VERY REGULAR ZERO SETS FOR THE BERGMAN SPACES

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Abstract. We obtain a condition which is both necessary and sufficient such that a sequence of regularly spaced points in the unit disc is a zero set for a Bergman space L^p_a.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the unit disc in the complex plane and $L^p_a(\mathbb{D})$, $p \geq 1$, be the Bergman space of functions $f(z)$ analytic in $\mathbb{D}$ such that

$$\|f\|_p^p = \iint_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where $dA(z)$ is normalised Lebesgue area measure on $\mathbb{D}$. A sequence of points $\{z_n\}$ in $\mathbb{D}$ is called a *zero set* for L^p_a if there exists $f \not\equiv 0$ which vanishes precisely on the z_n. It is an open problem to give a geometric classification of the zeros sets for the Bergman spaces [2].

The case for the Hardy spaces H^p is simpler [1]. A sequence $\{z_n\} \subset \mathbb{D}$ is a *Blaschke sequence* provided $\sum_{n=1}^{\infty}(1-|z_n|) < \infty$, and the zero sets for H^p are precisely the Blaschke sequences. Functions which vanish on the z_n are given by the Blaschke products:

$$B(z) = \prod_{n=1}^{\infty} \frac{z-a_n}{1-\overline{a}_n z}\frac{-|a_n|}{a_n}.$$

All H^p spaces have the same zero sets and a union of two zero sets is a zero set. Horowitz [3] showed that neither statement holds for the Bergman spaces.

For Bergman spaces we consider zero sets of the following form. Given an integer q, $q \geq 2$, and a positive number K, let $r_j = 1 - Kq^{-j}$, for $j \geq j_0 = \log K/\log q$, and on each circle $|z| = r_j$ take $\eta_j = q^j$ equally spaced points $z_{j,1}, z_{j,2}, \ldots, z_{j,\eta_j}$:

$$z_{j,k} = r_j e^{i\theta_{j,k}}, \quad k = 1, \ldots, \eta_j, \tag{1}$$

where $\theta_{j,k} = 2\pi ki/\eta_j$. We ask for conditions on q and K that make $A = \{z_{j,k}\}$ a zero set for some Bergman space $L^p_a(\mathbb{D})$.

A case to bear in mind is the following. With $q = 2$ and $K = 1$,

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$$z_{j,k} = \left(1 - 2^{-j}\right) e^{2\pi k i/\eta_j}, \quad k = 1, \ldots, \eta_j. \tag{2}$$

Such a set is separated in the pseudo-hyperbolic metric but is thick in the sense that a union of pseudo-hyperbolic balls with centres $z_{j,k}$ covers the disc [7]. It turns out that this set is not a zero set for L_a^1 although it is a zero set for L_a^p for p sufficiently small. By thinning the sequence we obtain zero sets for larger values of the parameter p.

We prove the following result:

Theorem 1. *Given an integer q, with $q \geq 2$, and a positive number K, let $r_j = 1 - Kq^{-j}$ and $\eta_j = q^j$, $j \geq j_0 = \log K/\log q$. Suppose that $A = \{z_{j,k}\}$ is the set of points defined by (1). Then A is a zero set for L_a^p if, and only if,*

$$p < \frac{\log q}{K}.$$

The sequences defined by (1) do not satisfy the Blaschke condition. However

$$\sum_{j,k} (1 - |z_{j,k}|)^2 < \infty,$$

and so the Weierstrass product

$$\Pi(z) = \prod_{j,k} \frac{z - z_{j,k}}{1 - \overline{z}_{j,k} z} \frac{-|z_{j,k}|}{z_{j,k}} e^{\left(1 + \frac{z - z_{j,k}}{1 - \overline{z}_{j,k} z} \frac{|z_{j,k}|}{z_{j,k}}\right)},$$

converges uniformly on compact subsets of $\mathbb{D}$ and vanishes precisely on the $z_{j,k}$ [8]. Part of the proof of Theorem 1 depends on estimates of the Bergman norm of a function related to $\Pi(z)$.

2. Proof of Theorem 1

2.1. **Necessity.** Suppose that $p \geq \frac{\log q}{K}$. We will show that, given $f(z)$, analytic in $\mathbb{D}$ and having A as its zero set, $\|f\|_p^p = +\infty$.

Let $n(t)$ be the counting function of zeros of $f(z)$ and

$$N(r) = \int_0^r \frac{n(t)}{t} dt.$$

According to Jensen's formula,

$$\log|f(0)| + N(r) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta, \tag{3}$$

so that, multiplying both sides by $p > 0$ and exponentiating,

$$|f(0)|^p e^{pN(r)} = e^{\frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|^p d\theta}.$$

Therefore, by Jensen's inequality,

$$|f(0)|^p e^{pN(r)} \leq \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta. \tag{4}$$

To show that $\|f\|_p^p = +\infty$ then, it is enough to show that $\int^1 re^{pN(r)}dr = +\infty$, or, what amounts to the same thing, $\int^1 e^{pN(r)}dr = +\infty$.

For $r_n \leq r < r_{n+1}$ we have

$$N(r) = \log r \sum_{j=j_0}^{n} \eta_j - \sum_{j=j_0}^{n} \eta_j \log r_j,$$

so that, since

$$r_j^{\eta^j} = (1 - Kq^{-j})^{q^j} < e^{-K}, \tag{5}$$

for all j, and $\log r \sum_{j=j_0}^{n} \eta_j = O(q^{n+1} \log r_{n+1} = O(1))$, we have

$$N(r) > Kn + O(1),$$

as $r \to 1$. It follows that

$$\int_{r_n}^{r_{n+1}} e^{pN(r)}dr > K(1-q^{-1})q^{-n}e^{pKn+O(1)} \geq K(1-q^{-1})e^{O(1)},$$

and therefore $\int^1 e^{pN(r)}dr = +\infty$.

2.2. **Sufficiency.** We will show if $p < (\log q)/K$, then, with j_0 as in the statement of the theorem,

$$f(z) = \prod_{j=j_0}^{\infty} \pi_j(z)\sigma_j(z)$$

belongs to L_a^p, where

$$\pi_j(z) = \prod_{k=1}^{\eta_j} \frac{z - z_{j,k}}{1 - \overline{z}_{j,k}z} \frac{-|z_{j,k}|}{z_{j,k}} e^{\left(1 + \frac{z - z_{j,k}}{1-\overline{z}_{j,k}z}\frac{|z_{j,k}|}{z_{j,k}}\right)}$$

and

$$\sigma_j(z) = \exp\left(-\frac{2Kr_j^{\eta_j}z^{\eta_j}}{1 - r_j^{\eta_j}z^{\eta_j}}\right).$$

Evidently $f(z)$ vanishes precisely at the points of A, so that A is a zero set for L_a^p.

Consider first $\pi_j(z)$. The general term in $\pi_j(z)$ has the form $(1-u)e^u$, where

$$u = 1 + \left(\frac{z - z_{j,k}}{1 - \overline{z}_{j,k}z}\right)\frac{|z_{j,k}|}{z_{j,k}} = (1 + z|z_{j,k}|/z_{j,k})\frac{1 - |z_{j,k}|}{(1 - \overline{z}_{j,k}z)},$$

and since $|(1-u)e^u| \leq e^{|u|^2/2}$, $|u| \leq 2$, we have

$$|\pi_j(z)| \leq e^{2\sum_{k=1}^{\eta_j}\left(\frac{1-|z_{j,k}|}{|1-\overline{z}_{j,k}z|}\right)^2} = e^{2\sum_{k=1}^{\eta_j}\left(\frac{1-r_j}{|1-\overline{z}_{j,k}z|}\right)^2}.$$

Now, with $z = re^{i\theta}$,

$$\left(\frac{1-r_j}{|1-\overline{z}_{j,k}z|}\right)^2 = \frac{(1-r_j)^2}{1-r_j^2r^2}\frac{1-r_j^2r^2}{|1-\overline{z}_{j,k}z|^2}$$
$$= \frac{(1-r_j)^2}{1-r_j^2r^2}\sum_{m=-\infty}^{\infty} r_j^{|m|}r^{|m|}e^{im(\theta_{j,k}-\theta)}.$$

Also $\theta_{j,k} = 2\pi kq^{-j}$, $k = 1, 2, \ldots, \eta_j = q^j$ are the η_jth roots of unity and so for m not an integer multiple of η_j, $\sum_{k=1}^{\eta_j} e^{im\theta_{j,k}} = 0$. For $m = l\eta_j$

$$\sum_{k=1}^{\eta_j} e^{im\theta_{j,k}} = \eta_j = q^j.$$

Therefore

$$\sum_{k=1}^{\eta_j}\left(\frac{1-r_j}{|1-\overline{z}_{j,k}z|}\right)^2 = \frac{K(1-r_j)}{1-r_j^2r^2}\sum_{l=-\infty}^{\infty} r_j^{|l|\eta_j}r^{|l|\eta_j}e^{-il\eta j\theta}$$
$$= \frac{K(1-r_j)}{1-r_j^2r^2}\frac{1-r_j^{2\eta_j}r^{2\eta_j}}{|1-r_j^{\eta_j}z^{\eta_j}|^2}$$
$$= \frac{K(1-r_j)}{1-r_j^2r^2}\Re\left(\frac{1+r_j^{\eta_j}z^{\eta_j}}{1-r_j^{\eta_j}z^{\eta_j}}\right),$$

and so

$$|\pi_j(z)| \le \exp\left(\frac{2K(1-r_j)}{1-r_j^2r^2}\Re\left(\frac{1+r_j^{\eta_j}z^{\eta_j}}{1-r_j^{\eta_j}z^{\eta_j}}\right)\right).$$

Supposing that $r_n \le r < r_{n+1}$, we consider two cases: j satisfying $1 \le j \le n$, and j satisfying $j \ge n+1$.

(i) $j \ge n+1$. Since

$$\frac{1-r_j}{1-r_j^2r^2} \le \frac{1-r_j}{1-r^2} \le \frac{1-r_j}{1-r} \le \frac{1-r_j}{1-r_{n+1}} = q^{n+1-j},$$

we have, taking account of (5),

$$|\pi_j(z)| \le \exp\left(2q^{n+1-j}\left(\frac{1+r_j^{\eta_j}r^{\eta_j}}{1-r_j^{\eta_j}r^{\eta_j}}\right)\right)$$
$$\le \exp\left(2q^{n+1-j}\left(\frac{1+r_j^{\eta_j}}{1-r_j^{\eta_j}}\right)\right)$$
$$\le \exp\left(2q^{n+1-j}\frac{e^K+1}{e^K-1}\right),$$

and therefore, for all large n,

$$\prod_{j=n+1}^{\infty} |\pi_j(z)| \leq \exp\left(\frac{2(e^K+1)}{(1-q^{-1})(e^K-1)}\right).$$

(ii) $j_0 \leq j \leq n$. Since

$$\frac{1-r_j}{1-r_j^2 r^2} \leq \frac{1-r_j}{1-r_j^2} = \frac{1}{1+r_j} = \frac{1}{2} + \frac{Kq^{-j}}{2-Kq^{-j}} < \frac{1}{2} + Kq^{-j}/2,$$

and

$$\Re\left(\frac{1+r_j^{\eta_j} z^{\eta_j}}{1-r_j^{\eta_j} z^{\eta_j}}\right) = 1 + \Re\left(\frac{2r_j^{\eta_j} z^{\eta_j}}{1-r_j^{\eta_j} z^{\eta_j}}\right),$$

and also, from (5),

$$\Re\left(\frac{2r_j^{\eta_j} z^{\eta_j}}{1-r_j^{\eta_j} z^{\eta_j}}\right) \leq \frac{2}{1-e^{-K}},$$

we have

$$|\pi_j(z)| \leq \exp\left(K + \Re\left(\frac{2Kr_j^{\eta_j} z^{\eta_j}}{1-r_j^{\eta_j} z^{\eta_j}}\right) + \frac{3K^2 q^{-j}}{1-e^{-K}}\right).$$

Recalling the definition of $\sigma_j(z)$, then,

$$\prod_{j=1}^{n} |\pi_j(z)\sigma_j(z)| \leq \exp\left(Kn + \frac{3K^2}{(1-q^{-1})(1-e^{-K})}\right).$$

Finally

$$\begin{aligned}
\prod_{j=n+1}^{\infty} |\sigma_j(z)| &\leq \exp\left(\frac{2Ke^{-K}}{1-e^{-K}} \sum_{j=n+1}^{\infty} r^{\eta_j}\right) \\
&\leq \exp\left(\frac{2Ke^{-K}}{1-e^{-K}} \sum_{j=n+1}^{\infty} (1-Kq^{-n-1})^{q^{n+1}Kq^{j-n-1}}\right) \\
&\leq \exp\left(\frac{2Ke^{-K}}{1-e^{-K}} \sum_{j=0}^{\infty} e^{-K^2 q^j}\right),
\end{aligned}$$

and, since $q^j > (q-1)j$ for all $j \geq 0$, this last series is convergent,and thus $\prod_{j\geq n+1} |\sigma_j(z)|$ is bounded independently of z.

Combining our results we obtain, for $r_n \leq r < r_{n+1}$, $|f(z)| \leq Me^{Kn}$, where $M = M(q,K)$ is a constant. Hence

$$\begin{aligned}
\int_{r_n \leq r < r_{n+1}} |f(z)|^p dA(z) &\leq 2\pi K M^p e^{Kpn}(q^{-n} - q^{-n-1}) \\
&= 2\pi K M^p (1-q^{-1}) e^{(Kp-\log q)n}.
\end{aligned}$$

Since $Kp < \log q$,

$$\int_{r_1 \le r < r_{n+1}} |f(z)|^p dA(z) = \sum_{n=1}^{\infty} \int_{r_n \le r < r_{n+1}} |f(z)|^p dA(z) < \infty,$$

and $f \in L^p_a$.

Remarks . (i) Let $r_j = 1 - K2^{-j}$. For $K = 1$, $A = \{z_{j,k}\}$ is a zero set for L^p_a with $p < 0.693\ldots$. For $K = 1/2$, A is a zero set for $p < 1.386\ldots$ and for $K = 1/4$, for $p < 2.772\ldots$.

(ii) We obtain easily zeros sets $A = \{z_{j,k}\}$ and $B = \{w_{j,k}\}$ whose union is not a zero set for the same L^p_a. For example take

$$\begin{aligned} z_{j,k} &= \left(1 - 1/2^j\right) e^{2\pi k i/2^j}, \\ w_{j,k} &= \left(1 - 1/2^j\right) e^{2\pi (k+1/2) i/2^j}. \end{aligned}$$

(iii) Plainly in order to obtain a geometric description of more general zero sets one has to consider also the angular distribution of the zeros. In two seminal papers [4], [5] Korenblum uses the Beurling-Carleson characteristic to give a geometric classification of certain growth spaces related to the Bergman spaces. In particular he characterised the zero sets for the union of all Bergman spaces.

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