Knots and Quandles

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Abstract

Quandles were introduced to Knot Theory in the 1980s as an almost complete algebraic invariant for knots and links. Like their more basic siblings, groups, they are difficult to distinguish so a major challenge is to devise means for determining when two quandles having different presentations are really different. This thesis addresses this point by studying algebraic aspects of quandles.

Following what is mainly a recapitulation of existing work on quandles, we firstly investigate how a link quandle is related to the quandles of the individual components of the link.

Next we investigate coset quandles. These are motivated by the transitive action of the operator, associated and automorphism group actions on a given quandle, allowing techniques of permutation group theory to be used. We will show that the class of all coset quandles includes the class of all Alexander quandles; indeed all group quandles.

Coset quandles are used in two ways: to give representations of connected quandles, which include knot quandles; and to provide target quandles for homomorphism invariants which may be useful in enabling one to distinguish quandles by counting homomorphisms onto target quandles.

Following an investigation of the information loss in going from the fundamental quandle of a link to the fundamental group, we apply our techniques to calculations for the figure eight knot and braid index two knots and involving lower triangular matrix groups.

The thesis is rounded out by two appendices, one giving a short table of knot quandles for knots up to six crossings and the other a computer program for computing the homomorphism invariants.

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Introduction

Quandles are algebraic objects which were originally introduced by David Joyce in [12]. It turned out that similar objects had been looked at under a variety of names, in particular racks, by J.H.Conway and G.Wraith, and kei by Mituhisa Takasaki in [27]. For a more complete history, see the introduction to Fenn and Rourke's paper [8].

Joyce introduced quandles to provide an invariant for knots and links and showed that these invariants are (almost) complete. Unfortunately the quandle invariants themselves are not easy to tell apart, and so it is worthwhile looking at the elementary structure of quandles, and ways to tell them apart. This is what this thesis investigates.

Contents

Chapter 1 contains the definition of, and elementary concepts to do with quandles, as well as some basic algebra regarding their structures, including the associated, operator and automorphism groups, which act on quandles and are of some importance in the study of quandles.

Chapter 2 deals with quandle presentations, a concept similar to group presentations. These are defined in several equivalent ways. Also a theorem is proved which gives a generating set for certain subgroups of the groups discussed in chapter 1.

Chapter 3 details the connection between quandles and links. It gives the definition of the fundamental quandle of a link and a method for writing down a presentation for this from a diagram of the link.

The first three chapters are mainly a reworking of known results. The exceptions are detailed in the introductory comments to each chapter. The remaining chapters are mostly my work, any exceptions to this will be pointed out as they arise within the chapter.

Chapter 4 gives an algebraic construction that, when applied to link quandles, enables the quandles of individual components of the link to be computed.

Chapter 5 introduces coset quandles, a class of quandles which includes all knot quandles. These are looked at in some detail.

Chapter 6 takes a close look at the stabilizer subgroups and centraliser subgroups of the associated and operator groups. These play a role in coset quandles. In the case of the associated group they also have a natural interpretation in terms of the topology of the knot.

Chapter 7 is a short chapter which discusses some ideas regarding the failure of the fundamental group of a knot to be a complete invariant.

Chapter 8 looks at an idea for telling whether two quandles are isomorphic or not.

Chapter 9 briefly looks at ideas for further research.

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