The error growth of some symplectic explicit Runge-Kutta Nyström methods on long N-body simulations

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August 24, 2001

Abstract

At one extreme, the global error for symplectic explicit Runge-Kutta Nyström (SERKN) methods consists entirely of truncation error and grows as t. At the other extreme, the global error consists entirely of random round-off error and grows stochastically as $t^{3/2}$. We use numerical testing to investigate how the global error grows for stepsizes between these two extremes. The testing is of representative SERKN methods of orders four to seven on three long N-body simulations of the Solar System. The work also provides an opportunity to introduce two new test problems for symplectic methods and to present comparisons of the efficiency of SERKN methods.

Keywords: Solar System, N-body, long simulations, explicit Nyström, symplectic, error growth, comparisons

2000 MSC: Primary - 65L05, secondary - 70F10

1 Introduction

Figure 1 contains plots of the L_2 norm of the global error for five one-million year simulations of the Sun and the gas giants (Jupiter, Saturn, Uranus and Neptune) using the order seven SERKN method of Calvo and Sanz-Serna [3]. The simulations were done in double precision using stepsizes of 4, 8, 16, 32 and 64 days. The reference solution was calculated in quadruple precision using a variable-stepsize integrator with a small tolerance.

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‡The work of the first author was partly supported by the University of Auckland Research Council. The work of the second author was partly supported by the Natural Sciences and Engineering Research Council of Canada.

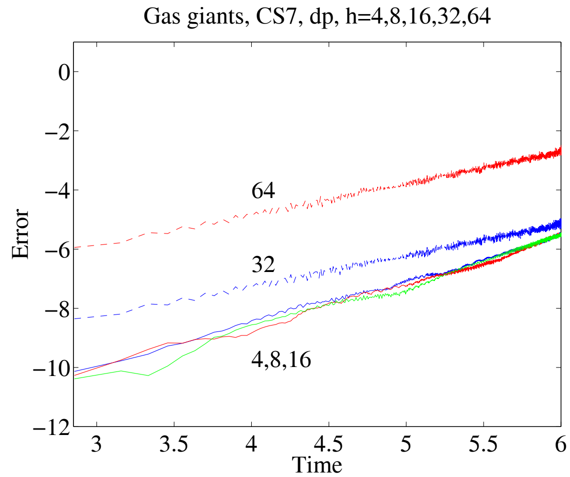

Figure 1: Base 10 log-log graph of the global error for a simulation of the Sun, Jupiter, Saturn, Uranus and Neptune, using the order seven SERKN method of Calvo and Sanz-Serna [3].

The norm of the global error for stepsizes of 32 and 64 days grows approximately as t. We used least squares to fit the power law αt^{β} and found β was 1.05 for the stepsize of 32 days and 1.07 for the stepsize of 64 days. These two values agree reasonably well with the value of one we would expect if the round-off error was insignificant.

The norm for the stepsizes of 4, 8 and 16 days grows faster than t. In addition, the plots for these stepsizes are almost coincident and have more undulations than those for the larger stepsizes. These three features suggest round-off error is a significant component of the global error. The exponent of the power law fit is 1.51, 1.60 and 1.68 for the stepsizes of 4, 8 and 16 days, respectively. Brouwer [1] proved that when the global error consists entirely of random round-off error, it grows stochastically as $t^{3/2}$. Since 1.51 is close to $3/2$, Brouwer's theorem suggests the global error for the stepsize of 4 days is dominated by random round-off error, although the agreement in exponents may be fortuitous.

The plots in Figure 1 raise interesting questions about how the growth of the global error for SERKN methods depends on the stepsize and the order, and how Møller's technique [7, 8] changes the growth.

We present numerical comparisons of representative SERKN methods of orders four to seven on three long N-body simulations of the Solar System. The simulations are a one million year simulation of the Sun and gas giants, a 22,000 year simulation of the Sun and nine planets, and a 11,000 year simulation of three Saturnian satellites.

A simulation of one million years may seemed contrived because it is far longer than normally used in testing SERKN methods. In fact, a simulation of one million years is short by modern standards of computational astronomy, 10^9 years being common. For example, Grazier, Newman, Kaula and Hyman [5] simulated the Sun, the gas giants and 10^5 particles for 10^9 years, and Kuchner and Holman [6] simulated 1500 trans-Neptunian particles for 4.5×10^9 years.

We begin in §2 with a few definitions and briefly discuss the selection of the SERKN methods we used in our testing. Then in §3 and 4 we give the comparisons when Møller's technique is not used (§3) and is used (§4). We end in §5 with a summary of our results.

2 Definitions and methods

The initial value problems for the three simulations can be written as

$$\ddot{y}(t) = f(y(t)), \quad y(t_0) = y_0, \quad \dot{y}(t_0) = \dot{y}_0, \tag{1}$$

where the dot operator denotes differentiation with respect to t, $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, and f is sufficiently smooth.

The explicit Runge-Kutta Nyström methods we test calculate order p approximations y_i and $\dot{y}_i$ to $y(t_i)$ and $\dot{y}(t_i)$, respectively, $i = 1, 2, \ldots$, using the formulae

$$y_i = y_{i-1} + h\dot{y}_{i-1} + h^2 \sum_{j=1}^{s} b_j f_j, \quad \dot{y}_i = \dot{y}_{i-1} + h \sum_{j=1}^{s} b'_j f_j, \tag{2}$$

where $h = t_i - t_{i-1}$ and $f_j = f(t_{i-1} + c_j h, y_{i-1} + c_j h \dot{y}_{i-1} + h^2 \sum_{k=1}^{j-1} a_{jk} f_k)$, $j = 1, \ldots, s$.

Method (2) is symplectic if

$$a_{jk} = (c_j - c_k) b'_k, \quad k = 1, \ldots, j-1, \quad j = 2, \ldots, s. \tag{3}$$

Hence, once c_j, b_j, and b'_j, $j = 1, \ldots, s$, are known, the remaining coefficients a_{jk} are.

If $c_1 = 0$, $c_s = 1$ and $a_{sj} = b_j, j = 1, \ldots, s-1$, the last stage f_s can be re-used as the first stage of the next step. This property, commonly called FSAL, means the number of f evaluations on all but the first step is $s - 1$.

Yoshida [14] gave order eight symplectic methods that can be written as 16-stage FSAL SERKN methods. Okunbor and Skeel [9] gave families of one-, two- and three-stage SERKN methods. Calvo and Sanz-Serna presented a five-stage, order four FSAL method in [2] and a 13-stage, order seven FSAL method in [3]. Both methods have minimised error coefficients. Okunbor and Skeel [10] performed numerical searching and found four five-stage, order five methods, and sixteen seven-stage, order six methods. Chou and Sharp [4] presented a seven-stage, order five FSAL method with minimised error coefficients. Tsitouras [13] gave a 33-stage order 10 non-FSAL method.

It is impractical to present comparisons of all of the above methods. Our preliminary testing showed the results for the methods of orders one, two, three, eight and ten added little to the results for the methods of orders four to seven. We also found the results for the four order five methods of [10] were similar, as were the results for the sixteen order six methods of [10].

These observations meant we could reduce the number of methods to five: the five-stage, order four FSAL method of Calvo and Sanz-Serna [2], the seven-stage, order five FSAL method of Chou and Sharp [4], one of the five-stage, order five methods in Table 1 of [10] (we chose the third method), one of the seven-stage, order six methods in Table 2 of [10] (we chose the first method), and the thirteen-stage, order seven FSAL method of

$c_1 = 0.0000000000000000000$	$c_2 = 0.2051776615422863869$	$c_3 = 0.6081989431465009739$
$c_4 = 0.4872780668075869657$	$c_5 = 1.0000000000000000000$	$b'_1 = 0.0617588581356263250$
$b'_2 = 0.3389780265536433551$	$b'_3 = 0.6147913071755775662$	$b'_4 = -0.1405480146593733802$
$b'_5 = 0.1250198227945261338$		

$c_1 = 0.0000000000000000$	$c_2 = 0.2179621390175646$	$c_3 = 0.4424703708255242$
$c_4 = 1.478460559438898$	$c_5 = 0.3400000000000000$	$c_6 = 0.7000000000000000$
$c_7 = 1.000000000000000$	$b'_1 = 0.06281213570268329$	$b'_2 = 0.3788983131252575$
$b'_3 = 0.2754528515261340$	$b'_4 = -0.001585299574780513$	$b'_5 = -0.1785704038527618$
$b'_6 = 0.3479995834198831$	$b'_7 = 0.1149928196535844$	

$c_1 = 0.69883375727544694289$	$c_2 = 0.20413810365459889029$	$c_3 = 1.02055757000418534370$
$c_4 = 0.36292800323075291580$	$c_5 = 0.30508610893167564804$	$b'_1 = 0.40090379269664777606$
$b'_2 = 0.95997088013412390506$	$b'_3 = 0.08849515812721633901$	$b'_4 = 1.22143909234910252870$
$b'_5 = -1.67080892330709041000$		

$c_1 = 1\text{-}c_7$	$c_2 = 1\text{-}c_6$	$c_3 = 1\text{-}c_5$
$c_4 = 0.5$	$c_5 = 1.43531315933193655010$	$c_6 = -0.24517048359575719767$
$c_7 = 0.88961673353684493504$	$b'_1 = b'_7$	$b'_2 = b'_6$
$b'_3 = b'_5$	$b'_4 = 0.00024286040977501724$	$b'_5 = 0.08191385007043372004$
$b'_6 = -0.23158642248235284281$	$b'_7 = 0.64955114220703161414$	

$c_1 = 0.00000000000000000000$	$c_2 = 0.60715821186110352503$	$c_3 = 0.96907291059136392378$
$c_4 = -0.10958316365513620399$	$c_5 = 0.05604981994113413605$	$c_6 = 1.30886529918631234010$
$c_7 = -0.11642101198009154794$	$c_8 = -0.29931245499473964831$	$c_9 = -0.16586962790248628655$
$c_{10} = 1.22007054181677755238$	$c_{11} = 0.20549254689579093228$	$c_{12} = 0.86890893813102759275$
$c_{13} = 1.00000000000000000000$		
$b'_1 = c_2/2$	$b'_j = (c_{j+1} - c_{j-1})/2,\ j = 2, \ldots, 12$	$b'_{13} = (1 - c_{12})/2$

Table 1: The coefficients of the five SERKN methods we used in our testing: top – the five-stage, order four FSAL method of Calvo and Sanz-Serna [2], top middle – the seven-stage, order five FSAL method of Chou and Sharp [4], middle – a seven-stage, order six non-FSAL method of Okunbor and Skeel [10], bottom middle – a five-stage, order five non-FSAL method of Okunbor and Skeel [10], bottom – the thirteen-stage, order seven FSAL method of Calvo and Sanz-Serna [3].

Calvo and Sanz-Serna [3]. We have included two methods of order five to illustrate how the global error can depend on the number of stages.

We denote the methods by CS4, C5, OS5, OS6 and CS7, respectively. The c_j and b'_j for the methods are listed in Table 1; the a_{jk} are then calculated using (3) and the b_j using $b_j = (1 - c_j)b'_j$, $j = 1, \ldots, s$.

3 No round-off error control

In this section we present the comparisons when Møller's technique for reducing the round-off error is not used. The units of distance, time and mass are one Astronomical Unit (denoted by au), one Julian day (denoted by day) and one solar mass.

3.1 Gas Giants

The Sun and the gas giants (Jupiter, Saturn, Uranus and Neptune) collectively drive much of the dynamics of the Solar System. For example, they control the motion of asteroids, short-period comets, and trans-Neptunian objects.

Let $\mathbf{r}_i$, $i = 1, 2, 3, 4, 5$, be the position of the i-th body, where the bodies are ordered Sun, Jupiter, Saturn, Uranus, Neptune and the coordinates are Cartesian with the origin at the barycentre of the five bodies. The equations of motion for the i-th body are

$$\ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{5} \frac{\mu_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}}, \quad i = 1, \ldots, 5, \tag{4}$$

where $r_{ij} = ||\mathbf{r}_j - \mathbf{r}_i||_2$ and μ_j is G times the mass of the j-th body, G being the gravitational constant. We used $\mu_1 = (0.01720209895)^2$, $\mu_2 = \mu_1/1047.355$, $\mu_3 = \mu_1/3498.5$, $\mu_4 = \mu_1/22869.0$, $\mu_5 = \mu_1/19314.0$ and the initial conditions listed in the Appendix.

We did a simulation of one million years using the five SERKN methods with stepsizes of 4, 8, 16, 32 and 64 days. These stepsizes were chosen because the round-off error is insignificant for the larger ones and significant for the smaller ones. Figure 2 gives the plots of the norm of the global error for CS4, C5, OS5 and OS6 (the plots for CS7 are in Figure 1). Table 2 lists the exponent of the least squares power law fit to the norm.

Method	$h = 4$	$h = 8$	$h = 16$	$h = 32$	$h = 64$
CS4	1.0575	1.0232	1.0226	1.0227	1.0232
C5	1.4249	1.5628	0.9066	1.0051	1.0229
OS5	1.7145	1.0510	1.0192	1.0261	1.0323
OS6	1.9680	1.4041	1.1935	1.1871	1.1762
CS7	1.5098	1.5994	1.6797	1.0491	1.0686

Table 2: Gas Giants simulation without Møller's technique – the exponent for the least squares power law fit. The stepsizes are in days.

The plots in Figure 2 show the global error for CS4 grows approximately as t for all five stepsizes. The exponents of the power law fit are all close to one, supporting the observation. The global error for C5, OS5 and CS7 grows approximately as t for larger stepsizes, and faster than t and with undulations for smaller stepsizes. There is one exception - the growth for C5 with a stepsize of 16 days levels out slightly for larger t. The global error for OS6 grows faster than t for all five stepsizes, a result we find puzzling.

We had hoped for the tidy conclusion that the round-off error became significant at increasingly larger stepsizes as the order increased. This conclusion is well supported by the results for the order four and seven methods, but not by the order five and six methods.

The growth of the global error for the two order five methods provides an interesting contrast. When the stepsize is 8 days, the growth for C5 is affected by round-off error, whereas the growth for OS5 is not, suggesting OS5 is the better method. This reasoning ignores the fact that a solution for C5 is of similar accuracy to the solution for OS5 with a stepsize half the size. This means the most accurate solutions possible with C5 and

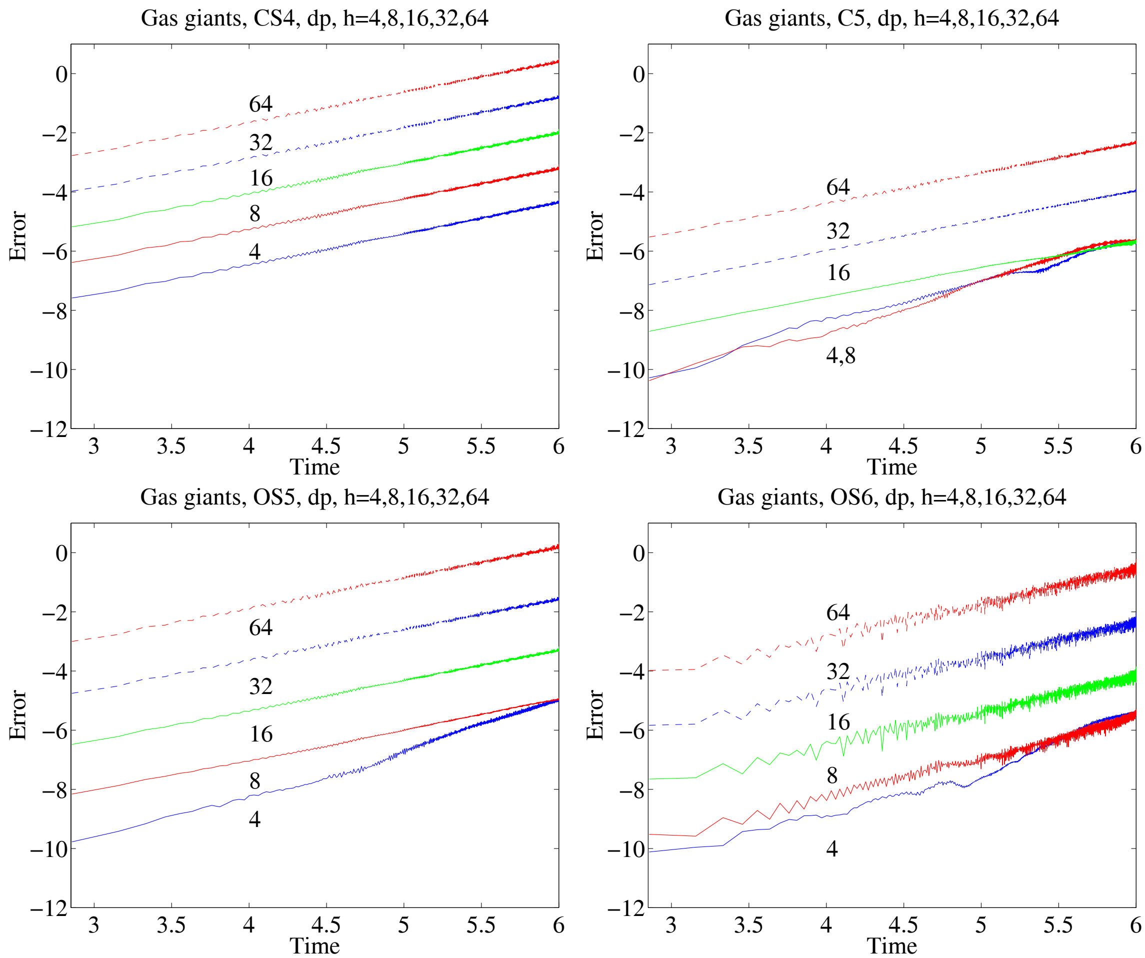

Figure 2: Gas Giants simulation without Møller's technique – base 10 log-log graph of the norm of the global error against time in years.

OS5 are of similar accuracy. OS5 retains a small advantage because it uses one evaluation fewer than C5 per step.

For C5, OS5, OS6 and CS7, a factor of no more than two separates the stepsizes for which the growth is nearly linear from the stepsizes for which the growth is faster than linear.

3.2 Nine Planets

The next set of simulations are of the Sun and nine planets. The equations of motion are those for the previous simulations except there are ten bodies instead of five. The bodies were ordered as the Sun, Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto. We used $\mu_1 = (0.01720209895)^2$, $\mu_2 = \mu_1/6023600.0$, $\mu_3 = \mu_1/408523.5$, $\mu_4 = \mu_1/328900.53$, $\mu_5 = \mu_1/3098710.0$, $\mu_6 = \mu_1/1047.355$, $\mu_7 = \mu_1/3498.5$, $\mu_8 = \mu_1/22869.0$, $\mu_9 = \mu_1/19314.0$, $\mu_{10} = \mu_1/3000000.0$ and the initial conditions listed in the Appendix.

To make the simulation more realistic, we included the mass of the Moon with that of the Earth and took the Earth-Moon barycentre as the position of the Earth.

We did a simulation of 8,192,000 days, approximately 22,428 years, with stepsizes of 2^{-i}, $i = 4, \ldots, -1$ days. The results are given in Figure 3 and Table 3.

Method	$h = 2^{-4}$	$h = 2^{-3}$	$h = 2^{-2}$	$h = 2^{-1}$	$h = 1$	$h = 2$
CS4	1.0293	1.0064	1.0043	1.0042	1.0038	0.7498
C5	1.5102	1.4826	0.8930	0.9961	1.0007	1.0020
OS5	1.4796	0.9893	1.0015	1.0022	1.0034	0.3296
OS6	1.4098	1.2256	1.0145	1.0105	1.0103	0.9702
CS7	1.4620	1.5597	1.6097	1.0257	1.0125	1.0119

Table 3: Nine Planets simulation without Møller's technique – the exponent for the least squares power law fit. The stepsizes are in days.

The results in Figure 3 and Table 3 confirm most of the observations for the Gas Giants simulations. One difference is that the global error for OS6 now grows approximately as t for the larger stepsizes. There are other differences, such as the large dip in the global error for OS5 with the largest stepsize, but these differences are probably not caused by round-off error.

3.3 Saturnian Satellites

Sinclair and Taylor [12] used numerical integration to analyse the orbits of the Saturnian satellites Titan, Hyperion and Iapetus. The equations of motion included terms for the oblateness of Saturn, perturbations from the Sun and the inner Saturnian satellite Rhea.

Let $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$ be the position of Titan, Hyperion, Iapetus and Rhea at time t, the coordinates being Cartesian with origin at the centre of mass of Saturn and its inner satellites (excluding Rhea). The equations of motion in [12] were

$$\begin{aligned} \ddot{\mathbf{r}}_i &= -\frac{GM(1+m_i)\mathbf{r}_i}{||\mathbf{r}_i||_2^3} + \sum_{j=1, j\neq i}^{4} GMm_j \left(\frac{\mathbf{r}_j - \mathbf{r}_i}{||\mathbf{r}_j - \mathbf{r}_i||_2^3} - \frac{\mathbf{r}_j}{||\mathbf{r}_j||_2^3} \right) + \\ & GM_s \left(\frac{\mathbf{r}_s - \mathbf{r}_i}{||\mathbf{r}_s - \mathbf{r}_i||_2^3} - \frac{\mathbf{r}_s}{||\mathbf{r}_s||_2^3} \right) + \nabla_i R_i, \quad i = 1, 2, 3, \end{aligned}$$

where m_j is the mass of the j-th satellite divided by the mass of Saturn, M is the mass of Saturn, M_s is the mass of Sun, G is the gravitational constant, and $\mathbf{r}_s$ is the position of the Sun.

The term $\nabla_i R_i$ is the effect on the i-th satellite of the oblateness of Saturn. This term is

$$\nabla_i R_i = A\mathbf{r}_i + B\hat{\mathbf{z}},$$

where $\hat{\mathbf{z}}$ is the unit vector in the z-direction. The coefficients A and B are

$$A = \frac{GM}{r_i^3} \sum_{n=2}^{4} J_n \frac{a_0^n}{r_i^n} P'_{n+1}(z_i/r_i), \quad B = -\frac{GM}{r_i^2} \sum_{n=2}^{4} J_n \frac{a_0^n}{r_i^n} P'_n(z_i/r_i),$$

where $r_i = ||\mathbf{r}_i||_2$, a_0 is the equatorial radius of Saturn, J_n are non-dimensional constants, and P_n is the Legendre polynomial of degree n.

Sinclair and Taylor assumed Rhea moved in a fixed circular orbit in the equatorial plane of Saturn. Rhea's coordinates were given as $x_4 = a\cos L$, $y_4 = a\sin L$, $z_4 = 0$, where $L = 231^\circ.761 + 79^\circ.69004007(t - 2411093.0)$ and a is a constant. The position of the Sun was calculated from the planetary ephemeris JPL DE200 using interpolation with Chebyshev polynomials.

The values for the parameters in the equations of motion were $a = 0.0035232$ au, $GM = 8.45945 \times 10^{-8}$ au^3 day^{-2}, $a_0 = 60,000$ km, $J_2 = 0.01675$, $J_3 = 0$, $J_4 = -0.001$, $m_1 = 2.36777 \times 10^{-4}$, $m_2 = 0$, $m_3 = 3.30000 \times 10^{-6}$, $m_4 = 4.4 \times 10^{-6}$. Sinclair and Taylor did not specify the length of one astronomical unit - we used 149,596,000.0 kilometres.

The period of Rhea, Titan, Hyperion and Iapetus is 4.51750 days, 15.94545 days, 21.27666 days and 79.33082 days, respectively ([11], p 478). We did a simulation of 4,096,000 days, approximately 11,214 years, using stepsizes of 2^{-i}, $i = 6, \ldots, 1$ days. To reduce the amount of CPU time required for the simulations, we omitted the Sun. This simplifies the equations of motion, but they retain enough features to make them interesting. The results are given in Figure 4 and Table 4. The results confirm the observations for the Nine Planets simulation except when the exponent is less than one.

Method	$h = 2^{-6}$	$h = 2^{-5}$	$h = 2^{-4}$	$h = 2^{-3}$	$h = 2^{-2}$
CS4	1.0263	1.0010	1.0003	1.0000	0.9784
C5	0.7971	1.9186	1.3747	0.9440	0.9992
OS5	1.4666	0.8740	1.0029	1.0010	0.9930
OS6	1.7136	1.6147	1.1075	0.9946	0.9897
OS7	1.5305	1.6548	1.8823	0.7884	0.9994

Table 4: Saturnian Satellites simulation without Møller's technique – the exponent for the least squares power law fit. The stepsizes are in days.

4 With round-off error control

Møller's technique for reducing the round-off error replaces the update formula for y_i by

$$\tau = \sum_{j=1}^{s} b_j f_j - \epsilon, \quad y_t = y_{i-1} + h\dot{y}_{i-1} + \tau, \quad \epsilon = y_t - (y_{i-1} + h\dot{y}_{i-1}), \quad y_i = y_t, \qquad (5)$$

where $\epsilon = 0$ at $t = t_0$. The formula for $\dot{y}_i$ is modified in a similar way.

We re-did the simulations of the previous section with Møller's technique applied to the update formulae. Figures 5, 6 and 7 contain the plots of the global error for C5, OS5, OS6 and CS7, and Tables 5, 6 and 7 the exponents for the power law fits. We excluded CS4 because the results were almost identical to those in the previous section, indicating the contribution of the round-off error to the global error was small.

Møller's technique was very effective with OS5, giving near linear growth for the smallest stepsize except near the end of the simulation of the Saturnian satellites. The technique was effective near the start of the simulations with the remaining three methods, reducing the global error for the smallest stepsize by at least an order of magnitude. The reduction decreased as t increased and was small at the end of the simulations. The reductions were usually smaller for the second to smallest stepsize.

A number of the exponents β for C5, OS6 and CS7 with the smaller stepsizes were close to two. For example, in the simulation of the Saturnian satellites, β for CS7 was 1.97, 2.01 and 2.00 for stepsizes of 2^{-6}, 2^{-5} and 2^{-4}, respectively. One explanation is that Møller's technique introduced a systematic error proportional to t^2 and this error dominated the global error.

Method	$h = 4$	$h = 8$	$h = 16$	$h = 32$	$h = 64$
C5	1.9733	1.8906	0.8072	1.0048	1.0229
OS5	1.0330	1.0133	1.0202	1.0261	1.0323
OS6	1.9552	1.4091	1.1923	1.1871	1.1762
CS7	1.8208	1.9525	1.9297	1.0362	1.0688

Table 5: Gas Giants simulation with Møller's technique – the exponent for the least squares power law fit. The stepsizes are in days.

Method	$h = 2^{-4}$	$h = 2^{-3}$	$h = 2^{-2}$	$h = 2^{-1}$	$h = 1$	$h = 2$
C5	2.1049	1.6020	0.9161	0.9959	1.0007	1.0020
OS5	0.9270	1.0016	1.0012	1.0022	1.0034	0.3296
OS6	1.8878	1.4040	1.0116	1.0105	1.0103	0.9702
CS7	1.9011	1.9227	2.0176	1.0084	1.0129	1.0119

Table 6: Nine Planets simulation with Møller's technique – the exponent for the least squares power law fit. The stepsizes are in days.

Method	$h = 2^{-6}$	$h = 2^{-5}$	$h = 2^{-4}$	$h = 2^{-3}$	$h = 2^{-2}$
C5	1.9678	2.1154	1.4185	0.9524	0.9993
OS5	1.2299	0.9970	1.0012	1.0010	0.9930
OS6	1.9833	1.6922	1.0885	0.9943	0.9897
CS7	1.9677	2.0084	2.0023	0.8091	0.9994

Table 7: Saturnian Satellites simulation with Møller's technique – the exponent for the least squares power law fit. The stepsizes are in days.

5 Summary

We compared five representative SERKN methods of orders four to seven on three long N-body simulations of the Solar System. The simulations were of the Sun and gas giants, the Sun and nine planets, and three Saturnian satellites. Our aim was to investigate how the growth of the global error depended on the stepsize and order, and how the growth was affected when Møller's technique was used to reduce the round-off error. The stepsizes were chosen so the round-off error was typically insignificant for the larger stepsizes and significant for the smaller stepsizes. All testing was done in double precision.

For larger stepsize, the global error usually grew as $t^{1+\epsilon}$ where $||\epsilon|| \ll 1$, confirming the asymptotic results for the growth of the global error in the absence of round-off error. One exception was for the order six method of Okunbor and Skeel [10] on the simulation of the Sun and nine planets - the exponent of the power law was approximately 1.18.

For smaller stepsizes, the global error usually grew as t^β, $\beta > 1$, when Møller's technique was not used. The value of β varied with the problem, the method and the stepsize. When Møller's technique was used, the global error was reduced. For the order six and order seven methods and one of the order five methods, the reduction decreased with t. For the other order five method, the reduction produced a near linear growth in the global error, although it is possible the linear growth would not occur for stepsizes smaller than we used. Our results suggests Møller's technique can introduce a systematic error that varies as t^2.

A factor of no more than two separated the stepsizes for which the growth was nearly linear from those for which the growth was faster than linear. This sharp transition could be useful if it is necessary to find the smallest stepsize for which the growth is nearly linear.

Our testing provided insight about the efficiency of the order seven method relative to the lower order methods. The relative efficiency clearly depends on the stepsize, but for the stepsizes we used, we can make general statements when the growth of the global error is nearly linear. The solution for the order seven method with a stepsize of $2h$ is at least as accurate as the solution for the order six method with a stepsize of h. This means when the number of f evaluations is used as the measure of work, that the order seven method is more efficient because it uses only $12/7$ as many evaluations as the order six method. If the CPU time is used as the measure of work, the efficiency of the order seven method is reduced because it has greater overhead. Reasoning in a similar way, the order five method of [4] is more efficient than the order seven method for the larger stepsizes and less efficient for the smaller stepsizes, and the order four method of [3] is less efficient for all stepsizes we used.

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Appendix

Tables 8, 9 and 10 lists the initial conditions for the Gas Giants, Nine Planets and Saturnian Satellites simulations. The first half of the rows in each table list the initial positions, and the second half the initial velocities. The initial time for the Gas Giants and Nine Planets simulations is taken as 0, that for the Saturnian Satellites simulation as 2442000.5.

	x	y	z
Sun	0.9209498686328694e-03	0.2304166030756204e-02	0.9127217048523883e-03
Jupiter	0.3350285173564643e+01	-0.3471457282981824e+01	-0.1571236964688948e+01
Saturn	-0.8971584118413477e+01	0.2281986174163616e+01	0.1331251331416312e+01
Uranus	-0.1002083045458687e+01	0.1732581263930256e+02	0.7605737768120762e+01
Neptune	-0.2919365978874257e+02	-0.7716981025967714e+01	-0.2426332656583918e+01
Sun	-0.4665246664531984e-05	-0.3149154564335707e-05	-0.1269852543206254e-05
Jupiter	0.5580977902917778e-02	0.4959111982658174e-02	0.1991007074196164e-02
Saturn	-0.1862917203661242e-02	-0.4987007735981776e-02	-0.1981527265350456e-02
Uranus	-0.3959919409682252e-02	-0.3790629396772767e-03	-0.1101198438310003e-03
Neptune	0.8160828111535530e-03	-0.2775247414144566e-02	-0.1157385882979126e-02

Table 8: The initial conditions for the Gas Giants simulation.

	x	y	z
Sun	0.9301259103994515e-03	0.2292733100662641e-02	0.9059057664779422e-03
Mercury	0.3448565760800415e+00	0.4790821305397614e-01	-0.1001813144545456e-01
Venus	0.1438953102536455e+00	0.6492977991345496e+00	0.2833883064268579e+00
Earth	-0.1354345700443955e+00	0.8956906559576626e+00	0.3883642504058149e+00
Mars	-0.1368903850273021e+01	0.8454279811185666e+00	0.4247388123779079e+00
Jupiter	0.3350294349606409e+01	-0.3471468715911917e+01	-0.1571243780627322e+01
Saturn	-0.8971574942371711e+01	0.2281974741233523e+01	0.1331244515477938e+01
Uranus	-0.1002073869416921e+01	0.1732580120637246e+02	0.7605730952182388e+01
Neptune	-0.2919365061270080e+02	-0.7716992458897807e+01	-0.2426339472522292e+01
Pluto	-0.2623272065610510e+02	0.2056426815315656e+02	0.1444546303354718e+02
Sun	-0.4559774360194479e-05	-0.3150250493626429e-05	-0.1274328432609927e-05
Mercury	-0.8471091819370054e-02	0.2561145505678817e-01	0.1458557100780699e-01
Venus	-0.1989837205370269e-01	0.3109969215624964e-02	0.2658171477313190e-02
Earth	-0.1732455862288979e-01	-0.2247454982261186e-02	-0.9746354441906539e-03
Mars	-0.7389123605631364e-02	-0.9480508889767826e-02	-0.4152929465094740e-02
Jupiter	0.5581083375222116e-02	0.4959110886728884e-02	0.1991002598306760e-02
Saturn	-0.1862811731356904e-02	-0.4987008831911066e-02	-0.1981531741239860e-02
Uranus	-0.3959813937377914e-02	-0.3790640356065674e-03	-0.1101243197204039e-03
Neptune	0.8161882834578905e-03	-0.2775248510073856e-02	-0.1157390358868530e-02
Pluto	-0.1320448472641354e-02	-0.2623278455987146e-02	-0.4283576834589079e-03

Table 9: The initial conditions for the Nine Planets simulation.

	x	y	z
Titan	-0.0075533871	0.0025250254	-0.0000462204
Hyperion	-0.0006436995	0.0099145485	0.0000357506
Iapetus	0.0219653473	-0.0071369083	0.0062333851
Titan	-0.0010017342	-0.0031443009	0.0000059503
Hyperion	-0.0029182723	0.0000521415	-0.0000356145
Iapetus	0.0006187633	0.0017696165	0.0000439292

Table 10: The initial conditions for the Saturnian Satellites simulation.

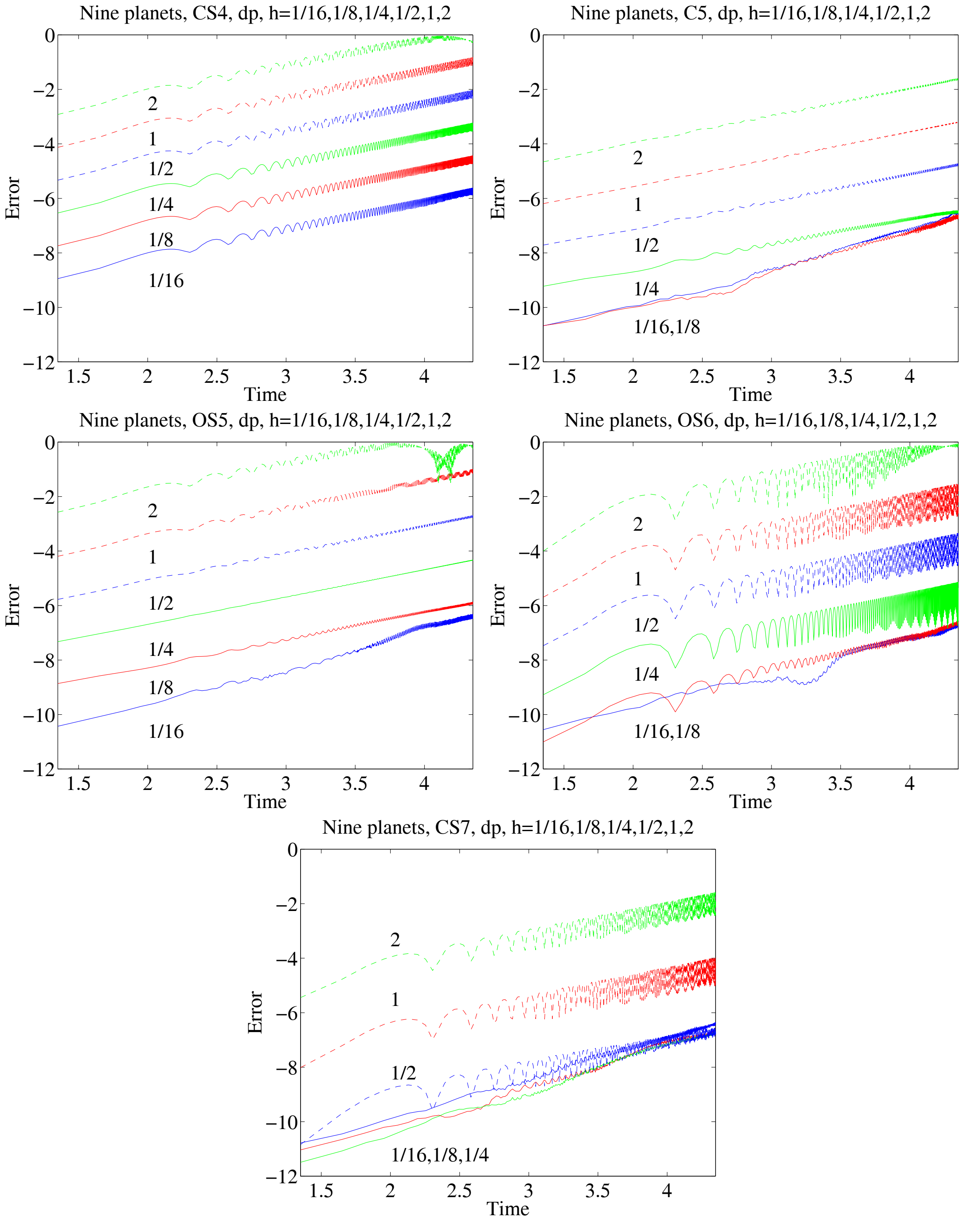

Figure 3: Nine Planets simulation without Møller's technique – base 10 log-log graph of the norm of the global error against time in years.

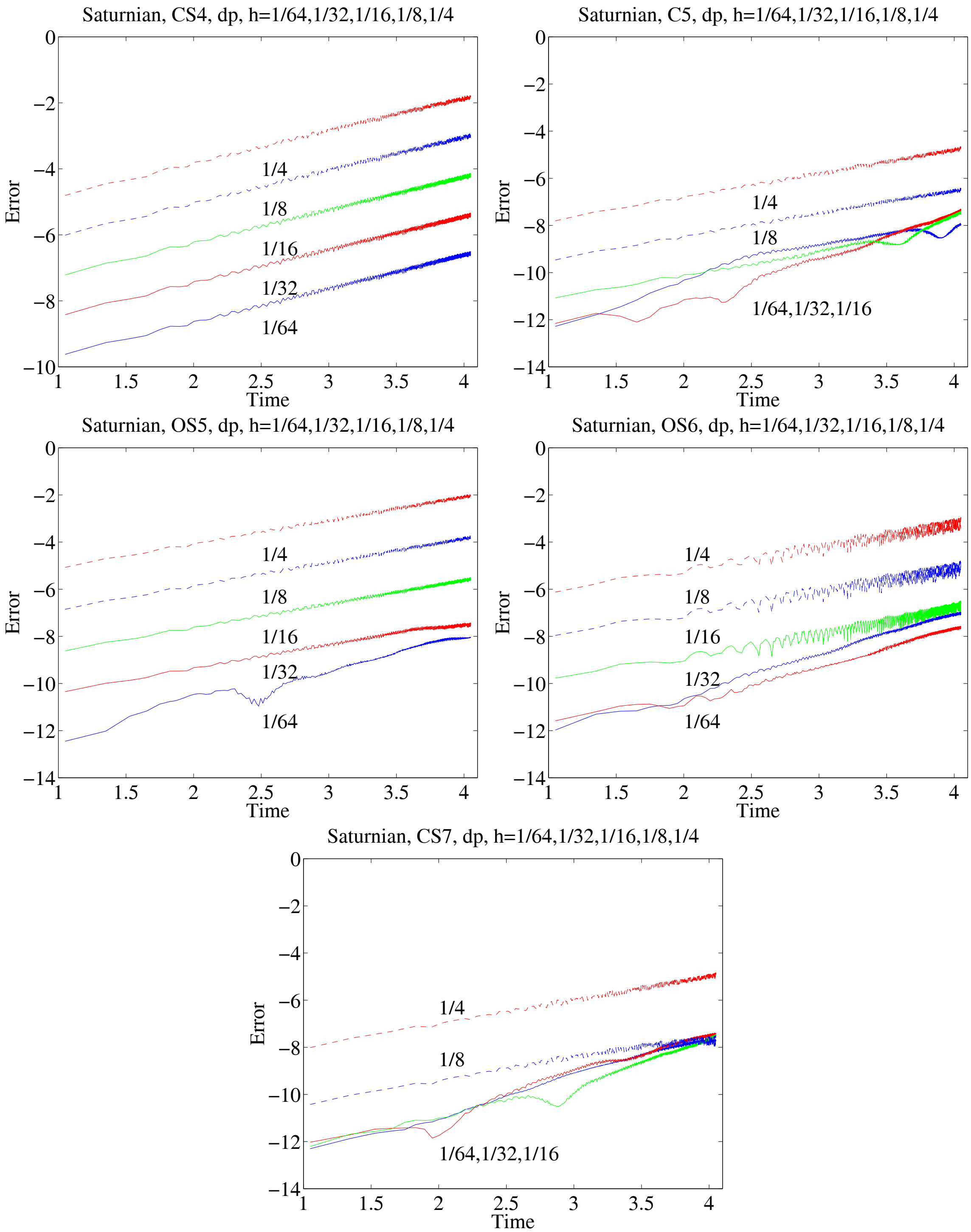

Figure 4: Saturnian Satellites simulation without Møller's technique – base 10 log-log graph of the norm of the global error against time in years.

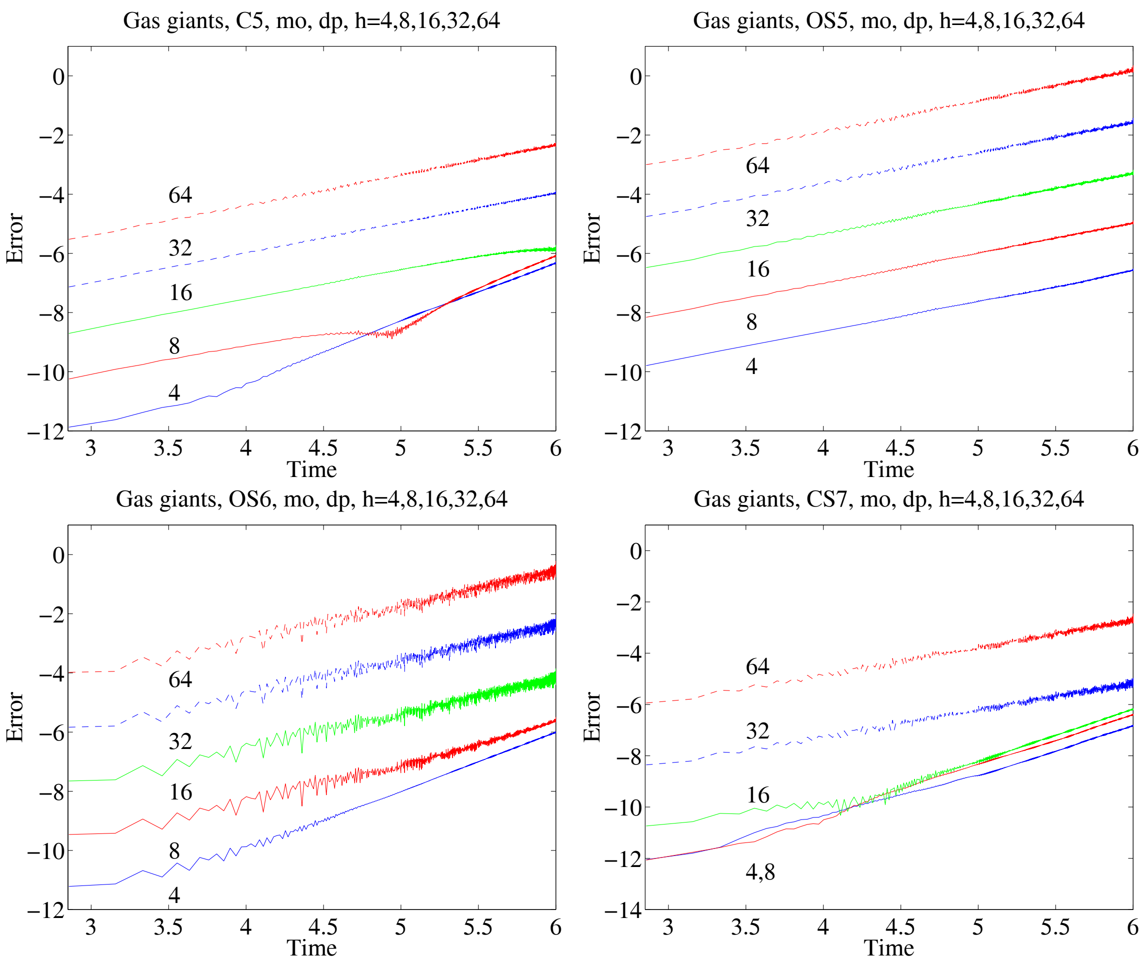

Figure 5: Gas Giants simulation with Møller's technique – base 10 log-log graph of the norm of the global error against time in years.

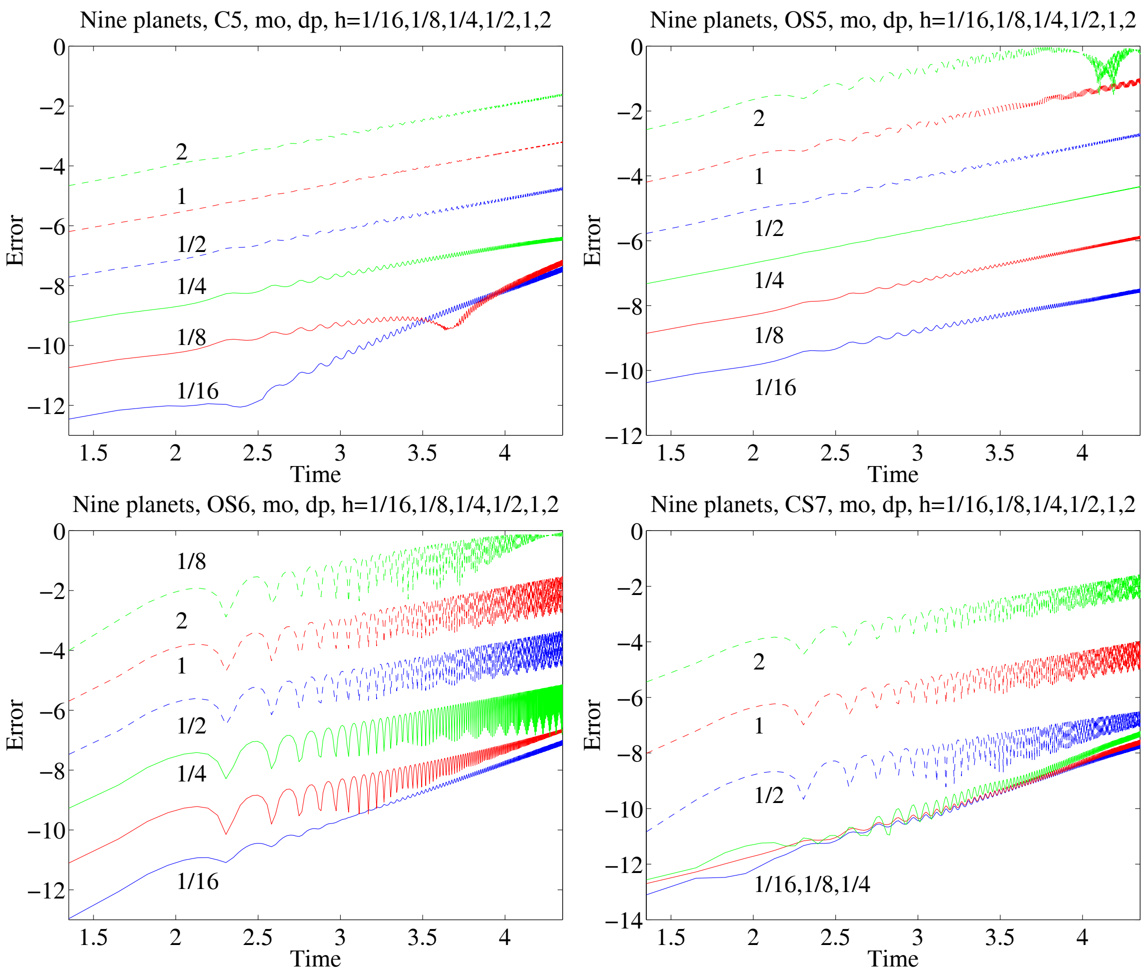

Figure 6: Nine Planets simulation with Møller's technique – base 10 log-log graph of the norm of the global error against time in years.

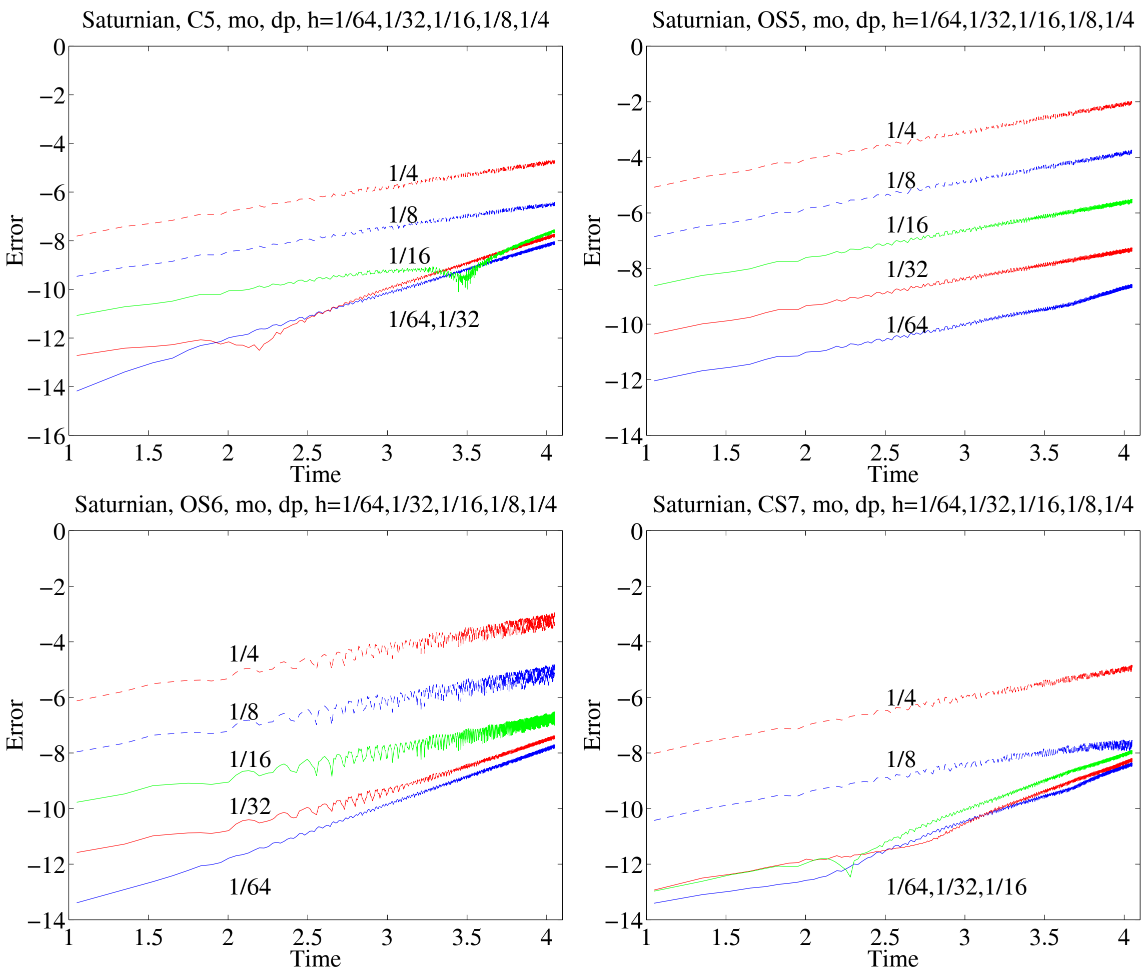

Figure 7: Saturnian Satellites simulation with Møller's technique – base 10 log-log graph of the norm of the global error against time in years.