $u$ and $v$ if $w_{A}=u$ and $w_{B}=v$, i.e., the restriction of $w$ onto $A$ is equal to $u$ and the restriction of $w$ onto $B$ is equal to $v$.

For example, 516723849 is a shuffle of 1234 and 56789 .

Given two linear orders $u$ and $v$, we define domain $u \oplus v$ as the set of all shuffles of $u$ and $v$. It is clear from definition that $u \oplus v=v \oplus u$. The cardinality of this domain is $|u \oplus v|=\binom{n+m}{m}$. We believe this domain corresponds to what is depicted in Figure 2 outside of $T$ and $T^{\prime}$.

Now we combine the two domains together.
Theorem 1 Let $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ be two Condorcet domains on disjoint sets of alternatives $A$ and $B$. Let $u \in \mathcal{D}_{1}$ and $v \in \mathcal{D}_{2}$ be arbitrary linear orders. Then

$$
\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(u, v):=\left(\mathcal{D}_{1} \odot \mathcal{D}_{2}\right) \cup(u \oplus v)
$$

is a Condorcet domain. Moreover, if $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are peak-pit domains, so is $\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(u, v)$.
Proof Let us fix $u$ and $v$ in this construction and denote $\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(u, v)$ as simply $\mathcal{D}_{1} \otimes \mathcal{D}_{2}$. If $a, b, c \in A$, then $\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)_{\{a, b, c\}}=\left(\mathcal{D}_{1}\right)_{\{a, b, c\}}$, i.e., the restriction of $\mathcal{D}_{1} \otimes \mathcal{D}_{2}$ onto $\{a, b, c\}$ is the same as the restriction of $\mathcal{D}_{1}$ onto $\{a, b, c\}$. Hence $\mathcal{D}_{1} \otimes \mathcal{D}_{2}$ satisfies the same never condition for $\{a, b, c\}$ as $\mathcal{D}_{1}$. For $x, y, z \in B$ the same thing happens.

Suppose now $a, b \in A$ and $x \in B$. Then $\left(\mathcal{D}_{1} \odot \mathcal{D}_{2}\right)_{\{a, b, x\}} \subseteq\{a b x, b a x\}$. Let also $u_{\{a, b\}}=\{a b\}$. Then $(u \oplus v)_{\{a, b, x\}}=\{a b x, a x b, x a b\}$, hence

$$
\begin{equation*}
\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)_{\{a, b, x\}} \subseteq\{a b x, b a x, a x b, x a b\} \tag{3}
\end{equation*}
$$

thus $\mathcal{D}_{1} \otimes \mathcal{D}_{2}$ satisfies $a N_{\{a, b, x\}} 3$. For $a \in A$ and $x, y \in B$ we have $\left(\mathcal{D}_{1} \odot \mathcal{D}_{2}\right)_{\{a, x, y\}} \subseteq\{a x y, a y x\}$. Let also $v_{\{x, y\}}=\{x y\}$. Then $(u \oplus v)_{\{a, x, y\}}=$ $\{a x y, x a y, x y a\}$, hence

$$
\begin{equation*}
\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)_{\{a, x, y\}} \subseteq\{a x y, a y x, x a y, x y a\} \tag{4}
\end{equation*}
$$

thus $\mathcal{D}_{1} \otimes \mathcal{D}_{2}$ satisfies $y N_{\{a, x, y\}} 1$.
Note: The inequalities (3) and (4) become equalities if for any $i \in\{1,2\}$ and any $a, b \in \mathcal{D}_{i}$ we have $\left(\overrightarrow{\mathcal{D}}_{i}\right)_{\{a, b\}}=\{a b, b a\}$, i.e., if $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are ample.
Proposition 3 If $|A|=m$ and $|B|=n$, then for any $u \in \mathcal{D}_{1}$ and $v \in \mathcal{D}_{2}$

$$
\begin{equation*}
\left|\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(u, v)\right|=\left|\mathcal{D}_{1}\right|\left|\mathcal{D}_{2}\right|+\binom{n+m}{m}-1 \tag{5}
\end{equation*}
$$

Proof We have $\left|\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right|=\left|\mathcal{D}_{1}\right|\left|\mathcal{D}_{2}\right|$ and $|u \oplus v|=\binom{n+m}{m}$. These two sets have only one linear order in common which is $u v$. This proves (5).
Proposition 4 Let $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ be of maximal width with $u, \bar{u} \in \mathcal{D}_{1}$ and $v, \bar{v} \in$ $\mathcal{D}_{2}$. Then $\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(u, v)$ is also of maximal width. If $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are semiconnected, then so is $\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(u, v)$.
Proof Since $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are of maximal width, we have $\bar{u} \in \mathcal{D}_{1}$ and $\bar{v} \in$ $\mathcal{D}_{2}$. Hence $\bar{u} \bar{v} \in \mathcal{D}_{1} \odot \mathcal{D}_{2}$. We also have $v u \in u \oplus v$, and $\overline{v u}=\bar{u} \bar{v}$, hence $\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(u, v)$ has maximal width. To prove the last statement we note that $\bar{u} \bar{v}$ can be connected to $u v$ (which belongs both to $\mathcal{D}_{1} \otimes \mathcal{D}_{2}$ and to $u \oplus v$ ) by a geodesic path and $u v$ in turn can be connected to $v u$ by a geodesic path within $u \oplus v$.

If both $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ have maximal width, it is not true, however, that $\left(\mathcal{D}_{1} \otimes\right.$ $\left.\mathcal{D}_{2}\right)(u, v)$ will have maximal width for any $u \in \mathcal{D}_{1}$ and $v \in \mathcal{D}_{2}$. Let us take, for example, $\mathcal{D}_{1}=\{x=a b, \bar{x}=b a\}$ and $\mathcal{D}_{2}=\{u=c d e, v=d e c, w=d c e, \bar{u}=$ $e d c\}$. Then $\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(x, u)$ has maximal width while $\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(x, v)$ does not since $\bar{v} \notin \mathcal{D}_{2}$. In particular,

$$
\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(x, u) \not \not 二\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(x, v)
$$

This indicates that the construction of the tensor product may be useful in description of Condorcet domains which do not satisfy the requirement of maximal width.

Proposition 5 Let $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ be two Condorcet domains on disjoint sets of alternatives $A$ and $B$. Let $u \in \mathcal{D}_{1}$ and $v \in \mathcal{D}_{2}$ be arbitrary linear orders. Then
(i) $\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(u, v)$ is connected, whenever $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are;
(ii) $\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(u, v)$ is copious, whenever $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are.

Proof (i) If $\mathcal{D}=\left(\mathcal{D}_{1} \otimes \mathcal{D}_{2}\right)(u, v)$ is connected, then $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are connected too. Suppose now that $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are connected, suppose $w, w^{\prime} \in \mathcal{D}$ which are neighbours in $\Gamma_{\mathcal{D}}$. Since all neighbours in $\mathcal{D}_{1} \otimes \mathcal{D}_{2}$ are neighbours in the permutahedron and so are neighbours in $\mathcal{D}_{1} \oplus \mathcal{D}_{2}$, it is enough to consider the case when $w \in \mathcal{D}_{1} \odot \mathcal{D}_{2}$ and $w^{\prime} \in \mathcal{D}_{1} \oplus \mathcal{D}_{2}$. But $u v$ is on the shortest path from $w$ to $w^{\prime}$ and it is in $\mathcal{D}$. Hence either $w=u v$ or $w^{\prime}=u v$ and either $\left\{w, w^{\prime}\right\} \subseteq \mathcal{D}_{1} \otimes \mathcal{D}_{2}$ or $\left\{w, w^{\prime}\right\} \subseteq \mathcal{D}_{1} \oplus \mathcal{D}_{2}$. This proves (i).
(ii) This part follows from (3) and (4) since, as was noted before, when $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are copious these inequalities become equalities.

Proposition 6 The following isomorphism holds

$$
\begin{equation*}
\left(F_{2}(a, b) \otimes F_{2}(c, d)\right)(a b, c d) \cong F_{4}(b, a, d, c) \tag{6}
\end{equation*}
$$

Proof We list orders of this domain as columns of the following matrix

$$
\left[F_{2}(a, b) \odot F_{2}(c, d) \mid a b \oplus c d\right]=\left[\begin{array}{lllllllll}
a & a & b & b & a & a & c & c & c \\
b & b & a & a & c & c & a & a & d \\
c & d & c & d & b & d & b & d & a \\
d & c & d & c & d & b & d & b & b
\end{array}\right] .
$$

We see that the following never conditions are satisfied: $a N_{\{a, b, c\}} 3, a N_{\{a, b, d\}} 3$, $d N_{\{a, c, d\}} 1, d N_{\{b, c, d\}} 1$. Hence the mapping $1 \rightarrow b, 2 \rightarrow a, 3 \rightarrow d$ and $4 \rightarrow c$ is an isomorphism of $F_{4}$ onto the tensor product $\left(F_{2}(a, b) \otimes F_{2}(c, d)\right)(a b, c d)$.

The isomorphism (6) is very nice but unfortunately for larger $m, n$ we have $F_{m} \otimes F_{n} \not \neq F_{m+n}$. Moreover, it appears that for two maximal Condorcet domains $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ on sets $A$ and $B$, respectively, $\mathcal{D}_{1} \otimes \mathcal{D}_{2}$ may not be maximal on $A \cup B$. Here is an example.

Example 2 Let us calculate $\mathcal{E}:=F_{3}(1,2,3) \otimes F_{2}(4,5)(321,54)$ :

$$
\left[\begin{array}{llllllll|lllllllll}
1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 3 & 3 & 5 & 3 & 3 & 3 & 5 & 5 & 5 \\
2 & 1 & 3 & 2 & 2 & 1 & 3 & 2 & 2 & 5 & 3 & 2 & 5 & 5 & 3 & 3 & 4 \\
3 & 3 & 1 & 1 & 3 & 3 & 1 & 1 & 5 & 2 & 2 & 5 & 2 & 4 & 2 & 4 & 3 \\
4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 1 & 1 & 1 & 4 & 4 & 2 & 4 & 2 & 2 \\
5 & 5 & 5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right] .
$$

There are 17 linear orders in this domain. It is known, however, that $F_{5}$ has 20 (Fishburn, 1996) but this fact alone does not mean non-maximality of $\mathcal{E}$. By Proposition 5 this domain is copious. By its construction it satisfies just three inversion triples:

$$
[1,2,4], \quad[1,3,4], \quad[2,3,4] .
$$

Now we see that there are two more linear orders 23514 and 23541 that satisfy these conditions. Hence $\mathcal{E}$ is not maximal.

### 3.2 On Fishburn's hypothesis

We will further write $\left(F_{k} \otimes F_{m}\right)(u, v)$ simply as $F_{k} \otimes F_{m}$, when $u \in F_{k}$ and $v \in F_{m}$ are chosen so that $\left(F_{k} \otimes F_{m}\right)(u, v)$ has maximal width. We note that equation (6) is just a one of a kind since $F_{2} \otimes F_{3} \neq F_{5}$ already.

Our calculations, using formulas (2) and (5) show that

$$
\left|F_{n} \otimes F_{n}\right|<\left|F_{2 n}\right|
$$

for $2<n \leq 19$ but $4611858343415=\left|F_{20} \otimes F_{20}\right|>\left|F_{40}\right|=4549082342996$. Earlier, Danilov et al. (2012) showed that $\left|F_{21} \otimes F_{21}\right|>\left|F_{42}\right|$ disproving an old Fishburn's hypothesis that $F_{n}$ is the largest peak-pit Condorcet domain on $n$ alternatives (Fishburn, 1996; Galambos \& Reiner, 2008).

## 4 Conclusion and further research

Operations over Condorcet domains are useful in many respects. The Danilov-Karzanov-Koshevoy construction is especially useful since it converts smaller peak-pit Condorcet domains into larger peak-pit domains. Fishburn's replacement scheme (Fishburn, 1996) also produces larger Condorcet domains from smaller ones but without preserving peak-pittedness. Using it Fishburn proved that $f(16)>\left|F_{16}\right|$ and since he believed that $g(n)=\left|F_{n}\right|$ this would imply that $f(n)>g(n)$ for large $n$. Now that we know that $g(n)>\left|F_{n}\right|$, the question whether or not $f(n)=g(n)$ comes to the fore. Another interesting question is to find the smallest positive integer $n$ for which $g(n)>\left|F_{n}\right|$.

## References

Abello, J. (1991). The weak Bruhat order of $S_{\Sigma}$, consistent sets, and Catalan numbers. SIAM Journal on Discrete Mathematics, 4(1), 1-16.
Condorcet, M. d. (1785). Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix. Paris.
Danilov, V., Karzanov, A., \& Koshevoy, G. (2012). Condorcet domains of tiling type. Discrete Applied Mathematics, 160(7-8), 933-940.
Elkind, E. (2018). Restricted preference domains in social choice: Two perspectives. In International symposium on algorithmic game theory (pp. 12-18).
Fishburn, P. (1996). Acyclic sets of linear orders. Social Choice and Welfare, 14(1), 113-124.
Fishburn, P. (2002). Acyclic sets of linear orders: A progress report. Social Choice and Welfare, 19(2), 431-447.
Galambos, A., \& Reiner, V. (2008). Acyclic sets of linear orders via the Bruhat orders. Social Choice and Welfare, 30(2), 245-264.
Kemeny, J. (1959). Mathematics without numbers. Daedalus, 88, 577-591.
Kemeny, J., \& Snell, L. (1960). Mathematical models in the social sciences. Ginn.
Kim, K., Roush, F., \& Intriligator, M. (1992). Overview of mathematical social sciences. The American Mathematical Monthly, 99(9), 838-844.
Monjardet, B. (2006). Condorcet domains and distributive lattices. Annales du LAMSADE, 6, 285-302.
Monjardet, B. (2009). Acyclic domains of linear orders: A survey. In S. Brams, W. Gehrlein, \& F. Roberts (Eds.), The mathematics of preference, choice and order (p. 139-160). Springer Berlin Heidelberg.
Mulder, H. M. (1978). The structure of median graphs. Discrete Math., 24 , 197-204.
Puppe, C. (2017). The single-peaked domain revisited: A simple global characterization. (Tech. Rep.). Karlsruhe Institute of Technology.
Puppe, C. (2018). The single-peaked domain revisited: A simple global characterization. Journal of Economic Theory, 176, 55-80.
Puppe, C., \& Slinko, A. (2019). Condorcet domains, median graphs and the single-crossing property. Economic Theory, 67(1), 285-318.
Slinko, A. (2019). Condorcet domains satisfying Arrow's single-peakedness. Journal of Mathematical Economics, 84, 166-175.


[^0]:    Arkadii Slinko
    Department of Mathematics, University of Auckland, Auckland, New Zealand E-mail: a.slinko@auckland.ac.nz

[^1]:    1 A profile, unlike the domain, can have several identical linear orders.

[^2]:    ${ }^{2}$ We use the same notation for both mappings since there can be no confusion.

