GRAPHS EMBEDDED IN THE PLANE WITH FINITELY MANY ACCUMULATION POINTS

C. PAUL BONNINGTON AND R. BRUCE RICHTER

Abstract. Halin's Theorem characterizes those infinite connected graphs that have an embedding in the plane with no accumulation points, by exhibiting the list of excluded subgraphs. We generalize this by obtaining a similar characterization of which infinite connected graphs have an embedding in the plane (and other surfaces) with at most k accumulation points. Thomassen [7] provided a different characterization of those infinite connected graphs that have an embedding in the plane with no accumulation points as those for which the $\mathbf{Z}_2$-vector space generated by the cycles has a basis for which every edge is in at most two members. Adopting the definition that the cycle space is the set of all edge-sets of subgraphs in which every vertex has even degree (and allowing restricted infinite sums), we prove a general analogue of Thomassen's result, obtaining a cycle space characterization of a graph having an embedding in the sphere with k accumulation points.

1. Introduction

A 1-*path* in a graph G is a non-trivial connected subgraph in which every vertex has degree two except for a specified *origin* that has degree one. Thus, 1-paths are necessarily infinite. A 2-*path* (or *2-way infinite path*) is a connected infinite 2-regular subgraph. Evidently, a 2-path is the one-point union of two 1-paths.

A *surface* is a compact connected 2-manifold (that may be non-orientable). An *embedding* ϕ of a graph G in a surface S assigns to each vertex v of G a point $\phi(v)$ of S and to each edge e of G a homeomorph of the open interval $(0,1)$ whose closure in S consists of $\phi(e) \cup \{\phi(v) \mid v \text{ is incident with } e\}$. If x and y are distinct elements of $V(G) \cup E(G)$, then $\phi(x)$ and $\phi(y)$ are disjoint.

In the context of embedding infinite graphs into surfaces, there will inevitably be accumulation points. We require that these not be points of G. (Thus, if we think of G as an abstract 1-dimensional complex, we are really embedding the complex into the surface.) A *vertex accumulation point* (or *VAP*) is a point a in the surface such that every local neighbourhood of a contains an infinite number of vertices of G.

For any graph G, let $G[e]$ denote the graph obtained from G by deleting the edge $e = vw$ and attaching the origin of a 1-path (that is otherwise disjoint from G) to each of v and w. Similarly, let $G[v]$ denote the graph obtained from G by deleting the vertex v and attaching to each neighbour of v the origin of a 1-path. Halin [4] proved the following.

Research supported by the Marsden Fund (Grant Number UOA825) administered by the Royal Society of New Zealand.
Submitted for publication, 16 August 2001.

Theorem 1.1. *Let G be a connected planar graph. Then G has an embedding in the plane with no accumulation points if and only if G does not contain a subdivision of any of $K_5[e]$, $K_5[v]$, $K_{3,3}[e]$ and $K_{3,3}[v]$.*

Our research began with the question: what about forbidding two accumulation points? Our first result is the following.

Theorem 1.2. *Let G be a connected graph and let k be a non-negative integer. Then there is a set $\mathcal{F}(k)$ of pairs (K, W), with K a finite graph, $W \subseteq V(K)$, such that G has an embedding in the plane with at most k accumulation points if and only if G does not contain a subdivision of any graph obtained from $(K, W) \in \mathcal{F}(k)$ by attaching to each vertex of W the origin of a 1-path.*

There is another characterization of graphs that have a planar embedding with no accumulation points. The *cycle space* $\mathcal{C}(G)$ of a graph G is the set of all symmetric differences of the edge-sets of finitely many cycles of G. It is easy to see that this is a vector space over $\mathbb{Z}_2$ (where $0\vec{v} = \emptyset$ and $1\vec{v} = \vec{v}$). Thomassen [7] proved the following.

Theorem 1.3. *Let G be a 2-connected graph. Then G has a planar embedding with no accumulation points if and only if there is a basis B of $\mathcal{C}(G)$ consisting of cycles such that every edge is in at most two elements of B.*

The *even cycle space* $\mathcal{Z}(G)$ of a graph G is the set of all edge-sets of subgraphs H of G such that every vertex has even degree in H. It is a simple matter to show that $\mathcal{C}(G) \subseteq \mathcal{Z}(G)$ and to observe that if G is a 2-path, then $\mathcal{C}(G) = \{\emptyset\}$ and $\mathcal{Z}(G) = \{\emptyset, E(G)\}$, and therefore the inclusion can be strict.

In the context of the even cycle space, it is natural to allow restricted infinite sums. In order to determine if an edge e is in the infinite symmetric difference $\sum E(H)$, it is necessary to decide if e is in an even or an odd number of the $E(H)$ occurring in the sum. Thus, infinite sums are allowed only when every edge occurs in only finitely many summands.

A motivating example is planar graphs. Let G_1 be the usual 4-valent 4-covalent grid in the plane with no accumulation points. Now let G be obtained by identifying one quadrilateral from each of two copies of G_1. Then G can be embedded in the plane with just one accumulation point (and it must have at least one in every embedding by Halin's Theorem 1.1). Now every cycle of G can be expressed as a sum of face boundaries — the sum can be taken to be finite exactly when the accumulation point of G is on the unbounded side of the cycle.

For a connected graph G embedded in the plane, let $\mathcal{B}(G)$ denote the set of all symmetric differences of edge-sets of face boundaries of G. Our other main result is the following.

Theorem 1.4. *Let G be embedded in the plane with k accumulation points such that G is an unbounded subset of the plane. Then $dim(\mathcal{Z}(G)/\mathcal{B}(G)) = k$.*

We can also get a converse of Theorem 1.4. Let G be a 2-connected graph. A *circuit* of G is a connected 2-regular subgraph of G, i.e., is either a cycle or a 2-path of G. A set B of circuits of G is a 2-*basis* of G if: (i) every edge of G is in exactly two members of B; and (ii) every cycle of G is in the span of B (allowing infinite sums).

Theorem 1.5. *Let G be a locally-finite 2-connected graph with a 2-basis B, generating the subspace $\mathcal{B}$ of $\mathcal{Z} = \mathcal{Z}(G)$. Then G has an embedding in the plane for which the face boundaries are precisely the elements of B. Furthermore, if $dim(\mathcal{Z}/\mathcal{B}) = k$, then G has an embedding in the plane with k accumulation points.*

The next section contains a proof of Theorem 1.2 while Section 3 contains the proof of Theorem 1.4. Theorem 1.5 is proved in Section 4, and in Section 5 is devoted to a discussion of related matters and suggestions for future work.

2. Proof of Theorem 1.2

Throughout this section G is used to denote an infinite locally-finite graph and K a finite subgraph of G.

A vertex of a graph is assumed to be colored *black* unless explicitly colored *white*. A *colored* graph is any graph that has a (possibly empty) subset of its vertices colored white. A *colored minor* of a colored graph is any colored graph obtain by a finite sequence of one ore more of the following operations: turning a white vertex black, deleting an isolated vertex, deleting an edge, contracting an edge (ensuring that if any end vertex is white, then the resulting new vertex is white).

A *K-bridge* B of G is a subgraph spanned by either an edge not in K but with both ends in K, or a connected component of $G - V(K)$ together with all edges (and their ends) that have one end in this component and the other end in K. The vertices of $V(B) \cap V(K)$ are *vertices of attachment* (of B). Of interest are the K-bridges that are infinite. Color white the vertices of K that are vertices of attachment for infinite K-bridges. We say that the resulting colored graph (that we also denote by K) is a *core* of G. Hence cores of G are finite subgraphs together with some structural information about their location within G.

Let A be a finite set of distinct points in a surface S (such as a finite set of vertex accumulation points arising from a graph embedding). A face of an embedding of a finite graph in $S \setminus A$ is called an *accumulation face* if it contains a point of A.

An embedding of a core K of G in $S \setminus A$ is *regular* if every white vertex lies on an accumulation face. Two embeddings of K in $S \setminus A$ are *equivalent* if they are isomorphic embeddings with the same set of accumulation faces. Clearly, for any pair of equivalent embeddings ϕ, ψ, there is a homeomorphism $\sigma : S \to S$ such that $\phi \circ \sigma = \psi$. Furthermore, since A is finite, the number of equivalence classes is finite. Clearly, any core has only a finite number of non-equivalent regular embeddings.

A face is said to *cover* the vertices on its boundary. A colored graph K has a *k-white-cover* (in S) if there is an embedding of K in S with k faces that collectively cover all the white vertices.

The following proposition is immediate from the definitions.

Proposition 2.1. *Let A be a set of k points in the surface S and suppose K is a core of G. Then K has a regular embedding in $S \setminus A$ if and only if K has an k-white-cover in S.*

We now present a partial generalisation of Halin's Theorem 1.1.

Theorem 2.2. *Let G be a locally-finite infinite graph embeddable in the surface S. Then there is an embedding of G in S with at most k VAPs if and only if every core of G is k-white-covered.*

The proof of this theorem will follow from Proposition 2.4 and Proposition 2.5.

We note that the above theorem generalises only part of Halin's as we now explain. A *minor-minimal* non-k-white-coverable graph (in S) is one whose colored minors are all k-white-coverable (in S). By the above theorem, generalising Halin's work essentially amounts to determining the minor-minimal non-k-white-covered graphs for values of k and for surfaces S (a two-pronged generalisation): Halin focuses on $k = 1$ and S the sphere. The following easy proposition, together with Theorem 2.2, immediately gives Halin's theorem.

Proposition 2.3. *The minor-minimal non-*1*-white-covered graphs for the sphere are* $K_5[e], K_5[v], K_{3,3}[e]$ *and* $K_{3,3}[v]$.

Of course the real power of Theorem 2.2 is in reducing a problem about infinite graphs (namely, "which graphs embed in S with at most k VAPs?") to one about finite graphs: What are the minor-minimal k-white-covered "obstructions" for a surface S. For example, K_4 with every edge subdivided by a white vertex is a 2-white-covered obstruction for the sphere.

Bienstock and Dean [3] provide further discussion for such obstructions for the sphere. They considered colored graphs in the context of finding obstructions to small face covers. They proved that a minor-minimal non-k-white-coverable planar graph K satisfies $|V(K)| \leq k^3$ – hence, there are only finitely many obstructions.

Determining obstructions (or complete sets for fixed k and S) is beyond the scope of this paper and appears challenging. However, for $k = 1$ and S the projective plane, Archdeacon, Bonnington, Debowsky and Prestidge [1] give a construction for the complete set of obstructions. Returning to the sphere, and setting $k = 2$, Archdeacon, Bonnington and Širáň [2] present the complete set of cubic obstructions.

We now move on to the proof of Theorem 2.2. Some of what follows is an adaptation and extension of Halin's proof of Theorem 1.1.

Proposition 2.4. *Suppose K is a core of G that is not k-white-coverable in S. Then any embedding of G has at least $k+1$ VAPs.*

Proof. Suppose G has an embedding ϕ in S with at most k VAPs. Then $\phi|_K$ is an regular embedding of K in S with at most k accumulation faces. These k accumulation faces collectively cover the white vertices of K, a contradiction. Hence we conclude G has at least $k + 1$ VAPs in any embedding in S. □

Let K be a core of G. A *subcore* K' of K is a subgraph of K (ignoring vertex colors) that is also a core of G. We note that K' may not necessarily be a colored-minor of K. (For example, removing a white vertex attached to two black vertices is not a subcore unless both black vertices are colored white; colored minor operations can color only one white.) Evidently, if ϕ is a regular embedding of K in $S \setminus A$ then $\phi|_{K'}$ is a regular embedding of K'.

Let K be a core of G. We denote by $\overline{K}$ the core of G spanned by K and its finite bridges. We note that the white vertices of K and $\overline{K}$ are identical.

Proposition 2.5. *If every core of G has an k-white-cover in S, then there is an embedding of G with at most k VAPs in S.*

Proof. Let A be a set of k distinct points in S. By Proposition 2.1 it is sufficient to show that if every core of G has a regular embedding in $S \setminus A$, then there is an embedding of G with at most k VAPs.

Pick a vertex v_1 of G, and let $V(G) = \{v_1, v_2, \ldots\}$ so that whenever $1 < i < j$ then $d(v_1, v_i) \leq d(v_1, v_j)$. Now, let $K_1 = \overline{K_1}$ be the subgraph comprising the vertex v_1, and let $K_{i+1} = \overline{K_i + v_{f(i)}}$ where $f(i)$ is the smallest integer such that $v_{f(i)} \notin K_i$. Thus we have an increasing sequence

$$K_1 \subset K_2 \subset \ldots \subset K_i \subset \ldots$$

of cores of G such $\bigcup_{i=1}^{\infty} K_i = G$. Note that each of $G - V(K_i)$ consists of just infinite components.

Claim: For every positive integer i, there exists a regular embedding ϕ_i of K_i such that for an infinite number of integers $j > i$ there exists a regular embedding ψ_j of K_j such that $\psi_j|_{K_i}$ is equivalent to $\phi_i|_{K_i}$.

We prove this by induction on i. Note that every embedding of $K_1 = \langle v_1 \rangle$ is trivially a regular embedding. Furthermore, for every $j > 1$ there exists a regular embedding ψ_j of K_j such that $\psi_j|_{K_1}$ is equivalent to $\phi|_{K_1}$.

Now suppose that there exists a regular embedding ϕ_i of K_i such that for an infinite number of integers $j > i$ there exists a regular embedding ψ_j of K_j such that $\psi_j|_{K_i}$ is equivalent to $\phi_i|_{K_i}$.

There is an infinite number from among the ψ_j that induce the same regular embedding of K_{i+1} up to equivalence. From this equivalence class choose a regular embedding ψ of K_{i+1}; $\psi|_{K_i}$ and $\phi_i|_{K_i}$ are of course equivalent.

Consider the vertices in $W_i = V(K_{i+1}) - V(K_i)$; we claim that under the embedding ψ of K_{i+1}, these vertices lie in the accumulation faces of $\psi|_{K_i}$. Suppose otherwise, and let $v \in W_i$ lie in a non-accumulation face of $\psi|_{K_i}$; let F denote the boundary of this face. Let G' denote the infinite component of $G - V(K_i)$ containing v. By the embedding of K_{i+1} in S, all vertices of W_i that belong to G' lie in F. Let R be a 1-path in G' with origin v. Let u be the last vertex of R that is in W_i. Then u is white in K_{i+1} and does not belong to an accumulation face of ψ, contradicting the regularity of ψ.

Therefore, we have a regular embedding ψ_{i+1} of K_{i+1} equivalent to ψ, such that $\psi_i|_{K_i} = \psi_{i+1}|_{K_i}$. For an infinite number of the remaining ψ_j $(j > i+1)$ $\psi_j|_{K_{i+1}}$ and $\phi_{i+1}|_{K_{i+1}}$ are equivalent.

By iteration of the above proceedure, we get an infinite chain of regular embeddings such that the limit is the required embedding of G. □

3. Proof of Theorem 1.4

In this section we prove Theorem 1.4, i.e., if G is a connected graph embedded in the plane with k accumulation points and G is unbounded, then $dim(\mathcal{Z}(G)/\mathcal{B}(G)) = k$. Equivalently, we can suppose G is embedded in the sphere with $k+1$ accumulation points. As this form makes the proof flow slightly more neatly, we adopt this point of view.

Before we get involved with the proof, we make a few remarks about this "infinite linear algebra". It is not apparent to us (and Zorn's Lemma does not seem to apply) that any of our "spaces" (i.e., $\mathcal{Z}$, $\mathcal{B}$ and $\mathcal{Z}/\mathcal{B}$) necessarily has a basis. (We use the terms "spanning", "independent" and "basis" in their usual way, except we allow restricted infinite sums.) For us, then, to say that a space has a dimension is equivalent to having a basis of that size. Since we are interested only in finite dimensions, many of the problems disappear. In particular, it is easy (and we use this later) to prove that if S is a finite spanning set in a space V and I is a finite independent set in V, then $|I| \leq |S|$.

Let $a_0, a_1, \ldots, a_k$ be the distinct accumulation points of G. For each i, j, $0 \leq i < j \leq k$, it is easy to see that there is a 2-path $P_{i,j}$ in G having exactly a_i and a_j as its accumulation points.

We shall prove that if $z \in \mathcal{Z}(G)$, then z can be expressed in the form $b + \sum_{i=1}^{k} \alpha_i P_{0,i}$, where $b \in \mathcal{B}(G)$ and $\alpha_i \in \mathbb{Z}_2$.

A locally-finite graph is *even* if every vertex has even degree. A *circuit* is a connected 2-regular graph. Thus, a circuit is either a cycle or a 2-path.

Lemma 3.1. *Let G be an even locally-finite graph and let $e \in E(G)$. Then there is a circuit $C \subseteq G$ such that $e \in E(C)$.*

Proof. The result is trivial if e is a loop. Thus, we can assume e has distinct ends u and v.

Let H be the component of G containing e. If $H - e$ is connected, then there is a path P in $H - e$ joining u and v and $P + e$ is a cycle containing e.

If $H - e$ is not connected, then it has exactly two components, one containing u and the other containing v. If both components are infinite, then $H - e$ contains 1-paths P_u and P_v originating from u and v respectively. Then $(P_u \cup P_v) + e$ is a 2-path containing e.

If the component K of $H - e$ containing u is finite, then

$$\left(\sum_{v \in V(K)} deg_H(v) \right) - 2|E(K)|$$

is the number of edges with exactly one end in K. Since each $deg_H(v)$ is even and e has exactly one end in K, there must be another edge with exactly one end in K, a contradiction. □

An easy consequence of Lemma 3.1 is the following.

Lemma 3.2. *Let G be an even locally-finite graph Then there is a set $\mathcal{C}$ of circuits of G such that $\{E(C) : C \in \mathcal{C}\}$ is a partition of $E(G)$.*

An embedding of a graph G in the sphere is *countable* if there are at most countably many accumulation points. It is *pointed* if every 1-path has a unique accumulation point. It is an easy exercise to show that countable implies pointed, but pointed need not imply countable.

Lemma 3.3. *Let G be an even locally-finite graph and let $\mathcal{C}$ be a set of circuits of G such that $\{E(C) : C \in \mathcal{C}\}$ is a partition of $E(G)$. Suppose G is countably embedded in the sphere such that for all 2-paths $C \in \mathcal{C}$, then C has only one accumulation point. Then G is 2-face-colorable.*

Proof. Let f_1 and f_2 be two faces of G. The closure of each $C \in \mathcal{C}$ is a simple closed curve in the sphere and so either separates f_1 from f_2 or both are on the same side of C. We claim that there are only finitely many $C \in \mathcal{C}$ that separate f_1 from f_2.

To see this, it is easy to find uncountably many pairwise disjoint arcs in the sphere, each with one end in f_1 and the other end in f_2. If there are infinitely many $C \in \mathcal{C}$ separating f_1 and f_2, then each of the arcs must have an accumulation point of G, showing G has uncountably many accumulation points, a contradiction.

Let $n(f_1, f_2)$ denote the number of $C \in \mathcal{C}$ that separate f_1 from f_2. We claim that, for any three faces f_0, f_1, f_2, $n(f_0, f_2) \equiv n(f_0, f_1) + n(f_1, f_2) \pmod 2$.

To see this, for $i, j \in \{0, 1, 2\}$, let $\mathcal{C}_{i,j}$ denote the set of $C \in \mathcal{C}$ such that C separates f_i from f_j. If $\{i, j, k\} = \{0, 1, 2\}$, then $\mathcal{C}_{i,j}$ is the disjoint union of $\mathcal{C}_{i,j} \cap \mathcal{C}_{i,k}$ and $\mathcal{C}_{i,j} \cap \mathcal{C}_{j,k}$ (every $C \in \mathcal{C}$ that separates f_i from f_j either separates f_k from f_i or f_k from f_j, but not both).

It follows that $n(f_0, f_2) = |\mathcal{C}_{0,1} \cap \mathcal{C}_{0,2}| + |\mathcal{C}_{1,2} \cap \mathcal{C}_{0,2}|$. Similarly,

$$n(f_0, f_1) = |\mathcal{C}_{0,1} \cap \mathcal{C}_{0,2}| + |\mathcal{C}_{1,2} \cap \mathcal{C}_{0,1}|$$

and

$$n(f_1, f_2) = |\mathcal{C}_{1,2} \cap \mathcal{C}_{0,2}| + |\mathcal{C}_{1,2} \cap \mathcal{C}_{0,1}|.$$

Adding these last two equations shows $n(f_0, f_1) + n(f_1, f_2) \equiv n(f_0, f_2) \pmod 2$, as desired.

It follows that if we color any specified face f^* red, then we can color any other face f red if $n(f^*, f)$ is even and blue if $n(f^*, f)$ is odd. The result will follow by showing that if two faces f and f' share an edge e, then $n(f, f')$ is odd.

Let $C \in \mathcal{C}$ be such that $e \in E(C)$. Evidently C separates f and f', while if $C' \in \mathcal{C} \setminus \{C\}$, then e is on one side of C' and, therefore, both f and f' are on that side of C'. Therefore, not only is $n(f, f')$ odd, but $n(f, f') = 1$. □

Let H and K be subgraphs of a graph G. Then $H + K$ is the subgraph of G induced by the edges in the symmetric difference $E(H) \triangle E(K)$.

Corollary 3.4. *Let G be countably embedded in the sphere and let P and Q be 1-paths in G having the same origin. If P and Q have the same accumulation point, then $P + Q \in \mathcal{B}$.*

Proof. In the subgraph $P + Q$, every vertex has degree 2 or 4. By Lemma 3.3, its faces can be colored red and blue so that every edge separates a red face from a blue face. Thus, $P + Q$ is the symmetric difference of all the faces of G contained in red faces. □

Corollary 3.5. *Let G be countably embedded in the sphere and let P and Q be 2-paths in G, both having exactly two accumulation points. If the accumulation points of P and Q are the same, then $P + Q \in \mathcal{B}$.*

Proof. If P and Q are totally disjoint, then the closure of $P \cup Q$ in the sphere is a simple closed curve, and so $P + Q = P \cup Q \in \mathcal{B}$, since $P + Q$ is the symmetric difference of all the faces on one side of the closure of $P \cup Q$.

Otherwise P and Q have a vertex v in common. Let P_1, P_2, Q_1, Q_2 be the 1-subpaths, starting at v, of $P = P_1 \cup P_2$ and $Q = Q_1 \cup Q_2$, labelled so that, for $i = 1, 2$, P_i and Q_i have the same accumulation point. By Corollary 3.4, $P_1 + Q_1 \in \mathcal{B}$ and $P_2 + Q_2 \in \mathcal{B}$. Therefore, $P + Q = (P_1 + Q_1) + (Q_2 + P_2) \in \mathcal{B}$. □

Corollary 3.6. *Let G be countably embedded in the sphere and let P, Q, and R be 2-paths in G. If there are distinct points a, b and c in the sphere such that the accumulation points of P are a and b, the accumulation points of Q are b and c, and the accumulation points of R are c and a, then $P + Q + R \in \mathcal{B}$.*

Proof. If P, Q and R are pairwise disjoint, then the closure of $P \cup Q \cup R$ is a simple closed curve in the sphere and $P + Q + R$ is the symmetric difference of the faces on one side of this curve.

Otherwise, we can suppose P and Q have vertices in common. Write

$$P = (\dots, v_{-2}, e_{-1}, v_{-1}, e_0, v_0, e_1, v_1, e_2, v_2, \dots),$$

so that

$$\lim_{n\to-\infty} v_n = a \qquad \text{and} \qquad \lim_{n\to+\infty} v_n = b.$$

Since P has only a and b as accumlation points and a is not an accumulation point of Q, there is an N such that $n < N$ implies $v_n \notin V(Q)$.

Since P and Q have a vertex in common, there is a least n such that $v_n \in V(P) \cap V(Q)$. Let $P = P_1 \cup P_2$ and $Q = Q_1 \cup Q_2$, where the P_i and Q_i are 1-paths with origin v_n, labelled so that both P_2 and Q_2 have b as accumulation point. Then $P_1 \cap V(Q) = \{v_n\}$, so $P_1 \cup Q_1$ is a 2-path having a and c as accumulation points.

By Corollary 3.4, $P_2 + Q_2 \in \mathcal{B}$. By Corollary 3.5, $(P_1 \cup Q_1) + R \in \mathcal{B}$. Therefore, $P + Q + R = (P_2 + Q_2) + ((P_1 \cup Q_1) + R) \in \mathcal{B}$. □

We will prove Theorem 1.4 in two stages, as the first stage will be useful for the proof of Theorem 1.5.

Theorem 3.7. *Let G be a connected locally-finite graph countably embedded in the sphere. Let $\mathcal{B}$ denote the subspace of $\mathcal{Z}$ generated by the face boundaries. For any finite set $\{a_0, a_1, \dots, a_k\}$ of accumulation points, let $P_1, P_2, \dots, P_k$ be 2-paths with P_i having a_0 and a_i as accumulation points. Then $\{P_i + \mathcal{B} : i = 1, 2, \dots, k\}$ is linearly independent in $\mathcal{Z}/\mathcal{B}$.*

Proof. Suppose $\alpha_t \in \{0, 1\}$, $t = 1, 2, \dots, k$, are such that $\sum_{t=1}^{k} \alpha_t P_t \in \mathcal{B}$. We show all α_t are 0, so that $\{P_t + \mathcal{B} : t = 1, 2, \dots, k\}$ is independent. Suppose, to the contrary, that for some j, $\alpha_j \neq 0$.

Let $P_j = (\dots, v_{-2}, e_{-1}, v_{-1}, e_0, v_0, e_1, v_1, \dots)$, with the labelling chosen so that

$$\lim_{n\to+\infty} v_n = a_j.$$

There is an $\varepsilon > 0$ such that the circle $B(a_j, \varepsilon)$ of radius ε around a_j is disjoint from all the other P_t. Within this neighbourhood of a_j, we can assume P_j is connected, and so can be taken to be a straightline segment from a_j to the boundary of the circle. (More precisely, there is a homeomorphism $h : B(a_j; \varepsilon) \to B(a_j; \varepsilon)$ such that the component of $h(P_j)$ that has a_j in its closure consists of a straightline segment. Then we can reduce ε to get a disc that misses the other segments of P_j that might happen to be contained in $B(a_j; \varepsilon)$.)

Choose $\varepsilon > 0$ small enough so that, in addition, no other P_i intersects $B(a_j; \varepsilon)$. If the faces of G are colored red and blue in such a way that the edges of P_j have faces of different color on each side, then, within the circle above, there must be both red and blue faces. For every positive $\varepsilon' < \varepsilon$, the circle of radius ε' must cross from a red face to a blue face at a point that is not in P_j. Since there are uncountably many such ε' and only countably many vertices and accumulation points, some edge e of G not in P_j, but contained in the circle of radius ε must also have faces of different colors on each side. But then e is not in $\sum_{t=1}^{k} \alpha_t P_t$, so this sum cannot be in $\mathcal{B}$. □

It is interesting to note that even for pointed embeddings the conclusion of Theorem 3.7 is not true. If G is the binary tree pointedly embedded in the plane so that the interval $\{0\} \times [0, 1]$ is the set of limit points, then every 2-path in G is in $\mathcal{B}$, despite many having distinct ends. So we cannot even get one path independent of $\mathcal{B}$. The proof above breaks down because all the circles of radius ε' cross from red to blue at accumulation points.

We now move on to the second stage of the proof of Theorem 1.4.

Theorem 3.8. *Let G be embedded in the sphere with $k+1$ accumulation points, for some non-negative integer k. Then $dim(\mathcal{Z}/\mathcal{B}) = k$.*

Proof. By Theorem 3.7, $dim(\mathcal{Z}/\mathcal{B}) \geq k$ (i.e., there are k linearly independent elements in $\mathcal{Z}/\mathcal{B}$). Let $a_0, a_1, \ldots, a_k$ be the accumulation points and let $P_1, \ldots, P_k$ be paths as in Theorem 3.7. We must show $\{P_i + \mathcal{B} : i = 1, 2, \ldots, k\}$ spans $\mathcal{Z}/\mathcal{B}$.

Let $H \subseteq G$ have all vertices of even degree. By Lemma 3.2, $E(H)$ partitions into the edge-sets of circuits $C_1, C_2, \cdots$. For $0 \leq i, j \leq k$, let $\mathcal{C}_{i,j}$ denote the set of C_i that are 2-paths having a_i and a_j as accumulation points.

We claim that if $i \neq j$, then $|\mathcal{C}_{i,j}|$ is finite. To see this, let $\varepsilon > 0$ be smaller than the distance between a_i and a_j. For every positive ε' less than ε, consider the circle of radius ε' centred at a_i. If there were infinitely many elements in $\mathcal{C}_{i,j}$, then every such circle would have an accumulation point of G on it, showing G to have uncountably many accumulation points, a contradiction.

Now, let $H' = H - \cup_{i \neq j} E(\cup_{C \in \mathcal{C}_{i,j}} C)$. Then the partition of $E(H)$ yields a partition of $E(H')$ into edge-sets of circuits, all of which are either cycles or 2-paths with only one accumulation point. By Lemma 3.3, H' is 2-face colorable, which implies $E(H') \in \mathcal{B}$.

For $i \neq j$, let $\mathcal{C}_{i,j} = \{C_1, C_2, \ldots, C_m\}$. Taking the members of $\mathcal{C}_{i,j}$ in pairs, by Corollary 3.5 we see that if m is even, then $C_1 + C_2 + \cdots + C_m \in \mathcal{B}$, while if m is odd, then $C_2 + \cdots + C_m \in \mathcal{B}$.

In the case m is odd, if $0 \notin \{i, j\}$, then Corollary 3.6 implies $P_i + P_j + C_1 \in \mathcal{B}$. If $0 = i$, then Corollary 3.5 implies $P_i + C_1 \in \mathcal{B}$.

Therefore, in both cases of m even and m odd, we can express $\sum_{C \in \mathcal{C}_{i,j}} C$ in the form $B + \sum_{t=1}^{k} \alpha_t P_t$, for some $B \in \mathcal{B}$ and some $\alpha_t \in \{0, 1\}$, $t = 1, 2, \ldots, k$. Thus, $\{P_t + B : t = 1, 2, \ldots, k\}$ spans $\mathcal{Z}/\mathcal{B}$. □

4. Proof of Theorem 1.5

In this section we prove Theorem 1.5, which is essentially the converse of Theorem 1.4. It is not exactly the converse of Theorem 1.4, since the face boundaries of an embedding of a 2-connected graph need not be either cycles or 2-paths. Some faces might be bounded by unions of 2-paths. Thus, the faces of such an embedding do not make a 2-basis. However, if there are only finitely many accumulation points, we can find another embedding for which the faces are a 2-basis; $dim(\mathcal{Z}/\mathcal{B})$ will decrease.

Theorem 1.5. *Let G be a locally-finite 2-connected graph with a 2-basis B, generating the subspace $\mathcal{B}$ of $\mathcal{Z}$. Then:*

1. *G is planar;*
2. *there is an embedding of G in the sphere with face boundaries precisely the elements of B; and*
3. *if $dim(\mathcal{Z}/\mathcal{B}) = k$, then G has an embedding in the sphere with $k+1$ accumulation points.*

Proof. To prove the first part of the theorem, we show that if H is a 2-connected subgraph of G, then H also has a 2-basis. Since no subdivision of either $K_{3,3}$ or K_5 has a 2-basis (by MacLane's Theorem, for example), it follows from Kuratowski's Theorem that G is planar.

So let H be a 2-connected subgraph of G. Let $\sim$ be the relation defined on $E(G) \setminus E(H)$ by $e \sim e'$ if there is a $C \in B$ such that $e, e' \in E(C)$. Thus, $\sim$ is reflexive and symmetric, so the transitive closure $\simeq$ is an equivalence relation.

For each equivalence class A of $\simeq$, let

$$C_A = \sum_{C \in B, A \cap E(C) \neq \emptyset} C.$$

Clearly, every edge of C_A is in H and every vertex of H is incident with an even number of edges in C_A, so Lemma 3.2 implies C_A partitions into edge-disjoint circuits of H. Let B_H denote the set of all such circuits, over all the equivalence classes A of $\simeq$. We claim that B_H is a 2-basis for H.

Consider an edge e of H. Because H is 2-connected, there is a cycle C_e in H containing e. Now $C_e = \sum_{C \in B_e} C$, for some $B_e \subseteq B$. Since $e \in E(C_e)$, there is exactly one element of B_e that contains e. Moreover, B_e is the union of equivalence classes of $\simeq$, so e is in exactly one member of the corresponding C_A. Since we could equally well have used $B \setminus B_e$ in place of B_e, we have that e is in exactly two members of B_H, as required.

For (2), we begin by creating a 3-connected graph G'' containing G that also has a 2-basis. For the construction, subdivide every edge of G to get G'; the cycles of G' corresponding to the elements of B obviously make a 2-basis B' for G'.

For each cycle $C \in B'$, add a new vertex v_C, that is joined to every vertex of C. For every 2-path P in B', identify P with the x-axis in the upper half plane grid. (The *upper half plane grid* is the planar graph with vertices $v_{i,j}$ at the lattice points (i,j), $i \in \mathbb{Z}$, $j \in \mathbb{Z}$, $j \geq 0$, and $v_{i,j}$ is adjacent to $v_{i\pm1,j}$ and $v_{i,j\pm1}$, whenever the indices correspond to a vertex (e.g., $v_{10,0}$ is not adjacent to $v_{10,-1}$, since the latter is not a vertex of the upper half plane grid).)

Let G'' be the resulting graph. Clearly G'' is locally-finite. A 2-basis B'' for G'' is obtained from the triangles generated by adding a vertex joined to a cycle in B' and the quadrilaterals in the half grids attached to 1-paths in B'. To see that this is a 2-basis, first note that any edge of G'' that is not an edge of G' is in exactly two of these cycles. If e is an edge of G', there are exactly two elements of B' containing e and, therefore, exactly two elements of B'' containing e.

We must show that every cycle C of G'' is generated by cycles in B''. We observe that if e is an edge of G'' with both ends in G', then e is an edge of G'. If $C \subseteq G'$, then the result is easy, so we may assume C has a vertex not in G', Therefore, $C - V(G')$ consists of finitely many paths each of which extends in C to a path that has its ends in $V(G')$. It is easy to find $B''_C \subseteq B''$ so that $C' = C + \sum_{C'' \in B''_C} C''$ is a finite set of edge-disjoint cycles in G'.

Now use some elements of B' to generate C'; for each element X of B' used, there is a natural set of elements of B'' that generates X, namely the ones attached to X. This shows that every cycle of G'' is generated by elements of B''. Therefore, B'' is a 2-basis as required.

Now for (3). By (1), G'' has an embedding in the sphere. For every $C \in B''$, $G'' - V(C)$ is connected, so C bounds a face of G'' in any embedding of G'' in the sphere. Mohar [5] proves that there is a unique pointed embedding of G'' in the sphere and the accumulation points are in 1-1 correspondence with the ends of G''.

Consider the embedding of G induced from this embedding of G''. The faces of G are clearly bounded by the elements of B, proving (2).

Furthermore, every accumulation point of G'' is the limit of a nested sequence of cycles. For if a is an accumulation point of G'', a is not incident with a face of G'' (since every face is bounded by a cycle). Therefore, if C is any cycle of G and K denotes the block of $G'' - V(C)$ containing 1-paths coverging to a, then the boundary of the face of K containing C is a cycle of K that separates C from a.

We would like to prove there are precisely $dim(\mathcal{Z}/\mathcal{B}) + 1$ accumulation points. This will be done in the following two claims. (Here we denote the even cycle space of G'' by $\mathcal{Z}''$ and $\mathcal{B}''$ denotes the subspace generated by the elements of B''.)

(1) $dim(\mathcal{Z}''/\mathcal{B}'') = dim(\mathcal{Z}/\mathcal{B})$.
(2) If $a_0, a_1, \dots, a_m$ are accumulation points of G'', then $dim(\mathcal{Z}''/\mathcal{B}'') \geq m$.

For (1), let $X_1, \dots, X_k \in \mathcal{Z}$ be such that $\mathcal{X} = \{X_i + \mathcal{B} \mid i = 1, 2, \dots, k\}$ is a basis for $\mathcal{Z}/\mathcal{B}$. We shall show that $\mathcal{X}'' = \{X_i + \mathcal{B}'' \mid i = 1, 2, \dots, k\}$ is a basis for $\mathcal{Z}''/\mathcal{B}''$.

To see that $\mathcal{X}''$ is spanning, let $Z \in \mathcal{Z}''$. It is easy to find $B''_Z \in \mathcal{B}''$ such that $Z_1 = Z + B''_Z \subset G$ (with only a slight abuse of notation). Thus, $Z_1 = (\sum \alpha_i X_i) + B_Z$, for some $\alpha_i \in \{0, 1\}$ and some $B_Z \in \mathcal{B}$. But $\mathcal{B} \subset \mathcal{B}''$, so $Z = (\sum \alpha_i X_i) + (B''_Z + B_Z)$, as required.

For independence, suppose $\sum \alpha_i X_i \in \mathcal{B}''$. Since $\sum \alpha_i X_i \subset G$, we conclude $\sum \alpha_i X_i \in \mathcal{B}$, so that all α_i are 0, as required.

For (2), suppose $a_0, a_1, \dots, a_m$ are any accumulation points of the pointed embedding of G''. For $i = 1, 2, \dots, m$, let P_i be a 2-path in G'' having a_0 and a_i as accumulation points. We claim that $\sum_{i=1}^m P_i \notin \mathcal{B}''$.

If this claim is false, we can color the faces of G'' red and blue so that $\sum P_i$ is the symmetric difference of the red faces. Let C be a cycle of G'' so that a_1 is on one side of C (the "inside") and all of $P_2, \dots, P_m$ are on the outside of C. The path P_1 crosses C an odd number of times — it starts on the outside and ends on the inside.

Consider the faces of G'' inside C and incident with a vertex of C. Some of these faces are red and some are blue, and as we walk once around C, we must change an even number of times from one color to the other. But every change occurs as we cross an edge of P_1, and there are an odd number of such edges inside C and incident with a vertex of C. This contradiction proves (2). □

5. Remarks

We begin this section by noting that Thomassen's Theorem 1.3 is equivalent to the $k = 0$ case in 1.4 and 1.5. We had hoped to prove Theorem 1.3 from Theorem 1.4 and Theorem 1.5 but we do not see how. What is required to show is that if there is a set B of cycles of a 2-connected graph G such that every cycle of G is a finite symmetric difference of cycles of B, then we can find a 2-basis of G. It is reasonably clear what that 2-basis should be. Let $H = \sum_{B \in \mathcal{B}} B$. Then H is an even subgraph of G and so partitions into circuits, that we can add to B to make a 2-basis B'. It is also not very hard to prove that H contains no cycles and, therefore, all the circuits added to B are 2-paths.

What is missing is the rest of the proof, namely that $\mathcal{Z} = \mathcal{B}'$, where $\mathcal{B}'$ is the subspace of $\mathcal{Z}$ generated by the elements of B'. It would be nice (and should be possible) to provide a relatively simple proof of this.

It is a natural question to wonder if there is a theorem about whether or not G has an embedding in the sphere with only finitely many accumulation points.

A graph G is a *finitely disjoint sequence of Halin graphs* if there is a sequence $H_1, H_2, \cdots$, of subgraphs of G, each H_i one of the four graphs from Theorem 1.1 with the property that the finite parts of each of the H_i (i.e., the subgraph of either K_5 or $K_{3,3}$ to which the 1-paths are attached) are pairwise disjoint.

Conjecture 5.1. *Let G be a connected planar graph. Then G has no embedding in the sphere with only finitely many accumulation points if and only if G contains an infinite sequence of finitely disjoint Halin graphs.*

In fact we believe we know how to prove this, but it is quite long and technical, so we postpone it to a future paper. A relatively short, direct proof would be welcome.

In another vein, we have mainly considered pointed embeddings. An embedding produced from a 2-basis has another attractive property, that we put into the following definition. An embedding is *proper* if it is pointed and each face is incident with at most one accumulation point. We put forward the following conjecture.

Conjecture 5.2. *Every 3-connected planar graph has a unique proper embedding in the plane.*

We note that Richter and Thomassen [6] prove this in the case there are only finitely many accumulation points. But it is not too hard to come up with an example of a 3-connected locally-finite planar graph that has a proper embedding with uncountably many accumulation points. There is little theory available for dealing with such examples and we hope our conjecture will stimulate research on this topic.

Acknowledgements

The authors thank Dan Archdeacon, Bojan Mohar, Carsten Thomassen and Tom Tucker for many useful discussions on the topic. Both authors acknowledge the support of the Marsden Fund (Grant Number UOA-825) administered by the Royal Society of New Zealand. The second author acknowledges the support of the Natural Sciences and Engineering Research Council of Canada.

References

[1] D. Archdeacon, C. P. Bonnington, M. Debowsky and M. Prestidge, *Halin's theorem for the Möbius strip*, submitted for publication.

[2] D. Archdeacon, C. P. Bonnington, J. Širáň, *Halin's theorem for cubic graphs on an annulus*, submitted for publication.

[3] D. Bienstock and N. Dean, *On obstructions to small face covers in planar graphs*, J. Combin. Theory, Ser. B **55** (1992), 163-189.

[4] R. Halin, *Zur haüfungspunktfeien Darstellung abzählbarer Graphen in der Ebene*, Arch. Math. **17** (1966), 239–243.

[5] B. Mohar, *Embeddings of infinite graphs*, J. Combin. Theory, Ser. B **44** (1988), 29-43.

[6] R. B. Richter and C. Thomassen, 3-*connected planar spaces uniquely embed in the sphere*, preprint.

[7] C. Thomassen, *Planarity and duality of finite and infinite graphs*, J. Combin. Theory Ser. B **29** (1980), 244–271.

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1