Unequal mass spinor-spinor Bethe–Salpeter equation

B. J. Brennan

Department of Physics, University of Auckland, Auckland, New Zealand
(Received 1 April 1975)

Coupled radial equations are derived for the ladder approximation Bethe–Salpeter equation describing a system of two spin-(1/2) particles of unequal masses interacting to form a bound state of total mass zero. The numerical behavior of the coupling parameter λ as a function of the mass ratio is examined for known analytical equal-mass solutions. In addition a perturbation method is employed to investigate the behavior of λ for small values of the exchange mass.

I. INTRODUCTION

In this section we briefly recapitulate certain features of the Bethe-Salpeter (BS) equation for two bound spin-$\frac{1}{2}$ particles. We examine equal mass systems in Sec. II and perturbation of the mass of the exchange boson in Sec. III. In several contexts we need to refer to a paper by Brennan and Keam[1] and a series of papers by Keam.[2] The notations and conventions of the present paper are the same as for these references.

In configuration space, the ladder approximation, Wick-rotated BS equation describing the interaction of a spin-$\frac{1}{2}$ fermion of mass m_a and a spin-$\frac{1}{2}$ antifermion of mass m_b to form a bound state of total 4-momentum $P=(\mathbf{P}, iE)$, may be written

$$[\gamma\cdot(\partial+i\mu_a P)+m_a]f(x)[\gamma\cdot(\overleftarrow{\partial}-i\mu_b P)+m_b]=-\hat{\mathcal{V}}(R)f(x), \quad (1)$$

where x is the (Euclidean) relative coordinate, $R\,(=\sqrt{x^2})$ is the four-dimensional radius, $\hat{\mathcal{V}}$ describes the interaction of the two particles, and $\mu_a+\mu_b=1$. In the particular case where the interaction is due to the exchange of a boson of mass μ,[3] the potential $\hat{\mathcal{V}}$ takes the form [we use the label n.s.(j) to show that j is not summed up in the preceding expression]:

$$\hat{\mathcal{V}}^j=-\lambda^j\Gamma^j(4\mu/R)K_1(\mu R), \quad \text{n.s.}(j), \quad (2)$$

where λ^j, the coupling parameter, is given by[4]

$$\lambda^j=g_a^j g_b^j/(4\pi)^2, \quad \text{n.s.}(j). \quad (3)$$

Here j assumes the values 1, 2, 4, or 5 when the exchange boson is of scalar, vector, axial vector, or pseudoscalar type, respectively. In the notation of K1, Γ^j is of form $\epsilon_j\hat{\Gamma}_j$ [n.s.(j)], where ϵ_j is +1 for $j=1,4$ and −1 for $j=2,5$. g_i^j $(i=a,b)$ is the coupling constant for the interaction of particle i with the exchange boson.

When the particle of mass m_b is a fermion rather than an antifermion, the modified BS amplitude $f^c(p)=f(p)C^{-1}$, where C is the charge conjugation matrix, satisfies an equation of the same form as Eq. (1). The potential due to boson exchange is the same, with the exception that $\Gamma^2=+\hat{\Gamma}_2$.[5]

In relative momentum space, the transform of Eq. (1) may be written

$$[\gamma\cdot(p+\mu_a P)-im_a]f(p)[\gamma\cdot(p-\mu_b P)-im_b]$$
$$=(2\pi)^{-4}\int d^4k\hat{\mathcal{W}}(|p-k|)f(k), \quad (4)$$

where

$$\hat{\mathcal{W}}(K)=\int d^4x e^{ikx}\hat{\mathcal{V}}(R), \quad (5)$$

and

$$K^2=k^2.$$

For the exchange of a boson of "type" j,

$$\hat{\mathcal{W}}(K)=-\lambda^j(4\pi)^2(K^2+\mu^2)^{-1}\Gamma^j, \quad \text{n.s.}(j). \quad (6)$$

Throughout this paper we consider the BS equation for the centre of mass system with E vanishing; viz $P_\mu=0$.

II. UNEQUAL MASS SYSTEMS

A. Introduction

Unequal mass systems have not previously been considered in any detail. Brennan and Keam[B1] have shown that the coupling parameter is an even function of the mass difference, m_b-m_a. Keam[(K4)] developed a perturbation theory and applied this to an SV sector solution, giving the approximate behavior of λ for small values of the mass difference, while in Ref. K6 the same author examined mass symmetries of the equation and λ.

In Sec. II B we derive the coupled radial equations for unequal mass systems, and in Sec. II C and the appendix we examine the numerical behavior of λ as a function of m_b/m_a for the known analytic equal mass solutions of Keam[(K4)] and Kummer.[6] The latter task has been performed using numerical methods and, for Kummer's solutions, by the use of perturbation theory.

B. Reduction of the equation

We consider the configuration form of the BS equation [Eq. (1)] with $P_\mu=0$.[7] In this case the set

$$\{\mathcal{B}_0^{-1}\hat{\mathcal{V}}, \alpha, \beta^2, \mathbf{J}^2, J_z, \bar{\mathcal{P}}, \bar{\mathcal{C}}_1\} \quad (7)$$

is a commuting set of operators. Here $\mathcal{B}_0$ is the BS operator of the left member of Eq. (1); $\alpha, \beta^2, \mathbf{J}^2, J_z$ are the $O(4)$ operators constructed from the angular momentum operators $\mathcal{M}_{\mu\nu}$[K1]; $\mathcal{P}$ is the parity operator[K2,8]; and $\mathcal{C}_1$ is the generalized charge parity operator.[K6] Thus we may express the BS amplitude f as a simultaneous eigenfunction of these operators. For convenience we shall consider eigenfunctions of $\bar{\mathcal{C}}_1$ later and for the moment consider eigenfunctions of the remaining operators, viz, $f^{\bar{\mathcal{P}}}{}_{\alpha\beta^2 J_m}$. Four distinct classes exist, as follow. We use the notation of K4 for kets, that is, we write angular kets as $|\Gamma_i(l,s^+)j^+,(l,s^-)j^-;Jm\rangle$, though for brevity we omit the quantum numbers J and m. The angular momentum operators $\mathbf{S}^{\pm}, \mathbf{L}^{\pm}, \mathbf{J}^{\pm}$ are defined in K1. Parity eigenvalues in the following refer to fermion—antifermion systems.[8,9]

Class 1: $\alpha = 4j(j+1)$, $\beta^2 = 0$, $\overline{\mathcal{P}} = (-1)^J$

$$
\begin{aligned}
f_1 = & f_S \,|\, \Gamma_1(j,0)j,\ (j,0)j\rangle \\
& + f_{V_1} |\, \Gamma_2(j+\tfrac{1}{2},\tfrac{1}{2})j,\ (j+\tfrac{1}{2},\tfrac{1}{2})j\rangle \\
& + f_{V_2} |\, \Gamma_2(j-\tfrac{1}{2},\tfrac{1}{2})j,\ (j-\tfrac{1}{2},\tfrac{1}{2})j\rangle \\
& + f_{T^-}(1/\sqrt{2})(|\, \Gamma_3(j,1)j,\ (j,0)j\rangle - |\, \Gamma_3(j,0)j,\ (j,1)j\rangle).
\end{aligned} \tag{8}
$$

Class 2: $\alpha = 4j(j+1)$, $\beta^2 = 0$, $\overline{\mathcal{P}} = (-1)^{J+1}$

$$
\begin{aligned}
f_2 = & f_{T^+}(1/\sqrt{2})(|\, \Gamma_3(j,1)j,\ (j,0)j\rangle + |\, \Gamma_3(j,0)j,\ (j,1)j\rangle) \\
& + f_{A_1} |\, \Gamma_4(j+\tfrac{1}{2},\tfrac{1}{2})j,\ (j+\tfrac{1}{2},\tfrac{1}{2})j\rangle \\
& + f_{A_2} |\, \Gamma_4(j-\tfrac{1}{2},\tfrac{1}{2})j,\ (j-\tfrac{1}{2},\tfrac{1}{2})j\rangle \\
& + f_P |\, \Gamma_5(j,0)j,\ (j,0)j\rangle.
\end{aligned} \tag{9}
$$

Class 3: $\alpha = \beta^2 = 4(j+1)^2$, $\overline{\mathcal{P}} = (-1)^J$

$$f_3 = (1/\sqrt{2})(f_{jj+1Jm} + f_{j+1jJm}), \tag{10}$$

where

$$
\begin{aligned}
f_{jj+1Jm} = & f_V |\, \Gamma_2(j+\tfrac{1}{2},\tfrac{1}{2})j,\ (j+\tfrac{1}{2},\tfrac{1}{2})j+1\rangle \\
& + f_{T_1} |\, \Gamma_3(j+1,1)j,\ (j+1,0)j+1\rangle \\
& + f_{T_2} |\, \Gamma_3(j,0)j,\ (j,1)j+1\rangle \\
& + f_A |\, \Gamma_4(j+\tfrac{1}{2},\tfrac{1}{2})j,\ (j+\tfrac{1}{2},\tfrac{1}{2})j+1\rangle
\end{aligned} \tag{11}
$$

and

$$
\begin{aligned}
f_{j+1jJm} = & -f_V |\, \Gamma_2(j+\tfrac{1}{2},\tfrac{1}{2})j+1,\ (j+\tfrac{1}{2},\tfrac{1}{2})j\rangle \\
& + f_{T_1} |\, \Gamma_3(j+1,0)j+1,\ (j+1,1)j\rangle \\
& + f_{T_2} |\, \Gamma_3(j,1)j+1,\ (j,0)j\rangle \\
& + f_A |\, \Gamma_4(j+\tfrac{1}{2},\tfrac{1}{2})j+1,\ (j+\tfrac{1}{2},\tfrac{1}{2})j\rangle.
\end{aligned} \tag{12}
$$

Class 4: $\alpha = \beta^2 = 4(j+1)^2$, $\overline{\mathcal{P}} = (-1)^{J+1}$

$$f_4 = (1/\sqrt{2})(f_{jj+1Jm} - f_{j+1jJm}). \tag{13}$$

We consider potentials of the type

$$\hat{\mathcal{V}} = \sum_j \hat{\mathcal{V}}^j(R)\hat{\Gamma}_j \quad (j = 1,2,4,5) \tag{14}$$

and define

$$\overline{V}_j = \sum_i c_{ij}\hat{\mathcal{V}}^i(R), \tag{15}$$

where the c_{ij} are defined by

$$\hat{\Gamma}_i\Gamma_j = c_{ij}\Gamma_j, \quad \text{n.s.}(j) \tag{16}$$

[cf. K1, Eq. (16) and Table I].

We also define

$$D^{\pm}_{\alpha} = \frac{\partial}{\partial R} \pm \frac{2(j+\alpha)}{R}, \tag{17}$$

as in K1, whilst

$$m = \tfrac{1}{2}(m_a + m_b), \quad \Delta = \tfrac{1}{2}(m_a - m_b). \tag{18}$$

The coupled radial equations obtained on substitution in Eq. (1) are then as follows:

Class 1:

$$D^+_{3/2}D^-_0 s + 2m[D^+_{3/2}v_1 + D^-_{-1/2}v_2] = -[m^2 - \Delta^2 + \overline{V}_1]s, \tag{19a}$$

$$
\begin{aligned}
& D^-_0[D^+_{3/2}v_1 + 2(j+1)D^-_{-1/2}v_2 + 2(j+1)ms + 2i\Delta\overline{w}] \\
& \quad = -(2j+1)[m^2 - \Delta^2 + \overline{V}_2]v_1,
\end{aligned} \tag{19b}
$$

$$
\begin{aligned}
& D^+_1[-D^-_{-1/2}v_2 + 2jD^+_{3/2}v_1 + 2jms - 2i\Delta\overline{w} \\
& \quad = -(2j+1)[m^2 - \Delta^2 + \overline{V}_2]v_2,
\end{aligned} \tag{19c}
$$

$$-D^+_{3/2}D^-_0\overline{w} + 2i\Delta[-jD^+_{3/2}v_1 + (j+1)D^-_{-1/2}v_2] = -[m^2 - \Delta^2 + \overline{V}_3]\overline{w}, \tag{19d}$$

where

$$
\begin{aligned}
& s = (2j+1)^{1/2}f_s, \quad v_1 = -(j+1)^{1/2}f_{V_1}, \quad v_2 = j^{1/2}f_{V_2}, \\
& \overline{w} = [j(j+1)(2j+1)]^{1/2}f_{T^-}.
\end{aligned} \tag{20}
$$

Class 2:

$$D^+_{3/2}D^-_0 w + 2m[-jD^+_{3/2}a_1 + (j+1)D^-_{-1/2}a_2] = -[m^2 - \Delta^2 + \overline{V}_3]w, \tag{21a}$$

$$
\begin{aligned}
& D^-_0[-D^+_{3/2}a_1 - 2(j+1)D^-_{-1/2}a_2 - 2mw - 2i(j+1)\Delta p] \\
& \quad = -(2j+1)[m^2 - \Delta^2 + \overline{V}_4]a_1,
\end{aligned} \tag{21b}
$$

$$
\begin{aligned}
& D^+_1[D^-_{-1/2}a_2 - 2jD^+_{3/2}a_1 + 2mw - 2ij\Delta p] \\
& \quad = -(2j+1)[m^2 - \Delta^2 + \overline{V}_4]a_2,
\end{aligned} \tag{21c}
$$

$$-D^+_{3/2}D^-_0 p + 2i\Delta[D^+_{3/2}a_1 + D^-_{-1/2}a_2] = -[m^2 - \Delta^2 + \overline{V}_5]p, \tag{21d}$$

where

$$
\begin{aligned}
& w = [j(j+1)(2j+1)]^{1/2}f_{T^+}, \quad a_1 = -(j+1)^{1/2}f_{A_1}, \quad a_2 = j^{1/2}f_{A_2}, \\
& p = (2j+1)^{1/2}f_P.
\end{aligned} \tag{22}
$$

Class 3 & 4:

$$D^+_2D^-_{1/2}v + i\Delta[D^+_2t_1 - D^-_0t_2] = -[m^2 - \Delta^2 + \overline{V}_2]v, \tag{23a}$$

$$D^-_{-1/2}[D^-_0t_2 + 2ma + 2i\Delta v] = -[m^2 - \Delta^2 + \overline{V}_3]t_1, \tag{23b}$$

$$D^+_{3/2}[D^+_2t_1 + 2ma - 2i\Delta v] = -[m^2 - \Delta^2 + \overline{V}_3]t_2, \tag{23c}$$

$$D^+_2D^-_{1/2}a + m[D^+_2t_1 + D^-_0t_2] = -[m^2 - \Delta^2 + \overline{V}_4]a, \tag{23d}$$

where

$$v = f_V, \quad t_1 = \sqrt{2}f_{T_1}, \quad t_2 = \sqrt{2}f_{T_2}, \quad a = f_A. \tag{24}$$

For each of the above classes, eigenfunctions of $\overline{\mathcal{C}}_1$ may be extracted from f by expressing each radial function as

$$g(R) = g^+(R, \Delta^2) + \Delta g^-(R, \Delta^2). \tag{25}$$

Separating odd and even functions of Δ in the equation sets (19), (21), and (23) yields six sets of equations involving the radial functions tabulated in Table I.

For the particular case where the potential is due to the exchange of one type of particle only, certain symmetries in the equation sets (19), (21), and (23) are apparent.

Some of these arise from the fact that the operator $\mathcal{T}^b_5$, where

$$\mathcal{T}^b_5 f(p; m_a, m_b) = f(p; m_a, -m_b)\gamma_5 = f^b(p)\gamma_5, \tag{26}$$

TABLE I. Grouping of radial functions according to modified charge parity eigenvalues.

$\bar{\mathcal{C}}_1$	Class 1	Class 2	Classes 3 and 4
$(-1)^{2j}$	$s^+, v_1^+, v_2^+, \bar{w}^-$	w^-, a_1^-, a_2^-, p^+	v^+, t_1^-, t_2^-, a^-
$(-1)^{2j+1}$	s^-, v_1^-, v_2^-, w^+	w^+, a_1^+, a_2^+, p^-	v^-, t_1^+, t_2^+, a^+

commutes with $\mathcal{B}_0^{-1}\hat{\mathcal{V}}$. Using the relation[K6]

$$\lambda^j(m_a, -m_b) = \epsilon'_j \lambda^j(m_a, m_b), \tag{27}$$

where ϵ'_j is +1 for $j'=2$ or 4, and −1 for $j=1$ or 5, we obtain the result that if $f(p)$ is a solution of the BS equation, then $\mathcal{T}_5^b f(p)$ is also a solution, for the same values of λ, m_a, and m_b. We note that in the proof of Eq. (27), it is assumed that f and λ are analytic functions of m_b, and that certain integrals converge and are nonzero.

Thus $\mathcal{T}_5^b f_1$ is a class 2 solution, with w, a_1, a_2, and p, respectively, replaced by $-\bar{w}^b$, iv_1^b, iv_2^b, and s^b (the superscript ^{b} denotes that in functions parametrically dependent on m_b, m_b has been replaced by $-m_b$). Similarly, $\mathcal{T}_5^b f_2$ is a class 1 solution, with the converse replacement. $\mathcal{T}_5^b f_3$ is a class 4 solution with v, t_1, t_2, and a, respectively, replaced by $-ia^b$, $-t_1^b - t_2^b$, and iv^b, while $\mathcal{T}_5^b f_4$ is a class 3 solution with the converse replacement.

However, the new solutions predicted by the action of $\mathcal{T}_5^b$ on known solutions in practice vanish, and this is examined briefly in Sec. II C3.

The equation set (23) is of interest for $j=2$ or 4 (V or A type exchange), where a simplification occurs. In these cases $\bar{V}_3=0$, and the terms involving v and $D_2^+t_1 - D_0^-t_2$, are decoupled from those involving a and $D_2^+t_1 + D_0^-t_2$. Similarly, in the momentum space transforms of Eqs. (23) [cf. Eqs. (33)], the terms involving $v(P)$ and $t_1(P)-t_2(P)$ are decoupled from those involving $a(P)$ and $t_1(P)+t_2(P)$.[10]

This decoupling is related to the properties of the operator $\mathcal{S}$, defined in configuration space by

$$\mathcal{S}_x f(x) = \frac{1}{2\pi^2}\gamma\cdot\partial\left(\int d^4y\,\frac{f(y)}{(x-y)^2}\right)\gamma\cdot\overleftarrow{\partial}$$
$$= \frac{1}{2\pi^2}\int d^4y\,\frac{\gamma\cdot\partial f(y)\gamma\cdot\overleftarrow{\partial}}{(x-y)^2} \tag{28}$$

or in momentum space by its transform

$$\mathcal{S}_P f(p) = \frac{\gamma\cdot p}{P} f(p)\,\frac{\gamma\cdot p}{P}, \tag{29}$$

where $P^2=p^2$. We note that, in both representations,

$$\mathcal{S}^2 = 1 \tag{30}$$

and

$$[\mathcal{S}_x, \mathcal{B}_0] = [\mathcal{S}_p, \mathcal{A}_0] = 0, \tag{31}$$

where $-\mathcal{A}_0$ is the momentum space BS operator of the left member of Eq. (4). In both representations, the operators $\mathcal{M}_{\mu\nu}$, $\mathcal{P}$, $\bar{\mathcal{C}}_1$, and $\mathcal{R}$ commute with $\mathcal{S}$. $\mathcal{S}_p$ and $\mathcal{U}$ (or $\mathcal{S}_x$ and $\hat{\mathcal{V}}$) do not commute, though in the case of a class 3 or class 4 solution with V or or A type exchange,

$$[\mathcal{S}_x, \hat{\mathcal{V}}]f(x) = [\mathcal{S}_p, \mathcal{U}]f(p) = 0, \tag{32}$$

and hence the decoupling noted above occurs. In particular, in momentum space, the terms in the expression for $f(p)$ associated with a and t_1+t_2 form an eigenfunction of $\mathcal{S}_p$ with eigenvalue +1, while those associated with v and t_1-t_2 form an eigenfunction with eigenvalue −1.

C. Behavior of λ when $ma \neq mb$ for known solutions

In this section we consider the unequal mass generalizations of the equal-mass solutions of Keam and Kummer. In Sec. II C 1 we present a perturbation treatment of Kummer's solutions, and in Sec. II C 2 we present a numerical approach to the solution of the unequal mass equations. As in Sec. II B we consider the BS equation with $P_\mu=0$, and also set $\mu=0$.

1. Perturbation theory for Kummer's solutions

These are class 3 class 4 solutions for a fermion—fermion system with V type exchange, and $\bar{\mathcal{C}}_1=(-1)^{2j+1}$. If we allow negative values[4] of λ, the solutions are appropriate to both fermion—fermion and fermion—antifermion systems, for both V and A type exchange. The eigenvalues of λ are described by the two parameters j and q,[11] where $q \geq 2j+2$. The momentum space equations in this case are the transforms of Eq. (21), viz.:

$$d_2^+ d_{1/2}^-[(1-\delta^2-\sigma^2)a + i\sigma(t_1+t_2)] = -8\lambda a, \tag{33a}$$

$$\sigma^2 t_2 - (1-\delta^2)t_1 = 2i\sigma a - 2\delta\sigma v, \tag{33b}$$

$$\sigma^2 t_1 - (1-\delta^2)t_2 = 2i\sigma a + 2\delta\sigma v, \tag{33c}$$

$$d_2^+ d_{1/2}^-[(1-\delta^2+\sigma^2)v + \delta\sigma(t_1-t_2)] = 8\lambda v, \tag{33d}$$

where

$$\delta = \Delta/m, \quad \sigma = P/m, \quad \text{and} \quad d_\alpha^\pm = \frac{\partial}{\partial\sigma} \pm \frac{2(j+\alpha)}{\sigma}. \tag{34}$$

Eliminating v, t_1, and t_2 yields

$$d_2^+ d_{1/2}^- \frac{[\sigma^2+(1+\delta)^2][\sigma^2+(1-\delta)^2]}{\sigma^2-(1-\delta^2)}\, a = 8\lambda a. \tag{35}$$

We assume that the operand of the left member of Eq. (35) and λ may both be expanded as a convergent power series in δ^2. Equation (35) may then be written as

$$[\mathcal{D}_0 + \mathcal{D}_1\delta^2 + \cdots][\tau_0 + \tau_1\delta^2 + \cdots] = 0, \tag{36}$$

where

$$\tau = \tau_0 + \tau_1\delta^2 + \cdots = \frac{[\sigma^2+(1+\delta)^2][\sigma^2+(1-\delta)^2]}{\sigma^2-(1-\delta^2)}\, a \tag{37}$$

and the differential operators $\mathcal{D}_0, \mathcal{D}_1, \cdots$ depend on the terms $\lambda_0, \lambda_1, \cdots$ in the expansion

$$\lambda = \lambda_0 + \lambda_1\delta^2 + \cdots \tag{38}$$

Using as the independent variable

$$Z = (1+\sigma^2)^{-1}, \tag{39}$$

we obtain for $\mathcal{D}_0$ and $\mathcal{D}_1$ the expressions

$$\mathcal{D}_0 = Z(1-Z)^2\frac{d^2}{dZ^2} - 2Z^2(1-Z)\frac{d}{dZ} - (j+\tfrac{1}{2})(j+\tfrac{3}{2}) + 2\lambda_0(1-Z)(1-2Z) \tag{40a}$$

and

$$\mathcal{D}_1 = 2Z(1-2Z)\mathcal{D}_0 + 2\lambda_1(1-Z)(1-2Z) - 2\lambda_0 Z(1-Z)(1-8Z+8Z^2) \tag{40b}$$

From Eq. (37),

$$\mathcal{D}_0\tau_0=0 \tag{41a}$$

and

$$\mathcal{D}_0\tau_1+\mathcal{D}_1\tau_0=0 \tag{41b}$$

We act on Eq. (41b) with the operator $\mathcal{L}_1\{\tau_0,\cdot\}$, where

$$\mathcal{L}_1\{A,B\}=\int_0^1 dZ Z^{-2}A\cdot B. \tag{42}$$

Assuming the integrals converge, it may be shown that

$$\mathcal{L}_1\{\tau_0,\mathcal{D}_0\tau_1\}=\mathcal{L}_1\{\tau_1,\mathcal{D}_0\tau_0\}=0, \tag{43}$$

using (41a). Thus

$$\mathcal{L}_1\{\tau_0\mathcal{D}_1\tau_0\}=0, \tag{44}$$

and for given values of j and q, the known expressions for λ_0 and τ_0 may be used to yield a value for λ_1. In the simplest case $(q=2j+2)$,

$$\lambda_1=\frac{\partial\lambda}{\partial\delta^2}\bigg|_{\delta^2=0}=\left(\frac{3j+4}{8j+13}\right)\lambda_0, \tag{45a}$$

where

$$\lambda_0=\tfrac{1}{2}(2j+3)(4j+5) \tag{45b}$$

2. Numerical calculations

The analytic solutions of the relevant equations [viz. Eq. (35) for Kummer's solutions, and the momentum space analogs of Eqs. (19) for Keam's solution (with $j=0$)] is rather difficult, and consequently a numerical approach has been adopted based on that used by Keam in finding his solution.

(i) The method

We require a solution $f(p)$ of the BS equation to satisfy the boundary conditions[12]

$$a>-2, \quad b<-3, \tag{46a}$$

where the behaviour of any radial term $g(P)$ in the expression for $f(p)$ is given by

$$g(P)\sim P^a \quad \text{as } P\to 0,$$
$$g(P)\sim P^b \quad \text{as } P\to\infty. \tag{46b}$$

Consider a system of n coupled second order differential equations in n radial functions $\mathbf{f}=(f_1,\ldots,f_n)$. This is equivalent to a system of $2n$ coupled first order differential equations in the $2n$ functions $\mathbf{f}'=(df_1,\ldots,df_n,f_1,\ldots,f_n)$. Here the operator d denotes differentiation with respect to y, where

$$y=\sigma^2. \tag{47}$$

With y as the independent variable, we may evaluate the Frobenius series for which Eq. (46a) is satisfied as P (and y) $\to 0$. This yields n_1 vectors $\mathbf{f}_i'$ $(i=1,\ldots,n_1)$. Similarly, with the independent variable

$$u=(1+\delta^2+y)^{-1}, \tag{48}$$

we may evaluate the n_2 vectors $\mathbf{g}_j'(j=1,\ldots,n_2)$ for which Eq. (46a) is satisfied as $P\to\infty$.

The variable u is chosen so that, for the cases considered, there is a region of the complex y plane in which both sets of vectors are convergent series.

Hence if $\mathbf{h}'$ represents the radial functions (and their derivatives) of a solution of the BS equation, then in this region

$$\mathbf{h}'\equiv a_i\mathbf{f}_i'\equiv b_j\mathbf{g}_j' \tag{49}$$

where i and j are summed over their respective ranges, and not all of a_i or b_j are zero.

If, as in both cases considered in this section,

$$n_1+n_2=2n, \tag{50}$$

the condition that a nontrivial solution of Eq. (49) exists is that

$$d_\lambda(y)=\det\begin{bmatrix}\mathbf{f}_1'\\ \vdots\\ \mathbf{f}_{n_1}\\ \mathbf{g}_1'\\ \mathbf{g}_{n_2}'\end{bmatrix}\equiv 0 \tag{51}$$

for each eigenvalue λ.

If $n_1+n_2>2n$, Eq. (49) may be satisfied for all appropriate values of λ, while if $n_1+n_2=2n+1-m$ $(m\geq 2)$, m distinct conditions must be satisfied in order that an eigenvalue exist. The latter possibility is considered unlikely, and the former in not encountered for the potentials considered.

The series involved in the expression for $d_\lambda(y)$ were numerically summed, term by term, for two different values of y, until the magnitude of the terms fell below a cut off value, and the determinant $d_\lambda(y)$ was evaluated. Changes in the sign of $d_\lambda(y)$ as λ was varied were used to locate eigenvalues. The use of two values of y can distinguish cases where $d_\lambda(y)$ has a zero, at one value of y, but is not identically zero, or where d_λ approaches an asymptote, and also provides a check on the effects of computer roundoff. The programmes were run on the University of Auckland Burroughs B6700 computer.

(ii) Keam's solution

This is a class 1 solution with $j=J=0$, A type exchange, and $\bar{C}_1=+1$. The momentum space radial functions for this case are

$$d_{3/2}^+d_0^-[(1-\delta^2-\sigma^2)s+2i\sigma v_1]=-16\lambda s, \tag{52a}$$

$$d_2^+d_{1/2}^-[(1-\delta^2-\sigma^2)v_1+2i\sigma s]=-8\lambda v_1. \tag{52b}$$

The system has regular singular points at $y=0$, ∞ and $r-2\pm\sqrt{1-r}$, where

$$r=1-\delta^2. \tag{53}$$

Here $n_1=n_2=2$, and the match may be tested on the interval $y\in(0,(1-\sigma)^2)$ for $\delta>0$.

The eigenvalue was determined for values of $1/r$ up to 40 (note that for large values of $1/r$ the mass ratio $m_a/m_b\cong 4/r$, assuming $\delta>0$). A relative cutoff value |term/sum of series| of 10^{-12} was used for the Frobenius series involved.

For large values of $1/r$, the matching region becomes small, and consequently a new variable $y/[2(1-\delta)^2+y]$ was used to perform the evaluation of the series that are

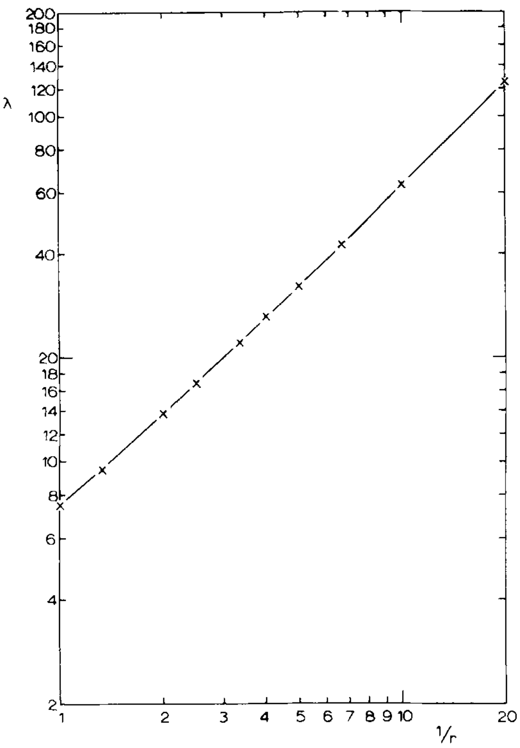

FIG. 1. The coupling parameter as a function of $1/r$ for Keam's solution.

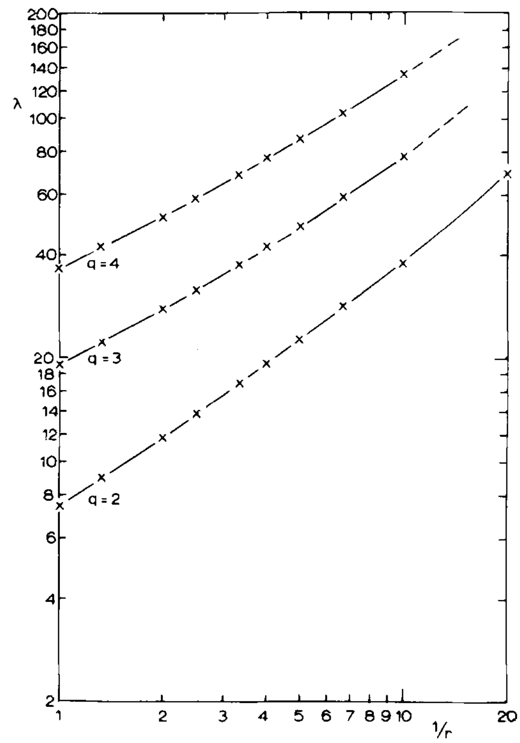

FIG. 2. The coupling parameter as a function of $1/r$ for Kummer's solutions, $j=0$.

valid as $y\to 0$. This provides more rapidly convergent series, with a matching region $y\in(0,\infty)$. When $1/r=40$, the errors due to roundoff were apparent (the B6700 has 22 significant figures in double precision). For all other values of $1/r$ the agreement between the zeroes of $d_\lambda(y)$ for the two different values of y was excellent, and for $1/r<10$, the two values agreed to within 1 in 10^8.

The results are summarized in Fig. 1, where λ is plotted against $1/r$ on a log-log scale. The numerical value of $\partial\lambda/\partial\delta^2|_{\delta^2=0}$ agress well with that obtained by Keam.[K4] It appears probable that $\lambda\to\infty$ as $1/r\to\infty$ (and $m_b\to 0$). This is consistent with the fact that no acceptable solutions to Eqs. (52) have been found for $\delta=1$.[13]

The graph of λ vs $1/r$ is very nearly linear. The gradient of the regression line of $\ln(\lambda-7.5)$ on $\ln(1/r-1)$ is 0.997, and the regression line of λ on $1/r$ yields estimates of λ that are accurate to within 0.05 for $1/r$ in the range $[1,20]$. Thus λ is approximately given by

$$\lambda = 1.357 + 6.194\,(1-\Delta^2/m^2)^{-1}. \qquad (54)$$

(iii) Kummer's solutions

Equation (35) has four regular singular points at the same values of y as for the system of Eqs. (52) for Keam's solution, and may be reduced to Heun's equation. We examine certain properties of Eq. (35), and present an alternative method of determining the eigenvalues, in the appendix.

We again consider the differential equation with dependent variable τ, rather than a. In this case $n_1=n_2=1$, and the match may be tested for $y\in(0,(1-\delta)^2)$. The zeroes of d were numerically determined for all cases in which $\lambda<50$ when $m_a=m_b$, and for $1/r\leq 10$. Again the agreement between the values of λ obtained for two different values of y is excellent.

The results are summarized in Figs. 2, 3, 4, and 5,

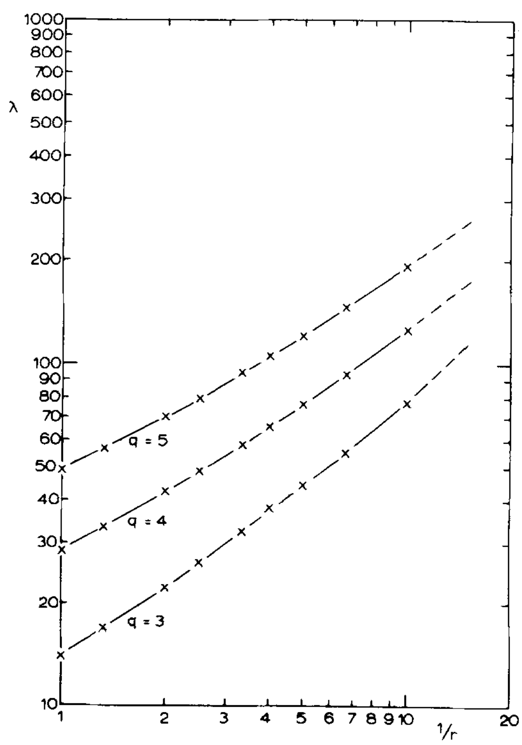

FIG. 3. The coupling parameter as a function of $1/r$ for Kummer's solutions, $j=1/2$.

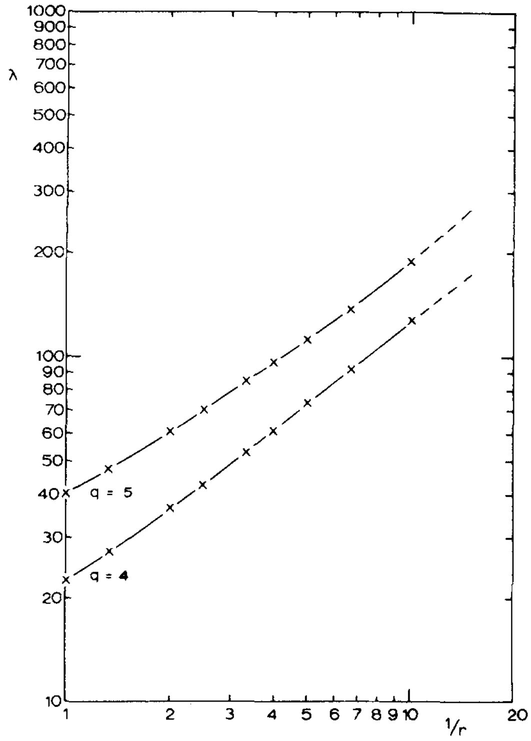

FIG. 4. The coupling parameter as a function of $1/r$ for Kummer's solutions, $j=1$.

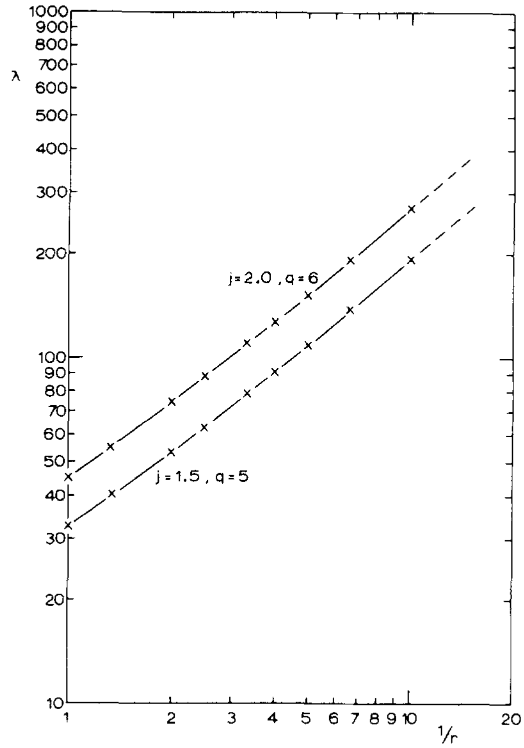

FIG. 5. The coupling parameter as a function of $1/r$ for Kummer's solutions, $j=3/2$ and 2.

again using a log—log scale. Qualitatively they are similar to the results for Keam's solution, though the departure from a linear relationship between λ and $1/r$ is more marked.

In the case $q=2j+2$, the value of $\partial\lambda/\partial\delta^2|_{\delta^2=0}$ agrees well with that given by Eq. (45). In all cases, it appears probable that $\lambda\to\infty$ as $m_b\to 0$. The author has investigated the hypergeometric equation obtained from Eq. (35) when $\delta=1$ (and $m_b=0$), and has found no cases in which acceptable solutions exist, with either positive or negative eigenvalues.

The regression lines of $\ln(\lambda-\lambda_0)$ on $\ln(1/r-1)$, where $\lambda_0=\lambda|_{\delta^2=0}$, yield estimates of λ accurate to within 2% for $1/r$ in the range $[1,10]$. These regression lines yield expressions of type

$$\lambda\approx\lambda_0+A(1/r-1)^B. \tag{55}$$

The values of λ_0, A and B are tabulated in Table II. B is always slightly less than 1.0, increasing with j for fixed q, and decreasing as q increases for fixed j. A increases with q, but is almost constant with respect to j, for fixed q.

3. Action of $\mathcal{T}_5^b$ on known solutions

As noted in Sec. IIB, we expect that $\mathcal{T}_5^b f$ be a solution of the BS equation, where f itself is a solution. We note that the radial equations for the radial functions of $\mathcal{T}_5^b f$ are equivalent to those for the radial functions of f, with m_b replaced by $-m_b$, and λ^j replaced by $\epsilon_i'\lambda^j$. Applying the methods of the previous section to Eqs. (52) and Eq. (35) with m_b negative yields no eigenvalues. Equation (A4) of the appendix has also been analyzed by splitting the operator and the operand into odd and even functions of m_b/m_a and solving the resulting pair of coupled equations numerically, as in the previous section. This yields the expected solutions and eigenvalues for m_b positive, but the solution vanishes identically for m_b negative.

Thus it appears that $\mathcal{T}_5^b f\equiv 0$ for the solutions considered in this section, and hence the integrals involved in K6, Eqs. (23), (25), and (26) vanish, so that Eq. (27) of Sec. IIB is not valid.

III. PERTURBATION OF EXCHANGE MASS

A. Introduction

Previous studies of the spinor—spinor BS equation with nonzero exchange mass have been performed by Narayanaswamy and Pagnamenta,[14] who numerically solved the eigenvalue problem in λ using a high momentum cutoff, and Guth,[15] who performed the same task without a momentum cutoff by the addition of regulating

TABLE II. Parameters λ_0, A and B in the expressions $\lambda_{jq}=\lambda_0+A(1/r-1)^B$.

j	q	λ_0	A	B
0	2	7.500	4.212	0.9172
0	3	19.187	8.456	0.8916
0	4	36.682	14.792	0.8766
$\frac{1}{2}$	3	14.000	8.308	0.9308
$\frac{1}{2}$	4	28.651	13.694	0.9118
$\frac{1}{2}$	5	49.083	21.118	0.8976
1	4	22.500	13.816	0.9347
1	5	40.127	20.375	0.9122
$\frac{3}{2}$	5	33.000	20.715	0.9416
2	6	45.500	29.024	0.9431

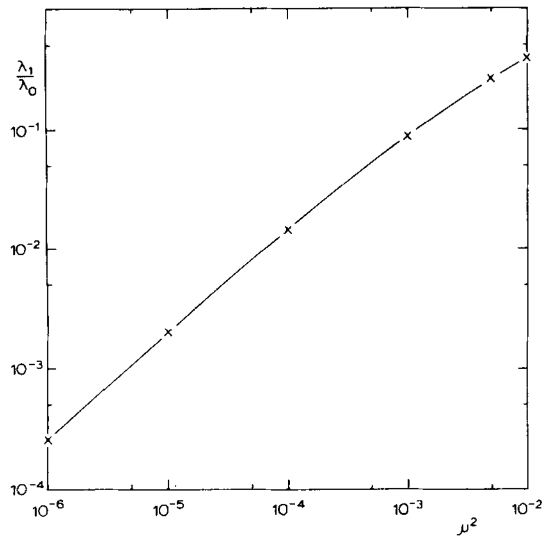

FIG. 6. The fractional change in the coupling parameter as a function of the exchange mass for Keam's solution.

terms to the one particle exchange propagator. The author's approach has been to apply perturbation theory, and use numerical methods to evaluate the integrals encountered. This analysis has been applied to Keam's solution and to the cases $j=0,\ q=2,\ J^P=1^\pm$ of Kummer's solutions.

B. The method

We consider the momentum space form Eq. (4) of the BS equation for $ma=mb$, $P_\mu=0$. When the exchange mass μ is nonzero, the right member involves terms of type $(\mu^2/m^2)\ln(\mu^2/m^2)$ and thus a perturbation expansion in powers of μ^2 is not valid. Rather we assume

$$f(p)=f_0(p)+f_1(\mu^2,p), \tag{56a}$$

$$\lambda=\lambda_0+\lambda_1(\mu^2), \tag{56b}$$

where $f_0(p)$ and λ_0 are appropriate to the case $\mu^2=0$, and $f_1(\mu^2,p)$ and $\lambda_1(\mu^2)\to 0$ as $\mu^2\to 0$. We assume also that the integrals encountered are convergent,[16] and set

$$\int d^4k\,\frac{\Gamma^j f_\alpha(k)}{(p-k)^2+\mu^2}=\int d^4k\,\frac{\Gamma^j f_\alpha(k)}{(p-k)^2}+\Delta I_\alpha(\mu^2,p) \qquad (\alpha=0,1) \tag{57}$$

where Γ^j is appropriate to the interaction type considered. It is assumed that for small $\mu\,\Delta I_1(\mu^2,p)$ and $\lambda_1(\mu^2)\Delta I_0(\mu^2,p)$ may be neglected.

Equating the remaining terms involving μ^2 yields

$$(\gamma\cdot p-im)f_1(\mu^2,p)(\gamma\cdot p-im)+\frac{\lambda_0}{\pi^2}\int d^4k\,\frac{\Gamma^j f_1(\mu^2,k)}{(p-k)^2}$$
$$=-\frac{\lambda_0}{\pi^2}\cdot\Delta I_0(\mu^2,p)-\frac{\lambda_1}{\pi^2}\int d^4k\,\frac{\Gamma^j f_0(k)}{(p-k)^2}; \tag{58}$$

we multiply Eq. (58) on the left with the adjoint $\bar f(p)$ where[K4]

$$\bar f(\mathbf{p},p_4)=\gamma_4 f^*(\mathbf{p},\,-p_4)\gamma_4, \tag{59}$$

take the trace and integrate over momentum space. The left member yields zero [cf. K4, Eq. (52)]. We therefore obtain, with the use of Eq. (57) for $\alpha=0$,

$$\frac{\lambda}{\lambda_0}=\frac{\lambda_0+\lambda_1(\mu^2)}{\lambda_0}=2-\mathcal{J}(\mu^2)/\mathcal{J}(0), \tag{60}$$

where

$$\mathcal{J}(\mu^2)=\int d^4p\,d^4k\,Tr[\bar f_0(p)\Gamma^j f_0(k)]/[(p-k)^2+\mu^2]. \tag{61}$$

For the solutions considered in Sec. IIIC the evaluation of the integrals in $\mathcal{J}(\mu^2)$ is most conveniently performed by numerical integration. In particular, we consider the integrals

$$A_r(P)=\int d^4k\{[(p-k)^2+\mu^2][1+K^2/m^2]^r\}^{-1}$$
$$=m^2\,\frac{\pi^2}{r-1}\int_0^1 dx\left(\frac{x}{z}\right)^{r-1}, \tag{62}$$

where

$$z=x+(1-x)(\mu^2/m^2)+x(1-x)(P^2/m^2), \tag{63}$$

and

$$b_r^\mu(P)=\int d^4k\,k_\mu\{[(p-k)^2+\mu^2][1+K^2/m^2]^r\}^{-1}$$
$$=p_\mu B_r(P), \tag{64}$$

where

$$B_r(P)=m^2\,\frac{\pi^2}{r-1}\int_0^1 dx(1-x)\left(\frac{x}{z}\right)^{r-1}. \tag{65}$$

The latter forms for $A_r(P)$ and $B_r(P)$ are obtained by using the Feynman method.[17]

C. Application to known solutions

(i) Keam's solution

This may be written,[K4] to within a normalization factor,

$$f_0(p)=s(P)+v_1(P)\gamma\cdot p/P, \tag{66}$$

where

$$s(P)=2u^7(1-14u+56u^2-84u^3+42u^4), \tag{67a}$$

$$v_1(P)=7i(K/m)u^8(1-2u)(1-6u+6u^2), \tag{67b}$$

and

$$u=(1+P^2/m^2)^{-1}. \tag{67c}$$

In this case

$$\bar f_0(p)=f_0(p). \tag{68}$$

After some simplification, we obtain

$$\mathcal{J}(\mu^2)=8\pi^2\int_0^\infty P^3dP\{8s(P)[A_7(P)+14A_8(P)+56A_9(p)$$
$$-84A_{10}(P)+42A_{11}(P)]+14iv_1(P)(P/m)[B_8(P)$$
$$-8B_9(P)+18B_{10}(P)-12B_{11}(P)]\}. \tag{69}$$

The double integral in the right member of Eq. (69) was evaluated by Euler–Romberg integration on the University of Auckland Burroughs B6700 computer, until successive estimates agreed to within one in 10^5. For very small μ^2/m^2 the range of integration for x was divided into two, to allow for the rapid change in x/z as $x\to 0$.

The results are summarized in Fig. 6, where λ_1/λ_0 is plotted against μ^2/m^2 on a log–log scale. The approximate expression

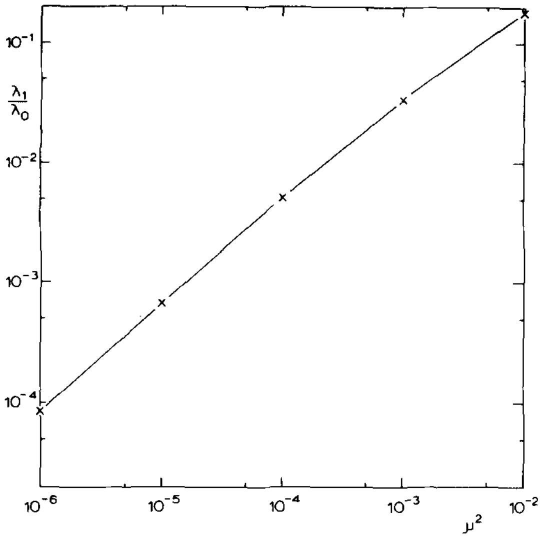

FIG. 7. The fractional change in the coupling parameter as a function of the exchange mass for Kummer's solutions ($j=0, q=2$).

$$\lambda \cong 7.5 - \frac{\mu^2}{m^2}\left(84.27 + 24.89 \ln \frac{\mu^2}{m^2}\right), \tag{70}$$

which is fitted to the values of λ when $\mu^2/m^2 = 0$, 10^{-6} and 10^{-5}, gives good agreement in the range $\mu^2/m^2 < 0.005$ (i.e., $\mu/m < 0.07$). For larger values of μ^2, $\mathcal{G}(\mu^2)$ is substantially smaller than $\mathcal{G}(0)$, and thus the perturbation assumption is no longer appropriate.

We note that expression of Eq. (70) for λ varies more rapidly with small μ^2 than is the case for the simplest solution to the scalar–scalar BS equation.[18] This arises principally because the integrals A_r and B_r vary more rapidly with μ^2 as r increases (for the scalar–scalar solution, $r=3$).

(ii) Kummer's solution

The solutions for $j=0$, $q=2$, $J=1$, $J_3=0$, $\overline{P}=\pm 1$, may be written, to within a normalization factor (the superscript denoting the eigenvalue of $\overline{P}$, for a fermion–fermion system), as

$$f_0^+ = -2iu^7[(\gamma_1 p_1 + \gamma_2 p_2)(\gamma_4 p_3 - \gamma_3 p_4) + (p_1^2 + p_2^2)\gamma_4\gamma_3] + [2u^7 - u^6][p_2\gamma_5\gamma_1 - p_1\gamma_5\gamma_2] \tag{71}$$

and

$$f_0^- = -2iu^7[(\gamma_1 p_2 - \gamma_2 p_1)(\gamma_3 p_3 + \gamma_4 p_4) - (p_3^2 + p_4^2)\gamma_1\gamma_2] + [2u^7 - u^6][p_4\gamma_5\gamma_3 - p_3\gamma_5\gamma_4]. \tag{72}$$

Only the A sector terms contribute to $\mathcal{G}(\mu^2)$, since $\Gamma^2(=\hat{\Gamma}_2)$ yields zero when acting on a T sector matrix. For both f_0^+ and f_0^-,

$$\mathcal{G}(\mu^2) = 8\pi^2 \int_0^\infty P^3 dP P^2 (2u^7 - u^6)(A_6 - 2A_7). \tag{73}$$

Using the same methods as for Keam's solution, we obtain the results summarized in Fig. 7. In this case λ may be approximately expressed as

$$\lambda \cong 7.5 - \frac{\mu^2}{m^2}[15.14 + 7.20 \ln(\mu^2/m^2)] \tag{74}$$

for $\mu^2/m^2 < 10^{-2}$. We note that the variations of λ with μ^2 is less rapid, and the expression is a more accurate approximation, than is the case with Eq. (70).

ACKNOWLEDGMENT

I should like to thank Professor R. F. Keam for his interest and for many helpful discussions.

APPENDIX: HEUN'S EQUATION AND KUMMER'S SOLUTIONS

We transform Eq. (35), using as independent variable

$$u' = (1 + P^2/m_a^2)^{-1} = (1+\delta)^2[(1+\delta)^2 + \sigma^2]^{-1}, \tag{A1}$$

and defining F by

$$\tau = (u')^{(\mu+1)/2}(1-u')^{-(2j+3)/2}F(u'), \tag{A2}$$

where

$$\mu = [8\lambda + (2j+2)^2]^{1/2} \tag{A3}$$

Thus we obtain Heun's equation[19]

$$\frac{d^2F}{du'^2} + \left(\frac{\gamma}{u'} + \frac{\delta}{u'-1} + \frac{\epsilon}{u'-b}\right)\frac{dF}{du'} + \frac{\alpha\beta u' - q}{u'(u'-1)(u'-b)} = 0 \tag{A4}$$

where

$$b = \left[1 - \frac{m_b}{m_a}\right]^2 - 1,$$
$$\alpha = \frac{\mu}{2} - j, \quad \beta = \frac{\mu}{2} - (j+1),$$
$$\gamma = \mu + 1, \quad \delta = -(2j+1),$$
$$\epsilon = 0, \quad q = b\left[\alpha\beta - 2\lambda\frac{m_b}{m_a}\left(1 + \frac{m_b}{m_a}\right)\right]. \tag{A5}$$

We consider a solution of form [cf. Eq. (10) of Ref. (19)].

$$F(u') = (u'-1)^{1-\delta}\sum_{m=0}^{\infty} a_m (u')^m, \quad a_0 = 1. \tag{A6}$$

If the series converges for $u' \in [0,1]$ then this solution gives acceptable behaviour of $a(P)$ for $P \to 0$ and $P \to \infty$. The recurrence relation for the series is

$$A_m a_{m-1} + B_m a_m + C_{m+1}a_{m+1} = 0 \tag{A7}$$

where

$$A_m = (m-1)(m-2) + (2+\gamma-\delta)(m-1) + \alpha\beta + \gamma(1-\delta),$$
$$m = 1, 2, \cdots, \tag{A8a}$$
$$B_m = -\{(a+1)m(m-1) + [a(2+\gamma-\delta)+\gamma]m + q + a\gamma(1-\delta)\},$$
$$m = 0, 1\cdots, \tag{A8b}$$
$$C_{m+1} = a(m+1)(m+\gamma), \quad m = 0, 1, \cdots. \tag{A8c}$$

The condition for convergence for $u' \in [0,1]$ is[19]

$$B_0 = q_1 C_1 \tag{A9a}$$

where

$$q_1 = -\frac{A_1}{B_1 - \dfrac{A_2 C_2}{B_2 - \dots}}. \tag{A9b}$$

The infinite continued fraction q_1 may be evaluated approximately by numerical means, and a search made for zeroes in $B_0 - q_1 C_1$. This was done for several values of m_b/m_a and j, and in each case the eigenvalue λ agreed with that obtained in Sec. IIB (iii) to the accuracy expected.

[1] B. J. Brennan and R. F. Keam, Prog. Theor. Phys. **49**, 1679 (1973). Hereafter referred to as B1.

[2] R. F. Keam, J. Math. Phys. **9**, 1962 (1968); **10**, 594 (1969); **11**, 394 (1970); **12**, 515 (1971); Prog. Theor. Phys. **50**, 957 (1973); **50**, 967 (1973). Hereafter referred to as K1 to K6, respectively.

[3] The notation used for higher transcendental functions is that of *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).

[4] We note that negative values of λ are acceptable, as g_a and g_b need not have the same sign.

[5] N. Nakanishi, Prog. Theor. Phys. Suppl. 43, 61 (1969).

[6] W. Kummer, Nuovo Cimento 31, 219 (1964); Erratum 34, 1840 (1964).

[7] We note that since $E < |(m_b - m_a)|$, the more massive particle is unstable with respect to the decay $a \rightarrow a\bar{b} + b$ or $\dot{a}b + \bar{b}$. We assume that our solutions have analytic behavior in the region around $E = |(m_b - m_a)|$, and that there are no pathological consequences of the above.

[8] Note that the parity operator for fermion—fermion systems is given by $\mathcal{P} f^c(x) = -\gamma_4 f^c(-\mathbf{x}, x_4)\gamma_4$, with the opposite sign to that for fermion—antifermion systems.

[9] There are some differences in notation to K1; viz. (i) a_1, a_2, w_2, and $\bar{w}$ are defined in a different manner to the same functions in K1 and (ii) in class 4, t_1 and t_2 are interchanged by comparison with case B, Sec. 5 of K1.

[10] We note that the (Fourier) transform $f(p)$ of a configuration space solution $f(x)$ involves the same angular kets, with radial terms that are related to those of $f(x)$ by a Hankel transform [cf. K3, Eqs. (11a) and (11b)].

[11] See Eq. 2.34 of Ref. 3, where $n = 2j + 1$, $\lambda' = 4\lambda$.

[12] See K3, Sec. 2, and B1, Sec. 2.

[13] R. F. Keam, Private Communication.

[14] P. Narayanaswamy and A. Pagnamenta, Nuovo Cimento 53A, 635 (1968).

[15] A. H. Guth, Ann. Phys. 82, 407 (1974).

[16] We note that the boundary condition Eq. (46) is a sufficient condition for this to be so.

[17] R. P. Feynman, Phys. Rev. 76, 769 (1949).

[18] See Ref. (11), Eqs. (19) and (19).

[19] A. Erdélyi *et al.*, *Higher Transcendental Functions* (McGraw Hill, New York, 1955), Vol. III, p. 57 *et seq.*