MAXIMAL EMBEDDINGS OF DIRECTED MULTI-CYCLES

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Abstract. We consider embeddings of Eulerian digraphs that have in-arcs alternating with out-arcs in the rotation schemes at each vertex. We define the multicycle $C_n^{l,m}$ to be the digraph on the vertex set $\{v_1, v_2, \ldots, v_n\}$, with arcs comprising l copies of the cycle $(v_1, v_2, \ldots, v_n)$ and m copies of the cycle $(v_n, v_{n-1}, \ldots, v_1)$. We consider maximal embeddings of multicycles and show that all except the bracelet digraphs $C_n^{1,1}$ are upper-embeddable. We find that some multicycles have the maximum possible genus range, being both upper-embeddable and planar, and some multicycles have a genus range of zero.

1. Introduction and Definitions

This report builds on work by Bonnington, Conder, Morton and McKenna [1] on embeddings of Eulerian digraphs.

The language used in this report is, as far as possible, faithful to the definitions and conventions used in [1]. Embeddings considered are 2-cell embeddings of loopless digraphs on compact connected orientable two-manifolds or *surfaces*, as they will be called. In all discussions of digraphs, the implicit assumption is that they are connected. One convention is dropped: in this report digraphs with in-deg(x) = out-deg$(x) = 1$ at some vertex x will be counted as embeddable digraphs, rather than excluded from consideration. The reason is that the circuit graphs occupy a natural position within the family of multicycles whose embedding properties will be studied.

An *embedding* of a digraph on a surface is an embedding in which the arcs and vertices of the digraph are placed on the surface with arcs meeting at mutually incident vertices in such a way that the orientation of a region is consistent with the orientation of the arcs which make up its boundary. When vertex rotation schemes are employed to represent an embedding, the conditions on the placing of arcs and vertices are equivalent to the condition that in-neighbours alternate with out-neighbours in the rotation schemes at each vertex. As with graph embeddings, the *regions* of an embedding are the components of the complement of the digraph on the surface. The term *face* is reserved for regions enclosed by arcs that form an anti-clockwise cycle, with the corresponding term *antiface* for regions enclosed by a clockwise cycle. The genus and maximum genus of an embeddable digraph D are denoted $\gamma(D)$ and $\gamma_M(D)$ respectively.

Adjoining a collection Q of arcs to a digraph $D = (V, A)$ will mean creating the digraph $D' = (V, A \cup Q)$. When the arcs in Q form a directed cycle C, the result of adjoining C to D will be denoted $D + C$.

We say that a digraph is *upper-embeddable* if it can be embedded with two or three regions; no digraph embedding has fewer than two regions. This is analogous to the usual concept of upper-embeddable graphs, applied to those graphs having one-region or two-region embeddings. This report considers the problem of upper-embeddability, describing certain types of directed cycle that may be adjoined to an embedding such that upper-embeddings are preserved. The results are applied to a particular family of embeddable digraphs, the multicycles, introduced in section 2. It is shown that all members of the family except the bracelet digraphs are upper-embeddable.

Questions about upper-embeddability are closely related to questions about maximum genus. For a survey of results on the maximum genus of graphs the reader is referred to an article by Ringeisen [3]. It lists some problems and questions about the upper-embeddability of graphs which can be analogously posed for embeddable digraphs. One question asked in Ringeisen's article is: Can the genus and maximum genus of a connected graph be equal? If we ask the question of embeddable digraphs, Section 5 shows that some multicycles do have this property.

2. The Directed Multicycles $C_n^{l,m}$

The notation $C_n^{l,m}$ will be used to denote the directed graph on the vertex set $\{v_1, v_2, ..., v_n\}$, whose edges consist of l cycles of the form $(v_1, v_2, \ldots, v_n)$, and m cycles of the form $(v_n, v_{n-1}, \ldots, v_1)$. We call this three parameter graph a *multicycle*. It should be noted that either l or m may be zero, but not both.

3. Adjoining Cycles to Digraph Embeddings

We begin with two lemmas that will form the basis for an inductive construction of 2 and 3-region embeddings of the multicycles $C_n^{l,m}$.

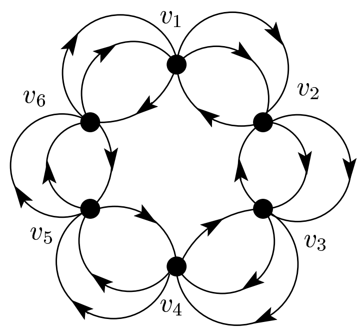

FIGURE 1. *The directed multicycle* $C_6^{2,1}$.

An inductive approach is natural for this family of digraphs, since all multicycles can be constructed by repeatedly adjoining directed cycles to a directed circuit.

The lemmas and their proofs are based on the section on Maximal Embeddings in the book by Gross and Tucker [2].

Lemma 3.1. *Let D be an embedded digraph. The edges of C_2 may be adjoined between any two vertices u and v of D, such that the number of regions is preserved.*

Proof. In a digraph embedding, every vertex meets at least two distinct regions, so let R_1 and R_2 be distinct regions containing u and v respectively. When the arc (u, v) is placed on a handle between R_1 and R_2, the two regions are replaced by a single region R. When (v, u) is placed such that the rotation schemes at u and v have in-arrows alternating with out-arrows, it lies across R, replacing R with two distinct regions. The operation is complete, and the original number of regions is preserved. □

It follows inductively that any number of copies of C_2 may be adjoined to a digraph embedding while preserving the number of regions.

The next statement describes a cycle that can be adjoined such that the number of regions is minimally changed.

Lemma 3.2. *Let D be a digraph embedded with N regions, and R_V and R_W be two regions of this embedding, bounded by the directed cycles $C_V = (v_1, v_2, \ldots, v_m)$ and $C_W = (w_1, w_2, \ldots, w_n)$ respectively. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ be a subsequence of C_V, and $w_{j_1}, w_{j_2}, \ldots, w_{j_s}$ a subsequence of C_W. The directed cycle $C = (v_{i_1}, v_{i_2}, \ldots, v_{i_r}, w_{j_1}, w_{j_2}, \ldots, w_{j_s})$ has the property that $D + C$ can be embedded as a digraph with N regions if r and s are odd, or $N + 1$ regions if r and s have opposite parity.*

Proof. Without loss of generality we can assume r is odd. We can assume, further, that R_V is a face; a reversal of arrows in the same arguments will cover the case when R_V is an antiface.

The assumption on r gives the path $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ an even number of arcs. Consider the graph D' consisting of D with $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ adjoined. The embedding of D becomes an embedding of D' when the arcs of the adjoined path are inserted into the rotation schemes at $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ as follows (or see Figure 2):

$$v_{i_1}\colon \quad \ldots v_{(i_1-1)} \mathbf{v_{i_2}} v_{(i_1+1)} \ldots$$
$$v_{i_k}\colon \quad \ldots v_{i_k-1} \mathbf{v_{i_{(k+1)}}} \mathbf{v_{i_{(k-1)}}} v_{i_k+1} \ldots \text{ for } k = 2, \ldots, (r-1)$$
$$v_{i_r}\colon \quad \ldots v_{i_r-1} \mathbf{v_{i_{(r-1)}}} v_{i_r+1} \ldots$$

where bold-face entries represent vertices inserted into the original rotation scheme.

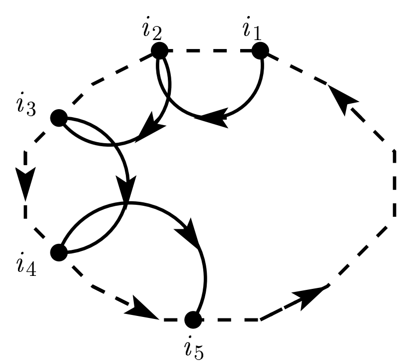

FIGURE 2. *The path $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}$ adjoined to a face R_V.*

This embedding of D' has the property that all regions of the original embedding of D are preserved except R_V, in whose stead is another region, R'_V say. (One can directly check that this is the case for a path with two edges, and note that the case for longer paths of even length builds inductively on this.) Moreover, in-arrows alternate with out-arrows at all vertices except v_{i_1} and v_{i_r}.

Now consider the path $w_{j_1}, w_{j_2}, \ldots, w_{j_s}$:

Case 1: s is odd
Then $w_{j_1}, w_{j_2}, \ldots, w_{j_s}$ contains an even number of arcs; hence it may be adjoined to D' in the same way that $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ was adjoined to D. This embedding of D'', as we shall call D' with the new path adjoined, has the property that all regions of the embedding of D' are preserved

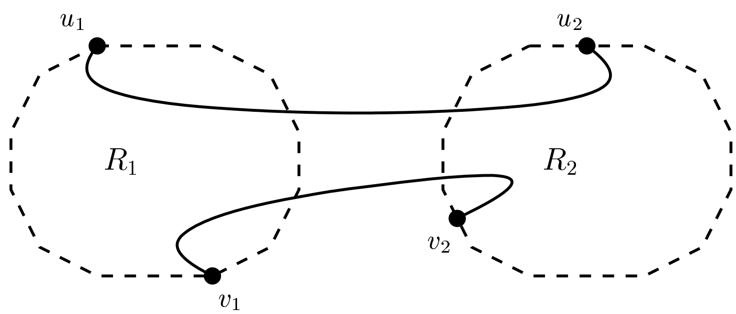

FIGURE 3. *A pair of edges adjoined to an embedding across distinct regions R_1 and R_2, replacing them with two new regions.*

except R_W, which is replaced by a region R'_W say. Thus, it has the property that all regions of the original embedding of D are preserved, except R_V and R_W. Moreover, in-arrows alternate with out-arrows at every vertex of D'' except at the four vertices $v_{i_1}, v_{i_r}, w_{j_1}$ and w_{j_s}.

Suppose the arcs (v_{i_r}, w_{j_1}) and (w_{j_s}, v_{i_1}) are adjoined to this embedding of D'' to satisfy the condition of alternating in- and out-arrows in the rotation schemes of these four vertices. The result is clearly a digraph embedding of $D+C$. To see that the number of regions is still N, we observe that when two edges are adjoined to a graph such that each is placed across regions R_1 and R_2, the regions R_1 and R_2 are replaced by two new regions, and the other regions of the embedding are preserved (see Figure 3). Hence, this embedding of $D+C$ has N regions.

Case 2: s is even

Now the path $w_{j_1}, w_{j_2}, \ldots, w_{j_s}$ has an odd number of arcs. If the arc (w_{j_1}, w_{j_2}) is momentarily ignored, it follows from the previous case that the paths $w_{j_2}, w_{j_3}, \ldots, w_{j_s}$ and $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ and the arcs (v_{i_r}, w_{j_1}) and (w_{j_s}, v_{i_1}) may be adjoined to D such that the resulting embedding preserves all regions of the original embedding except R_V and R_W. Also, in-arrows alternate with out arrows at all vertices except w_{j_1} and w_{j_2}. Thus, if (w_{j_1}, w_{j_2}) is inserted into the rotation schemes of w_{j_1} and w_{j_2} such that in-arrows alternate with out-arrows, the result is a digraph embedding of $D+C$. The operation changes the number of regions in one of two ways – either it increases by 1, or it decreases by 1. If it decreases by 1, the only regions affected are R'_V and R'_W, which are replaced by a region R. This results in the region R lying on both sides of the arcs of C, which contradicts the embedding being a digraph embedding. Hence, this embedding of $D + C$ has $N + 1$ regions. □

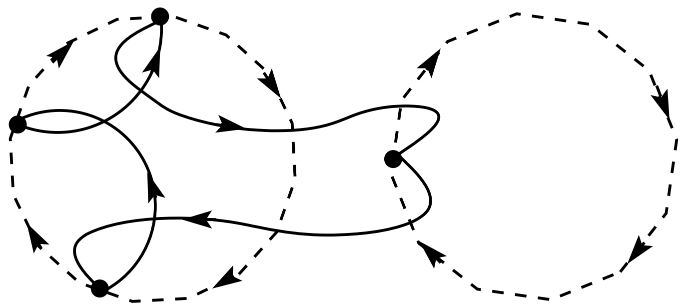

FIGURE 4. *Adjoining a cycle of length 4 between two antifaces of an embedding. The antifaces are replaced by a face and antiface.*

4. MAXIMAL EMBEDDINGS OF $C_n^{l,m}$

Lemma 3.1 implies that all graphs formed by adjoining copies of C_2 to an upper-embeddable graph are upper-embeddable too. A multicycle $C_n^{l,m}$ contains $n \times \text{Min}(l, m)$ copies of the directed cycle C_2, which simplify the problem of finding maximal embeddings.

If $C_n^{k,0}$ is upper-embeddable for all k, it follows that $C_n^{k+l,l}$ is upper-embeddable for all l, hence all graphs $C_n^{l,m}$ with $l > m$; vice-versa for $l < m$. Similarly, if $C_n^{2,2}$ is upper-embeddable for all n, then all graphs $C_n^{m,m}$ with $m \geq 2$ are upper-embeddable.

With these considerations in mind, we will show that the bracelet digraphs on n vertices, $C_n^{1,1}$, are the exception to an otherwise general rule: for all pairs (l, m) that are neither $(1, 1)$ nor $(0, 0)$, and all $n \geq 2$, the multicycles $C_n^{l,m}$ are upper-embeddable.

Proposition 4.1. *The directed graphs $C_n^{k,0}$ (and the graphs $C_n^{0,k}$ by symmetry) are upper-embeddable.*

Proof. This can be shown by a form of induction on k – showing first that the digraphs $C_n^{2l+1,0}$ have a 2-region embedding, and then that the digraphs $C_n^{2l+2,0}$ have a 2-region embedding if n is even, and a 3-region embedding if n is odd.

We show by induction on l that the graphs $C_n^{2l+1,0}$ have a 2-region embedding. Taking $l = 0$ as the base step, the digraph $C_n^{1,0}$ is a circuit on n vertices and has a 2-region embedding in the plane. For the inductive step, suppose $C_n^{2l+1,0}$ has a 2-region embedding. Then the face and the antiface are each enclosed by a cycle made up of one or more copies of the n-cycle $(v_1, v_2, \ldots, v_n)$, because these are the only directed cycles the digraph has. Suppose n is even: the vertices $v_1, v_2, \ldots, v_{n-1}$ are encountered, in that order, along the cycle enclosing the antiface, and the vertex v_n is on the cycle enclosing the face. By Lemma 3.2 the n-cycle $(v_1, v_2, \ldots, v_n)$ may be adjoined to the embedding such that the number of regions is preserved. The same conditions hold with the new digraph, so a further n-cycle may be adjoined such that the number of regions is preserved, yielding a 2-region embedding of $C_n^{2(l+1)+1,0}$. If n is odd, the vertices $v_1, v_2, \ldots, v_n$ sit, in that order, on the cycle enclosing the face, and the vertices $v_1, v_2, \ldots, v_n$ sit, in that order, on the cycle enclosing the antiface. By Lemma 3.2 the cycle $(v_1, v_2, \ldots, v_n, v_1, v_2, \ldots, v_n)$ may be adjoined to the embedding such that the number of regions is preserved; hence the graph of $C_n^{2(l+1)+1,0}$ has a 2-region embedding.

For graphs of the form $C_n^{2l+2,0}$, let the graph $C_n^{2l+1,0}$ be embedded with 2 regions. The directed cycles in this digraph consist of one or more copies of $(v_1, v_2, \ldots, v_n)$, so the vertices $v_2, v_3, \ldots, v_n$ sit, in that order, on the cycle enclosing the antiface, and the vertex v_1 is on the cycle enclosing the face. By Lemma 3.2 the cycle $(v_1, v_2, \ldots, v_n)$ may be adjoined to give a 2-region embedding of $C_n^{2l+2,0}$ if n is even, or a 3-region embedding if n is odd. □

Corollary 4.2. *For $l \neq m$, the graphs $C_n^{l,m}$ are upper-embeddable.*

Proposition 4.3. *For all n the graphs $C_n^{2,2}$ are upper-embeddable.*

Proof. Consider the embedding of $C_n^{1,1}$ that has clockwise rotation schemes at $n-1$ of the vertices, and an anticlockwise rotation scheme on the n^{th} vertex (see Figure 5). This embedding has n regions. With reference to the labelling of the regions used in Figure 5, consider the effect of adjoining the cycle $(v_n, v_{n-1}, \ldots, v_1)$ to the embedding by placing each arc (v_k, v_{k+1}) on a handle between region R_k and region R_{k+1}. When $n-1$ of the arcs are placed in this way, the regions $R_1, \ldots, R_n$ are amalgamated into a single region, giving the underlying graph a 1-region embedding. Adjoining the n^{th} arc results in a 2-region embedding.

The following rotation scheme, derived from the original rotation scheme, has that effect:

$$v_k\text{:}\quad \ldots v^+_{(k-1)} \mathbf{v}^-_{(\mathbf{k+1})} \mathbf{v}^+_{(\mathbf{k-1})} v^-_{(k-1)} \ldots \text{ for } k = 1, \ldots, n-1$$
$$v_n\text{:}\quad \ldots v^-_1 \mathbf{v}^+_{(\mathbf{n-1})} \mathbf{v}^-_{\mathbf{1}} v^+_1 \ldots$$

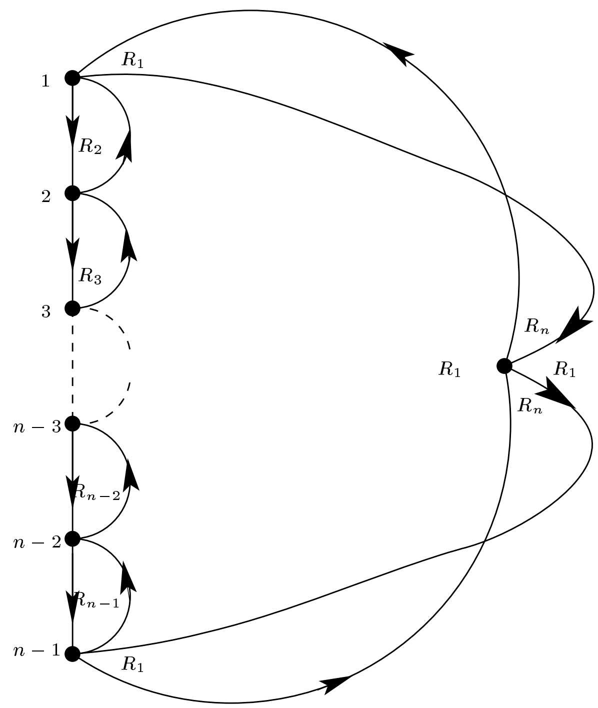

FIGURE 5. *The embedding of $C_n^{1,1}$ used in Proposition 4.3. The n regions of this embedding have been labelled $R_1, \ldots, R_n$.*

(Bold-face entries again represent vertices inserted into the old rotation scheme. When $k = 1$ the subscript $k - 1$ should be understood as n, and the symbols + and - are to distinguish out-neighbours from in-neighbours respectively.)

In this embedding in-neighbours alternate with out-neighbours in the rotation schemes at each vertex, so it is 2-region digraph embedding of $C_n^{1,2}$. Moreover, the vertices $v_1, v_2, \ldots, v_{n-1}$ sit, in that order, along the cycle enclosing the antiface, so by Lemma 3.2 the directed cycle $(v_1, v_2, \ldots, v_n)$ may be adjoined in such a way that the number of regions is 2 if n is even, and 3 if n is odd. Hence the graphs $C_n^{2,2}$ are upper-embeddable. □

Corollary 4.4. *The graphs $C_n^{m,m}$ for $m \geq 2$ are upper-embeddable.*

Proposition 4.5. *A maximal embedding of $C_n^{1,1}$, the bracelet digraph on n vertices, has n regions.*

Proof. The authors of [1] proved the case when n is even. It is also true for n odd. The following proof is independent of the parity of n.

Let the colouring of a vertex black or white again represent anti-clockwise or clockwise rotation schemes at that vertex. In this way a particular 2-colouring of the vertices corresponds to a particular embedding of the digraph.

Starting with a bracelet digraph coloured with alternating black and white vertices, we show by induction that if one black vertex is substituted for an adjacent pair of black vertices, giving an embedding of a bracelet digraph on one more vertex, the new embedding has one more antiface than the original embedding, and the same number of faces. Likewise the process of substituting a white vertex with an adjacent pair of white vertices, the new embedding has one more face and the same number of antifaces. It is clear than any black and white vertex-colouring of a bracelet digraph that has at least one vertex of each colour can be derived from a bracelet digraph on fewer vertices, vertex-coloured black and white alternately, with a series of such substitutions.

As the basis step, let a bracelet digraph on $2k$ vertices be vertex-coloured black and white alternately – giving k black vertices, and k white vertices. To each black-white-black sequence of three vertices in the bracelet there corresponds one face (see Figure 6), so there are $\frac{2k}{2} = k$ faces in the embedding, and to each white-black-white sequence there corresponds one antiface, giving k antifaces.

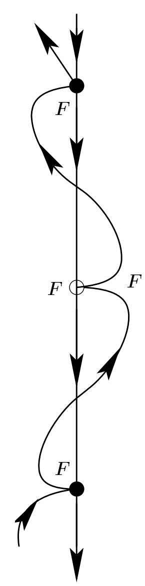

Figure 6. *The face F corresponding to a black-white-black sequence of vertices in the bracelet digraph.*

As the inductive step, suppose that various substitutions led to a bracelet on $2k + r + s$ vertices, with $k + r$ vertices coloured black corresponding to $k + r$ antifaces, and $k + s$ vertices coloured white corresponding to $k + s$ faces. Let one vertex, let us assume a black one, be replaced by a pair of adjacent black vertices, u and v. The new bracelet has $k+r+1$ black vertices and $k+s$ white vertices. The arcs (u, v) and (v, u) enclose between them an antiface, call it A. If u, v, the arcs between and the antiface A are contracted into a single vertex, the remaining regions of the embedding are still preserved and the rotation scheme at the contracted vertex corresponds to a black colour. By the inductive hypothesis, these regions consist of $k + r$ antifaces and $k + s$ faces. Thus, the embedding has $k+r+1$ antifaces, which is the number of black vertices, and $k+s$ faces, which is the number of white vertices. This completes the induction.

An embedding described by a 1-colouring of the vertices is planar, with $n+2$ regions. An embedding of $C_n^{1,1}$ that is described by a colouring with at least 1 vertex of each colour has n regions; hence it is a maximal embedding. □

The maximum genus γ_M of all graphs $C_n^{l,m}$ can now be stated as a consequence of Euler's formula:

Proposition 4.6. *For all graphs $C_n^{l,m}$ on $n \geq 2$ vertices, with $l, m \geq 0$ (excluding $l = m = 0$),*

$$\gamma_M(C_n^{l,m}) = \begin{cases} 1, & \text{if } (l,m) = (1,1), \text{ and} \\ \left[\frac{1}{2}n(l+m-1)\right] & \text{otherwise} \end{cases}$$

where square brackets denote the greatest integer function.

5. Comments on Genus and Genus Range

Restriction on the genus range was demonstrated by the authors of [1] to be a property of the spoke digraphs on $n = 2k+1$ vertices, which have genus range 1. The multicycle family contains members whose genus range is even more restricted, and is in fact zero:

Proposition 5.1. *For odd values of $n > 2$, the multicycles $C_n^{0,2}$ and $C_n^{2,0}$ have $\gamma = \gamma_M = \frac{n-1}{2}$.*

Proof. From the previous section we know that $C_n^{0,2}$ is upper-embeddable and has a 3-region embedding. We show it is a minimal embedding by showing that it does not have a 5-region embedding. For, with a view to contradiction, suppose it did have a 5-region embedding. The shortest directed cycles in the graph have length n, so such an embedding would require the digraph to have at least $\frac{5n}{2}$ arcs – this number comes from counting at least n arcs for each cycle enclosing a region, and in doing so counting each arc twice. The digraph has $2n$ arcs, and $2n < \frac{5}{2}n$; a contradiction. The genus follows from Euler's formula. $\square$

For odd n, Lemma 3.2 can be used to derive an upper bound for the genus of a multicycle $C_n^{0,k}$. Starting with a 2-region embedding of a circuit in the plane, copies of the cycle $(v_1, v_2, \ldots, v_n)$ can be inductively adjoined, each time increasing the number of regions by 1. Hence $C_n^{0,k}$ has an embedding with $k+1$ regions, and by Euler's formula the corresponding surface has genus $\frac{(k-1)(n-1)}{2}$.

For even n, the genus of the graphs $C_n^{0,k}$ and $C_n^{k,0}$ is found via embeddings that have n-cycles enclosing all regions:

Proposition 5.2. *For even values of n, and for $k \geq 1$, a minimal embedding of $C_n^{0,k}$ has $2k$ regions, and*

$$\gamma(C_n^{0,k}) = \gamma(C_n^{k,0}) = \frac{(n-2)(k-1)}{2}.$$

Proof. By induction on k. For $k = 1$, the circuit $C_n^{0,1}$, has a 2-region embedding in the plane, with each region enclosed by an n-cycle. As the inductive step suppose $C_n^{0,k}$ has an embedding with $2k$ regions, with each region enclosed by an n-cycle. Let the cycle $(v_1, v_2, \ldots, v_n)$ enclose a face F of this embedding. Suppose the cycle $(v_1, v_2, \ldots, v_n)$ is adjoined to the embedding by inserting its arcs into the existing rotation schemes at $v_1, \ldots, v_n$ as follows:

$$v_i\colon \quad \ldots v_{i+1}^{+} \mathbf{v_{i-1}^{-}} \mathbf{v_{i+1}^{+}} v_{i-1}^{-} \ldots \quad \text{for } i = 1, \ldots, n$$

where (as subscripts) $1 - 1 \equiv n$, and $n + 1 \equiv 1$.

By face-tracing (see Figure 7), and considering vertices v_{i-1}, v_i and v_{i+1}, there are three possible regions in place of the original face F: two faces, which have been labelled F_1 and F_2, and one antiface, labelled A. The antiface A is clearly enclosed by an n-cycle. The faces F_1 and F_2 follow arcs that alternate between the original arcs and the adjoined arcs; because n is even, they retrace themselves after n arcs. Hence F_1 and F_2 are distinct faces, each enclosed by an n-cycle. The number of regions in this embedding of $C_n^{0,k+1}$ is $2k - 1 + 3 = 2(k+1)$, and all regions are enclosed by n-cycles. The genus follows from Euler's formula. $\square$

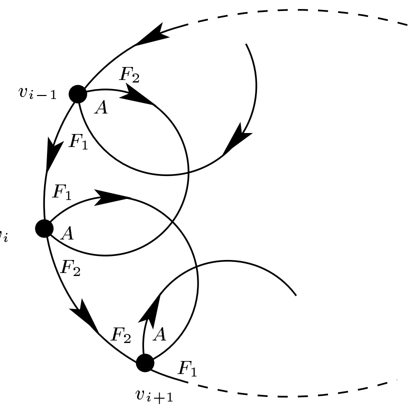

FIGURE 7. *The rotation scheme at* v_{i-1}, v_i, v_{i+1}. *The regions replacing* F *have been labelled* F_1, F_2 *and* A.

The digraphs $C_n^{l,l}, C_n^{l,l+1}, C_n^{l+1,l}$ obviously have planar embeddings. With the exception of $C_n^{1,1}$, they are also upper-embeddable, so in a sense they have the greatest possible genus range.

Finally, it is clear that a copy of the cycle C_2 can be adjoined between adjacent vertices while preserving the genus. If $l > m$, the graphs $C_n^{l,m}$ can have all its m cycles in one direction paired off with m cycles in the other direction, and treated as a colletion of copies of C_2. Hence, for $l > m$ the graphs $C_n^{l,m}$ have genus less than or equal to the genus of $C_n^{l-m,0}$. For the multicycles with $m > l$, the genus is bounded above by the genus of $C_n^{0,m-l}$.

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