VERY REGULAR ZERO SETS FOR THE BERGMAN SPACES

N.F. DUDLEY WARD AND P.C. FENTON

Abstract. We obtain a condition which is both necessary and sufficient such that a sequence of regularly spaced points in the unit disc is a zero set for a Bergman space $L_b^p$.

1. Introduction

Let $\mathbb{D} = \{ z : |z| < 1 \}$ denote the unit disc in the complex plane and $L_b^p(\mathbb{D})$, $p \geq 1$, be the Bergman space of functions $f(z)$ analytic in $\mathbb{D}$ such that

$$\|f\|_p^p = \int \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where $dA(z)$ is normalised Lebesgue area measure on $\mathbb{D}$. A sequence of points $\{z_n\}$ in $\mathbb{D}$ is called a zero set for $L_b^p$ if there exists $f \neq 0$ which vanishes precisely on the $z_n$. It is an open problem to give a geometric classification of the zeros sets for the Bergman spaces [2].

The case for the Hardy spaces $H^p$ is simpler [1]. A sequence $\{z_n\} \subset \mathbb{D}$ is a Blaschke sequence provided $\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$, and the zero sets for $H^p$ are precisely the Blaschke sequences. Functions which vanish on the $z_n$ are given by the Blaschke products:

$$B(z) = \prod_{n=1}^{\infty} \frac{z - a_n}{\overline{a_n} z},$$

All $H^p$ spaces have the same zero sets and a union of two zero sets is a zero set. Horowitz [3] showed that neither statement holds for the Bergman spaces.

For Bergman spaces we consider zero sets of the following form. Given an integer $q$, $q \geq 2$, and a positive number $K$, let $r_j = 1 - K q^{-j}$, for $j \geq j_0 = \log K/\log q$, and on each circle $|z| = r_j$ take $\eta_j = q^j$ equally spaced points $z_{j,1}, z_{j,2}, \ldots, z_{j,\eta_j}$:

$$z_{j,k} = r_j e^{i\theta_{j,k}}, \quad k = 1, \ldots, \eta_j,$$

where $\theta_{j,k} = 2\pi k i / \eta_j$. We ask for conditions on $q$ and $K$ that make $A = \{z_{j,k}\}$ a zero set for some Bergman space $L_b^p(\mathbb{D})$.

A case to bear in mind is the following. With $q = 2$ and $K = 1$,
(2) \[ z_{j,k} = (1 - 2^{-j}) e^{2\pi ki/n_j}, \quad k = 1, \ldots, n_j. \]

Such a set is separated in the pseudo-hyperbolic metric but is thick in the sense that a union of pseudo-hyperbolic balls with centres \( z_{j,k} \) covers the disc \([7]\). It turns out that this set is not a zero set for \( L^1_a \) although it is a zero set for \( L^p_a \) for \( p \) sufficiently small. By thinning the sequence we obtain zero sets for larger values of the parameter \( p \).

We prove the following result:

**Theorem 1.** Given an integer \( q \), with \( q \geq 2 \), and a positive number \( K \), let \( r_j = 1 - Kq^{-j} \) and \( n_j = q^j \), \( j \geq j_0 = \log K/\log q \). Suppose that \( A = \{ z_{j,k} \} \) is the set of points defined by (1). Then \( A \) is a zero set for \( L^p_a \) if, and only if,

\[ p < \frac{\log q}{K}. \]

The sequences defined by (1) do not satisfy the Blaschke condition. However,

\[ \sum_{j,k} (1 - |z_{j,k}|)^2 < \infty, \]

and so the Weierstrass product

\[ \Pi(z) = \prod_{j,k} \frac{z - z_{j,k}}{1 - \frac{z_{j,k}}{z}} \left( 1 + \frac{|z_{j,k}|}{z} \right)^{1/(1 - |z_{j,k}|/z)}, \]

converges uniformly on compact subsets of \( \mathbb{D} \) and vanishes precisely on the \( z_{j,k} \) \([8]\). Part of the proof of Theorem 1 depends on estimates of the Bergman norm of a function related to \( \Pi(z) \).

2. **Proof of Theorem 1**

2.1. **Necessity.** Suppose that \( p \geq \frac{\log q}{K} \). We will show that, given \( f(z) \), analytic in \( \mathbb{D} \) and having \( A \) as its zero set, \( \|f\|_p \leq +\infty \).

Let \( n(t) \) be the counting function of zeros of \( f(z) \) and

\[ N(r) = \int_0^r n(t) \frac{dt}{t}. \]

According to Jensen’s formula,

\[ \log |f(0)| + N(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta, \]

so that, multiplying both sides by \( p > 0 \) and exponentiating,

\[ |f(0)|^p e^{pN(r)} = e^{\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})|^p d\theta}. \]

Therefore, by Jensen’s inequality,
(4)  \[ |f(0)|^p e^{pN(r)} \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta. \]

To show that \( \|f\|_p = +\infty \) then, it is enough to show that \( \int_1^r e^{pN(r)} \, dr = +\infty \), or, what amounts to the same thing, \( \int_1^r e^{pN(r)} \, dr = +\infty \).

For \( r_n \leq r < r_{n+1} \) we have

\[
N(r) = \log r \sum_{j=0}^{n} \eta_j - \sum_{j=0}^{n} \eta_j \log r_j,
\]

so that, since

\[
\eta_j n_j = (1 - Kq^{-j}) q^j < e^{-K},
\]

for all \( j \), and \( \log r \sum_{j=0}^{n} \eta_j = O(q^{n+1} \log r_{n+1} = O(1)) \), we have

\[
N(r) > Kn + O(1),
\]

as \( r \to 1 \). It follows that

\[
\int_{r_n}^{r_{n+1}} e^{pN(r)} \, dr > K(1 - q^{-1}) q^{-n} e^{pK_n + O(1)} \geq K(1 - q^{-1}) e^{O(1)},
\]

and therefore \( \int_1^r e^{pN(r)} \, dr = +\infty \).

2.2. **Sufficiency.** We will show if \( p < (\log q)/K \), then, with \( j_0 \) as in the statement of the theorem,

\[
f(z) = \prod_{j=j_0}^{\infty} \pi_j(z) \sigma_j(z)
\]

belongs to \( L^p_a \), where

\[
\pi_j(z) = \prod_{k=1}^{\eta_j} \left( \frac{z - z_{j,k}}{1 - \overline{z}_{j,k} z} \frac{|z_{j,k}|}{z_{j,k}} \right) \left( 1 + \frac{z - z_{j,k}}{1 - \overline{z}_{j,k} z} \frac{|z_{j,k}|}{z_{j,k}} \right)
\]

and

\[
\sigma_j(z) = \exp \left( -\frac{2Kr_j^j z^{n_j}}{1 - r_j^j z^{n_j}} \right).
\]

Evidently \( f(z) \) vanishes precisely at the points of \( A \), so that \( A \) is a zero set for \( L^p_a \).

Consider first \( \pi_j(z) \). The general term in \( \pi_j(z) \) has the form \((1-u)e^u\), where

\[
u = 1 + \left( \frac{z - z_{j,k}}{1 - \overline{z}_{j,k} z} \right) \frac{|z_{j,k}|}{z_{j,k}} = (1 + z|z_{j,k}|/z_{j,k}) \frac{1 - |z_{j,k}|}{1 - \overline{z}_{j,k} z}.
\]

and since \(|(1-u)e^u| \leq e^{|u|/2} |u| \leq 2 \), we have

\[
|\pi_j(z)| \leq e^{2 \Sigma_{j=1}^{n_j} \left( \frac{1 - |z_{j,k}|}{|z_{j,k}|} \right)^2} = e^{2 \Sigma_{j=1}^{n_j} \left( \frac{1 - r_j^j}{r_j^j} \right)^2}.
\]
Now, with \( z = re^{i\theta} \),
\[
\left( \frac{1 - r_j}{1 - \frac{r^2}{j}} \right) = \frac{(1 - r_j)^2}{1 - \frac{r^2}{j}} \frac{1 - \frac{r^2}{j}}{1 - \frac{r^2}{j} z^2} = \frac{(1 - r_j)^2}{1 - \frac{r^2}{j} z^2} \sum_{m=-\infty}^{\infty} r_j^{m\eta_j} e^{im\theta - \theta}.
\]
Also \( \theta_j = 2\pi k q^{-j} \), \( k = 1, 2, \ldots \), \( \eta_j = q^j \) are the \( \eta_j \)th roots of unity and so for \( m \) not an integer multiple of \( \eta_j \), \( \sum_{k=1}^{\eta_j} e^{im\theta_j} = 0 \). For \( m = l\eta_j \)
\[
\sum_{k=1}^{\eta_j} e^{im\theta_j} = \eta_j = q^j.
\]
Therefore
\[
\sum_{k=1}^{\eta_j} \left( \frac{1 - r_j}{1 - \frac{r^2}{j} z^2} \right)^2 = \frac{K(1 - r_j)}{1 - \frac{r^2}{j} z^2} \sum_{l=-\infty}^{\infty} r_j^{l\eta_j} \eta_j e^{-il\eta_j \theta}
\]
\[
= \frac{K(1 - r_j) (1 - r_j^{2\eta_j} r^{2\eta_j})}{1 - \frac{r^2}{j} z^2} \frac{1 - \frac{r^2}{j} z^{2\eta_j}}{1 - \frac{r^2}{j} z^{2\eta_j} \eta_j^2} = \frac{K(1 - r_j) \eta_j}{1 - \frac{r^2}{j} z^2} \frac{1 + r_j^{2\eta_j} z^{2\eta_j}}{1 - r_j^{2\eta_j} z^{2\eta_j}},
\]
and so
\[
|\pi_j(z)| \leq \exp \left( \frac{2K(1 - r_j)}{1 - \frac{r^2}{j} z^2} \frac{1 + r_j^{2\eta_j} z^{2\eta_j}}{1 - r_j^{2\eta_j} z^{2\eta_j}} \right).
\]
Supposing that \( r_n \leq r < r_{n+1} \), we consider two cases: \( j \) satisfying \( 1 \leq j \leq n \), and \( j \) satisfying \( j \geq n + 1 \).
(i) \( j \geq n + 1 \). Since
\[
\frac{1 - r_j}{1 - \frac{r^2}{j}} \leq \frac{1 - r_j}{1 - r^2} \leq \frac{1 - r_j}{1 - r_{n+1}} = q^{n+1-j},
\]
we have, taking account of (5),
\[
|\pi_j(z)| \leq \exp \left( 2q^{n+1-j} \left( \frac{1 + r_j^{2\eta_j} r^{2\eta_j}}{1 - r_j^{2\eta_j} r^{2\eta_j}} \right) \right) \leq \exp \left( 2q^{n+1-j} \left( \frac{1 + r_j^{2\eta_j}}{1 - r_j^{2\eta_j}} \right) \right) \leq \exp \left( 2q^{n+1-j} \frac{e^K + 1}{e^K - 1} \right),
\]
and therefore, for all large \( n \),
\[
\prod_{j=n+1}^{\infty} |\pi_j(z)| \leq \exp \left( \frac{2(e^K + 1)}{(1 - q^{-1})(e^K - 1)} \right).
\]

(ii) \( j_0 \leq j \leq n \). Since
\[
\frac{1 - r_j}{1 - r_j^2} \leq 1 - r_j = 1 + r_j = 1 + \frac{Kq^{-j}}{2 - Kq^{-j}} < 1 + \frac{1}{2} Kq^{-j/2},
\]
and
\[
\Re \left( \frac{1 + r_j^{n_j} z_j^{n_j}}{1 - r_j^{n_j} z_j^{n_j}} \right) = 1 + \Re \left( \frac{2r_j^{n_j} z_j^{n_j}}{1 - r_j^{n_j} z_j^{n_j}} \right),
\]
and also, from (5),
\[
\Re \left( \frac{2r_j^{n_j} z_j^{n_j}}{1 - r_j^{n_j} z_j^{n_j}} \right) \leq \frac{2}{1 - e^{-K}},
\]
we have
\[
|\pi_j(z)| \leq \exp \left( K + \Re \left( \frac{2Kq^{-j} r_j^{n_j}}{1 - r_j^{n_j} z_j^{n_j}} \right) + \frac{3K^2 q^{-j}}{1 - e^{-K}} \right).
\]

Recalling the definition of \( \sigma_j(z) \), then,
\[
\prod_{j=1}^{n} |\pi_j(z)\sigma_j(z)| \leq \exp \left( Kn + \frac{3K^2}{(1 - q^{-1})(1 - e^{-K})} \right).
\]

Finally
\[
\prod_{j=n+1}^{\infty} |\sigma_j(z)| \leq \exp \left( \frac{2Ke^{-K}}{1 - e^{-K}} \sum_{j=n+1}^{\infty} r_j^{n_j} \right)
\]
\[
\leq \exp \left( \frac{2Ke^{-K}}{1 - e^{-K}} \sum_{j=n+1}^{\infty} (1 - Kq^{-n_j - 1})q^{r_j - n_j - 1} \right)
\]
\[
\leq \exp \left( \frac{2Ke^{-K}}{1 - e^{-K}} \sum_{j=0}^{\infty} e^{-Kq^j} \right),
\]
and, since \( q^j > (q - 1)j \) for all \( j \geq 0 \), this last series is convergent, and thus \( \prod_{j=n+1}^{\infty} |\sigma_j(z)| \) is bounded independently of \( z \).

Combining our results we obtain, for \( r_n \leq r < r_{n+1} \), \(|f(z)| \leq Me^{Kn}\), where \( M = M(q, K) \) is a constant. Hence
\[
\int_{r_n \leq r < r_{n+1}} |f(z)|^p dA(z) \leq 2\pi KMPe^{Kn}(q^{-n} - q^{-n-1})
\]
\[
= 2\pi KMP(1 - q^{-1})e^{(Kp-\log q)n}.
\]
Since $Kp < \log q$,

$$
\int_{r_1 \leq r < r_{n+1}} |f(z)|^p dA(z) = \sum_{n=1}^{\infty} \int_{r_n \leq r < r_{n+1}} |f(z)|^p dA(z) < \infty,
$$

and $f \in L^p_a$.

Remarks. (i) Let $r_j = 1 - K2^{-j}$. For $K = 1$, $A = \{ z_{j,k} \}$ is a zero set for $L^p_a$ with $p < 0.693 \ldots$. For $K = 1/2$, $A$ is a zero set for $p < 1.386 \ldots$ and for $K = 1/4$, for $p < 2.777 \ldots$.

(ii) We obtain easily zero sets $A = \{ z_{j,k} \}$ and $B = \{ w_{j,k} \}$ whose union is not a zero set for the same $L^p_a$. For example take

$$
z_{j,k} = (1 - 1/2^j) e^{2\pi ki/2^j},
$$

$$
w_{j,k} = (1 - 1/2^j) e^{2\pi (k+1/2)j/2^j}.
$$

(iii) Plainly in order to obtain a geometric description of more general zero sets one has to consider also the angular distribution of the zeros. In two seminal papers [4], [5] Korenblum uses the Beurling-Carleson characteristic to give a geometric classification of certain growth spaces related to the Bergman spaces. In particular he characterised the zero sets for the union of all Bergman spaces.

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References

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE, GOWER STREET, LONDON, WC1E 6BT, UNITED KINGDOM.

Current address: Department of Mathematics, Auckland University, Private Bag 92019, Auckland, NEW ZEALAND.

E-mail address: mfdu@math.ucl.ac.uk

DEPARTMENT OF MATHEMATICS, Otago University, PO Box 56, Dunedin, NEW ZEALAND.

E-mail address: pfenton@maths.otago.ac.nz