Knots and Quandles

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Abstract

Quandles were introduced to Knot Theory in the 1980s as an almost complete algebraic invariant for knots and links. Like their more basic siblings, groups, they are difficult to distinguish so a major challenge is to devise means for determining when two quandles having different presentations are really different. This thesis addresses this point by studying algebraic aspects of quandles.

Following what is mainly a recapitulation of existing work on quandles, we firstly investigate how a link quandle is related to the quandles of the individual components of the link.

Next we investigate coset quandles. These are motivated by the transitive action of the operator, associated and automorphism group actions on a given quandle, allowing techniques of permutation group theory to be used. We will show that the class of all coset quandles includes the class of all Alexander quandles; indeed all group quandles.

Coset quandles are used in two ways: to give representations of connected quandles, which include knot quandles; and to provide target quandles for homomorphism invariants which may be useful in enabling one to distinguish quandles by counting homomorphisms onto target quandles.

Following an investigation of the information loss in going from the fundamental quandle of a link to the fundamental group, we apply our techniques to calculations for the figure eight knot and braid index two knots and involving lower triangular matrix groups.

The thesis is rounded out by two appendices, one giving a short table of knot quandles for knots up to six crossings and the other a computer program for computing the homomorphism invariants.

Contents

Introduction

Quandles are algebraic objects which were originally introduced by David Joyce in [12]. It turned out that similar objects had been looked at under a variety of names, in particular racks, by J.H.Conway and G.Wraith, and kei by Mituhisa Takasaki in [27]. For a more complete history, see the introduction to Fenn and Rourke's paper [8].

Joyce introduced quandles to provide an invariant for knots and links and showed that these invariants are (almost) complete. Unfortunately the quandle invariants themselves are not easy to tell apart, and so it is worthwhile looking at the elementary structure of quandles, and ways to tell them apart. This is what this thesis investigates.

Contents

Chapter 1 contains the definition of, and elementary concepts to do with quandles, as well as some basic algebra regarding their structures, including the associated, operator and automorphism groups, which act on quandles and are of some importance in the study of quandles.

Chapter 2 deals with quandle presentations, a concept similar to group presentations. These are defined in several equivalent ways. Also a theorem is proved which gives a generating set for certain subgroups of the groups discussed in chapter 1.

Chapter 3 details the connection between quandles and links. It gives the definition of the fundamental quandle of a link and a method for writing down a presentation for this from a diagram of the link.

The first three chapters are mainly a reworking of known results. The exceptions are detailed in the introductory comments to each chapter. The remaining chapters are mostly my work, any exceptions to this will be pointed out as they arise within the chapter.

Chapter 4 gives an algebraic construction that, when applied to link quandles, enables the quandles of individual components of the link to be computed.

Chapter 5 introduces coset quandles, a class of quandles which includes all knot quandles. These are looked at in some detail.

Chapter 6 takes a close look at the stabilizer subgroups and centraliser subgroups of the associated and operator groups. These play a role in coset quandles. In the case of the associated group they also have a natural interpretation in terms of the topology of the knot.

Chapter 7 is a short chapter which discusses some ideas regarding the failure of the fundamental group of a knot to be a complete invariant.

Chapter 8 looks at an idea for telling whether two quandles are isomorphic or not.

Chapter 9 briefly looks at ideas for further research.

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Chapter 1

Preliminaries - Basic Definitions And Lemmas

We start with some basic algebra.

This chapter consists mostly of old results. My own new work lies in

- A characterisation of normal subquandles on page 11.
- A lemma showing that free decompositions of a quandle are essentially unique on page 15.
- A method for finding free decompositions of a given quandle on page 16
- Lemma 1.2.6 which shows that the associated and operator groups of a free composition of two quandles is the direct sums of the associated and operator groups of the composition quandles.

1.1 Definition and Examples

1.1.1 Basic Definitions

Definition 1.1.1 *A* quandle *is a set Q together with two binary operations $\triangleright$ and $\triangleright^{-1}$ obeying the following axioms.*

1. $q \triangleright q = q : \forall q \in Q$ *: The* idempotency *axiom*
2. $(q \triangleright p) \triangleright^{-1} p = (q \triangleright^{-1} p) \triangleright p = q : \forall q, p \in Q$ *: The* invertibility *axiom*

3. $(q \rhd p) \rhd r = (q \rhd r) \rhd (p \rhd r) : \forall q, p, r \in Q$ *: The (right)* distributivity *axiom*

There have been several different forms of notation. One notable alternative was introduced in [8], *exponential notation* where $q \rhd p$ is written q^p, and $q \rhd^{-1} p$ is written as $q^{\bar{p}}$.

Associativity doesn't hold in general, but we shall adopt the convention of 'bracket from the left' ie. $q \rhd p \rhd r = (q \rhd p) \rhd r$.

Homomorphisms, isomorphisms, and so on, are all defined in the obvious way.

It is easy to see that

$$q \rhd^{-1} q = q : \forall q \in Q$$

$$(q \rhd^{-1} p) \rhd^{-1} r = (q \rhd^{-1} r) \rhd^{-1} (p \rhd^{-1} r) : \forall q, p, r \in Q$$

and hence there is a duality between the two operations. That is, if $\{Q, \rhd, \rhd^{-1}\}$ is a quandle, then so is $\{Q, \rhd^{-1}, \rhd\}$.
It is also easy to see the following useful equations hold.

$$(q \rhd^{-1} p) \rhd r = (q \rhd r) \rhd^{-1} (p \rhd r) : \forall q, p, r \in Q$$

$$(q \rhd p) \rhd^{-1} r = (q \rhd^{-1} r) \rhd (p \rhd^{-1} r) : \forall q, p, r \in Q$$

$$(q \rhd p) \rhd q = q \rhd (p \rhd q) : \forall q, p \in Q$$

$$q \rhd (p \rhd r) = ((q \rhd^{-1} r) \rhd p) \rhd r : \forall q, p, r \in Q \tag{1.1}$$

These are mostly not explicitly used here, but are handy for 'hands on' manipulation of quandles.

It is often helpful to think of the quandle operations as elements acting on the right, that is $q \rhd p$ is 'p acting on q'. The distributivity axiom and the first of the above equations can be interpreted as stating that these actions are homomorphisms, and the invertibility axiom states that the actions are invertible, and so are automorphisms. The last equation is particularly useful for several reasons: for the moment we'll just observe that it means that any quandle sum can be rewritten in the form $q_1 \rhd^{\pm 1} q_2 \rhd^{\pm 1} q_3 \cdots \rhd^{\pm 1} q_n$. We will call this a *normal form* for a quandle element.

1.1.2 Examples

1. A set with just one element, with the only possible quandle structure on it. This is the *unit quandle*.

2. The empty set. This is the *empty quandle*.

   The above two quandles are interesting only because in any natural categorical treatment of quandles, they are respectively terminal and initial objects.

3. For any set S define $q \rhd p = q \rhd^{-1} p = q : \forall q, p \in S$. Such a quandle is called *trivial*. Notice that for two trivial quandles T and T', any function between them is a homomorphism, hence they are isomorphic precisely when they have the same cardinality. T_n will be used to denote the trivial quandle on n elements.

4. For any group G define $q \rhd p = q \rhd^{-1} p = pq^{-1}p$. When G is the cyclic group of order n, this is denoted C_n.

5. Take the set of circles and straight lines in the extended plane, and put $q \rhd p = q \rhd^{-1} p$ equal to the element q inverted in p.

6. For any module M and non-singular linear operator T, define a quandle on the points of M by setting $q \rhd p = T(q-p)+p$, $q \rhd^{-1} p = T^{-1}(q-p)+p$. Such a quandle is called an *Alexander Quandle*

7. As a generalization of the last example, let G be any group, and ψ any automorphism of G, then for elements q, p of G, let $q \rhd p = \psi(qp^{-1})p$ and $q \rhd^{-1} p = \psi^{-1}(qp^{-1})p$.

8. Let G be any group, and ψ any automorphism of G, and this time for elements q, p of G, let $q \rhd p = \psi(p^{-1}q)p$ and $q \rhd^{-1} p = p\psi^{-1}(qp^{-1})$. In the special case that ψ is the identity, this reduces to example 4.

9. For any (tame) link L there is defined *The Fundamental Quandle* of L computable from its diagram. See [12], or chapter 3. This was the example that quandles were first examined for. This construction has been extended to arbitrary codimension two manifold pairs, see [8].

10. Pick any group G for the underlying set, and some integer n then put $q \rhd p = p^{-n}qp^{n}$ and $q \rhd^{-1} p = p^{n}qp^{-n}$. This is called *the n'th conjugacy quandle structure* on G. When $n = 1$ it is called simply the conjugacy quandle structure on G.

The final example is particularly important, partly due to the fact that out of all of the above examples, these are, for $n \geq 2$, the only ones where $q \rhd p = q \nRightarrow p \rhd q = p$, but mainly because in the $n = 1$ case there is an important connection with the operator groups of quandles. This will be covered in section 1.1.4.

1.1.3 The Automorphism and the Operator Groups

As with any algebraic structure, the automorphisms of a quandle Q form a group under composition. This is called the *automorphism group* of Q, and denoted $Aut(Q)$. As mentioned above, the set of right actions by elements of the quandle are automorphisms, and so generate a subgroup of the automorphism group called the *operator group* of Q or $Op(Q)$ following [8]. In [12] this is called the *Inner Automorphism Group* and denoted $Inn(Q)$. We will denote by $\bar{q}$ the operator $(p)\bar{q} = p \rhd q$, and in general an element of the operator group will be denoted with a bar, $\bar{g}$. $Op(Q)$ can be presented as a group by taking generators $\bar{q}$ corresponding to each element of $q \in Q$, and relations $w = id$ for each word w in the generators that leave all elements of Q unchanged, a fact we will dignify in a lemma.

Lemma 1.1.2 $Op(Q) \cong \langle \bar{q} \text{ for } q \in Q | \bar{q}_1^{\pm 1} \bar{q}_2^{\pm 1} \cdots \bar{q}_n^{\pm 1} = id \text{ where } p \rhd^{\pm 1} q_1 \rhd^{\pm 1} q_2 \cdots \rhd^{\pm 1} q_n = p \ \forall p \in Q \rangle$ $\square$

Notice that, from eqn.(1.1) the relations in $Op(Q)$ include the following.

$$\{\overline{q_1 \rhd q_2} = \bar{q}_2^{-1} \bar{q}_1 \bar{q}_2 \forall q_1, q_2 \in Q\}$$

It has been shown by H.J.Ryder [22] that any group can arise as the operator group of some quandle. Extending the previous notation, we will use 'barred' notation for any element of the automorphism group.

Lemma 1.1.3 *$Op(Q)$ is normal in $Aut(Q)$.*

Proof.
$Op(Q)$ is generated by $\{\bar{q} | q \in Q\}$ so we need only show that $\bar{g}^{-1}\bar{q}\bar{g} \in Op(Q)$ for $\bar{g} \in Aut(Q), q \in Q$, but by (1.1), $\bar{g}^{-1}\bar{q}\bar{g} = \overline{qg}$.$\square$

Since we have groups acting on Q we will have occasion to use the theory and language of group actions.

Definition 1.1.4 *If G is a group acting on a quandle Q on the right, then two elements $q_1, q_2 \in Q$ are called G-*connected* if and only if there exists an element $g \in G$ s.t. $q_1 g = q_2$. This is clearly an equivalence relation,* *and the equivalence classes are called G-*orbits*. If the only G-orbit is the whole of Q, then Q is also called G-connected. For any $q \in Q$, the G-orbit containing q is denoted $Orb_G(q)$. If G is the operator group (resp. automorphism group), then we say just* orbit *$Orb(q)$, and* connected *(resp.* quasi-connected *or* q-connected*). In this case each orbit will be closed under the quandle operations, and so will form quandles in their own right.*

As an example, consider the quandle Q with four elements, q_1, q_2, p_1, p_2 with the quandle structure given by the following table, where, for example, the entry in the third column of the first row is $q_1 \rhd p_1$.

$\rhd$	q_1	q_2	p_1	p_2
q_1	q_1	q_1	q_2	q_2
q_2	q_2	q_2	q_1	q_1
p_1	p_2	p_2	p_1	p_1
p_2	p_1	p_1	p_2	p_2

This does define a quandle, with $q \rhd^{-1} p = q \rhd p$. The axioms are easily checked, although the third is somewhat tedious. This quandle has two orbits under the operator group, $Q_1 = Orb(q_1) = \{q_1, q_2\}$ and $Q_2 = Orb(p_1) = \{p_1, p_2\}$. Notice that the orbit Q_1 is not connected, it is $Op(Q)$-connected, but not $Op(Q_1)$-connected. In this case, the orbits of Q_1 are connected, they are just single elements! This does not necessarily always happen, as has been studied in [7].

Lemma 1.1.5 *Let $\phi : Q_1 \to Q_2$ be a quandle homomorphism. Then ϕ takes connected subsets of Q_1 to connected subsets of Q_2. Hence any homomorphic image of a connected quandle is connected.*

Proof.
Let q_1, q_2 be connected in Q_1, so there exist $p_1, \cdots, p_n$ in Q such that $q_2 = q_1 \rhd^{\pm 1} p_1 \rhd^{\pm 1} \cdots \rhd^{\pm 1} p_n$ and so $q_2\phi = (q_1\phi) \rhd^{\pm 1} (p_1\phi) \rhd^{\pm 1} \cdots (p_n\phi)$, hence $q_1\phi$ and $q_2\phi$ are connected. $\Box$

It is possible to have a connected quandle with a subquandle that is not even quasi-connected. Let S_n be the quandle which isthe group of permutations of the numbers one through n with the conjugacy quandle structure on it. Let Q_n be that subquandle of S_n consisting of all 2-cycles. Now let $P_1 \subset Q_7$ be the quandle of all 2-cycles on the symbols 1 - 3, $P_2 \subset Q_7$ be the quandle of all 2-cycles on the symbols 4 - 7 and $P = P_1 \cup P_2$. Then Q_7 is clearly connected, P is a subquandle of Q_7 which is not even quasi-connected. To see this last, notice that there cannot be an automorphism which takes the cycle $(4,5)$ to $(1,2)$ as there are four elements of P which act non-trivially on $(4,5)$, namely $(4,6),(4,7),(5,6),(5,7)$, but there are only two elements which act non-trivially on $(1,2)$.

1.1.4 The Associated Groups of a Quandle

For any quandle Q there are associated groups $A_n(Q)$ (if $n = 1$ then just $A(Q)$). These can be given by presentations

$$A_n(Q) = \langle \hat{q}, \text{for } q \in Q | \widehat{(q_1 \rhd q_2)} = \hat{q}_2^{-n} \hat{q}_1 \hat{q}_2^n \rangle$$

These have the universal property that any homomorphism ϕ from Q to a group G with the n'th conjugacy quandle structure on it can be factored into $\psi_n \theta$ where ψ_n is the canonical map from Q to $A_n(Q)$, and θ is a group homomorphism (and hence a quandle homomorphism). In particular, $Op(Q)$ is a quotient of $A(Q)$ and so $A(Q)$ acts on Q in a natural way. For this reason $A(Q) = A_1(Q)$ is by far the most important of the associated groups. In future the canonical homomorphism from $A(Q)$ to $Op(Q)$ will be denoted π_Q although the subscript will usually be dropped.

Question 1.1.6 *What is the kernel of π_Q?*

In [8] Fenn and Rourke call this the *excess* of the quandle.

It is easy to see that A_n can be extended to a functor, if f is a homomorphism that maps $Q_1 \to Q_2$ then define $A_n(f)(\hat{q}) = \widehat{f(q)}$, and extend linearly to $A_n(Q_1)$. In the $n = 1$ case this functor respects the action of $A(Q)$ on Q. That is, if we denote $A(f)$ by $\hat{f}$, then we have for any quandle homomorphism f that

$$f(qg) = f(q)\hat{f}(g)$$

Note that $A_n(Q)$ made abelian will be a free abelian group with one generator for each orbit of Q. An element q in Q maps to the generator of its corresponding orbit. In particular, the identity of $A_n(Q)$ will never be in the image of ψ_n.

Example. If Q is trivial, then $A(Q)$ is simply the free abelian group on the underlying set of Q. Notice that in this case Q injects into $A(Q)$. It transpires that for a link, the associated group of the fundamental quandle is just the fundamental group of the link. This will be covered in chapter 3. The groups $A_n(Q)$ are invariants of the quandle Q and so also of links, via their fundamental quandle, and it may be asked how good these invariants are. This has not been explored here, but it has been shown in [20] that these distinguish the (unoriented) knot type in the $n = 2$ case.

A point of notation: while elements of the associated group of a quandle are usually written with hats, the hats will often be written in the 'wrong place' for aesthetic reasons. For example we will write $\hat{q}'_{1,2}$ rather than $\widehat{q'_{1,2}}$.

1.1.5 Congruences, Quotient Quandles and Normal Subquandles

For any algebraic structure there are defined *congruence classes* (see any standard textbook on universal algebra eg. [5]). These are equivalence relations used for 'factoring out' the structure, and defined in terms of the algebra. Congruences are entirely standard objects and have been discussed in the case of racks in far greater detail by Ryder in her doctoral thesis [22]. In the case of quandles, the requirements for the equivalence relation $\approx$ to be a congruence are that if $q \approx q'$ and $p \approx p'$ then $q \rhd p \approx q' \rhd p'$ and $q \rhd^{-1} p \approx q' \rhd^{-1} p'$. Alternatively, for any quandle Q and some homomorphism ϕ to some other quandle Q', the relation '$q \approx p$ if and only if $\phi(q) = \phi(p)$' is a congruence, with each congruence class being the preimage of some element in Q'.

For a congruence $\approx$ let $[q]_\approx$ denote the congruence class containing q. Where the congruence is obvious, this will be shortened to $[q]$. This means that they are equivalence relations with the property that the equivalence classes are respected by elements of the operator group, and so form a quandle under the (well defined) operation $[q] \rhd [p] = [q \rhd p]$.

We note the following definitions.

Definition 1.1.7 *Let Δ_Q be the congruence on $Q : q\Delta_Q p$ if and only if $q = p$. Let ∇_Q be the congruence on $Q : q\nabla_Q p$ $\forall q, p \in Q$. When Q is clear, we suppress the subscript. These are called the trivial congruences.*

Definition 1.1.8 *A quandle Q is called* simple *if and only if the only congruences are trivial. Equivalently, if and only if the only homomorphisms from Q to some other quandle Q' are either injective or map the whole of Q to one element of Q'.*

Lemma 1.1.9 *For any congruence relation $\approx$ on some quandle Q*

- *$[q]$ is a quandle $\forall q \in Q$*
- *The functions $\phi^q(p) = p \rhd q$ and $\phi_q(p) = p \rhd^{-1} q$ provide isomorphisms from $[q_0]$ to $[q_0 \rhd q]$ and $[q_0 \rhd^{-1} q]$ respectively, for all q_0, q in Q.*

- *Hence the functions $\phi^{\bar{g}}(p) = pg$ provide isomorphisms from $[q_0]$ to $[q_0 g]$ for all q_0 in Q and $\bar{g}$ in $Op(Q)$.*

- *The function $\psi^{\bar{g}}([q]) = [qg]$ provides an automorphism of $Q/\approx$ for all g in the operator group of Q.*

- *Hence $\psi : Op(Q) \to Op(Q/\approx)$, $\psi(\bar{g}) = \psi^{\bar{g}}$ is a homomorphism.*

Proof.
The first statement is a consequence of the idempotency axiom.

As for the second, that the given ϕ's take congruence classes to congruence classes is part of the definition of a congruence, their status as a homomorphism is a consequence of the distributivity axiom and that they are bijective is precisely the invertibility axiom.

The third statement is a direct consequence of the second.

For the fourth statement, that ψ^g is well defined is a consequence of the definition of a congruence. That it is a homomorphism is a consequence of the distributivity axiom. That it is invertible is obvious.

For the final statement, that ψ is a homomorphism from the operator group of Q to the automorphism group of $Q/\approx$ is obvious from the above. To see that the image of ψ is in the operator group of $Q/\approx$, note that for any q in Q, $\psi(\bar{q}) = \overline{[q]}$. Since the set of $\bar{q}$ for q in Q generate $Op(Q)$, the whole of the image of $Op(Q)$ is in $Op(Q/\approx)$. $\square$

Although any congruence class is a sub-quandle, the reverse is not true. Eg. Take $Q =$ the group $D_4 = \langle a, b | a^4 = b^2 = e, b^{-1}ab = a^{-1}\rangle$ with the conjugacy quandle structure on it. Then $Q_1 = \{a, a^2\}$ is a subquandle, but $Q_1 \rhd b = \{a^2, a^3\}$ is not disjoint from Q_1, hence there cannot be a congruence relation with Q_1 as a class. Call a subquandle *normal* if it is a congruence class for some congruence. Write $Q_1 < Q$ for Q_1 is a subquandle of Q, and $Q_1 \lhd Q$ for Q_1 is a normal subquandle of Q.

We have the following characterisation of normal subquandles.

Theorem 1.1.10 *For a subquandle Q_1 of a quandle Q the following are equivalent.*

1. *Q_1 is normal in Q.*
2. *(a) $\forall \bar{g}, \bar{g}' \in Op(Q)$, $Q_1\bar{g} = Q_1\bar{g}'$ or $Q_1\bar{g} \cap Q_1\bar{g}' = \emptyset$ and*

   (b) $\forall \bar{g} \in Op(Q), q, p \in Q_1$, $(Q_1\bar{g}) \rhd q = (Q_1\bar{g}) \rhd p$

   (c) $\forall \bar{g} \in Op(Q), q, p \in Q_1$, $(Q_1\bar{g}) \rhd^{-1} q = (Q_1\bar{g}) \rhd^{-1} p$

Proof.

$1 \Longrightarrow 2$

From the above comments $Q_1 g$ and $Q_1 g'$ are both congruence classes under the same congruence, hence all three statements follow.

$2 \Longrightarrow 1$

Define an equivalence relation $\approx$ on Q by the following.
$q \approx p$ if and only if

- For all $\bar{g}$ in $Op(Q)$, $q \in Q_1\bar{g}$ whenever $p \in Q_1\bar{g}$, and

- For all $\bar{g}$ in $Op(Q)$, $Q_1\bar{g} \rhd q = Q_1\bar{g} \rhd p$.

Notice that by substituting $\bar{g} = \bar{g}'\bar{q}^{-1}$ in the second condition, we obtain that $q \approx p \Rightarrow \forall \bar{g}' \in Op(Q), Q_1\bar{g}' \rhd^{-1} q = Q_1\bar{g}' \rhd^{-1} p$.
This is clearly an equivalence relation. I claim it is also a congruence, and furthermore, that for $q \in Q_1$, $[q] = Q_1$.

The relation $\approx$ is a congruence.

Let $q \approx q'$ and $p \approx p'$. There are two conditions that must be shown.

- $q \rhd p \in Q_1\bar{g} \Leftrightarrow q' \rhd p' \in Q_1\bar{g}$ for all $\bar{g}$ in $Op(Q)$.

- $Q_1\bar{g} \rhd (q \rhd p) = Q_1\bar{g} \rhd (q' \rhd p')$ for all $\bar{g}$ in $Op(Q)$.

First we show that $\forall \bar{g} \in Op(Q)$, $(q \rhd p) \in Q_1\bar{g} \Leftrightarrow (q' \rhd p') \in Q_1\bar{g}$.

$$\begin{array}{rrcl} & (q \rhd p) & \in & Q_1\bar{g} \\ \Leftrightarrow & q & \in & (Q_1\bar{g}) \rhd^{-1} p \\ & & = & Q_1\bar{g} \rhd^{-1} p' \\ \Leftrightarrow & q' & \in & Q_1\bar{g} \rhd^{-1} p' \\ \Leftrightarrow & (q' \rhd p') & \in & Q_1\bar{g} \end{array}$$

Next we show that for all $\bar{g} \in Op(Q)$, $Q_1\bar{g} \rhd (q \rhd p) = Q_1\bar{g} \rhd (q' \rhd p')$. To this end, take $c \in Q_1\bar{g} \rhd (q \rhd p)$

$$\begin{array}{rcll} c & = & q_0\bar{g} \rhd (q \rhd p) & \text{for some } q_0 \in Q_1 \\ & = & ((q_0\bar{g} \rhd^{-1} p) \rhd q) \rhd p & \\ & = & ((q_1\bar{g} \rhd^{-1} p') \rhd q) \rhd p & \text{for some } q_1 \in Q_1 \\ & = & ((q_2\bar{g} \rhd^{-1} p') \rhd q') \rhd p' & \text{for some } q_2 \in Q_1 \\ & = & (q_2\bar{g}) \rhd (q' \rhd p') & \\ & \in & Q_1\bar{g} \rhd (q' \rhd p') & \end{array}$$

So $\approx$ is a congruence relation.

For $q \in Q_1, [q] = Q_1$.

$Q_1 \subset [q]$
Pick $p \in Q_1$. Then $p \in Q_1\bar{g} \Leftrightarrow p \in Q_1 \cap Q_1\bar{g} \Leftrightarrow Q_1\bar{g} = Q_1 \Leftrightarrow q \in Q_1\bar{g}$ so the first condition for $p \approx q$ is satisfied. The second condition is just the second condition in the statement of the theorem.
$[q] \subset Q_1$
Pick $p \approx q$ and take $\bar{e} = id \in Op(Q)$ then $q \in Q_1\bar{e} = Q_1$ so from the first condition on $\approx$, $p \in Q_1$.

We have given a congruence on Q which has Q_1 as a congruence class and so Q_1 is normal in Q. $\square$

Notice that the intersection of two normal subquandles is itself normal.

The Orbit Congruence

Definition 1.1.11 *For a quandle Q, let $\doteq$ be the relation on Q: $q_1 \doteq q_2$ if and only if $q_2 = q_1\bar{g}$ for some $\bar{g}$ in $Op(Q)$, that is, if and only if q_1 and q_2 are in the same orbit of $Op(Q)$. This is called the* orbit *congruence.*

Lemma 1.1.12 *$\doteq$ is a congruence. Furthermore the congruence factor quandle is a trivial quandle.*

Proof.
$\doteq$ is a congruence.
That $\doteq$ is an equivalence relation is clear. If $q = q'\bar{g}$ and $p = p'\bar{g}'$ with q, q', p, p' in Q and $\bar{g}, \bar{g}'$ in $Op(Q)$ then $q \triangleright p = (q' \triangleright p')\bar{h}$, where $\bar{h} = \bar{p}'^{-1}\bar{g}\bar{p}$, an element of $Op(Q)$, and so $\doteq$ is a congruence.
$Q/\doteq$ is trivial
To see this simply notice that $q \triangleright p \doteq q \quad \forall q, p \in Q$ so $[q]_{\doteq} \triangleright [p]_{\doteq} = [q \triangleright p]_{\doteq} = [q]_{\doteq}$ for all q and p in Q. $\square$

Given a quandle Q, the factor quandle under the orbit congruence is denoted $\bar{Q}$, and the canonical homomorphism is denoted $\bigcirc : Q \to \bar{Q}$.

Lemma 1.1.13 *$\bigcirc$ has the following universal property. For any quandle, Q and any homomorphism ϕ from Q to a trivial quandle T, $\bigcirc$ factors ϕ uniquely. That is, there exists a unique homomorphism $\psi : \bar{Q} \to T$ s.t. $\phi = \bigcirc\psi$.*

Proof.

Pick a homomorphism ϕ from Q to a trivial quandle. Given q in $\bar{Q}$, and p, p' in $\bigcirc^{-1}(q)$ then $p' = p\bar{g}$ for some $\bar{g}$ in $Op(Q)$, so

$$\begin{aligned}(p')\phi &= (p\bar{g})\phi = ((p \rhd^{\pm 1} q_1) \rhd^{\pm 1} \cdots \rhd^{\pm 1} q_n)\phi \\ &= (((p)\phi \rhd^{\pm 1} (q_1)\phi) \rhd^{\pm 1} \cdots \rhd^{\pm 1} (q_n)\phi) = (p)\phi\end{aligned}$$

So $(q)\psi = (\bigcirc^{-1}(q))\phi$ is well defined, and is the required function. It takes a trivial quandle to a trivial quandle, and is hence necessarily a homomorphism. That ψ is unique with this property is clear. $\square$

Notice that Q is not necessarily the free union of the congruence classes, defined as the preimages of elements of $\bar{Q}$. If P_1 and P_2 are two such classes the elements of P_1 leave P_2 set-wise fixed, but not necessarily point-wise fixed.

The function that takes quandles to their associated groups is functorial, so $\bigcirc$ maps to a homomorphism $A(\bigcirc)$ of groups.

Theorem 1.1.14 *$A(\bigcirc)$ maps $A(Q)$ to its abelianization.*

Proof.

The homomorphism $\bigcirc$ acting on Q, factors out Q by the congruence generated by the relations $q\bar{g} = q : \ \forall q \in Q, \bar{g} \in Op(Q)$. These in turn are generated by the relations $q \rhd p = q : \ \forall q, p \in Q$. Under A these become the relations $\bar{p}^{-1}\bar{q}\bar{p} = \bar{q} : \ \forall q, p \in Q$. These relators generate the commutator subgroup of $A(Q)$. $\square$

Lemma 1.1.15 *The following are equivalent -*

1. *Q is connected.*
2. *There exists a q_0 in Q with the property that for all q in Q there exists g in $Op(Q)$ such that $q = q_0 g$*
3. *$\bar{Q}$ is the unit quandle.*

Proof.

1 $\Rightarrow$ *2* Pick elements q, q'. There is a $q_0 \in Q$ and $g, g' \in Op(Q)$ with $q = q_0 g$ and $q' = q_0 g'$, then $q' = qg^{-1}g'$.
2 $\Rightarrow$ *3* and *3* $\Rightarrow$ *1* These are simple consequences of the definition of $\bar{Q}$. $\square$

1.2 Compositions

1.2.1 The Free Composition

In this section the $Q_i = \{Q_i, \triangleright_i, \triangleright_i^{-1}\}$'s are mutually disjoint quandles with quandle operators $\triangleright_i$ and $\triangleright_i^{-1}$.

Definition 1.2.1 *The* free composition *of two disjoint quandles, $Q_1 \sqcup Q_2$ is the (disjoint) union of the elements of Q_1 and Q_2 together with the quandle structure defined by*

$$q \triangleright p = \left\{ \begin{array}{lcl} q \triangleright_i p & : & q, p \text{ from the same } Q_i \\ q & : & q, p \text{ from different } Q_i \end{array} \right\}$$

$q \triangleright^{-1} p$ is defined similarly.

This composition is clearly both commutative and associative on quandles.

Definition 1.2.2 *A quandle Q is called* indecomposable *if and only if $Q \cong Q_1 \sqcup Q_2 \Rightarrow$ one of the Q_i is the empty quandle (and hence the other is isomorphic to Q).*

We wish to decompose quandles into disjoint unions of indecomposable quandles, but first we will show uniqueness of any such decomposition.

Lemma 1.2.3 *Any decomposition of a quandle Q into a free union of indecomposable, non-empty quandles is unique up to ordering. That is, if $Q = \sqcup Q_i = \sqcup P_i$ where the quandles Q_i and P_i are indecomposable, then there is a bijection θ between the sets $\{Q_i\}$ and $\{P_i\}$ such that $Q_i \cong \theta(Q_i) \forall i$.*

Proof.
For each Q_i, $Q_i = \sqcup_j (Q_i \cap P_j)$ is a free union. Hence $Q_i = Q_i \cap P_j$ for some unique j. Likewise $P_j = P_j \cap Q_i$. $\square$

Given a quandle Q, how do we maximally decompose it ? That is, how do we find a collection of subquandles $\{Q_i\}$ s.t. $Q = \bigsqcup Q_i$ and each Q_i is indecomposable? We give a partial answer.

Definition 1.2.4 *For a given quandle Q and a subset $S \subset Q$, define the following.*
$N_Q(S) = \{q \in Q | q \triangleright q' \neq q$ for some $q' \in S\} \cup S$.
$M_Q(S) = \{q \in Q | q' \triangleright q \neq q'$ for some $q' \in S\} \cup S$.
That is $N_Q(S)$ is the set S together with all elements of Q which are acted on in a non-trivial fashion by some element of S. $M_Q(S)$ is the set S together

with all elements of Q which act non-trivially on some element of S.
$J_Q(S) = M_Q(N_Q(S))$.
$\hat{J}_Q(S) = \bigcup_{n<\infty} J_Q^n(S)$.
In practice, the subscript Q is omitted.

Theorem 1.2.5 *If $\bigsqcup Q_i$ is a decomposition of a quandle Q and $q_0 \in Q_j$ for some j, then $Q_j = \hat{J}(q_0)$*

Proof.

We prove this by first showing that $\hat{J}(q_0)$ is contained in Q_j, and then showing that $Q = \hat{J}(q_0) \sqcup (Q - \hat{J}(q_0))$ is a free composition of Q. That is, if we pick q in $\hat{J}(q_0)$ and any p in $Q - \hat{J}(q_0)$ then q and p act trivially on each other. These two facts together, are enough to show that Q_j must equal $\hat{J}(q_0)$.

$\hat{J}(q_0)$ is contained in Q_j.

Pick any q in Q_j. Any element q' that is acted on non-trivially by q must also be in Q_j. Similarly any element q' which acts non-trivially on q must also be in Q_j. Hence if S is a subset of Q_j then $M(S)$ and $N(S)$ are also subsets Q_j and so $\hat{J}(S)$ is likewise a subset of Q_j. In particular, for any $q_0 \in Q$ $\hat{J}(q_0) \subseteq Q_j$ as claimed.

$Q = \hat{J}(q_0) \sqcup (Q - \hat{J}(q_0))$ is a free composition of Q.

Pick q from $Q - \hat{J}(q_0)$. We show that q acts trivially on all elements of $\hat{J}(q_0)$, and conversely, that any element of $\hat{J}(q_0)$ acts trivially on q.

q acts trivially on all elements of $\hat{J}(q_0)$.
Since q is not in $\hat{J}(q_0)$ it cannot be in $J^{n+1}(q_0) = (MNJ^n)(q_0)$ for any integer n. Hence q must act trivially on all elements of $(NJ^n)(q_0)$ and in particular on any element of $J^n(q_0)$. Since this holds for all n, it is true for $\hat{J}^n(q_0)$.

Any element of $\hat{J}(q_0)$ acts trivially on q.
Since q is not in $\hat{J}(q_0)$ it follows that q cannot be in $(NJ^n)(q_0)$ for any integer n. Hence

$$\begin{aligned} q &\notin (NJ^n)(q_0) \quad \forall n \\ \Rightarrow q \rhd q' &= q \ \forall q' \in J^n(q_0) \quad \forall n \\ \Rightarrow q \rhd q' &= q \ \forall q' \in \hat{J}(q_0) \end{aligned}$$

Since q was an arbitrary element of $Q - \hat{J}(q_0)$, we have finished the proof. $\square$

Notice that for all $\bar{g} \in Aut(Q)$, since $q \rhd p \neq q \Rightarrow q\bar{g} \rhd p\bar{g} = (q \rhd p)\bar{g} \neq q\bar{g}$ we have $(N(q))\bar{g} = N(q\bar{g})$, and similarly for M. Hence $\hat{J}(S)\bar{g} = \hat{J}(S\bar{g})$.
This can be rewritten as 'For any quandle Q and any element $q \in Q$, the

indecomposable component of Q containing q is $\cap_{S \ni q, S \text{ closed under } J} S$, or the smallest subset of Q containing q and closed under J'.

This is a good start, but the functions N and M are not in general easy to compute, although they are used later in Chapter 5 on coset quandles.

As another attack on the problem, consider for some quandle Q the quandle Q' constructed with one element for each indecomposable subquandle of Q and with the trivial quandle structure imposed. Then decomposing Q into its free union can be viewed as a homomorphism from Q to Q'. Hence it may be easier to compute the indecomposable components of Q by computing $\hat{J}(Q')$ where Q' is some component of $\bar{Q}$.

Now we show the associated and operator groups of a free composition.

Proposition 1.2.6

1. $A(Q_1 \sqcup Q_2) \cong A(Q_1) \oplus A(Q_2)$
2. $Op(Q_1 \sqcup Q_2) \cong Op(Q_1) \oplus Op(Q_2)$

Proof.

1) $A(Q_1 \sqcup Q_2)$ has as generators the elements of $Q_1 \sqcup Q_2$ which is to say, the disjoint union of the elements of Q_1 with those of Q_2. Its defining relations will be -

$$\begin{array}{rcll} \widehat{q_i \rhd q_j} & = & \hat{q}_j^{-1}\hat{q}_i\hat{q}_j & : q_i, q_j \in Q_1 \\ \widehat{p_i \rhd p_j} & = & \hat{p}_j^{-1}\hat{p}_i\hat{p}_j & : p_i, p_j \in Q_2 \\ \hat{q}_i\hat{p}_j & = & \hat{p}_j\hat{q}_i & : q_i \in Q_1, p_j \in Q_2 \end{array}$$

Now define a function $\phi : A(Q_1 \sqcup Q_2) \to A(Q_1) \oplus A(Q_2)$ by defining ϕ on the generators

$$\phi(\hat{q}) = \begin{cases} (\hat{q}, \hat{id}) & q \in Q_1 \\ (\hat{id}, \hat{q}) & q \in Q_2 \end{cases}$$

and extending linearly. The relations are all easily seen to carry over, and this is a well defined homomorphism. It is also easy to see that it is surjective.

ϕ is injective.

To show injectivity, suppose there exists $\hat{r} \in A(Q_1 \sqcup Q_2)$ such that $\phi(\hat{r}) = (\hat{id}, \hat{id})$. Then $\hat{r}$ can be written as $\hat{r} = \hat{r}_1\hat{r}_2 \cdots \hat{r}_n$ where $r_i \in Q_1 \sqcup Q_2$. Now using the third set of defining relations of $A(Q_1 \sqcup Q_2)$ it is possible to rearrange the $\hat{r}_i$ to obtain a word, $\hat{w}$ that represents $\hat{r}$ and such that the $r_i \in Q_1$ appear at the beginning of $\hat{w}$, and the r_i from Q_2 appear at the end of $\hat{w}$: thus we assume without loss of generality that $r_1, \cdots, r_m \in Q_1$ and $r_{m+1}, \cdots, r_n \in Q_2$. Then $\phi(\hat{r}) = \phi(\hat{r}_1 \cdots \hat{r}_m)\phi(\hat{r}_{m+1} \cdots \hat{r}_n) = (\hat{r}_1 \cdots \hat{r}_m, \hat{r}_{m+1} \cdots \hat{r}_n) = (\hat{id}, \hat{id})$.

So $\hat{r}_1 \cdots \hat{r}_m = \hat{id} \in A(Q_1)$ and $\hat{r}_{m+1} \cdots \hat{r}_n = \hat{id} \in A(Q_2)$. These are both necessarily implied by the relations in the definition of $A(Q_1)$ and $A(Q_2)$ respectively, which are the first and second set of relations in the definition of $A(Q_1 \sqcup Q_2)$ and so hold there. That is $\hat{r} = \hat{id} \in A(Q_1 \sqcup Q_2)$ and ϕ is surjective. $\square$

2) Let $\phi : A(Q_1 \sqcup Q_2) \to A(Q_1) \oplus A(Q_2)$ be the isomorphism defined above, and let $\pi_1, \pi_2, \pi_{Q_1 \sqcup Q_2}$ be the canonical homomorphisms from the associated groups of quandles to their respective operator groups. Then it is sufficient to show that $\phi(ker(\pi_{Q_1 \sqcup Q_2})) = ker(\pi_1) \oplus ker(\pi_2)$.

$\phi(ker(\pi_{Q_1 \sqcup Q_2})) \subseteq ker(\pi_1) \oplus ker(\pi_2)$**.**

Suppose $\hat{r} \in ker(\pi_{Q_1 \sqcup Q_2})$, so $q\hat{r} = q$ for all $q \in Q_1$ and $p\hat{r} = p$ for all $p \in Q_2$. Then, as in the above proof, we can write $\hat{r} = \hat{r}_1 \cdots \hat{r}_m \hat{r}_{m+1} \cdots \hat{r}_n$, with $r_1, \cdots, r_m \in Q_1$ and $r_{m+1}, \cdots, r_n \in Q_2$. Now $\hat{r}_1 \cdots \hat{r}_m$ acts trivially on Q_2 by the definition of free composition, and so $\hat{r}_{m+1} \cdots \hat{r}_n$ also acts trivially and so is in the kernel of π_2. Similarly $\hat{r}_1 \cdots \hat{r}_m$ is in the kernel of π_1. Hence $\phi(\hat{r}) = (\hat{r}_1 \cdots \hat{r}_m, \hat{r}_{m+1} \cdots \hat{r}_n) \in (ker(\pi_1), ker(\pi_2))$.

$ker(\pi_1) \oplus ker(\pi_2) \subseteq \phi(\ker(\pi_{Q_1 \sqcup Q_2}))$**.**

Let $\hat{r} = (\hat{r}_1 \cdots \hat{r}_m, \hat{r}_{m+1} \cdots \hat{r}_n) \in ker(\pi_1) \oplus ker(\pi_2)$ where $r_1, \cdots, r_m \in Q_1$ and $r_{m+1}, \cdots, r_n \in Q_2$. Then $\hat{r}_1 \cdots \hat{r}_m$ acts trivially on Q_1 when considered as an element of $A(Q_1 \sqcup Q_2)$, and also acts trivially on Q_2 by the definition of $Q_1 \sqcup Q_2$. Similarly $\hat{r}_{m+1} \cdots \hat{r}_n$ when considered as an element of $A(Q_1 \sqcup Q_2)$ acts trivially on $Q_1 \sqcup Q_2$, so $\hat{r}_1 \cdots \hat{r}_m \hat{r}_{m+1} \cdots \hat{r}_n \in ker(\pi_{Q_1 \sqcup Q_2})$. Since $\phi(\hat{r}_1 \cdots \hat{r}_m \hat{r}_{m+1} \cdots \hat{r}_n) = \hat{r}$ the proposition is proved. $\square$

1.2.2 The Direct Product

Definition 1.2.7 *The* direct product *of Q_1 and Q_2, $Q_1 \otimes Q_2$ is the cartesian product of the elements of the Q_i together with the quandle structure defined by the following rule.*

$$(q_1, q_2) \rhd (q'_1, q'_2) = (q_1 \rhd_1 q'_1, q_2 \rhd_2 q'_2)$$
$$(q_1, q_2) \rhd^{-1} (q'_1, q'_2) = (q_1 \rhd_1^{-1} q'_1, q_2 \rhd_2^{-1} q'_2)$$

This is clearly, up to isomorphism, both commutative and associative on quandles. The direct product has the universal property that makes it the categorical product. Define the projections $\pi_i : Q_1 \otimes Q_2 \rightarrow Q_i$ by $\pi_i(q_1, q_2) = q_i$. Then we have for any quandle Q and any pair of homomorphisms $f_i : Q \rightarrow Q_i$ a unique homomorphism $g : Q \rightarrow Q_1 \otimes Q_2$ with the property that the following diagram commutes.

$$\begin{tikzcd} Q_1 & Q_1 \otimes Q_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Q_2 \\ & Q \arrow[ul, "f_1"] \arrow[u, "g"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

Simply define $g(q) = (f_1(q), f_2(q))$.
This easily generalises to the direct product of finite collections of quandles, and can be generalised to arbitrary collections in the standard way, although that will not be used here.

Definition 1.2.8 *A quandle Q is called* irreducible *if and only if $Q \cong Q_1 \otimes Q_2 \Rightarrow Q_1$ or Q_2 is the unit quandle. Otherwise it is called* reducible.

Lemma 1.2.9 *The following are equivalent for a quandle Q -*

1. *Q is reducible.*
2. *There exists a pair of non-trivial congruences $\approx_1, \approx_2$ such that for any pair of congruence classes Q_1 a congruence class of $\approx_1$, and Q_2 a congruence class of $\approx_2$, $Q_1 \cap Q_2$ contains precisely one element.*
3. *There exists a pair of non-trivial congruences $\approx_1, \approx_2$ such that for any pair of congruence classes Q_1 a congruence class of $\approx_1$, and Q_2 a congruence class of $\approx_2$, $Q_1 \cap Q_2$ contains precisely one element. Furthermore, for any such pair, and for any $q \in Q$*
   $Q \cong Q_1 \otimes Q_2$ where $Q_1 = Q/\approx_1 \cong [q]_{\approx_2}$ and $Q_2 = Q/\approx_2 \cong [q]_{\approx_1}$

Proof.
$1 \Rightarrow 2$
Q is reducible so $Q \cong Q_1 \otimes Q_2$ for some non-unit quandles Q_1 and Q_2. Let ψ be an appropriate isomorphism. Then for $i = 1, 2$, define the congruences $\approx_i$ on Q by $p \approx_i q$ if and only if $(p)\psi$ and $(q)\psi$ agree in the i'th coordinate. These provide the required congruences.

$2 \Rightarrow 3$
To show that for any such pair of congruences, $Q/\approx_2 \cong [q]_{\approx_1}$ use the isomorphism, $\phi : Q/\approx_2 \rightarrow [q]_{\approx_1}$, $\phi([p]_{\approx_2}) = [p]_{\approx_2} \cap [q]_{\approx_1}$ This is well defined

by hypothesis and a homomorphism, injective and surjective by definition of congruence. Similarly $Q/\approx_1 \cong [q]_{\approx_2}$.
Let $Q_i = Q/\approx_i$. Then we claim that $\psi : Q_1 \otimes Q_2 \to Q$ given by $\psi([q_1]_{\approx_1}, [q_2]_{\approx_2}) = [q_1]_{\approx_1} \cap [q_2]_{\approx_2}$ is an isomorphism.

$$\begin{array}{rcl}
\psi(([q_1]_{\approx_1}, [q_2]_{\approx_2})) \rhd \psi(([p_1]_{\approx_1}, [p_2]_{\approx_2})) & = & ([q_1]_{\approx_1} \cap [q_2]_{\approx_2}) \rhd ([p_1]_{\approx_1} \cap [p_2]_{\approx_2}) \\
& \in & [q_1]_{\approx_1} \rhd [p_1]_{\approx_1} \\
\text{and also} & \in & [q_2]_{\approx_2} \rhd [p_2]_{\approx_2} \\
\text{hence} & \in & ([q_1]_{\approx_1} \rhd [p_1]_{\approx_1}) \cap ([q_2]_{\approx_2} \rhd [p_2]_{\approx_2}) \\
& = & \psi(([q_1]_{\approx_1} \rhd [p_1]_{\approx_1}), ([q_2]_{\approx_2} \rhd [p_2]_{\approx_2})) \\
& = & \psi(([q_1]_{\approx_1}, [q_2]_{\approx_2}) \rhd ([p_1]_{\approx_1}, [p_2]_{\approx_2}))
\end{array}$$

Similarly for $\rhd^{-1}$. So ψ is a homomorphism. ψ is clearly injective, due to uniqueness of intersections, and surjective, as every element of Q must belong to some congruence class for each of the congruences. Hence ψ is an isomorphism.

$3 \Rightarrow 1$/ Obvious. $\square$

1.2.3 Free Sum

Let $\{Q_i\}$ be some set of pairwise disjoint quandles with associated groups A_i for each i. Then the *free sum* $\{Q_i\}$, denoted $\oplus Q_i$ is constructed as follows. Let $S = \bigcup |Q_i|$ where $|Q|$ is the underlying set of the quandle Q, and let $A = \otimes A_i$ be the free product of the associated groups. We shall identify each associated group A_i with its canonical image in A. Let $S \times A$ be the cartesian product of S and A and let $\sim$ be the relation $(q, gg') \sim (qg, g')$ for q in some Q_i, g in the corresponding $A(Q_i)$, and g' in A. That this is an equivalence relation is immediate. Then define

$$\bigoplus Q_i = \frac{S \times A}{\sim}$$

together with the quandle operations

$$(q_i, g_i) \rhd (q_j, g_j) = (q_i, g_i g_j^{-1} \hat{q}_j g_j) \text{ and } (q_i, g_i) \rhd^{-1} (q_j, g_j) = (q_i, g_i g_j^{-1} \hat{q}_j^{-1} g_j).$$

That this is well defined on equivalence classes is easily seen. That the operations obey the quandle axioms is also routine.

Definition 1.2.10 *The above construction on the quandles* $\{Q_i\}$ *is called the* free sum *of the* Q_i*.*

There are the obvious mappings ι_i from each Q_i to $\oplus Q_i$ defined by $\iota_i(q) = (q, id)$.

Proposition 1.2.11 *The free sum, together with the mappings* ι_i *has the universal properties that make it the categorical sum. That is, for any set of homomorphisms,* $f_i : Q_i \to Q$*, there is a unique* $f : \oplus Q_i \to Q$ *with the property that the following diagram commutes.*

$$\begin{tikzcd} Q_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & \oplus Q_i \arrow[d, "f"] \\ & Q \end{tikzcd}$$

Proof.

Definition of f.

For each Q_i, define $\hat{f}_i : A(Q_i) \to A(Q)$ by putting $\hat{f}_i(\hat{q}) = \widehat{f_i(q)}$ for each q in Q_i and extending linearly. To see that this is well defined, note that, by definition, each $A(Q_i)$ has as a defining set of relators, all relators of the form $\widehat{q_1\hat{q}_2} = \hat{q}_2^{-1}\hat{q}_1\hat{q}_2$. Hence we require that $\widehat{f_i(q_1\hat{q}_2)} = \widehat{f_i(q_2)}^{-1}\widehat{f_i(q_1)}\widehat{f_i(q_2)}$ in $A(Q)$. But this follows from the fact that the f_i are quandle homomorphisms and from the definition of $A(Q)$.

Note that since f_i is a quandle homomorphism, it follows that $f_i(q)\hat{f}_i(g) = f_i(q\hat{g})$ for all q in Q_i and g in $A(Q_i)$.

Now by the definition of free product of groups there is a unique homomorphism $\hat{f}$ from $A = \otimes A(Q_i)$ to $A(Q)$ such that $\hat{f}_i = \hat{f}\hat{\iota}_i$ where $\hat{\iota}_i$ is the canonical injection from $A(Q_i)$ to A mentioned above.

A typical element q of $\oplus Q_i$ is (q_0, g), where q_0 belongs to some particular Q_i. We claim that $f(q_0, g) = f_i(q_0)\hat{f}(g)$ is well defined and is the desired function f in the statement of the proposition.

The function f is well defined.

Let $(q, g_1g_2) = (qg_1, g_2)$, where q is in some Q_i and g_1 is in the corresponding $A(Q_i)$, be a typical element of $\oplus Q_i$. Then

$$f(q, g_1g_2) = f_i(q)\hat{f}_i(g_1)\hat{f}(g_2) = f_i(q)\widehat{f_i(g_1)}\hat{f}(g_2) = f_i(q\hat{g}_1)\hat{f}(g_2) = f(qg_1, g_2).$$

The proof for the operation $\lhd^{-1}$ is identical.

The function f is a quandle homomorphism.

Pick (q_1, g_1) with q_1 in Q_i and (q_2, g_2) with q_2 in Q_j. Then

$$\begin{aligned} f(q_1, g_1) \rhd f(q_2, g_2) &= (f_i(q_1)\hat{f}(g_1)) \rhd (f_j(q_2)\hat{f}(g_2)) \\ &= f_i(q_1)\hat{f}(g_1)\hat{f}(g_2)^{-1}\hat{f}(q_2)\hat{f}(g_2) \\ &= f(q_1, g_1 g_2^{-1}\hat{q}_2 g_2) \\ &= f((q_1, g_1) \rhd (q_2, g_2)). \end{aligned}$$

That this makes the above diagram commute is immediate. Since the elements (q, id) generate $\oplus Q_i$ as a quandle, f must be unique with this property. $\square$.

Proposition 1.2.12 *For some set of quandles $\{Q_i\}$, we have the following for the associated group of the free sum.*

$$A(\oplus Q_i) = \otimes A(Q_i)$$

Proof.

Since A is a functor, we may apply it to the diagram of the above proposition 1.2.11 to obtain the following commuting diagram of groups.

$$\begin{tikzcd} A(Q_i) \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & A(\oplus Q_i) \arrow[d, "f"] \\ & A(Q) \end{tikzcd}$$

It is well known that the unique group with this property for all $A(Q)$ is $\otimes A(Q_i)$. $\square$

Of course $A = \otimes A(Q_i)$ is just the second part of the cartesian product, $S \times A$ which defines the free sum in the first place, and the the action of $A(\oplus Q_i)$ on $\oplus Q_i$ is simply given by $(q, g).g' = (q, gg')$.

A special case occurs when each quandle in the set is a unit quandle, that is, if it consists of just one element.

Definition 1.2.13 *Let $S = \{Q_i\}$ be a set of unit quandles, $Q_i = \{q_i\}$. Then $FQ(S) = \bigoplus Q_i$ is called the* free quandle *on S.*

The associated group of a unit quandle $Q = \{q\}$ is just the free group on the generator $\hat{q}$, and so $A(FQ(S))$, the associated group on the free quandle, is $FG(S)$ the free group on the set $\{\hat{q}_i\}$. The free quandle will be used later in the chapter on quandle presentations.

From now on we will forget the brackets, and write (q, g) as qg. The equivalence relation ensures that there is no danger of ambiguity. If g is the identity then it will usually be dropped. In the next chapter the 'bracket notation' will be recycled in the special case of free quandles and used to mean just an ordered pair (q, g) with $q = |Q_i| = q_i$ for some Q_i and g in $FG(S)$.

Chapter 2

Finitely Presented Quandles

One class of quandles are the *finitely presented* or *fp-quandles*. These are of interest, not least because they include knot and link quandles. They have been treated elsewhere, by V.Manturov in [18] (pg 50 - 51). In [8] R.Fenn and C.Rourke give an elegant account of rack presentations. These are remarkably similar to quandle presentations, and most of the first section below is a more or less direct copy of their work, with the word 'rack' replaced by the word 'quandle' throughout. The one small exception is the characterisation of equivalence expressed in Definition 2.1.4 Proposition 2.1.5, which is my own, albeit only one small step from the original. Subsequent sections are my own work unless noted.

What are called presentations here, are called primary presentations by Fenn and Rourke. They go on to give several layers of presentation. These can equally well be applied to quandles, and for the sake of completeness they will be briefly recapped here after the basic definitions have been given, although they will not be mentioned again in this thesis. This is again lifted straight from Fenn and Rourke's work.

2.1 Definitions.

Definition 2.1.1 *A* presentation *for a quandle consists of two sets, a set S of symbols (the generating set) and a set R (the set of relators). A typical element of R is an ordered pair (x, y) where x, y are elements of $FQ(S)$ the free quandle on the set S, which we shall usually write as an equation: $x = y$ or $(x = y)$.*

The presentation defines a quandle $\langle S|R\rangle$ as follows: Define the congruence $\approx$ on $FQ(S)$ to be the smallest congruence containing R (i.e. such that

$x \approx y$ whenever $(x = y)$ is an element of R). Then

$$\langle S|R\rangle = \frac{FQ(S)}{\approx}$$

We will frequently identify a quandle presentation with the quandle it defines when this is unlikely to cause confusion.

We can describe $\approx$ more constructively as follows.

Proposition 2.1.2 *Given a set T of relators, consider the following set of* relator moves*, methods for generating new relators.*

1. *Add a* trivial *relator $qg = qg$ to T for some qg in $FQ(S)$.*
2. *If $(q_1g_1 = q_2g_2)$ is in T then add the relator $(q_2g_2 = q_1g_1)$.*
3. *If $(q_1g_1 = q_2g_2)$ and $(q_2g_2 = q_3g_3)$ are in T then add $(q_1g_1 = q_3g_3)$.*
4. *If $(q_1g_1 = q_2g_2)$ is in T then add $(q_1g_1g = q_2g_2g)$ for some element g of A.*
5. *If $(q_1g_1 = q_2g_2)$ is in T then add $(qgg_1^{-1}\bar{q}_1g_1 = qgg_2^{-1}\bar{q}_2g_2)$ for some qg in $FQ(S)$.*

For any set T of relators, let T^+ be the set T together with all extra relators which can be obtained by applying one of the above moves to T. Let $R_0 = R$, the set of defining relators of the presentation, and let $R_{i+1} = R_i^+$. Define the set of consequences *of R to be the union of all such statements. $\langle R\rangle = \bigcup_{i=0}^{\infty} R_i$. Then $x \approx y$ in $FQ(S)$ precisely when $x = y$ is a consequence of R.*

Proof.

The first three of the above moves say precisely that '=' is an equivalence relation, whilst the fourth and fifth say precisely that it respects the quandle operations. The result follows. $\square$

This definition shows how to generate relations in $\langle S|R\rangle$. We shall give a third definition which comes from a slightly different angle. The elements of $\langle S|R\rangle$ are equivalence classes of elements of $FQ(S)$, which in turn are equivalence classes of ordered pairs (q, g) with q taken from S and g taken from $FG(S)$. In practice an element of $\langle S|R\rangle$ is written down as a representative $qg = (q, g)$, and we are interested in telling when two such pairs represent the same element of $\langle S|R\rangle$. It is easy to compare two such pairs, they are identical precisely when their elements are equal. The first element is just an element of S and these can be told apart by direct inspection. The second element is an element of the free group on S, and to tell if these are equal it is only necessary to freely reduce, and then inspect them.

After a definition formalising the language, we give a definition which takes a step in the right direction. This will lead ultimately to an algorithm to give, for a given $q = [(q, id)]$ in $Q = \langle S|R\rangle$, a generating set for the stabilizer subgroup of q in the associated group $A(Q)$.

Definition 2.1.3 *For some quandle presentation $Q = \langle S|R\rangle$, a* pair *for Q is an ordered pair (q, g) with $q \in S$ and $g \in FG(S)$. For such a (q, g), q is called the* primary part *of the pair, and g is called the* secondary part *of the pair. The set of all such pairs will be denoted $PFQ(S)$. The pair (q, g) is said to represent the element qg of Q.*

Definition 2.1.4 *Given a quandle presentation $Q = \langle S|R\rangle$, consider the following word moves for changing a pair (q, g).*

1. *$(q, g) \leftrightarrow (q, \hat{q}g)$, or $(q, \hat{q}g) \leftrightarrow (q, g)$, for any q in S and g in $FG(S)$.*
2. *If $q_1g_1 = q_2g_2$ is a defining relator in R, then $(q_1, g_1g) \leftrightarrow (q_2, g_2g)$ or $(q_2, g_2g) \leftrightarrow (q_1, g_1g)$, for any g in $FG(S)$.*
3. *If $q_1g_1 = q_2g_2$ is a defining relator in R, then $(q, gg_1^{-1}\hat{q}_1g_1g') \leftrightarrow (q, gg_2^{-1}\hat{q}_2g_2g')$ or $(q, gg_2^{-1}\hat{q}_2g_2g') \leftrightarrow (q, gg_1^{-1}\hat{q}_1g_1g')$ for any (q, g) in $PFQ(S)$, and any g' in $FG(S)$.*
4. *If $q_1g_1 = q_2g_2$ is a defining relator in R, then $(q, gg_1^{-1}\hat{q}_1^{-1}g_1g') \leftrightarrow (q, gg_2^{-1}\hat{q}_2^{-1}g_2g')$ or $(q, gg_2^{-1}\hat{q}_2^{-1}g_2g') \leftrightarrow (q, gg_1^{-1}\hat{q}_1^{-1}g_1g')$ for any (q, g) in $PFQ(S)$ and any g' in $FG(S)$.*

Then for (q, g) and (q', g') in $PFQ(S)$, define the relation $(q, g) \sim (q', g')$ if and only if there is a finite sequence of elements of $PFQ(S)$, $(q_1, g_1), \cdots, (q_n, g_n)$ such that $(q_1, g_1) = (q, g)$, $(q_n, g_n) = (q', g')$ and $(q_i, g_i) \leftrightarrow (q_{i+1}, g_{i+1})$ for each (q_i, g_i) in the sequence. Such a sequence will be called a connecting sequence *for (q, g) and (q', g').*

Proposition 2.1.5 *For a given quandle presentation $Q = \langle S|R\rangle$, and for two pairs for Q, (q, g) and (q', g'), $(q, g) \sim (q', g')$ if and only if $qg \approx q'g'$. That is, two pairs for Q, (q, g) and (q', g') represent the same element of Q if and only if there exists a connecting sequence for them.*

Before we give a proof of this proposition, we shall contrast it with the proposition before, by giving a worked example. Let Q be the quandle presentation

$$Q = \langle q_1, q_2, q_3, q_4 | q_1 = q_4\hat{q}_2, q_2 = q_1\hat{q}_3^{-1}, q_3 = q_2\hat{q}_4, q_4 = q_3\hat{q}_1^{-1}\rangle,$$

a presentation for the fundamental quandle of the figure 8 knot (see appendix). Then Proposition 2.1.2 can be used to generate the following consequences:

$q_1\hat{q}_3 = q_4\hat{q}_2\hat{q}_3$ – Use the fourth relator move and first defining relator.
$q_1\hat{q}_3 = q_1\hat{q}_4^{-1}\hat{q}_2\hat{q}_4$ – Fifth relator move, third defining relator.
$q_1\hat{q}_4^{-1}\hat{q}_2\hat{q}_4 = q_1\hat{q}_3$ – Second relator move, the previous consequence.
$q_1\hat{q}_4^{-1}\hat{q}_2\hat{q}_4 = q_4\hat{q}_2\hat{q}_3$ – Third relator move, previously generated consequences.
$q_4\hat{q}_2^{-1}\hat{q}_4\hat{q}_1\hat{q}_4^{-1}\hat{q}_2\hat{q}_4 = q_4\hat{q}_3^{-1}\hat{q}_2^{-1}\hat{q}_4\hat{q}_2\hat{q}_3$ – Fifth relator move, previous consequence.

Whereas Proposition 2.1.5 could be used thus:

$(q_4, \hat{q}_2^{-1}\hat{q}_4\hat{q}_1\hat{q}_4^{-1}\hat{q}_2\hat{q}_4)$	$\leftrightarrow$	$(q_4, \hat{q}_2^{-1}\hat{q}_4\hat{q}_1\hat{q}_3)$	Using the third word move and the third defining relator.
	$=$	$(q_4, \hat{q}_4\hat{q}_4^{-1}\hat{q}_2^{-1}\hat{q}_4\hat{q}_1\hat{q}_3)$	
	$\leftrightarrow$	$(q_4, \hat{q}_4^{-1}\hat{q}_2^{-1}\hat{q}_4\hat{q}_1\hat{q}_3)$	Using the first move.
	$\leftrightarrow$	$(q_4, \hat{q}_3^{-1}\hat{q}_1\hat{q}_3)$	Fourth move, third relator.
	$\leftrightarrow$	$(q_4, \hat{q}_3^{-1}\hat{q}_2^{-1}\hat{q}_4\hat{q}_2\hat{q}_3)$	Third move, first relator.

Therefore $q_4\hat{q}_2^{-1}\hat{q}_4\hat{q}_1\hat{q}_4^{-1}\hat{q}_2\hat{q}_4 = q_4\hat{q}_3^{-1}\hat{q}_2^{-1}\hat{q}_4\hat{q}_2\hat{q}_3$ in Q.

Notice that in the second case there is a straight chain of equivalences. At any stage, only relators actually in the definition of the quandle are used, whereas in the first example we had to look back to use an old generated consequence. This means in particular that inductive methods are more easily applied. Inductive methods can be used, and indeed have been used, see for example [8, Lemma 4.8]. However this only works in one direction. Induction has been used in this case to show that the desired equivalence can be reached via a finite sequence of consequences, and hence, is in fact a consequence itself. It does not assume that there is necessarily a finite sequence of such consequences. In short, the original formulation can be used to show that a finite sequence of consequences leads to a consequence. It does not say that any consequence can be reached via such a sequence.

Now for the proof of the proposition.

Proof.

It is clear that $\sim$ is an equivalence relation, since each word move is its own converse. We must show that $\sim$ is the equivalence relation $\approx$.

$(q, g) \sim (q', g')$ **implies that** $qg = q'g'$ **is a consequence of** R**.**

Since $\approx$ is an equivalence, it suffices to show that if $(q, g) \sim (q', g')$ via a connecting sequence of length just one, that is to say, if $(q, g) \leftrightarrow (q', g')$ by one of the above word moves then $(qg = q'g')$ is a consequence of R. We go through the word moves one at a time. Since the word moves come in pairs which are opposites of each other, we need only show that one of each pair yields a consequence, and then the other side follows by relator move two.

The relator $qg = q\hat{q}g$ *is a consequence of* R*.*

$qg = q\hat{q}g = q\hat{q}g$ in $FQ(S)$, so this is just a trivial relator.

If $(q_1, g_1) = (q_2, g_2)$ *is in* R*, then* $q_1g_1g = q_2g_2g$ *is a consequence.*

This is immediate from the fourth relator move.

If $q_1g_1 = q_2g_2$ *is in* R*, then* $qgg_1^{-1}\hat{q}_1g_1g' \leftrightarrow qgg_2^{-1}\hat{q}_2g_2g'$ *is a consequence of* R*.*

This is just an application of the fifth relator move, and then finishing off with the fourth relator move.

If $q_1g_1 = q_2g_2$ *is a defining relator in* R*, then* $qg_1'g_1^{-1}\hat{q}_1^{-1}g_1g_2' = qg_1'g_2^{-1}\hat{q}_2^{-1}g_2g_2'$ *is a consequence of* R*.*

Start with the consequence $qg_1'g_1^{-1}\hat{q}_1^{-1}g_1 = qg_1'g_1^{-1}\hat{q}_1^{-1}g_1$. Then use the fifth relator move to obtain the consequence $qg_1'g_1^{-1}\hat{q}_1^{-1}g_1g_2^{-1}\hat{q}_2g_2 = qg_1'g_1^{-1}\hat{q}_1^{-1}g_1g_1^{-1}\hat{q}_1g_1$. Finally use the fourth relator move with $g = g_2^{-1}\hat{q}_2^{-1}g_2g_2'$ to see that $qg_1'g_1^{-1}\hat{q}_1^{-1}g_1g_2' = qg_1'g_2^{-1}\hat{q}_2^{-1}g_2g_2'$ is a consequence of R.

This covers all word moves so we have shown that $qg \sim q'g'$ implies that $qg = q'g'$ a consequence of R.

If $qg = q'g'$ **is a consequence of** R **then** $(q, g) \sim (q', g')$**.**

In this direction, instead of using Proposition 2.1.2, we shall use the original definition of $\approx$. That is, we shall show that the relation $\sim$ is a congruence, and so contains the relation $\approx$.

That $\sim$ is an equivalence is immediate. It is also a congruence. The second word move guarantees that $\sim$ respects an equivalence in the first argument, and the third and fourth move guarantee that $\sim$ respects an equivalence in the second argument. $\square$

As mentioned before, these definitions correspond to Fenn and Rourke's concept of primary presentation. They go on to give two more types of rack presentation, both of which could be applied to quandles. We will briefly mention them, although they will not be used subsequently.

First of all, they introduce *operator relations.* These are a set of relations that hold on the group $A = A(Q)$. That is, a set of relators $R_O = \{g_i = 0\}$ in A such that the relation $qg'g_i = qg'$ will (by definition) hold for all qg' in the defined quandle Q. A presentation in this definition looks like $\langle S|R_P; R_O\rangle$ where the set S is the set of generators, the same as in the primary presentation, the set R_P is the set of primary relators, and is just the set R of a primary presentation, and the set R_O is the new set of operator relations. They then proceed to show that any operator relation can be replaced with $|S|$ primary relators and so this concept is in fact, for finite presentations, no more general than primary presentations, although they may allow such presentations to be written in a more elegant and economical way. The proof carries over to the quandle case in its entirety and is short enough that it is worth including here.

Lemma 2.1.6 *An operator relator is equivalent to n primary relators where $n = |S|$.*

Proof.

For $S = \{q_1, \cdots, q_n\}$ we show that the operator relation $g \equiv 1$ is equivalent to the n primary relators, $q_i g = q_i$ for each q_i in S. Since $g \equiv 1$ includes each of these as a special case, we need only prove the converse.

Pick an element qg' from $FQ(S)$. Then $g' = \hat{q}_1^{\pm 1} \cdots \hat{q}_m^{\pm 1}$ for some $q_1, \cdots, q_m$ in S. Use induction on m.

Using the fact that $q_m g = q_m$ is a primary relator, and since $q\hat{q}_1^{\pm 1} \cdots \hat{q}_{m-1}^{\pm 1} g = q\hat{q}_1^{\pm 1} \cdots \hat{q}_{m-1}^{\pm 1}$ by induction hypothesis, we have that by an application of relator moves five and then four

$$\begin{aligned}
(q\hat{q}_1^{\pm 1} \cdots \hat{q}_{m-1}^{\pm 1}\hat{q}_m^{\pm 1})g &\equiv (q\hat{q}_1^{\pm 1} \cdots \hat{q}_{m-1}^{\pm 1} g)(g^{-1}\hat{q}_m g) \\
&= (q\hat{q}_1^{\pm 1} \cdots \hat{q}_{m-1}^{\pm 1} g)(\widehat{q_m g}) \\
&= q\hat{q}_1^{\pm 1} \cdots \hat{q}_m^{\pm 1}
\end{aligned}$$

and we are done. $\square$

There is a last species of presentation that is introduced by Fenn and Rourke, which they call *general presentations.* These have not only operator relations, but whole new operator generators also. A presentation of this sort looks like $\langle S_P; S_O | R_P; R_O \rangle$. Elements of this quandle are again equivalence classes of objects of the form (q, g) but now these are from the cartesian product $S_P \times FG(S_P \cup S_O)$ rather than from $S_P \times FG(S_P)$. This is a genuine generalisation, the example given in [8] works equally well for quandles. Take the general presentation $Q = \langle q; u|; \rangle$, then Q is the set of elements of the form qg with $g \in FG(\hat{q}, \hat{u})$, with the equivalence $qg = q\hat{q}^n g$ and the quandle structure $q_1 g_1 \widehat{q_2 g_2} = q_1 g_1 g_2^{-1} \hat{q}_2 g_2$. Then for $q_1 g_1$ to be in the same orbit as $q_2 g_2$ requires that g_1 and g_2 have the same total degree in u, hence there are infinitely many orbits and so Q cannot be written down as a (finite) primary presentation.

We take one last item from Fenn and Rourke, a 'Tietze Theorem' for quandle presentations.

Theorem 2.1.7 *Tietze Theorem*
Consider the following moves we can apply to a quandle presentation $\langle S|R \rangle$.

1. *Add to R a consequence of the relations in R.*
2. *Delete from R a relation which is a consequence of the remaining relations.*
3. *Introduce to R a new generator x and a new relation $x = q\bar{g}$ where x does not appear in S, or delete such a pair, if x appears nowhere else in the relations.*

These moves are called Tietze Moves.

Two presentations, $Q = \langle S|R \rangle$ and $Q' = \langle S'|R' \rangle$ are isomorphic if and only if they are related by a finite sequence of Tietze moves.

2.2 Associated Groups, Orbits, Normal Forms, and Stabilizers

Given a finitely presented quandle Q, it is very easy to write down a group presentation for the associated group of Q, and the orbits are also very easy to find. Furthermore, for any given element q_0 of Q, it is easy to rewrite the presentation in a form where a generating set for the stabilizer of q_0 in $A(Q)$ or $Op(Q)$ can be read off, a normal form for q_0. Here are the details

2.2.1 Associated Groups of Finitely Presented Quandles

For a given finitely presented quandle $Q = \langle S|R \rangle$ what is the associated group of Q, $A(Q)$? Recall that the function A taking a quandle to its associated group is functorial. The quandle Q is, by definition, $FQ(S)$ factored out by the congruence generated by the relations R. Hence $A(Q)$ will be $A(FQ(S)) = FG(S)$ factored out by the normal subgroup of $FG(S)$ generated by the image of R in $FG(S)$. Of course the image of an element $q_1 g$ of Q in $A(Q)$ is simply $g^{-1}\hat{q}g$.

Example.

As an example, consider the presentation of the fundamental quandle of the figure 8 knot considered earlier.

$$Q = \langle q_1, q_2, q_3, q_4 | q_1 = q_4\hat{q}_2, q_2 = q_1\hat{q}_3^{-1}, q_3 = q_2\hat{q}_4, q_4 = q_3\hat{q}_1^{-1} \rangle,$$

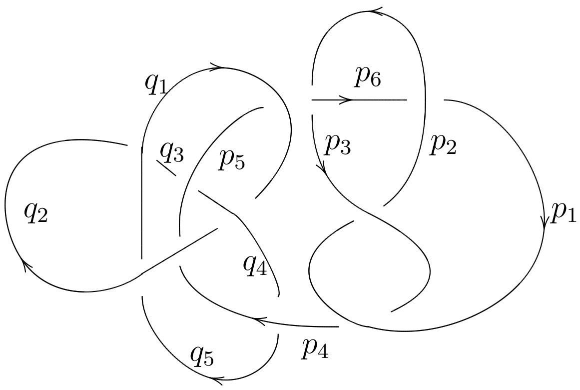
Then this has an associated group with the group presentation

$$Q = \langle \hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_4 | \hat{q}_1 = \hat{q}_2^{-1}\hat{q}_4\hat{q}_2, \hat{q}_2 = \hat{q}_3\hat{q}_1\hat{q}_3^{-1}, \hat{q}_3 = \hat{q}_4^{-1}\hat{q}_2\hat{q}_4, \hat{q}_4 = \hat{q}_1\hat{q}_3\hat{q}_1^{-1} \rangle.$$

2.2.2 Orbits

The orbit congruence of a finitely presented quandle $Q = \langle S|R \rangle$ is easy to calculate from a presentation. Any element qg of Q will be put in the same equivalence class as its primary element q, so we need only look at which generators are made equivalent, hence for each relator of the form $q_1 = q_2 g$ we can read $q_1 \doteq q_2$. The resulting classes of generators will each represent one element in $\bar{Q}$, and the orbits are, for each such class of generators $q_1 \cdots q_n$, the set of elements of the form $q_i g$, $g \in A(Q)$. Notice that two generators in A, q_1, q_2 belong to the same orbit of Q if and only if there is some sequence of generators $p_1, \cdots p_n$ such that $p_1 = q_1, p_n = q_2$ and for each $1 \leq i \leq n-1$ there is some relator of the form $p_i g_1 = p_{i+1} g_2$. Furthermore, such a sequence can always be chosen such that $p_i \neq p_j$ $\forall i \neq j$. We use as an example the quandle arising from the link pictured. This will be used as a running example from now on.

Example.



It will be shown on page 7 that this link has a fundamental quandle with the following presentation.

$Q = \langle q_1, q_2, q_3, q_4, q_5, p_1, p_2, p_3, p_4, p_5, p_6 | q_1 = q_5\hat{q}_2, q_2 = q_1\hat{q}_4, q_3 = q_2\hat{q}_1, q_4 = q_3\hat{p}_5,$

$q_5 = q_4\hat{p}_4^{-1}, p_1 = p_6\hat{p}_2, p_2 = p_1\hat{p}_3^{-1}, p_3 = p_2\hat{p}_6, p_4 = p_3\hat{p}_1^{-1}, p_5 = p_4\hat{q}_2^{-1}, p_6 = p_5\hat{q}_1^{-1}\rangle$

Then, denoting the orbit equivalence class containing q_i by $[q_i]$,

$$\begin{aligned}
\hat{Q} \;&=\; \langle [q_1], [q_2], [q_3], [q_4], [q_5], [p_1], [p_2], [p_3], [p_4], [p_5], [p_6] | [q_1] = [q_5], [q_2] = [q_1],\\
&\quad [q_3] = [q_2], [q_4] = [q_3], [q_5] = [q_4], [p_1] = [p_6], [p_2] = [p_1], [p_3] = [p_2],\\
&\quad [p_4] = [p_3], [p_5] = [p_4], [p_6] = [p_5] \ ; \ [q]\hat{g} = [q] \ \ \forall [q] \in \bar{Q}, \ g \in A(\bar{Q})\rangle\\
&=\; \langle [q_1], [p_1] | [q_1] \rhd [p_1] = [q_1], [p_1] \rhd [q_1] = [p_1]\rangle
\end{aligned}$$

the two element quandle, and so Q has two orbits, the set of elements of the form $\{q_i\hat{g}\}$ and the set of elements of the form $\{p_i\hat{g}\}$. Notice that these correspond to the two components of the link from which the quandle arose.

2.2.3 Normal Forms

We will now define normal forms. These are useful for finding generating sets for stabilizer subgroups. After defining them we will show some basic facts about them. In the next subsection it will be shown how to read off a generating set for a stabilizer subgroup from them.

Definition 2.2.1 *Given a quandle presentation $P = \langle S|R\rangle$ for a quandle Q, and a generator $q_0 \in S$, letting O_{q_0} be the orbit of Q containing q_0, then P is in* normal form for q_0 *if and only if*

1. *All defining relators in P having primary elements in O_{q_0} have the form $q_i = q_0 g$ where $q_i \in S$ and $g \in A(FQ(S)) = FG(S)$.*

2. *Every generator of P in O_{q_0} other than q_0 appears as the primary element of the left hand side of at most one, and hence precisely one, defining relator.*

A quandle presentation P is in completely normal form (for $q_0, q_1, \cdots, q_n \in A$) *if each orbit of P contains an element q_i such that P is in normal form for q_i.*

Lemma 2.2.2 *Given a quandle presentation $P = \langle S|R\rangle$ of a quandle Q and $q_0 \in S$ there is a presentation P' of Q which is in normal form for q_0.*

Suppose we wish to find a normal form of P for some q in Q which is not in S, that is, which is not a generator, $q = q_i g$ say. Then by the third Tietze move we can simply add a new generator q_0 and a relation $q_0 = q_i g$, enabling us to find a presentation in normal form for $q = q_0$.

Proof.

Replace each relator in R with primary elements in O_{q_0} with a relator of the form $q_i = q_0 g$.

Let O_{q_0} be the orbit of Q containing q_0. The first step is to replace the set R with an equivalent set R' such that each relator in R' with primary elements in O_{q_0} is of the form $q_i = q_0 g$. To this end let T be the set of defining relators in R that have primary elements in O_{q_0}.

Each r in T is of the form $q_i g_i = q_j g_j$ with two primary elements, q_i and q_j, both of which are contained in O_{q_0}. If either (or both) of them is q_0 then it is a simple matter to put that side on the right, and shuffle all operators to the right. That is, replace $q_0 g = q_i g'$ with $q_i = q_0 g {g'}^{-1}$ using each of the first two Tietze moves.

So suppose that r has two primary elements $r \equiv (q_1 g = q_2 g')$ possibly equal, but neither q_0. Then since q_2 is in the same orbit as q_0 there is a sequence of defining relators

$$q_0 g_0 \equiv p_0 g_0 = p_1 g_1',\ p_1 g_1 = p_2 g_2', \cdots, p_{n-1} g_{n-1} = p_n g_n' \equiv q_2 g_n'$$

with $p_0 = q_0$ and $p_n = q_2$, $p_i \neq q_2, i < n$. If $p_j = q_1$ for some j then for the first such j take the sequence $p_0 g_0 = p_1 g_1', p_1 g_1 = p_2 g_2', \cdots, p_{j-1} g_{j-1} = p_j g_j' \equiv q_1 g_j'$ where none of the p_i are q_2, and so there exists a sequence of relators connecting one of q_1, q_2 to q_0 which does not contain the relator r. For simplicity assume q_2.

From this we obtain the consequence $q_2 = q_0 g''$ where $g'' = g_0 {g'}_1^{-1} g_1 {g'}_2^{-1} \cdots g_{n-1} {g'}_n^{-1}$. Now, returning to the original relator r, we can use this to give the consequence $(q_1 g = q_0 g'' g')$ and then $(q_1 = q_0 g'' g' g^{-1})$. Add this relator to the set R using the first Tietze move. Conversely, the original relator $r \equiv q_1 g = q_2 g'$ is a consequence of the new relator and the sequence of relators used in its construction. Since the relator r is not in the given sequence, it can be deleted from the set R using the second Tietze move. We have replaced the relator r with a relator of the desired form.

Go through each relator in T in this fashion, replacing each one with one of the form $q_i = q_0 g$, and we have obtained a presentation $P' = \langle T | R' \rangle$ of Q in which every relator with primary element in O_{q_0} is of the form $q_i = q_0 g$.

Ensure that no two defining relator $q_i = q_0 g_1$ and $q_i = q_0 g_2$ have the same left hand side for $q_i \neq q_0$.

Now in the set of defining relators in R' that have primary elements in O_{q_0}, we may have q_i appearing as the primary element of the left hand side of more than one relator, $q_i \neq q_0$ and $q_i = q_0 g_1$, $q_i = q_0 g_2$ say. In this case replace the second relator with $q_0 g_1 = q_0 g_2$ and then replace this with $q_0 = q_0 g_1 g_2^{-1}$ using the first and second Tietze moves. $\square$

We will illustrate this by putting the quandle in the previous example into normal form for q_1. The quandle has a presentation

$$Q = \langle q_1, q_2, q_3, q_4, q_5, p_1, p_2, p_3, p_4, p_5, p_6 | q_1 = q_5 \hat{q}_2, q_2 = q_1 \hat{q}_4, q_3 = q_2 \hat{q}_1,$$

$$q_4 = q_3 \hat{p}_5, q_5 = q_4 \hat{p}_4^{-1}, p_1 = p_6 \hat{p}_2, p_2 = p_1 \hat{p}_3^{-1}, p_3 = p_2 \hat{p}_6, p_4 = p_3 \hat{p}_1^{-1},$$
$$p_5 = p_4 \hat{q}_2^{-1}, p_6 = p_5 \hat{q}_1^{-1} \rangle$$

The relators for the orbit containing q_1 are those with primary elements q_i. These must first be rearranged to be of the form $q_i = q_1 g$ for some g in $FG(S)$. The first relator, $q_1 = q_5 \hat{q}_2$ may be rearranged to give $q_5 = q_1 \hat{q}_2^{-1}$. The second relator is already in the required form. The third relator, $q_3 = q_2 \hat{q}_1$ may be replaced with $q_3 = q_1 \hat{q}_4 \hat{q}_1$, using the second relator. This in turn allows us to replace the fourth relator, $q_4 = q_3 \hat{p}_5$, with $q_4 = q_1 \hat{q}_4 \hat{q}_1 \hat{p}_5$, and finally the fifth generator can be replaced using the (new) first generator to give $q_1 \hat{q}_2^{-1} = q_4 \hat{p}_4^{-1}$, and hence $q_4 = q_1 \hat{q}_2^{-1} \hat{p}_4$. We now have the presentation

$$Q = \langle q_1, q_2, q_3, q_4, q_5, p_1, p_2, p_3, p_4, p_5, p_6 | q_5 = q_1 \hat{q}_2^{-1}, q_2 = q_1 \hat{q}_4, q_3 = q_1 \hat{q}_4 \hat{q}_1,$$

$$q_4 = q_1 \hat{q}_4 \hat{q}_1 \hat{p}_5, q_4 = q_1 \hat{q}_2^{-1} \hat{p}_4, p_1 = p_6 \hat{p}_2, p_2 = p_1 \hat{p}_3^{-1}, p_3 = p_2 \hat{p}_6,$$
$$p_4 = p_3 \hat{p}_1^{-1}, p_5 = p_4 \hat{q}_2^{-1}, p_6 = p_5 \hat{q}_1^{-1} \rangle$$

This is still not in normal form because there are two relators of the form $q_4 = q_1 g$. Use the second of these to replace the first with $q_1 \hat{q}_2^{-1} \hat{p}_4 = q_1 \hat{q}_4 \hat{q}_1 \hat{p}_5$ and then $q_1 = q_1 \hat{q}_4 \hat{q}_1 \hat{p}_5 \hat{p}_4^{-1} \hat{q}_2$. We now have the presentation

$$Q = \langle q_1, q_2, q_3, q_4, q_5, p_1, p_2, p_3, p_4, p_5, p_6 | q_5 = q_1 \hat{q}_2^{-1}, q_2 = q_1 \hat{q}_4, q_3 = q_1 \hat{q}_4 \hat{q}_1,$$
$$q_4 = q_1 \hat{q}_2^{-1} \hat{p}_4, q_1 = q_1 \hat{q}_4 \hat{q}_1 \hat{p}_5 \hat{p}_4^{-1} \hat{q}_2, p_1 = p_6 \hat{p}_2, p_2 = p_1 \hat{p}_3^{-1}, p_3 = p_2 \hat{p}_6,$$
$$p_4 = p_3 \hat{p}_1^{-1}, p_5 = p_4 \hat{q}_2^{-1}, p_6 = p_5 \hat{q}_1^{-1} \rangle \tag{2.1}$$

This is now in normal form for q_1.

From the relator $q_1 = q_1 \hat{q}_4 \hat{q}_1 \hat{p}_5 \hat{p}_4^{-1} \hat{q}_2$ we can see that $\hat{q}_4 \hat{q}_1 \hat{p}_5 \hat{p}_4^{-1} \hat{q}_2$ is in the stabilizer subgroup of q_1, also $\hat{q}_1$ is in the stabilizer subgroup of q_1 by the definition of quandle. The theorem in the next section will show that in this example, these generate the whole of this stabilizer subgroup.

The next lemma is an adaption of Proposition 2.1.5, and again gives, for two pairs (q_0, g_0) and (q_0, g_0) which represent the same element in a quandle presentation in normal form, a connecting sequence of pairs, (q, g_i) such that consecutive pairs are related by the word moves given in Definition 2.1.4. The key difference is that this time the primary part of each pair in the connecting sequence is the same for all pairs.

This lemma only works for presentations in normal form, which is why this form has been introduced in the first place.

Lemma 2.2.3 *Let $Q = \langle S|R \rangle$ be a quandle presentation in normal form for some generator q_0. Suppose (q_0, g_0) and (q_0, g_0') represent the same element of Q. Then there is a connecting sequence of pairs for (q_0, g_0) and (q_0, g_0'), $W_i \equiv (q_0, g_i), 1 \leq i \leq n$ such that the primary part of each term in the sequence is q_0 in all cases.*

Proof.

That there exists a connecting sequence $\{W_i\}$ from (q_0, g_0) to (q_0, g_0') is guaranteed by Proposition 2.1.5. If all primary parts of the W_i in this sequence are q_0 then we are done, so assume otherwise. The lemma will be proved by showing that in this case we can construct another such sequence containing fewer words with primary parts not q_0, then invoking induction.

Let W_m be the first word in the sequence with primary part not q_0, so the word move taking W_{m-1} to W_m must be a move of type 2 using the defining relator $q_i = q_0 g_i$ say. For the sake of clarity let $i = 1$, so

$$W_{m-1} \equiv (q_0, g_1 g) \leftrightarrow W_m \equiv (q_1, g) \leftrightarrow W_{m+1} \equiv (q_j, g') \tag{2.2}$$

We will go through each possible word move $W_m \to W_{m+1}$ and show in each case that a connecting sequence with fewer words with primary parts not q_0 can be constructed. Specifically we will show in each case a subsequence which will replace the above subsequence 2.2 with just one word with primary element not q_0.

$W_m \leftrightarrow W_{m+1}$ **via a word move of type 1.**
First sort. $(q, g) \leftrightarrow (q, \hat{q}g)$.
In this case we have that $W_{m+1} \equiv (q_1, \hat{q}_1 g)$. Consider the following sequence.

$$\begin{array}{rcll} W_{m-1} \equiv (q_0, g_1 g) & \leftrightarrow & (q_0, \hat{q}_0 g_1 g) & \text{Using a move of type 1.} \\ & = & (q_0, g_1 g_1^{-1} \hat{q}_0 g_1 g) & \\ & \leftrightarrow & (q_0, g_1 \hat{q}_1 g) & \text{Using a move of type 3.} \\ & \leftrightarrow & (q_1, \hat{q}_1 g) \equiv W_{m+1} & \text{Using a move of type 2.} \end{array}$$

This can be used to replace the subsequence 2.2 to produce a connecting sequence from (q_0, g_0) to (q_0, g_0') with fewer elements with primary part not q_0.

Second sort. $(q, \hat{q}g) \leftrightarrow (q, g)$.
In this case, the terms in the subsequence 2.2 are

$$W_{m-1} \equiv (q_0, g_1 g) \leftrightarrow W_m \equiv (q_1, g) \equiv (q_1, \hat{q}_1 g') \leftrightarrow W_{m+1} \equiv (q_1, g') \tag{2.3}$$

So $g = \hat{q}_1 g'$. We use the following sequence.

$$\begin{array}{rcll} W_{m-1} \equiv (q_0, g_1 \hat{q}_1 g') & \leftrightarrow & (q_0, g_1 g_1^{-1} \hat{q}_0 g_1 g') & \text{Using a move of type 3 with} \\ & & & \text{the relator } q_1 = q_0 g_1. \\ & = & (q_0, \hat{q}_0 g_1 g') & \\ & \leftrightarrow & (q_0, g_1 g') & \text{Using a move of type 1.} \\ & \leftrightarrow & (q_1, g') & \text{Using a move of type 2.} \end{array}$$

$W_m \leftrightarrow W_{m+1}$ **via a word move of type 2.**
The only defining relator with q_1 as a primary element of either side is the relator $q_1 = q_0 g_1$ and so $W_{m+1} \equiv W_{m-1}$. Hence we can simply delete two terms in the subsequence to leave the subsequence W_{m-1}.

$W_m \leftrightarrow W_{m+1}$ **via a word move of type 3 or 4.**
In this case the $W_{m+1} \equiv (q_1, g')$ and the last $W_m \leftrightarrow W_{m+1}$ move changes g to g'. These moves can be swapped round to give the sequence -

$$W_{m-1} \equiv (q_0, g_1 g) \leftrightarrow W_m' \equiv (q_0, g_1 g') \leftrightarrow (q_1, g')$$

which is the desired replacement subsequence with one less term with primary element not q_0.

In each case given a connecting sequence from (q_0, g_0) to (q_0, g_0') with some term with primary part not q_0 we have constructed a new connecting sequence containing fewer terms with primary part not q_0. By induction therefore, we can construct a sequence with no terms with primary part not q_0. $\square$

Corollary 2.2.4 *Let $P = \langle A|R\rangle$ be a quandle presentation in normal form for some generator q_0. Suppose (q_1, g_0) and (q_2, g_0') represent the same element of P in the orbit containing q_0. Then there is a connecting sequence $W_i \equiv (p_i, g_i), 1 \leq i \leq n$ s.t. $(p_i, g_i) \leftrightarrow (p_{i+1}, g_{i+1})$ for each $1 \leq i \leq n-1$ where $\leftrightarrow$ is the relation introduced in Definition 2.1.4, $(p_1, g_1) \equiv (q_1, g_0)$, $(p_n, g_n) \equiv (q_2, g_0')$ and $p_i = q_0$ for all $2 \leq i \leq n-1$.*

Proof.
If $q_1 = q_0$ then set $W \equiv (q_0, g_0)$. Otherwise by normality of P there is a relation of the form $q_1 = q_0 g_1 \in R$: set $W \equiv (q_0, g_1 g_0)$. Then $(q_1, g_0) \leftrightarrow W$ by a word move of type two. Similarly set $W' \equiv (q_0, g_0')$ or $(q_0, g_2 g_0')$ according to whether $q_2 = q_0$ or not.

By the above lemma there exists a connecting sequence of pairs $W \equiv W_2, \cdots, W_{n-1} \equiv W'$ from W to W'. Put $W_1 \equiv (q_1, g_0)$ and $W_n \equiv (q_2, g_0')$ and the sequence $W_i \;\; 1 \leq i \leq n$ provides the required connecting sequence from (q_1, g_0) to (q_2, g_0'). $\square$

Since two pairs (q_0, g) and (q_0, g') represent the same element of Q if and only if they can be joined by a sequence of words $q_0 w_1, \cdots, q_0 w_n$ linked by the moves listed in Definition 2.1.4, and since by the above lemma in this case a connecting sequence can be found such that only the secondary parts in each term of the sequence are affected, we can rewrite Proposition 2.1.5 for this special case so that the only moves involved leave the primary part, of a pair, q_0, unchanged. This is the content of the following corollary. In the next section this will be used to give a generating set for the stabilizer subgroups of the operator and associated groups of a quandle.

Corollary 2.2.5 *Suppose $\langle S|R\rangle$ is a quandle presentation in normal form for some element q_0. Define a relation $\leftrightarrow$ on $FG(S)$ via the following secondary moves.*

1. *$g \leftrightarrow \hat{q}_0 g$ and $\hat{q}_0 g \leftrightarrow g \;\; \forall g \in FG(S)$.*

2. *If* $q_0 = q_0 g_0$ *is a relator in* R, *then* $g \leftrightarrow g_0 g$ *and* $g_0 g \leftrightarrow g \ \ \forall g \in FG(S)$.

3. *If* $q_1 g_1 = q_2 g_2$ *is a relator in* R, *then*
   $g'_1 g_1^{-1} \hat{q}_1 g_1 g'_2 \leftrightarrow g'_1 g_2^{-1} \hat{q}_2 g_2 g'_2$,
   $g'_1 g_2^{-1} \hat{q}_2 g_2 g'_2 \leftrightarrow g'_1 g_1^{-1} \hat{q}_1 g_1 g'_2$,
   $g'_1 g_1^{-1} \hat{q}_1^{-1} g1 g'_2 \leftrightarrow g'_1 g_2^{-1} \hat{q}_2^{-1} g_2 g'_2$, *and*
   $g'_1 g_2^{-1} \hat{q}_2^{-1} g_2 g'_2 \leftrightarrow g'_1 g_1^{-1} \hat{q}_1^{-1} g_1 g'_2$.
   $\forall g'_1, g'_2 \in FG(S)$.

Define an equivalence relation, $\approx$ *on* $FG(S), g \approx g'$ *if and only if there is a finite sequence of elements of* $FG(S)$, $\{g_i\}_{i=1}^n$ *such that* $g_1 = g, g_n = g'$, *and* $g_i \leftrightarrow g_{i+1}$. *Then two pairs* (q_0, g) *and* (q_0, g') *represent the same quandle element if and only if* $g \approx g'$.

2.3 Stabilizers And Finitely Presented Quandles

Recall the following definition.

Definition 2.3.1 *For some quandle* Q, *some group* G *acting on* Q *and some element* $q \in Q$, *the stabilizer of* q *in* G *is the subgroup* $St_G(q)$ *of* G *of elements whose actions leave* q *fixed.*

$$St_G(q) = \{g \in G | qg = q\}$$

Stabilizer subgroups arise in several places in quandles. In Chapter 3 we will quote a theorem of Fenn and Rourke giving a topological interpretation of the stabilizer subgroups of associated groups of knot quandles. The above methods can be used to give a generating set for the stabilizer in $A(Q)$ or $Op(Q)$ of a particular element in Q.

Theorem 2.3.2 *Let* $Q = \langle S|R \rangle$ *be a quandle presentation with a generator* $q_0 \in S$ *such that* Q *is in normal form for* q_0. *Let* $T = \{g \in FG(S) | (q_0 = q_0 g) \in R\}$ *so* T *is the set of* g *in* $FG(S)$ *such that* $q_0 = q_0 g$ *is a defining relator of* Q. *Let* $\hat{T}$ *be the image of* T *in the associated group* $A(Q)$ *and* $\bar{T}$ *be the image of* T *in the operator group* $Op(Q)$.

Then $St_{A(Q)}(q_0) = \Sigma\{\hat{T}, \hat{q_0}\}$, *and* $St_{Op(Q)} = \Sigma\{\bar{T}, \bar{q}_0\}$. *That is, the stabilizer of* q_0 *in* $A(Q)$ *is generated by* $\hat{q}_0$ *and the elements of* $\hat{T}$, *and the stabilizer of* q_0 *in* $Op(Q)$ *is generated by* $\bar{q}_0$ *and the elements of* $\bar{T}$

Again, the restriction to generators of Q is of minor importance: if we wish to find the stabilizer of some element q, then simply pick some new symbol, q_0 say, and add this to the presentation together with a relator defining it to be q, then put into normal form for q_0.

Proof.

That $\hat{q}_0$ and the elements of $\hat{T}$ are all in $St_G(q_0)$ is obvious. We must show the converse.

Let g be some element of $St_{A(Q)}(q_0)$, and let g' be in the preimage of g in $FG(S)$ so (q_0, g') (q_0, id) represent the same element of Q. Such a g' must exist since $A(Q)$ is a quotient of $FG(S)$. Then by Corollary 2.2.5 there is some connecting sequence $\{g_i\}, i = 1 \cdots n$ of elements of $FG(S)$ with $g_1 = id$ and $g_n = g'$ such that $g_i \leftrightarrow g_{i+1}$ for each term in the sequence.

Now by consideration of Section 2.2.1, two elements of $FG(S)$, g'_1 and g'_2 are connected by a sequence consisting exclusively of moves of the third type precisely when they represent the same element of $A(Q)$. Moves of type one and two left multiply the g_i by $\hat{q}_0$ or its inverse, and by an element of T or its inverse respectively. Hence if we take the image of the connecting sequence in $A(Q)$ we find that each move multiplies g_i by an element of $\Sigma\{\hat{T}, \hat{q_0}\}$ or else leaves it unchanged. Hence $g \in \Sigma\{\hat{T}, \hat{q_0}\}$.

For the statement about the operator group, simply note that the associated group acts on the quandle Q via the operator group. $\square$

Returning to the running example, the Presentation 2.1 is in normal form and this theorem states that the stabilizer subgroup of q_1 is generated by $\hat{q}_1$ and $\hat{q}_4\hat{q}_1\hat{p}_5\hat{p}_4^{-1}\hat{q}_2$, as promised.

In the case of knot quandles, which will be looked at in the next Chapter, it is well known that from any knot diagram there can be written down a quandle presentation, similar to the Wirtinger presentation for the fundamental group, in which there is one generator for each arc, and one relator for each crossing. Since there are the same number of arcs as crossings in a (non-trivial) knot diagram there will be the same number of relations as generators in a presentation of the knot quandle. When put in normal form for some generator q_0, there will be one 'spare' relator of the form $q_0 = q_0 w$ to give a generator $\hat{g}$ for the stabilizer subgroup of q_0. This is as expected, as the stabilizer subgroup is in fact the fundamental group of a torus which is a regular neighbourhood of the knot, and so is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. When we pass to the associated group of the quandle, that is the fundamental group of the knot, this relator becomes the group relation $\hat{q}_0 = \hat{w}^{-1}\hat{q}_0\hat{w}$. However, as is well known, any one relation in the Wirtinger presentation of a knot is a consequence of the rest, and so we can treat this relation as superfluous.

This is one way in which information is lost in the passage from knot quandle to knot group.

2.4 Free Sums Of Quandle Presentations

Lemma 2.4.1 *If*

$$Q_1 \cong \langle q_1, \cdots, q_n | r_1, \cdots, r_m \rangle \text{ and } Q_2 \cong \langle p_1, \cdots, p_{n'} | s_1, \cdots, s_{m'} \rangle$$

then

$$Q_1 \oplus Q_2 \cong \tilde{Q} := \langle q_1, \cdots, q_n, p_1 \cdots, p_{n'} | r_1, \cdots, r_m, s_1, \cdots, s_{m'} \rangle$$

Proof.

It must be shown (see Lemma 1.2.3 and the following comments) that there exist projections $\iota_i : Q_i \to Q_1 \oplus Q_2$ with the property that for any quandle Q, and any pair of homomorphisms, $f_i : Q_i \to Q$, there exists a unique $g : \tilde{Q} \to Q$ with the property that the following diagram commutes.

$$\begin{tikzcd} Q_1 \arrow[r, "\iota_1"] \arrow[dr, "f_1"'] & \tilde{Q} \arrow[d, "g"] & Q_2 \arrow[l, "\iota_2"'] \arrow[dl, "f_2"] \\ & Q & \end{tikzcd}$$

Define the ι_i in the obvious way: $\iota_1(q_i) = q_i$ and $\iota_2(p_i) = p_i$. That relations in the Q_i carry over to $\tilde{Q}$ is obvious, and so these are homomorphisms. Now for given f_i define $g(q_i) = f_1(q_i)$ and $g(p_i) = f_2(p_i)$. That this satisfies the conditions is clear. $\square$

Chapter 3

Knot And Link Quandles

One of the motivations in studying quandles is their role in knot theory. It has been shown by Joyce in [12] how to construct an invariant quandle for a link, the so called fundamental quandle, and that when this is restricted to the case of knots, this is an (almost) complete invariant; The 'almost' arising from the fact that it cannot tell equivalent knots apart. That is two knots which are mirror images of each other, save with opposite orientations, yield isomorphic quandles. In [8] Fenn and Rourke have extended the construction to any manifold with a codimension 2 submanifold, and it has been shown that for a wide class of links in 3-manifolds, this is a complete invariant of both the whole manifold, and the embedding of the submanifold. However here we stick to links in $\mathbb{R}^3$, where Fenn and Rourke's result can be interpreted as showing that the fundamental quandle is, subject to the above restrictions, a complete invariant of non-split links, ie. links which are not the disjoint union of non-empty sublinks. Joyce goes on to show how to write down a presentation of the fundamental quandle from any diagram of the knot, which we will include here for the sake of completeness, as we will be using it on occasion. A list of the fundamental quandles of some knots is given in an appendix. Throughout the following we work with tame links only, so all links are ambient isotopic to a PL link made up of a finite number of straight line segments.

This chapter consists entirely of well known results, so proofs will not be given here as they would take up too much space.

3.1 The Fundamental Quandle Of A Link

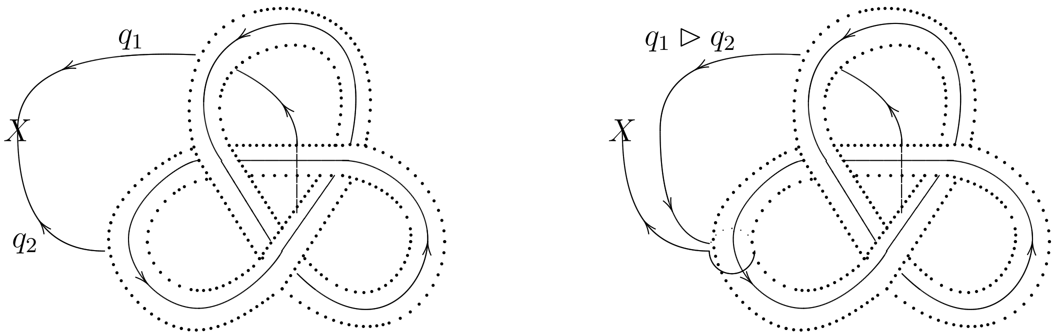
Definition 3.1.1 *Let L be an oriented link, $N(L)$ a regular neighbourhood of L, and X a fixed base point in $\mathbb{R}^3 - int(N(L))$. The* fundamental quandle *of L, $Q(L)$, is the quandle with underlying set, the set of homotopy classes of paths with initial point on $\partial(N(L))$ and end point X, where the initial point can wander freely over the whole of $\partial(N(L))$ during the course of the homotopy, and the end point is fixed.*

Each point p on $\partial(N(L))$ lies on a unique meridian circle of $N(L)$. Let m_p be the path that goes once round this meridian in a positive direction. Define the quandle operation of $Q(L)$ thus: for $[t_1], [t_2] \in Q(L)$ represented by paths t_1, t_2, and p the initial point of t_2, let

$$[t_1] \rhd [t_2] = [t_1 \circ \hat{t_2} \circ m_p \circ t_2]$$

Informally, $[t_1 \rhd t_2]$ is represented by the path that travels 'down' t_1, 'up' t_2, around the meridian of the link based at the initial point of t_2, and back 'down' t_2.

The following diagram illustrates this.



The Quandle Action In The Quandle Of The Trefoil Knot

Of course this definition depends on the basepoint X, and the regular neighbourhood $N(L)$, but these can be shown to be irrelevant, in the sense that different choices of X and $N(L)$ will produce isomorphic quandles.

Next the fundamental theorem of knot quandles.

Theorem 3.1.2 *Let L_1 and L_2 be two non-split links. Then L_1 is equivalent to L_2 if and only if $Q(L_1) \cong Q(L_2)$.*

For details of the proof see [8] p.380 onwards.

Next the relation between the fundamental quandle of a link, and its fundamental group. Again, full details can be found in [8].

Theorem 3.1.3 *Let L be a link, with fundamental quandle Q, and fundamental group $F(L)$. Then $F(L) \cong A(Q)$.*

This is an appropriate place to give a theorem shown by Fenn and Rourke in [8] p.362 giving a topological interpretation of the stabilizer subgroups of the associated groups of link quandles, ie. the fundamental groups of the links. Recall that for some $q \in Q$, the stabilizer subgroup of q in $A(Q)$ is the subgroup of that $\hat{g}$ in $A(Q)$ such that $q\hat{g} = q$.

Definition 3.1.4 *Let L be a link. Then an element of the fundamental group of L, $F(L)$, represented by a loop of the form $\hat{\alpha} \circ \gamma \circ \alpha$ where γ lies in $\partial N(M)$ and α represents an element a of $Q(L)$ is called* a-peripheral. *For fixed a, the set of a-peripheral elements of $F(L)$ forms a subgroup called the* a-peripheral *subgroup.*

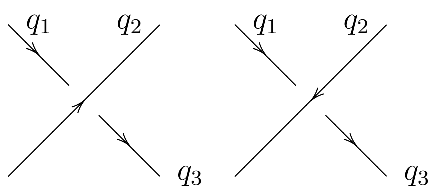
Theorem 3.1.5 *The stabilizer of a in the fundamental group, is the a-peripheral subgroup.*

3.2 Some Consequences

We next quote the theorem giving a method for writing down a presentation for the link quandle of a link given a diagram the link. For details and a proof see [12], sections 2 and 12.

In the rest of this chapter, 'arc' will be used to refer to an arc of a link diagram: the line passing from one under crossing to another, rather than a topological arc.

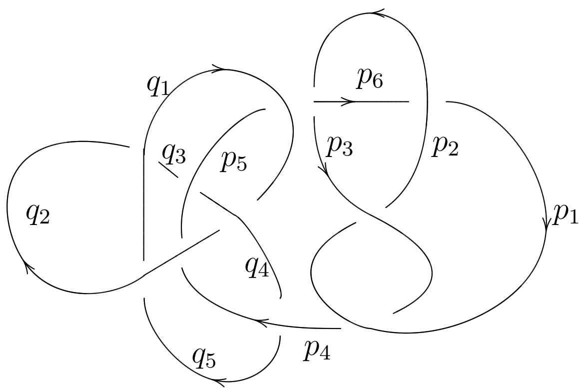
Theorem 3.2.1 *Suppose a link L has a diagram D. Label all arcs of D with distinct labels $q_1 \cdots q_n$. These form the generators of the presentation. For each crossing, write down a relation r_i according to the following rule.*



$q_3 = q_1 \triangleright q_2 \qquad q_3 = q_1 \triangleright^{-1} q_2$

Then $Q(L) \cong \langle q_1, \cdots, q_n | r_1, \cdots, r_m \rangle$

For an informal explanation of this, note that a diagram of a link is a projection of the link onto $\mathbb{R}^2$ such that the only singularities are transverse double points. If a base point is picked that projects to a point X_0 far away from the projection of the link, then a representative for a quandle element can be picked such that its projection only crosses the link projection transversely. Then one of the given set of generators is represented by a path going from the labeled arc, over all other arcs to the point X_0. A general element of the quandle $q_i\hat{q}_{j_1}^{\pm 1}\cdots\hat{q}_{j_n}^{\pm 1}$ is represented by a path starting at the arc labeled by q_i and successively going under the arcs labeled by the q_{j_k} in a direction indicated by the ± 1 until finally arriving at X_0. The relations then represent relations that hold between paths starting at successive arcs in the diagram.
We can now see that the quandle presentation given for the running example is indeed the quandle of the following link.



Then it is routine to check that a presentation for the fundamental quandle if this link is given by the following.

$$Q = \langle q_1, q_2, q_3, q_4, q_5, p_1, p_2, p_3, p_4, p_5, p_6 | q_1 = q_5\hat{q}_2, q_2 = q_1\hat{q}_4, q_3 = q_2\hat{q}_1, q_4 = q_3\hat{p}_5,$$

$$q_5 = q_4\hat{p}_4^{-1}, p_1 = p_6\hat{p}_2, p_2 = p_1\hat{p}_3^{-1}, p_3 = p_2\hat{p}_6, p_4 = p_3\hat{p}_1^{-1}, p_5 = p_4\hat{q}_2^{-1}, p_6 = p_5\hat{q}_1^{-1}\rangle$$

The following can also be shown. Recall that a link L is the split union of two links L_1, L_2 whenever L is the union of the two links, and there exists a ball in the ambient space of L such that L_1 is contained within the ball and L_2 is exterior to the ball.

Lemma 3.2.2 *Let a link L be the split union of two links, L_1, L_2. Then $Q(L) = Q(L_1) \oplus Q(L_2)$. Hence by induction if L is the disjoint union of n links, $L_1, \cdots, L_n$ then $Q(L) = \bigoplus_{i=1}^{n} Q(L_i)$.*

Proof.

To find a presentation for the fundamental quandles of L_1 and L_2, draw a diagram for each, and then write down a generator for each arc, and a relation for each crossing. A presentation for L is found in a similar fashion, but for a suitable diagram for L we can just use the disjoint union of diagrams for L_1 and L_2. This would give a presentation for L which would have as generators the union of the generators of the presentations for L_1 and L_2, and as relations the union of the relations for L_1 and L_2. Hence by Lemma 2.4.1, the fundamental quandle of L is the free sum of the fundamental quandles of L_1 and L_2. $\square$

The following is also well known.

Lemma 3.2.3 *A link quandle Q has precisely one orbit for each component of the link, in particular, a knot quandle is connected.*

Recall that our running example is the fundamental quandle of the knot shown, above, and that back on page 31 it was shown that this quandle has two orbits, as expected.

Finally, we quote a result proved by Ryder in her doctoral thesis [22], (Corollary 2.27, pg 33), and also [24].

Theorem 3.2.4 (Ryder) *The fundamental quandle of a knot injects into the associated group if and only if the knot is prime.*

Chapter 4

Nucleii Of Operator Orbits

Given the link quandle of some link, it is possible (and in fact quite easy) to compute the quandles of the components of the link. Each component of course corresponds to some orbit of the quandle under the associated group, however each orbit is more than just the quandle of the link component, it is also acted on by elements of the associated group corresponding to the other components of the link. Factor out by these actions, and the quandle remaining is just the fundamental quandle of the component. We formalise this construction, prove that this does give the quandles of the components, and then show how to apply this to a quandle presentation.

4.1 Definition

Definition 4.1.1 *Let Q be a quandle and Q_1 be the union of a collection of operator orbits of Q. Then define $A_Q(Q_1)$ to be the subgroup of $A(Q)$ generated by the image of Q_1 in $A(Q)$.*

Lemma 4.1.2 *$A_Q(Q_1)$ is normal in $A(Q)$*

Proof.
$A_Q(Q_1)$ is generated by $\{\hat{q} | q \in Q_1\}$ so we need only show that $\hat{g}^{-1}\hat{q}\hat{g}$ is in $A(Q_1)$ for all q in Q_1, and $\hat{g}$ in $A(Q)$. Recall from the definition of associated group, page 9, that $\hat{g}^{-1}\hat{q}\hat{g} = \widehat{q\hat{g}}$, and since $q\hat{g}$ is in the same orbit of Q as q, and so is in Q_1, it must be that $\widehat{q\hat{g}}$ is in $A_Q(Q_1)$. □

Lemma 4.1.3 *Let Q be some quandle, and let Q_1 be the union of a collection of operator orbits of Q. Then the relation $\approx_{Q_1}$ on Q defined by*

$$q_1 \approx_{Q_1} q_2 \text{ if and only if there exists } \hat{g} \in A_Q(Q_1) \text{ such that } q_1\hat{g} = q_2$$

respects the quandle operations. That is to say, the equivalence relation whose equivalence classes are the orbits of $A_Q(Q_1)$*, is a congruence.*

Proof.

$\approx_{Q_1}$ is clearly an equivalence relation, so it remains to be shown that if $q_1 \approx_{Q_1} q_1'$ and $q_2 \approx_{Q_1} q_2'$ then $q_1 \rhd q_2 \approx_{Q_1} q_1' \rhd q_2'$. So let $q_1' = q_1 g_1$ and $q_2' = q_2 g_2$ where g_1, g_2 are in $A_Q(Q_1)$. Then

$$\begin{aligned} q_1' \rhd q_2' &= (q_1 g_1) \rhd (q_2 g_2) \\ &= q_1 g_1 g_2^{-1} \hat{q}_2 g_2 \\ &= (q_1 \rhd q_2) \hat{q}_2^{-1} g_1 g_2^{-1} \hat{q}_2 g_2. \end{aligned}$$

Since $A_Q(Q_1)$ is normal in $A(Q)$, $\hat{q}_2^{-1} g_1 g_2^{-1} \hat{q}_2 \in A_Q(Q_1)$. Hence $q_1 \rhd q_2 \approx_{Q_1} q_1' \rhd q_2'$. $\square$

Definition 4.1.4 *Let Q be a quandle. Let Q_1 and Q_2 be disjoint collections of orbits of Q, whose union is the whole of Q. Then Q_1 and Q_2 are called a* complementary pair *for Q.*

The Nucleus *of Q_1 in Q, ($Nu_Q(Q_1)$) is $Q_1/\approx_{Q_2}$.*

Lemma 4.1.5 *If Q_1, Q_2 are a complementary pair for some quandle Q, and Q_1 is a single orbit of Q, then $Nu_Q(Q_1)$ is connected.*

Proof.

Suppose $[q_1], [q_2] \in Nu_Q(Q_1)$, so $q_1, q_2 \in Q_1$. As Q_1 is connected, there exists g in $A(Q)$ such that $q_1 g = q_2$. Since $A_Q(Q_1)$ is normal in $A(Q)$, we may write $g = g_1 g_2$ with $g_i \in A_Q(Q_i)$ and g_1 can be written $g_1 = \hat{p}_1^{\pm 1} \cdots \hat{p}_m^{\pm 1}$ where each $p_i \in Q_1$. So

$$\begin{aligned} [q_2] &= [q_1 g_1 g_2] \\ &= [q_1 g_1] \\ &= [q_1 \rhd^{\pm 1} p_1 \rhd^{\pm 1} \cdots \rhd^{\pm 1} p_m] \\ &= [q_1] \rhd^{\pm 1} [p_1] \rhd^{\pm 1} \cdots \rhd^{\pm 1} [p_m]. \quad \square \end{aligned}$$

4.2 Application To Link Quandles

We must show now that this above construction does what we have claimed. That is the content of the following theorem.

Theorem 4.2.1 *Let L be a link with link quandle Q. Let L_1 be some collection of components of L, and let Q_1 be the orbit of Q corresponding to L_1. If L_1 is considered as a link in its own right, that is ignoring the rest of L, then $Q(L_1) \cong Nu_Q(Q_1)$.*

Before we prove the theorem, we give a small lemma that will be useful in the proof.

Lemma 4.2.2 *Let Q_1 and Q_2 be quandles and Q_1' a subquandle of Q_1 consisting of a collection of orbits of Q_1. Let S be a subset of Q_1' with the property that every orbit of Q_1 in Q_1' contains at least one element of S, so any q in Q_1' can be written as $q = q_0 g$ with q_0 in S and g in $A(Q_1)$. Let θ be a function $\theta : S \to Q_2$, and $\hat{\theta}$ a group homomorphism $\hat{\theta} : A(Q_1) \to A(Q_2)$ such that the following two conditions hold.*

- *Suppose that $q_2 = q_1 g$ for some q_1, q_2 in S and g in $A(Q_1)$. Then $q_2\theta = (q_1\theta)(g\hat{\theta})$.*

- *For all q in S we have that $\widehat{q\theta} = \hat{q}\hat{\theta}$.*

Then $\phi : Q_1' \to Q_2$ defined by $(qg)\phi = (q\theta)(g\hat{\theta})$ is a well defined homomorphism.

In the case that Q_1 is a finitely presented quandle, and S is the set of defining generators of Q_1 which are contained in Q_1', then these conditions can be replaced with the following.

- *For any defining relator $q_1 g_1 = q_2 g_2$ with primary elements in Q_1, we have that $(q_1\theta)(g_1\hat{\theta}) = (q_2\theta)(g_2\hat{\theta})$.*

- *For all q in S we have that $\widehat{q\theta} = \hat{q}\hat{\theta}$.*

Proof.

That the function ϕ is well defined is guaranteed by the first condition. That it is a homomorphism is guaranteed by the second condition.

For the statement about finitely presented quandles, it must be shown that the second pair of conditions implies the first pair. Since the second condition is identical in each pair, it remains only to show that the second pair of conditions implies the first condition of the first pair. Now Proposition 2.1.5 says that any relation of the form $q' = qg$ can be obtained as a sequence of pairs $(q_1, g_1), \cdots, (q_n, g_n)$ with $(q_1, g_1) \equiv (q', id)$, $(q_n, g_n) \equiv (q, g)$ and such that each (q_{i+1}, g_{i+1}) is related to (q_i, g_i) via one of the word moves in

Definition 2.1.4. It follows that we need only go through the word moves and show that in each case, $(q, g) \leftrightarrow (q', g')$ implies that $(q\theta)(g\hat{\theta}) = (q'\theta)(g'\hat{\theta})$.

For the first word move, we have that

$$(q\theta)(\hat{q}g)\hat{\theta} = (q\theta)(\hat{q}\hat{\theta})(g\hat{\theta}) = (q\theta)(\widehat{q\theta})(g\hat{\theta}) = (q\theta)(g\hat{\theta}).$$

In the case of the second move, we may assume that the defining relator being used has primary elements in S. In this case then,

$$(q_1\theta)(g_1\hat{\theta})(g\hat{\theta}) = (q_1g_1)\theta(g\hat{\theta}) = (q_2g_2)\theta(g\hat{\theta}) = (q_2\theta)(g_2g)\hat{\theta}.$$

The third and fourth moves necessarily hold since $\hat{\theta}$ is a homomorphism. $\square$

Now for the main proof.

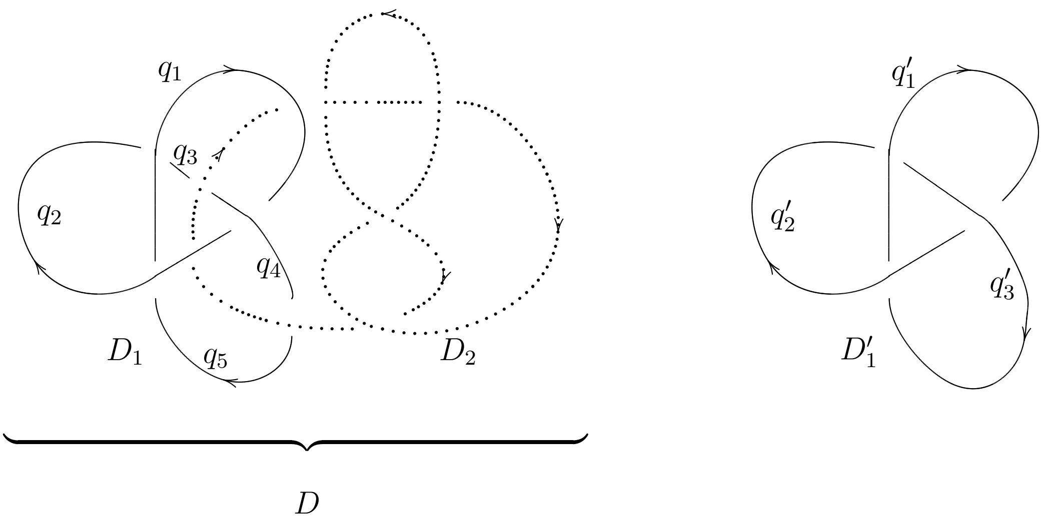
Proof of theorem 4.2.1.

We will prove this by setting up an explicit isomorphism from $Nu_Q(Q_1)$ to $Q(L_1)$.

Set up the notation

Let L_2 be the link L with the knot L_1 removed. Let D be a diagram for L using some projection ξ from $\mathbb{R}^3$ to $\mathbb{R}^2$, let D_1 be the collection of arcs of D corresponding to L_1, as a subset of L, and let D_2 be the collection of arcs corresponding to L_2. Finally let D_1' be the diagram of L_1 alone arising from the same projection ξ.

Let $Q(D)$ be the quandle presentation of L arising from the diagram D, and let $Q(D_1)$ and $Q(D_2)$ be the complementary pair for $Q(D)$ consisting of quandle elements with primary parts in D_1 and D_2 respectively. Finally let $Q(D_1')$ be the quandle presentation for L_1 arising from D_1'. It must be shown that $Q(D_1')$ is isomorphic to $Q(D_1)/\approx_{Q(D_2)}$.



By abuse of notation, we will also use D, D_1, D_2 and D_1' to denote the set of labels of arcs in these diagrams, as used for writing down a presentation for the link quandles.

Note that each arc in D_1 corresponds to a subarc of some arc in D_1'. Let θ be the function that takes a label of an arc in D_1 to the label of the corresponding arc in D_1'. In the example given by the above diagram then, $(q_1)\theta = q_1'$, $(q_2)\theta = q_2'$ and $(q_3)\theta = (q_4)\theta = (q_5)\theta = q_3'$. This function will be the foundation for the isomorphism from $Nu_{Q(D)}(Q(D_1))$ to $Q(D_1)$.

Define a homomorphism ϕ from $Q(D_1)$ to $Q(D_1')$

This will be done in stages using the above lemma.

Define a group homomorphism $\hat{\theta}$ from $A(Q)$ to $A(Q(L_1))$.

The group $A(Q)$ is generated by the set of $\hat{q}_i$ for q_i in D, so define $\hat{\theta} : A(Q) \to A(Qd(L_1))$ by defining it on the generators

$$(\hat{q})\hat{\theta} = \begin{cases} \widehat{(q)\theta} & : \ q \in D_1 \\ id & : \ q \in D_2 \end{cases}$$

and then extending linearly.

The homomorphism $\hat{\theta}$ is well defined.

The defining relators of $A(Q(D))$ are given by the crossings of the diagram D, and so to check that this is well defined, we will go through the crossings of D according as to whether the under and over lying arcs are in D_1 or D_2. For each crossing let q_1 and q_2 be the labels of the underpassing arcs, and q_3 be the label of the overpassing arc, so the given relation in $Q(D)$ will be $q_1 \triangleright q_3 = q_2$, and the corresponding relation $A(Q(D))$ will be $\hat{q}_3^{-1}\hat{q}_1\hat{q}_3 = \hat{q}_2$.

<u>The underpassing and overpassing arcs are all in D_1.</u>

In this case there is a corresponding crossing in D_1', and a corresponding relation in $Q(D_1')$ and so $(\hat{q}_3\hat{\theta})^{-1}(\hat{q}_1\hat{\theta})(\hat{q}_3\hat{\theta}) = (\hat{q}_2\hat{\theta})$ will be a relation in $A(Q(D_1'))$.

<u>The underpassing arcs are in D_1 and the overpassing arc is in D_2.</u>

In this case q_3 is in D_2 and $\hat{q}_3\hat{\theta} = \hat{id}$. Hence we require that $(\hat{id}^{-1}\hat{q}_1\hat{id})\hat{\theta} = \hat{q}_2\hat{\theta}$, but θ maps q_1 and q_2 to the same arc in D_1', and so the relation is satisfied.

<u>The underpasses are in D_2.</u>

In this case the requirement is that $(\hat{q}_3^{-1}\hat{id}\hat{q}_3)\hat{\theta} = id$, which is certainly satisfied.

Altogether then, we have that $\hat{\theta}$ is well defined. It is a homomorphism by construction.

Define a quandle homomorphism $\phi : Q(D_1) \to Q(D_1')$

This is where the above lemma is used. It is clear that D_1 contains at least one element from each orbit of $Q(D_1)$. For the first condition we need to check the defining relators of $Q(D_1)$ with primary elements in D_1. These are all of the form $q_1 = q_2\hat{q}_3^{\pm 1}$ and arise from crossings in the diagram with underpassing arcs in D_1.

If the overpassing arc is also in D_1, then all the arcs are taken by θ to the arcs of the same crossing in D_1', and so the relation is taken to an identical relation and the condition holds.

If on the other hand the overpassing arc q_3 is in D_2, then $\hat{q}_3\hat{\theta} = id$, $q_2\theta = q_1\theta$, and again the condition holds.

Lemma 4.2.2 then states that the function ϕ defined on a typical element $q = q'g$ of $Q(D_1)$ by

$$q\phi = (q')\theta(\hat{g})\hat{\theta}$$

is a quandle homomorphism.

The homomorphism ϕ is constant on equivalence classes of $\approx_{Q(D_2)}$ and hence induces a homomorphism from $Nu_{Q(D)}(Q(D_1))$ to $Q(D_1')$.

Suppose that $q_1 \approx_{Q(D_2)} q_2$ for some q_1, q_2 in $Q(D_1)$ so there exists g in $A_Q(Q(D_2))$ such that $q_2 = q_1 g$. Then

$$q_2\phi = (q_1 g)\phi = (q_1\theta)(g\hat{\theta}) = q_1\theta$$

since $A_Q(Q(D_2))$ is generated by D_2 and so is taken to the identity by $\hat{\theta}$. Abusing notation somewhat, we will call the induced homomorphism from $Nu_{Q(D)}(Q(D_1))$ to $Q(D_1')$ by the name ϕ also.

The homomorphism ϕ is an isomorphism.

It remains to show that ϕ is an isomorphism. This will be done by providing an inverse. The preimage of each arc in D_1' under θ is a non-empty collection of arcs in D_1. These arcs come in a natural sequence, each connected to the next via an overpassing arc from D_2, and so the associated elements of $Q(D_1)$ are in the same element of $Nu_{Q(D)}(Q(D_1))$. Since the labels of the arcs of D_1' generate the whole of $Q(D_1')$, this means that there is a unique well defined function $\phi' : Q(D_1') \to Nu_Q(Q_1)$ s.t. $(q)\phi'\phi = q$. This can be extended linearly to the whole of $Q(D_1')$ in a well-defined way, since the defining relations of $Q(D_1')$ arise from the crossings of D_1', which all map onto identical crossings in D. It remains to show that ϕ and ϕ' are inverses, but they are inverses on the arcs of D_1 and D_1' which generate the respective quandles, and so are inverses everywhere. $\square$

4.3 Computations For Finitely Presented Quandles

Computing the nucleus of an orbit of a finitely presented quandle is very easy. Recall that finding the orbits from a presentation, amounts to finding the generators in each orbit, and then each relation will relate generators from the same orbit.

Theorem 4.3.1 *Let Q be a finitely presented quandle with generators $q_1, \cdots, q_n$. Let Q_1 be an orbit of Q corresponding to the generators $q_1, \cdots, q_m$ and relations $\{p_1 = p_1'\hat{g}_1, \cdots, p_k = p_k'\hat{g}_k\}$ with the $p_i, p_i' \in \{q_1, \cdots, q_m\}$, and the $\hat{g}_i = \hat{r}_{i,1}^{\pm 1} \cdots \hat{r}_{i,l_i}^{\pm 1}$, with $r_{i,j} \in \{q_1, \cdots, q_n\}$.*

Let $\hat{r}_{i,j}' = \begin{cases} \hat{r}_{i,j} & r_{i,j} \in Q_1 \\ \hat{id} & r_{i,j} \notin Q_1 \end{cases}$. Let $\hat{g}_i' = \hat{r}_{i,1}'^{\pm 1} \cdots \hat{r}_{i,l_i}'^{\pm 1}$.

Then $Nu_Q(Q_1) \cong Q' = \langle q_1, \cdots q_m | p_1 = p_1'\hat{g}_1' \cdots p_k = p_k'\hat{g}_k' \rangle$

Before we give the proof, we will show what the above says in the case of our running example. Recall that the link given on page 84 has a fundamental quandle with the following presentation.

$$Q = \langle q_1, q_2, q_3, q_4, q_5, p_1, p_2, p_3, p_4, p_5, p_6 | q_1 = q_5\hat{q}_2, q_2 = q_1\hat{q}_4, q_3 = q_2\hat{q}_1, q_4 = q_3\hat{p}_5,$$

$$q_5 = q_4\hat{p}_4^{-1}, p_1 = p_6\hat{p}_2, p_2 = p_1\hat{p}_3^{-1}, p_3 = p_2\hat{p}_6, p_4 = p_3\hat{p}_1^{-1}, p_5 = p_4\hat{q}_2^{-1}, p_6 = p_5\hat{q}_1^{-1}\rangle$$

From page 31 this has two orbits. One orbit topologically consists of the set of all elements with initial point on the trefoil part of the link, and algebraically consists of the set of elements of the form $q_i g$. We will call this orbit Q_1. The other orbit topologically consists of the set of all quandle elements with initial point on the 'figure 8 knot' part of the link, and algebraically consists of the set of all quandle elements of the form $p_i g$. This orbit will be called Q_2.

So now to compute the nuclei of the two orbits.

$$\begin{aligned} Nu_Q(Q_1) &\cong \langle q_1', q_2', q_3', q_4', q_5' | q_1' = q_5'\hat{q}_2', q_2' = q_1'\hat{q}_4', q_3' = q_2'\hat{q}_1', q_4' = q_3'\hat{id}, q_5' = q_4'\hat{id}\rangle \\ &\cong \langle q_1', q_2', q_3' | q_1' = q_3'\hat{q}_2', q_2' = q_1'\hat{q}_3', q_3' = q_2'\hat{q}_1'\rangle \end{aligned}$$

Upon comparison with the list of knot quandles in appendix 1 it is easily seen that this is the fundamental quandle of the trefoil knot, as expected.

$$\begin{aligned} Nu_Q(Q_2) &\cong \langle p'_1, p'_2, p'_3, p'_4, p'_5, p'_6 | p'_1 = p'_6\hat{p}'_2, p'_2 = p'_1\hat{p'}_3^{-1}, p'_3 = p'_2\hat{p}'_6, p'_4 = p'_3\hat{p'}_1^{-1}, \\ &\quad p'_5 = p'_4\hat{id}, p'_6 = p'_5\hat{id}\rangle \\ &\cong \langle p'_1, p'_2, p'_3, p'_4 | p'_1 = p'_4\hat{p}'_2, p'_2 = p'_1\hat{p'}_3^{-1}, p'_3 = p'_2\hat{p}'_4, p'_4 = p'_3\hat{p'}_1^{-1}\rangle \end{aligned}$$

Again this agrees with the fundamental quandle of the figure 8 knot given in the appendix. Now for the proof of the theorem.

Proof.

Let $Q_2 = Q - Q_1$. So $Nu_Q(Q_1)$ is Q_1 factored out by the equivalence relation $q_1 \approx q_2$ if and only if there exists $\hat{g} \in A_Q(Q_2), q_1\hat{g} = q_2$. There is an obvious bijection

$$\theta : \{\text{ `generators of } Q \text{ in } Q_1\text{' }\} \to \{\text{ `generators of } Q'\text{' }\}$$

Define $\phi : A(Q) \to A(Q')$ by defining ϕ on the generators of $A(Q)$

$$(\hat{q})\phi = \begin{cases} \widehat{\theta(q)} & : \hat{q} \in Q_1 \\ \hat{id} & : \hat{q} \in Q_2 \end{cases}$$

and extending linearly. That this respects the defining relators of $A(Q)$, can be seen by going through cases. For a relation $p = p'\hat{g}$ in Q, p is in Q_1 if and only if p' is, so for $p, p' \in Q_1$ we have

$$\hat{p} = \hat{g}^{-1}\hat{p}'\hat{g}$$

but $\hat{p}\phi = (\hat{g})\phi^{-1}(\hat{p}')\phi(\hat{g})\phi$ is a relation in the definition of $A(Q')$. For $p, p' \in Q_2$ we have

$$\hat{id} = \hat{p}\phi = (\hat{g}^{-1})\phi(\hat{p}')\phi(\hat{g})\phi = (\hat{g}^{-1})\phi(\hat{g})\phi$$

and this is trivially a relation in $A(Q')$.

Let $q \in Q_1$ be $q = q'\hat{g}$, where q' is a generator of Q_1 and $\hat{g} \in A(Q)$. Define $\theta' : Q_1 \to Q'$ by $\theta'(q) = \theta(q')(\hat{g}\phi)$. Then this is a homomorphism, since for any $q_1, q_2 \in Q_1,\ q_i = q'_i\hat{g}_i$ we have

$$\begin{aligned} \theta'(q_1 \triangleright q_2) &= \theta'(q'_1\hat{g}_1\hat{g}_2^{-1}\hat{q}'_2\hat{g}_2) \\ &= \theta(q'_1)(\hat{g}_1)\phi(\hat{g}_2)\phi^{-1}(\hat{q}'_2)\phi(\hat{g}_2)\phi \\ &= \theta(q'_1)(\hat{g}_1)\phi \triangleright (\theta(q'_2)(\hat{g}_2)\phi) \\ &= \theta'(q_1) \triangleright \theta'(q_2) \end{aligned}$$

If $q_1 \approx q_2$ then there exists $\hat{g} \in A_Q(Q_2), q_1 = q_2\hat{g}$ and so

$$\theta'(q_1) = \theta'(q_2\hat{g}) = \theta'(q_2)(\hat{g}\phi) = \theta'(q_2)$$

Thus θ' induces a homomorphism θ'' from $Nu_Q(Q_1) = Q_1/\approx$ to Q'. We show that this is an isomorphism.

Since θ is bijective, it can be used to give a homomorphism from the quandle defined by Q' to $Nu_Q(Q_1)$, let $\psi : Qd(Q) \to Nu_Q(Q_1)$ be defined as θ^{-1} on the generators of Q', extended linearly. This is well defined, as the relators in Q' are of the form $p = p'\hat{r}_1 \cdots \hat{r}_m$ with p, p' generators of Q' and the $r_i \in A(Q')$ arising from a relator in Q of the form $p = p'\hat{r}'_1\hat{r}_1 \cdots \hat{r}'_m\hat{r}_m\hat{r}'_m$, with the $r_i \in A_Q(Q_1)$ and the $r'_i \in A_Q(Q_2)$. Since $A_Q(Q_2)$ is normal in Q, this is equivalent to a relation $p = p'\hat{r}_1 \cdots \hat{r}_m g$ with $g \in A_Q(Q_2)$, hence in $Nu_Q(Q_1)$ we have $p = p'\hat{r}_1 \cdots \hat{r}_m g = p'\hat{r}_1 \cdots \hat{r}_m$.

ψ is an inverse to θ'' on the generators of Q' and $Nu_Q(Q_1)$, and so must be an inverse everywhere. Hence θ'' is an isomorphism. $\square$

Chapter 5

Coset Quandles

Coset quandles are an idea briefly mentioned by Joyce in his doctoral thesis [12]. Here we will approach them from a different direction which seems more natural to this writer, and is slightly more general. Joyce generalises the idea rather more considerably in a different direction. My definition includes no more isomorphism classes of quandles, it includes (up to isomorphism) precisely the quasi-connected quandles, a fact given here as corollary 5.2.8, but originally proved by Joyce in his thesis. We will take the idea rather further than Joyce. At the end of the chapter, it will be shown that they include isomorphic copies of all Alexander quandles, and, more generally, all group quandles.

For any quandle Q, the operator, associated and automorphism groups all act on Q. When these groups act in a transitive way, that is, if the quandle is connected (in the case of the operator or associated groups) or quasi-connected (in the case of the automorphism group) then the techniques of permutation group theory can be used. This can be used to define certain representations of the quandle. These fit into a general scheme, the coset quandles. These are defined and studied in this chapter.

Coset quandles will be used in this thesis in two different ways. First of all for their representations of connected quandles, including knot quandles. Another way in which coset quandles will be used is to provide target quandles for homomorphism invariants. These attempt to tell quandles apart by counting homomorphisms onto other, target quandles.

5.1 Construction And Basic Facts

Before we go into the details here are some useful preliminary definitions.

Definition 5.1.1

For any group G and any subset $S \subseteq G$ we denote by ΣS the subgroup of G generated by S, by $Z(G)$ the centre of G, and by $Z_G(g)$ the centraliser of g in G.
For any group G and any subgroup $H < G$, the normal core *(or just* core*) of H is $\cap_{g \in G} g^{-1}Hg$, the largest normal subgroup of G contained in H. Alternatively, $h \in core(H)$ if and only if $g^{-1}hg \in H \quad \forall g \in G$. If the core of H is trivial, then we refer to H as* corefree.
For any quandle Q, any element $q \in Q$ and any group G acting on Q the stabiliser *of q is $St_G(q) = \{\hat{g} \in G | q\hat{g} = q\}$. In the case $G = Op(Q), Aut(Q)$ or $A(Q)$, we refer to the operator stabilizer, $St_{Op}(Q)$, the automorphism stabilizer $St_{Aut}(Q)$, or the associated stabilizer $St_A(Q)$ respectively.*

Since much of the following applies to either the operator or associated (in the connected case) or automorphism (in the quasi-connected case) groups, in the rest of this section we will use $G(Q)$ to refer to any of these, and $St(Q)$ to refer to any of their stabilizer groups. When we are dealing with an unspecified group we will label the elements with a 'bar', although in the case of the associated group, a 'hat' is more correct.

Lemma 5.1.2 *For any $G(Q)$-connected quandle Q, and any $q_0 \in Q$, we have*

I) $\hat{q}_0 \in Z(St_{G(Q)}(q_0))$

II) $St_{Op}(q_0)$ *and* $St_{Aut}(q_0)$ *are corefree in* $Op(Q)$ *and* $Aut(Q)$ *respectively. In* $A(Q)$ *we have* $core(St_A(q_0)) = ker\big(\pi_Q : A(Q) \to Op(Q)\big)$.

For any connected quandle and $G(Q)$ the operator or associated group we have

III) $\sum\{\hat{g}^{-1}\hat{q}_0\hat{g} | \hat{g} \in G(Q)\} = G(Q)$

Proof.
I) Pick $\hat{g} \in St(q_0)$, and $q \in Q$ then

$$q\hat{q}_0\hat{g} = (q \triangleright (q_0\hat{g}^{-1}))\hat{g} = q\hat{g}\hat{q}_0$$

II) For $\hat{g} \in G(Q)$ we have that $\hat{g}^{-1}St(q_0)\hat{g} = St(q_0\hat{g})$, and so

$$core(H) = \bigcap_{\hat{g} \in G} \hat{g}^{-1}St(q_0)\hat{g} = \bigcap_{barg \in G} St(q_0\hat{g}).$$

Hence $\hat{h}$ is in $core(H)$ precisely when $q_0\hat{g}\hat{h} = q_0\hat{g}$ for all $\hat{g}$ in G. Since Q is $G(Q)$-connected, this means that $q\hat{h} = q$ for all q in Q.
In the case $G = Op(Q)$ or $Aut(Q)$ this means that $\hat{h} = \overline{id}$.
In the case $G = A(Q)$ this happens if and only if $\hat{h} \in ker(\pi_Q)$.

III) Pick $\hat{g}_0 \in G(Q)$ where $G(Q) = Op(Q)$ or $A(Q)$. $\hat{g}_0 = \hat{q}_1^{\pm 1}\hat{q}_2^{\pm 1}\cdots\hat{q}_n^{\pm 1}$ some $q_i \in Q$. Q is connected, so $q_i = q_0\hat{g}_i$ for some $\hat{g}_i \in G(Q)$. Then $\hat{q}_i = \hat{g}_i^{-1}\hat{q}_0\hat{g}_i$ and $\hat{g}_0 \in \Sigma_{\hat{g}\in G(Q)}\{\hat{g}^{-1}\hat{q}_0\hat{g}\}$ $\square$

For any $G(Q)$-connected quandle, and any $\hat{h}, \hat{h}' \in G(Q)$,

$$q_0\hat{h} = q_0\hat{h}' \Leftrightarrow \hat{h}'\hat{h}^{-1} \in St_{G(Q)}(q_0)$$

which is to say, the set of elements taking q_0 to $q_0\hat{h}$ is a right coset of $St_{G(Q)}(q_0)$. Denoting the right coset of $St_{G(Q)}(q_0)$ containing $\hat{g}$ by $[\hat{g}]_{G(Q)}$, or just $[\hat{g}]$ when $G(Q)$ is clear from context, then there is a bijection ξ_{q_0} from the right cosets of $St_{G(Q)}(q_0)$ in $G(Q)$ to Q defined by $\xi_{q_0}([\hat{g}]) = q_0\hat{g}$.
Now let $q, q' \in Q$, $q = q_0\hat{g}, q' = q_0\hat{g}'$ for some $\hat{g}, \hat{g}' \in G(Q)$. So $q \triangleright q' = (q_0\hat{g}) \triangleright (q_0\hat{g}') = q_0\hat{g}\hat{g}'^{-1}\hat{q}_0\hat{g}'$ and in terms of right cosets we have

$$\xi_{q_0}[\hat{g}] \triangleright \xi_{q_0}[\hat{g}'] = \xi_{q_0}[\hat{g}\hat{g}'^{-1}\hat{q}_0\hat{g}']$$

We have a converse to this.

Theorem 5.1.3 *Given any group G with subgroup $H < G$ and $g_0 \in H$ s.t. $g_0hg_0^{-1}h^{-1} \in core(H)$ $\forall h \in H$ then define operations on the right cosets of H in G.*

$$[g_1] \triangleright [g_2] = [g_1g_2^{-1}g_0g_2]$$
$$[g_1] \triangleright^{-1} [g_2] = [g_1g_2^{-1}g_0^{-1}g_2]$$

Then this construction is well defined and yields a quandle.

Proof.
Well defined.
The condition on g_0 implies that for any $h_2 \in H$ and any $g_1, g_2 \in G$

$$g_0h_2g_0^{-1}h_2^{-1} \in h_2g_2g_1^{-1}Hg_1g_2^{-1}h_2^{-1}$$

hence

$$\begin{aligned} h_2^{-1}g_0h_2g_0^{-1} &\in g_2g_1^{-1}Hg_1g_2^{-1} \quad \forall g_1, g_2 \in G, h_2 \in H \\ &= g_2g_1^{-1}hg_1g_2^{-1} \text{ say.} \end{aligned}$$

So let $h_1, h_2 \in H$ then

$$\begin{aligned}
[h_1g_1] \rhd [h_2g_2] &= [h_1g_1g_2^{-1}h_2^{-1}g_0h_2g_2] \\
&= [h_1hg_1g_2^{-1}g_0g_2] \\
&= [g_1g_2^{-1}g_0g_2] \\
&= [g_1] \rhd [g_2]
\end{aligned}$$

Well definedness of $\rhd^{-1}$ is similar.
Indempotency.
$[g] \rhd [g] = [gg^{-1}g_0g] = [g]$
Invertibility.
Obvious from the construction.
Distributivity.

$$\begin{aligned}
([g_i] \rhd [g_k]) \rhd ([g_j] \rhd [g_k]) &= [g_ig_k^{-1}g_0g_k] \rhd [g_jg_k^{-1}g_0g_k] \\
&= [g_ig_j^{-1}g_0g_jg_k^{-1}g_0g_k] \\
&= ([g_i] \rhd [g_j]) \rhd [g_k] \quad \square
\end{aligned}$$

Definition 5.1.4 *A* Q-data set *is a set $P = \{G, H, g_0\}$ where G is a group, H is a subgroup of G and g_0 is an element of H s.t. $g_0hg_0^{-1}h^{-1} \in core(P)\ \forall h \in H$. The quandle constructed from this data in Theorem 5.1.3 will be referred to as $Qd(P)$. In general such quandles will be called* coset quandles.
If $Q = Qd(P) \cong Q'$ then Q is called a coset representation *of Q'.*
The group G is called the base group *of Q or of P. The* core *of P, denoted $core(P)$ is the core of H (in G), and P is* corefree *if and only if H is corefree.*

Joyce comes to this definition from an entirely different direction. He starts with group quandles, quandles constructed from a group G together with some automorphism ψ of G, the quandle structure on these gadgets being defined by

$$g_1 \rhd g_2 = (g_1g_2^{-1})\psi g_2 \text{ and } g_1 \rhd^{-1} g_2 = (g_1g_2^{-1})\psi^{-1}g_2$$

He then observes that if H is some subgroup of G which is pointwise fixed by ψ then the set of right cosets of H in G inherits this quandle structure

$$Hg_1 \rhd Hg_2 = H(g_1g_2^{-1})\psi g_2 \text{ and } Hg_1 \rhd^{-1} Hg_2 = H(g_1g_2^{-1})\psi g_2$$

If ψ is taken to be an inner automorphism, conjugation by g_0 say, and if g_0 is contained in H, then we have

$$Hg_1 \rhd Hg_2 = Hg_1g_2^{-1}g_0g_2 \text{ and } Hg_1 \rhd^{-1} Hg_2 = Hg_1g_2^{-1}g_0^{-1}g_2$$

and we are back to the above definition, with the slightly stronger condition that g_0 must be in the centre of H, that is $g_0hg_0^{-1}h^{-1}$ must be the identity for all h in H, rather than just being in the core of H.

The distinction between the Q-data set and the quandle it gives rise to is necessary, different Q-data sets can give rise to isomorphic or even identical quandles, and still have certain useful properties which are different. Later on some of these will be mentioned, see definition 5.2.12 onwards.

From the remarks preceding Theorem 5.1.3, it is clear that, for any connected quandle Q, and any $q \in Q$,

$$Q \cong Qd\{Op(Q), St_{Op}(q), \hat{q}\} \cong Qd\{A(Q), St_A(q), \hat{q}\}$$

and that for any quasi-connected quandle, and any $q \in Q$

$$Q \cong Qd\{Aut(Q), St_{Aut}(q), \hat{q}\}$$

.

Definition 5.1.5 *For any quasi-connected quandle* Q *and for any* q_0 *in* Q*, the* automorphic representation *for* Q *at* q_0*, is* $Qd\{Aut(Q), St_{Aut}(q_0), \hat{q}_0\}$*. For any connected quandle* Q *and* q_0 *in* Q*, the* operator representation *for* Q *at* q_0*, is* $Qd\{Op(Q), St_{Op}(q_0), \hat{q}_0\}$*, and the* associated representation *for* Q *at* q_0*, is* $Qd\{A(Q), St_A(q_0), \hat{q}_0\}$*.*

The canonical isomorphism at q_0 *from* Q *to its automorphism representation (resp. operator representation, associated representation) is the isomorphism*

$$(q)\rho_{Aut}(resp.\ \rho_{Op}, \rho_A) = \{\hat{g} \in Aut(Q)(resp.\ Op(Q), A(Q))|q_0\hat{g} = q\}$$

So that for $\hat{g} \in Aut(Q), (Op(Q), A(Q))$*,* $(q_0\hat{g})\rho = [\hat{g}]$*. The subscript on* ρ *will normally be suppressed. In the case of the associated group, this isomorphism is defined via the standard projection of the associated group onto the operator group.*

Of these three representations, the associated representation of a connected quandle is arguably the most interesting. Given a quandle presentation of a connected quandle Q, it is easy to write down a group presentation of the associated group. It is equally easy to pick an element of Q and write down its image in $A(Q)$. A generating set for the stabilizer can be obtained using Theorem 2.3.2. In the case of knots, these items of data are the fundamental group of the knot, a meridian of the knot, and the peripheral subgroup of the meridian respectively.

5.2 Coset Morphisms And Coset Representations

In chapter 8, coset quandles are used as targets for homomorphism in an attempt to tell quandles apart. It is hoped that if the original quandles are also coset quandles such homomorphisms will be related to homomorphisms of the base groups. Hence, if the target quandle has $GL(n, \mathbb{C})$ as a base group say, then these homomorphisms will reduce to group representations, a well studied and understood subject. To facilitate this, the following will be considered.

Definition 5.2.1 *Let $P = \{G, H, g_0\}$ and $P' = \{G', H', g_0'\}$ be two Q-data sets.*

A coset homomorphism *from P to P' is a group homomorphism $\phi : G \to G'$ such that $H\phi \subseteq H'$ and $g_0\phi = g_0'$.*

A coset isomorphism *is a coset homomorphism ϕ with the additional properties that ϕ is an isomorphism on G and $H\phi = H'$.*

A coset homomorphism ϕ is called monic *if and only if $H'\phi^{-1} = H$.*

A coset homomorphism ϕ is called epic *if and only if $G\phi \cap H'g'$ is non-empty for all g' in G'.*

A coset homomorphism is called a coset semi-equivalence *if and only if it is both monic and epic.*

From now on, to distinguish between different types of homomorphism, the prefixes Q-, C-, and G- will sometimes be used to denote Quandle, Coset and Group respectively. So a G-homomorphism is a group homomorphism, a Q-isomorphism is a quandle isomorphism and so on. Since a C-homomorphism is a particular sort of G-homomorphism on the base group of a coset quandle, a C-homomorphism will often be given just in terms of a G-homomorphism when the extra conditions have been shown to be satisfied.

Now comes a lemma which justifies the definitions of epic, monic, and semi-equivalence.

Lemma 5.2.2 *Let $P = \{G, H, g_0\}$ and $P' = \{G', H', g_0'\}$ be Q-data sets, and let $\phi : P \to P'$ be a C-homomorphism. Then the function $\dot{\phi} : Qd(P) \to Qd(P')$ defined by $([g])\dot{\phi} = [(g)\phi]$ is a Q-homomorphism, called the* induced

Q-homomorphism from Q to Q'.
$\dot{\phi}$ is injective precisely when ϕ is monic. $\dot{\phi}$ is surjective precisely when ϕ is epic. In particular ϕ surjective implies that $\dot{\phi}$ is surjective.
As a consequence, $\dot{\phi}$ is a Q-isomorphism precisely when ϕ is a C-semi-equivalence.

Proof.

Let $\rhd$ and $\rhd'$ be the quandle operations on $Qd(P)$ and $Qd(P')$ respectively.
$\dot{\phi}$ is well defined.
Suppose $h \in H$.
Then $[hg]\dot{\phi} = [(hg)\phi] = [(h\phi)(g\phi)] = [g\phi] = [g]\dot{\phi}$.
$\dot{\phi}$ is a Q-homomorphism.

$$\begin{aligned}([g_1]_G \rhd [g_2]_G)\dot{\phi} &= ([g_1 g_2^{-1} g_0 g_2]_G)\dot{\phi} \\ &= [(g_1\phi)(g_2\phi)^{-1} g_0'(g_2\phi)]_{G'} \\ &= [g_1\phi]_{G'} \rhd' [g_2\phi]_{G'} \\ &= ([g_1]_G)\dot{\phi} \rhd' ([g_2]_G)\dot{\phi}\end{aligned}$$

Injectivity condition.
By the definition of Q-homomorphism, $H \subseteq H'\phi^{-1}$. Suppose that $H \neq H'\phi^{-1}$, so there exists $g \in H'\phi^{-1}, g \notin H$. Then $(Hg)\dot{\phi} = (H)\dot{\phi} = H'$ and $\dot{\phi}$ would not be injective, so $\dot{\phi}$ injective implies that $(H')\phi^{-1} = H$.

On the other hand if $\dot{\phi}$ is not injective then we have $([g])\dot{\phi} = ([g'])\dot{\phi}$ for some $[g] \neq [g']$. That is $H'(g\phi) = H'(g'\phi)$, $H' = H'(g'g^{-1})\phi$ and $H(g'g^{-1}) \subseteq (H')\phi^{-1}$. So $(H')\phi^{-1} \neq H$.

The surjectivity condition is easily seen since the $H'g'$ are the elements of Q'. The proof is complete. $\square$

Lemma 5.2.3 *Let $\phi_1 : P_1 \to P_2$ and $\phi_2 : P_2 \to P_3$ be C-homomorphisms. Then $\phi_1\phi_2 : P_1 \to P_3$ is a C-homomorphism.*
If ϕ_1 and ϕ_2 are both monic, epic, a semi-equivalence or isomorphisms then the same property holds for $\phi_1\phi_2$.
If $\phi_1\phi_2$ is monic then so is ϕ_1.
If $\phi_1\phi_2$ is epic then so is ϕ_2.
If $\phi_1\phi_2$ is a semi-equivalence then so are ϕ_1 and ϕ_2.
$(\phi_1\dot{\ }\phi_2) = \dot{\phi}_1\dot{\phi}_2$.

Proof.
All obvious.

If we consider the category of Q-data sets and C-homomorphisms, and the category of quandles and Q-homomorphisms then the function Qd which takes a Q-data set to the quandle which it defines and a C-homomorphism ϕ to $\dot{\phi}$ is a functor.

We give some examples of C-homomorphisms.

Example

In Lemma 1.1.9 it was shown how, for any quandle Q, and any congruence relation $\approx$ on Q, there is a homomorphism $\psi : Op(Q) \to Op(Q/\approx)$ defined by

$$[q](g\psi) = [g\psi]$$

This is an epic C-homomorphism from $P = \{Op(Q), St(q_0), \hat{q}_0\}$ to $\{Op(Q/\approx), St([q_0]), \overline{[q_0]}\}$. The conditions for ψ to be a C-homomorphism are easily seen to hold, and the epic condition is true, since the map from Q to $Q/\approx$ is surjective.

Example

Recall that the function A : Quandles $\to$ Groups is functorial. For a given $\phi : Q_1 \to Q_2$ for two connected quandles, then $A(\phi)$ is the G-homomorphism between $A(Q_1)$ and $A(Q_2)$ given by

$$(\hat{q})(A\phi) = \widehat{q\phi}$$

We show that $A(\phi)$ is a C-homomorphism between the associated representations of Q_1 and Q_2.

Lemma 5.2.4 *Let Q_1 and Q_2 be two connected quandles, and let $\phi : Q_1 \to Q_2$ be a Q-homomorphism. Then*

$$A(\phi) : \{A(Q_1), St_A(q_0), \hat{q}_0\} \to \{A(Q_2), St_A(q_0\phi), \widehat{q_0\phi}\}$$

is a C-homomorphism.

Proof.
Show that $A(\phi)$ is a C-homomorphism. That $(\hat{q}_0)(A(\phi)) = \widehat{q_0\phi}$ is true by definition. It remains to show that $(St_A(q_0))(A(\phi)) \subseteq St_A(q_0\phi)$.
Pick $\hat{g} = \hat{q}_1^{\pm 1} \cdots \hat{q}_n^{\pm 1} \in St_A(q_0)$. Then

$$\begin{aligned}(q_0\phi)(\hat{q}_1^{\pm 1} \cdots \hat{q}_n^{\pm 1})(A(\phi)) &= (q_0\phi)(\hat{q}_1^{\pm 1}\phi) \cdots (\hat{q}_n^{\pm 1}\phi) \\ &= (q_0\hat{q}_1^{\pm 1} \cdots \hat{q}_n^{\pm 1})\phi \\ &= q_0\phi \quad \Box\end{aligned}$$

However it should be noted that for $\phi : Q_1 \to Q_2$, the Q-homomorphism induced from $A(\phi)$ is not related to ϕ in any obvious way.

$$(q\rho_A)(A(\dot{\phi})) = \{\hat{g} \in A(Q_2) | (q_0\phi)\hat{g} = (q_0\hat{q})\phi\} \neq \{\hat{g} \in A(Q_2) | (q_0\phi)\hat{g} = q\phi\} = (q\phi)\rho_A$$

Different choices of defining element can lead to identical quandles.

Lemma 5.2.5 *Let $P = \{G, H, g_0\}$ and $P' = \{G, H, g_0'\}$ be coset quandles on the same base group. Then $Qd(P) = Qd(P') \Leftrightarrow g_0 g_0'^{-1} \in core(P)$.*

Proof.

Let $[g_1]$ and $[g_2]$ be arbitrary cosets of H. Then for $Qd(P)$ and $Q(P')$ to be the same quandle we need

$$\begin{array}{rrcl} & [g_1] \triangleright [g_2] = [g_1 g_2^{-1} g_0 g_2] & = & [g_1 g_2^{-1} g_0' g_2] = [g_1] \triangleright' [g_2] \\ \Leftrightarrow & g_1 g_2^{-1} g_0 g_2 & = & h g_1 g_2^{-1} g_0' g_2 \text{ for some } h \in H \\ \Leftrightarrow & g_1 g_2^{-1} g_0 g_0'^{-1} g_2 g_1^{-1} & \in & H \end{array}$$

But this has to be true for all $g_1, g_2 \in G$ so

$$\Leftrightarrow g_0 g_0'^{-1} \in core(P) \quad \square$$

This says something stronger than that the quandles are Q-isomorphic, it says they are the same. The elements are the same, they are right cosets of H in G, and the actions of elements on each other are the same, if $[g_1] \triangleright [g_2] = [g_3]$ in one quandle, then $[g_1] \triangleright [g_2] = [g_3]$ in the other also.

In the light of this result, it may be supposed that the definition of C-homomorphism could be generalised slightly so that it is only required that $(g_0)\phi \in g_0' core(P')$ rather than $(g_0)\phi = g_0'$. Unfortunately, if this course is pursued, then the composition of two C-homomorphism is not necessarily a C-homomorphism. As an example of this, consider some Q-data set $P = \{G, H, g_0\}$ where H is not normal in G, and $g_0 \notin core(P)$. Then $P_1 = \{H, H, g_0\}$ and $P_2 = \{H, H, id_G\}$ are both clearly Q-data sets. Furthermore, the inclusion homomorphism $\iota : P_1 \to P$ is a C-homomorphism and the identity homomorphism $id : P_2 \to P_1$ would be a C-homomorphism under the more general definition. Yet the composition $id \circ \iota : P_2 \to P$ would not be a C-homomorphism.

It is clear that two quasi-connected quandles are Q-isomorphic if and only if their automorphism representations are C-isomorphic. Similarly two connected quandles are Q-isomorphic if and only if their operator representations are C-isomorphic, if and only if their associated representations are C-isomorphic.

Now for some consequences of Lemma 5.2.2.

Corollary 5.2.6

- *Let K be a normal subgroup of G. Then the canonical C-homomorphism ϕ from G to G/K induces a surjective Q-homomorphism*

$$\dot{\phi} : Qd\{G, H, g_0\} \to Qd\{G/K, H\phi, Kg_0\}$$

  $\dot{\phi}$ is injective if and only if $K < H$.

- *The canonical C-homomorphism from G to $G/core(P)$ induces a Q-isomorphism from $Qd\{G, H, g_0\}$ to $Qd\{G/core(P), H/core(P), core(P)g_0\}$.*

- *A G-automorphism ϕ of G that takes H to itself induces a Q-automorphism $Qd\{G, H, g_0\} \to Qd\{G, H, (g_0)\phi\}$*

- *A G-automorphism of G that takes H to itself and takes g_0 to an element of $core(P)g_0$ induces a Q-automorphism of Q.*

- *Let g be some element of $N(H)$, the normaliser of H in G. Then the conjugation function that takes $[g'] \mapsto [g^{-1}g'g]$ is a Q-isomorphism from $Qd\{G, H, g_0\}$ to $Qd\{G, H, g^{-1}g_0g\}$.*

Lemma 5.2.7 *Let P be a Q-data set $P = \{G, H, g_0\}$ and let $Q = Qd(P)$. Then for any $g \in G$, the function $\eta_g : Q \to Q, ([g'])\eta_g = [g'g]$ is a Q-automorphism.*

Proof.

η_g is well defined. Let $[g'] = [hg']$ so g' and hg' are in the same right coset of H. Then $g'g$ and $hg'g$ are clearly also in the same right coset of H, so $[g']\eta_g = [g'g] = [hg'g] = [hg']\eta_g$.
η_g is a Q-homomorphism. $([g']\eta_g) \rhd ([g'']\eta_g) = [g'gg^{-1}g''^{-1}g_0g''g] = ([g'] \rhd [g''])\eta_g$.
That η_g is invertible and hence a Q-automorphism is obvious. $\square$

Corollary 5.2.8 *A quandle has a coset representation if and only if it is quasi-connected.*

Proof.

Any coset quandle is quasi-connected since given any two right cosets, $[g], [g']$, $\eta_{g^{-1}g'}$ provides an automorphism taking the former to the latter.
On the other hand, Definition (5.1.4) gives a coset representation for any quasi-connected quandle. $\square$

This was first shown by Joyce in his thesis, [12] using essentially the above proof. He then goes on to give a generalisation of coset quandles, constructed from a group using multiple (not necessarily distinct) subgroups, H_i, each one having its own defining element g_i. The quandle is then defined on the set of all right cosets of the H_i with the quandle operation

$$H_i h_i \triangleright H_j h_j = H_i h_i h_j^{-1} g_j h_j$$

Joyce shows that this definition provides isomorphic copies of any quandle Q at all, one needs an element q_i from each orbit of Q, and then the H_i and g_i are the stabilizer subgroups and images of the q_i respectively.

Definition 5.2.9 *Let $P = \{G, H, g_0\}$ and let Q be the coset quandle $Qd(P)$. Then define $\eta : G \to Aut(Q)$ to be the function $(g)\eta = \eta_g$.*

Lemma 5.2.10 *The function η is a G-homomorphism with $ker(\eta) = core(P)$.*

Proof.

That η is a homomorphism is obvious.
$ker(\eta) = core(P)$.

$$\begin{aligned} g \in ker(\eta) &\Leftrightarrow \forall g' \in G : \ [g'g] = [g'] \\ &\Leftrightarrow \forall g' \in G \ \exists h \in H \ s.t. \ g'g = hg' \\ &\Leftrightarrow \forall g' \in G : \ g \in {g'}^{-1} H g' \\ &\Leftrightarrow g \in core(P) \quad \square \end{aligned}$$

Corollary 5.2.11 *If the Q-data set $P = \{G, H, g_0\}$ is corefree then $\eta : G \to Aut(Q)$ is an injection.*

Definition 5.2.12 *Let P be a Q-data set $P = \{G, H, g_0\}$ and let Q be its quandle $Q = Qd(P)$. A* right Q-automorphism *is an element of the image of η in $Aut(Q)$.*

This is an example where the Q-data set is important: Q-isomorphic coset quandles can have non-isomorphic right automorphism groups. In the case of an automorphism representation, the right automorphism group is the whole of the automorphism group, however, for an operator representation, it is just the operator subgroup of the Q-automorphism group as the following lemma shows.

Lemma 5.2.13 *Let Q be a quandle, and let $G(Q)$ be the automorphism or operator group of Q. Let $Q' = Qd\{G(Q), St_G(q_0), \hat{q}_0\}$, be an automorphism or operator representation of Q, and let ρ be the canonical Q-isomorphism at q_0.*
Then for $\hat{g} \in G(Q)$, and $q \in Q'$ we have $(q)(\hat{g}\eta) = (q)(\rho^{-1}\hat{g}\rho)$.

What this states is that if $\hat{g}$ is some element of the automorphism or operator group of the quandle Q, then under the canonical isomorphism ρ from Q to its automorphism or operator group representation Q', the right automorphism of Q', $(\hat{g})\eta$, corresponds to the action of $\hat{g}$ on Q.

Proof.

Pick $\hat{g}$ in G. If $q \in Q', q = [\hat{g}']$ for some $\hat{g}' \in G(Q)$ then $q = \{\hat{h} \in G(Q) | q_0\hat{h} = q_0\hat{g}'\} = (q_0\hat{g}')\rho$. So

$$\begin{aligned} [\hat{g}'](\rho^{-1}\hat{g}\rho) &= (q_0\hat{g}')\hat{g}\rho \\ &= (q_0\hat{g}'\hat{g})\rho \\ &= [\hat{g}'\hat{g}] \\ &= [\hat{g}'](\hat{g}\eta) \end{aligned}$$

and we are done. $\square$

In the case that G is the automorphism group, this means that the right automorphisms of an automorphism representation, are precisely the actions of right multiplication by the elements of its base group, so the right automorphism group is the whole automorphism group. However for G the operator group, the right automorphism group is just the operator group, as stated. Hence for a connected quandle Q with operator group not the whole of the automorphism group, there exist coset representations with different right automorphism groups.

For an automorphic representation Q', of some quandle Q, where Q is itself a coset quandle $Q = \{G, H, g_0\}$, $\eta : G \to Aut(G)$ we will next show that Lemma 5.2.2 can be used to construct $\dot{\eta} : Q \to Q'$, and that $\dot{\eta}$ is ρ, the canonical isomorphism at H from Q to Q'.

First we show that $\dot{\eta}$ is well defined by showing that η is a C-homomorphism. Notice that $[g_0] = H = [id]$.

Lemma 5.2.14 *Let $P = \{G, H, g_0\}$ be a Q-data set. $H\eta \subseteq St_{Aut}([g_0])$, $g_0\eta = \overline{[g_0]}$ hence η is a C-homomorphism.*

Proof.

$$\begin{aligned} h \in H &\Leftrightarrow ([g_0])(h\eta) = [g_0 h] = [id] \\ &\Leftrightarrow h\eta \in St_{Aut}([g_0]) \end{aligned}$$

Pick $g \in G$. Then

$$\begin{aligned} ([g])(g_0\eta) &= [gg_0] \\ &= [gg_0^{-1}g_0g_0] \\ &= [g] \rhd [g_0] \\ &= [g]\overline{[g_0]} \quad \square \end{aligned}$$

So $\dot{\eta}$ is a well defined homomorphism from Q to Q' . Now we must show that it is the canonical Q-isomorphism.

Theorem 5.2.15 *$\dot{\eta}$ is ρ, the canonical Q-isomorphism at $[id]$.*

Proof.
Pick $g \in G$

$$\begin{aligned} [id]\eta_g = [g] &\Rightarrow \eta_g \in \{\hat{g}' \in Aut(Q) | [id]\hat{g}' = [g]\} = [g]\rho \\ &\Rightarrow [\eta_g] = [g]\rho \\ &\Rightarrow [g]\dot{\eta} = [g]\rho \end{aligned}$$

Since g was chosen arbitrarily then $\dot{\eta}$ and ρ are the same isomorphism. $\square$

Corollary 5.2.16 *η is a C-semi-equivalence.*

It has been shown (Corollary 5.2.11) that for any corefree Q-data set P, the base group of P injects into the automorphism group of $Qd(P)$. Conversely, the operator group of $Qd(P)$ injects into the base group.

Proposition 5.2.17 *Let $P = \{G, H, g_0\}$ with $Q = Qd(P)$ be a corefree coset quandle, and let $\mu : Op(Q) \to G$ be the function defined on the generators of $Op(Q)$ by $(\overline{[g]})\mu = g^{-1}g_0g$ and extended linearly. Then μ is a well defined injective G-homomorphism.*

Proof.
First show that this is well defined on the generators. Let $\overline{[g_2]} = \overline{[g_1]}$. Then

$$\begin{aligned}
\overline{[g_1]} = \overline{[g_2]} &\Rightarrow [g] \rhd [g_1] = [g] \rhd [g_2] \ \forall g \in G \\
&\Rightarrow [gg_1^{-1}g_0g_1] = [gg_2^{-1}g_0g_2] \\
&\Rightarrow gg_1^{-1}g_0g_1g_2^{-1}g_0^{-1}g_2g^{-1} \in H \ \forall g \in G \\
&\Rightarrow g_1^{-1}g_0g_1g_2^{-1}g_0^{-1}g_2 \in core(P) = \{id_G\} \\
&\Rightarrow g_1^{-1}g_0g_1 = g_2^{-1}g_0g_2
\end{aligned}$$

Next show that this is a well defined homomorphism, and injects.

$$\begin{aligned}
\overline{[g_1]}^{\pm 1} \cdots \overline{[g_n]}^{\pm 1} = \hat{id}_{Op} &\Leftrightarrow [g]\overline{[g_1]}^{\pm 1} \cdots \overline{[g_n]}^{\pm 1} = [g] \ \forall g \in G \\
&\Leftrightarrow [gg_1^{-1}g_0^{\pm 1}g_1 \cdots g_n^{-1}g_0^{\pm 1}g_n] = [g] \\
&\Leftrightarrow gg_1^{-1}g_0^{\pm 1}g_1 \cdots g_n^{-1}g_0^{\pm 1}g_ng^{-1} \in H \ \forall g \in G \\
&\Leftrightarrow g_1^{-1}g_0^{\pm 1}g_1 \cdots g_n^{-1}g_0^{\pm 1}g_n \in core(P) = \{id_G\} \\
&\Leftrightarrow (\overline{[g_1]}\phi)^{\pm 1} \cdots (\overline{[g_n]}\phi)^{\pm 1} = id_G \quad \square
\end{aligned}$$

The function μ is, in a sense, just a restricted inverse to η

Lemma 5.2.18 *$\mu\eta = \iota_Q$ where ι_Q is the inclusion map of $Op(Q)$ into $Aut(Q)$.*

Proof.
We must show that for any $\overline{[g_1]} \in Op(Q)$ and any $[g] \in Q$ we have that $([g])(\overline{[g_1]}\mu\eta) = [g]\overline{[g_1]}$, but

$$\begin{aligned}
([g])(\overline{[g_1]}\mu\eta) &= [g](g_1^{-1}g_0g_1)\eta \\
&= [gg_1^{-1}g_0g_1] \\
&= [g]\overline{[g_1]} \quad \square
\end{aligned}$$

Lemma 5.2.19 *Let Q be some connected quandle, and $\hat{q}_0 \in Op(Q)$. Then ι_Q is a coset semi-equivalence from the operator representation of Q at $\hat{q}_0$ to the automorphism representation of Q at $\hat{q}_0$.*

Proof.

There are three conditions which must be satisfied. $(\hat{q}_0)\iota = \hat{q}_0$, $(St_{Aut}(q_0))\iota^{-1} = St_{Op}(q_0)$ and $(Op(Q))\iota \cap St_{Aut}(q_0)\hat{g} \neq \emptyset \quad \forall \hat{g} \in Aut(Q)$. The first two are immediate.

For the last condition, pick $\hat{g} \in Aut(Q)$. Q is connected so there exists some $\hat{h} \in Op(Q)$ such that $q_0\hat{h} = q_0\hat{g}$. $\hat{h}\hat{g}^{-1} \in St_{Aut}(q_0)$ and $\hat{h} = (\hat{h}\hat{g}^{-1})\hat{g} \in (Op(Q))\iota \cap St_{Aut}(q_0)\hat{g}$ $\square$.

Corollary 5.2.20 *Let $Q = Qd(P)$ be some connected, corefree, coset quandle. Then μ_Q is a C-semi-equivalence.*

Consider two isomorphic coset quandles, $\theta : Qd\{G, H, g_0\} \to Qd\{G', H', g_0'\}$. There is no reason to suppose that θ respects the group structures of G and G' in any way. However something almost as good can be stated.

Proposition 5.2.21 *Let $P = \{G, H, g_0\}$ be some Q-data set, and Q some quandle. Then $Qd(P) \cong Q$ if and only if Q is quasi-connected and for some $q_0 \in Q$, there exists a C-semi-equivalence $\phi : P \to \{Aut(Q), St_{Aut}(q_0), \hat{q}_0\}$.*

Proof.

If there exists such a semi-equivalence, then $Qd(P)$ is Q-isomorphic to Q by Lemma 5.2.2.

Conversely, assume that $Qd(P) \cong Q$. Then there exists a C-isomorphism θ from the automorphism representation of $Qd(P)$ to $\{Aut(Q), St_{Aut}(q_0), \hat{q}_0\}$. There is also the C-semi-equivalence η from P to the automorphism representation of $Qd(P)$. The composition of these, $\eta\theta$ is the required C-semi-equivalence. $\square$

Proposition 5.2.22 *Let Q be a connected quandle, q_0 some element of Q, and $P = \{G, H, g_0\}$ a corefree Q-data set which gives rise to a connected quandle. Then $Q \cong Qd(P)$ if and only if there is a C-semi-equivalence from the operator representation Q-data set of Q at q_0 to P*

Proof.

If there is such a semi-equivalence then Lemma (5.2.2) shows that Q and $Qd(P)$ are isomorphic.

Conversely, assume that $Q \cong Qd(P)$. Then there is a C-semi-equivalence θ from the operator representation of Q at q_0 to that of $Qd(P)$ at $[id]$. Since μ is a C-semi-equivalence from the operator representation of $Qd(P)$ to P, we have that $\theta\mu$ is the required C-semi-equivalence. $\square$

We are attempting to reduce the study of homomorphisms between coset quandles to the study of homomorphisms between their base groups. The work so far is a start, but more needs to be done. In particular, it would be useful if we could extend the above collection of lemmas to include maps from associated representations of quandles, not just operator or automorphism representations, to include target coset quandles which are not necessarily corefree, and most importantly, to extend to include homomorphisms, not just isomorphisms.

The first problem is the easiest.

Lemma 5.2.23 *Let Q be a connected quandle, and q_0 some element of Q. Then the canonical homomorphism $\pi_Q : A(Q) \to Op(Q)$ induces a C-semi-equivalence from the associated representation of Q at q_0 to the operator representation of Q at q_0.*

Proof.

There are three conditions that must be satisfied, that $(\hat{q}_0)\pi = \hat{q}_0$, that $(St_{Op}(q_0))\pi^{-1} = St_A(q_0)$, and $A(Q)\pi \cap St_{Op}(q_0)\hat{g} \neq \emptyset \quad \forall g \in Op(Q)$. These are all immediate. $\square$

Proposition 5.2.24 *Let Q be a connected quandle, q_0 some element of Q, and $P = \{G, H, g_0\}$ a corefree Q-data set. Then $Q \cong Qd(P)$ if and only if there is a coset semi-equivalence from the associated representation Q-data set of Q at q_0 to P.*

Proof.

If there is such a semi-equivalence then Lemma (5.2.2) shows that Q and $Qd(P)$ are isomorphic. On the other hand there is a coset semi-equivalence from the associated representation of Q at q_0 to the operator representation of Q at q_0. Since by corollary 5.2.20 μ provides a semi-equivalence from the operator representation of Q to P, these can be composed to give a semi-equivalence from the associated representation to P. $\square$

5.2.1 Other Comments

Lemma 5.2.25 *Let $P = \{G, H, g_0\}$. Then $Q = Qd(P)$ is trivial if and only if $g_0 \in Core(P)$*

Proof.

$$\begin{array}{rcl} Q \text{ trivial} & \Leftrightarrow & [g_1] \rhd [g_2] = [g_1] \ \forall g_1, g_2 \in G \\ & \Leftrightarrow & g_1 g_2^{-1} g_0 g_2 g_1^{-1} \in H \ \forall g_1, g_2 \in G \\ & \Leftrightarrow & g_0 \in core(P) \quad \Box \end{array}$$

Although coset quandles are at first sight useful only for looking at quasi-connected quandles, it should be noted that they may contain subquandles which are not quasi-connected, see the example at the end of section 1.1.3. This raises the interesting question

Question 5.2.26 *When can a quandle be embedded in a quasi-connected quandle ?*

Answering this question may lead to useful facts about link quandles in general.

5.3 Orbits And Decompositions Of Coset Quandles

5.3.1 Orbits Of Coset Quandles

Theorem 5.3.1 *Let S be the orbit of $Q = Qd\{G, H, g_0\}$ containing H, and let $|S| = \bigcup_{p \in S} p$ where the elements of S are considered as subsets of G. Then $|S| = \Sigma \equiv \Sigma(H \cup_{g \in G} g^{-1} g_0 g)$, the subgroup of G generated by H and the conjugates of g_0.*

Proof.
$|S| \subset \Sigma$: Pick $g \in |S|$. Then

$$\begin{array}{rrcl} & [g] & = & [id] \rhd^{\pm 1} [g_1] \rhd^{\pm 1} \cdots \rhd^{\pm 1} [g_n] \text{ some } g_i \in G \\ & & = & [g_1^{-1} g_0^{\pm 1} g_1 \cdots g_n^{-1} g_0^{\pm 1} g_n] \\ \Rightarrow & g & = & h g_1^{-1} g_0^{\pm 1} g_1 \cdots g_n^{-1} g_0^{\pm 1} g_n \text{ some } h \text{ in } H \\ & & \in & \Sigma \end{array}$$

$\Sigma \subset |S|$: We need to show that for any $g \in \Sigma$, there are right cosets $[g_i]$ of H in G s.t. $[g] = [id] \rhd^{\pm 1} [g_1] \rhd^{\pm 1} \cdots \rhd^{\pm 1} [g_n]$
Now $g = h_0 g_1 h_1 \cdots g_n h_n$ where $h_i \in H$ and $g_i = g_{i,1}^{-1} g_0^{\pm 1} g_{i,1} \cdots g_{i,m_i}^{-1} g_0^{\pm 1} g_{i,m_i}$

$$\begin{array}{rcl} [g] & = & [g_1 h_1 \cdots g_n h_n] \\ & = & [g_1 h_1 \cdots g_{n-1} h_{n-1} h_n] \rhd^{\pm 1} [g_{n,1} h_n] \rhd^{\pm 1} \cdots \rhd^{\pm 1} [g_{n,m_n} h_n] \\ & = & [h_1 \cdots h_n] \rhd^{\pm 1} [g_{1,1} h_1 \cdots h_n] \rhd^{\pm 1} \cdots \rhd^{\pm 1} [g_{n,m_n} h_n] \end{array}$$

That is $[g]$ is in the same orbit of Q as $[id] = H$. Since $[g]$ is an arbitrary right coset of H in Σ, $\Sigma \subset |S|$. $\square$

Corollary 5.3.2 *$Qd\{G, H, g_0\}$ is connected $\Leftrightarrow$ $H \cup \{g^{-1}g_0 g\}_{g \in G}$ generates G.*

Lemma 5.3.3 *Consider $Q = Qd\{G, H, g_0\}$, and let S be a non-empty connected collection of right cosets of H that is, a collection of elements of Q, s.t. for any pair H_1, H_2 in S there exists $\hat{h} \in Op(Q)$ s.t. $(H_1)\hat{h} = H_2$. Then for any $g \in G$ the collection of cosets $Sg = \{H_1 g | H_1 \in S\}$ is connected.*

Proof.

Let $S_1 g, S_2 g \in Sg$ where $S_1, S_2 \in S$. S is connected, so there is some $\hat{h} \in Op(Q)$ with $S_2 = S_1\hat{h}$ or $S_2 g = S_1\hat{h}g$. But from Lemma 1.1.3 we have that $Op(Q)$ is normal in $Aut(Q)$, so $S_1\hat{h}g = S_1 g\hat{h}'$ for some $\hat{h}' \in Op(Q)$, and $S_1 g$ and $S_2 g$ are connected. $\square$

Corollary 5.3.4 *The orbits of $Qd\{G, H, g_0\}$ are precisely the right cosets of $\Sigma(\cup_{g\in G}\ g^{-1}g_0 g\ \cup H)$.*

5.3.2 Free Decomposition Of Coset Quandles

In order to find the free decomposition of a coset quandle, we first define the following.

Definition 5.3.5 *Let $G_{g_0} = \{g \in G | g g_0 g^{-1} \in H\} = \{g \in G | g_0 \in g^{-1}Hg\}$ Let $G^c_{g_0}$ be the complement of G_{g_0} in G.*

Proposition 5.3.6 *P, the indecomposable component of Q containing the element Hg, is the collection of right cosets of H of the form $Hg'g$ where $g' \in \Sigma G^c_{g_0}$.*

Proof.

We will use the methods of section 1.2.1, so let S_I = some collection of Hg_i as i ranges through I for some index set I. Then -

$$\begin{aligned}
N(S_I) &= \{Hg \in Q | Hg \triangleright Hg_i \neq Hg \text{ some } i \in I\} \cup S_I \\
&= \{Hg \in Q | g g_i^{-1} g_0 g_i g^{-1} \notin H \text{ some } i \in I\} \cup S_I \\
&= \{Hg \in Q | g g_i^{-1} \in G^c_{g_0} \text{ some } i \in I\} \cup S_I \\
&= \{Hg \in Q | g \in G^c_{g_0} g_i \text{ some } i \in I\} \cup S_I \\
&= \{Hg \in Q | g \in \cup_{i\in I} G^c_{g_0} g_i\} \cup S_I
\end{aligned}$$

Similarly $M(S_I) = \{Hg \in Q | g \in \cup_{i\in I}(G^c_q)^{-1}g_i\} \cup S_I$
So to find $P = \hat{J}(Hg)$, using Theorem 1.2.5 start the above process with $I = \{1\}, S_I = \{Hg\}, g_1 = g$. Each application of N or M takes a set S_I of $[g']$ and adds those obtained by left multiplying by an element of $G^c_{g_0}$ or by the inverse of an element of $G^c_{g_0}$ respectively. The collection of all of these, is precisely $\{[g'g] | g' \in \Sigma G^c_{g_0}\}$, so P is as stated in the theorem. □

So we have an alternative proof of Lemma 5.2.25.

Corollary 5.3.7 *Let $P = \{G, H, g_0\}$. Then $Q = Qd(P)$ is trivial if and only if $g_0 \in core(P)$.*

Proof.
If Q is trivial then H is an indecomposable component of Q so the above proposition implies that G^c_g is a subset of H. That is, $g \notin H$ implies that $g_0 \in g^{-1}Hg$. Since this automatically holds for $g \in H$ as well, we have that $g_0 \in core(P)$. Conversely, if $g_0 \in core(P)$ then $G^c_{g_0}$ is empty, and so the indecomposable component of Q containing quandle element Hg is just Hg. That is, Q is trivial. □

5.3.3 Other Compositions

We state and prove a fairly obvious lemma.

Lemma 5.3.8 *Let $Q_1 = Qd\{G_1, H_1, g_1\}$ and $Q_2 = Qd\{G_2, H_2, g_2\}$ be a pair of coset quandles. Then the direct product of Q_1 and Q_2 can also be written as a coset quandle.*

$$Q_1 \otimes Q_2 \cong Qd\{G_1 \oplus G_2, H_1 \oplus H_2, (g_1, g_2)\}$$

Proof.
Define $\psi : Q_1 \otimes Q_2 \to Qd\{G_1 \oplus G_2, H_1 \oplus H_2, (g_1, g_2)\}$.
$\psi((H_1g_1'), (H_2g_2')) = ((H_1 \oplus H_2)(g_1', g_2'))$. Showing that ψ is well-defined, and an isomorphism is routine. □

5.4 Group Quandles And Alexander Quandles As Coset Quandles

Recall that for any pair (G, γ) where G is some group, and γ some automorphism of G, there is a quandle structure defined on G by the rule

$g_1 \rhd g_2 = \gamma(g_1 g_2^{-1})g_2$. When G is abelian then these are Alexander quandles. Any such quandle is isomorphic to some coset quandle. We first show how these arise from coset quandles, and then show how to construct a coset quandle isomorphic to any given quandle of this form.

Lemma 5.4.1 *Given a coset quandle $Q = Qd\{G, H, g_0\}$, suppose that there is a subgroup G_0 of G with the properties that*

- $|G_0 \cap Hg| = 1 \quad \forall g \in G$.
- *G_0 is invariant under conjugation by g_0.*

Then $Q \cong (G_0, \gamma_0)$ where γ_0 is the automorphism defined by conjugation by g_0.

Proof.

Define $\phi : (G_0, \gamma_0) \to Q$ by $\phi(g) = [g]$. Then ϕ is a Q-homomorphism since

$$\begin{aligned}
\phi(g_1 \rhd g_2) &= [\gamma_0(g_1 g_2^{-1})g_2] \\
&= [g_0^{-1} g_1 g_2^{-1} g_0 g_2] \\
&= [g_1 g_2^{-1} g_0 g_2] \\
&= [g_1] \rhd [g_2] \\
&= \phi(g_1) \rhd \phi(g_2)
\end{aligned}$$

The first condition ensures that ϕ is a bijection and so an isomorphism. $\square$

Proposition 5.4.2 *Any group quandle (G_0, γ_0) is isomorphic to some coset quandle.*

Proof.

Let H_0 be a copy of $\mathbb{Z}$, generated by h_0, let $\theta : H_0 \to Aut(G_0)$ be given by $\theta(h_0) = \gamma_0$, and let $G = G_0 \ltimes_\theta H_0$. Let H be the natural embedding of H_0 in G, ie. $H = (id_{G_0}, H_0)$ and let $g_0 = (id_{G_0}, h_0)$. Since H is abelian, the set $\{G, H, g_0\}$ satisfies the conditions for a Q-data set. Let $Q = Qd\{G, H, g_0\}$ and identify G_0 with its natural embedding in G, $G_0 = (G_0, id_H)$.

We show that G_0 satisfies the hypotheses of the above lemma. For $h = (id_{G_0}, h') \in H$ and $g = (a, x) \in G$ we have $hg = (\theta(h')(a), h'x)$ so $Hg = \{(\theta(h')(a), h'x) | h' \in H\}$ and hence $G_0 \cap Hg = (\theta(x^{-1})(a), id_H)$, and the first condition is satisfied. The second is satisfied since by the semi-direct product construction, G_0 is normal in G. Since γ_0 on G_0 is conjugation by h_0, again by the semi-direct product construction, we have by the above lemma, that $Q \cong (G_0, \gamma_0)$. $\square$

Chapter 6

Stabilizers, Centralisers And The Associate Congruence

In view of their role in coset quandles, stabilizers hold some substantial interest. They turn out to be closely linked with centralisers, and indeed, in the case of the automorphism, operator and associated groups, will be the centralisers, whenever the quandles inject into the respective groups. There is a connection with knot quandles via Ryder's Theorem 3.2.4, as mentioned earlier, that knot quandles inject into their associated group precisely when the knot in question is prime, and also via a theorem of Fenn and Rourke, which was mentioned earlier (see Theorem 3.1.5), which gives a topological interpretation of the stabilizer subgroups in the fundamental groups of the links, which are of course the associated groups of the fundamental quandle.

Throughout this section, we use $\hat{g}$ to refer to an element of the associated group, and $\hat{q}$ to refer to the image of a quandle element q in $A(Q)$. We sometimes use $\hat{g}$ to refer to an element of the operator group, and in particular for some quandle element $q \in Q$, $\hat{q}$ is the image of q in $Op(Q)$. We sometimes use $\hat{g}$ to refer to an arbitrary element of G when we are proving something simultaneously for $G = A(Q), Op(Q)$ or $Aut(Q)$. Which use should be clear from the context.

6.1 The Associate And Operator Congruence

We wish to study the injectivity or otherwise of quandles into their various groups.

Definition 6.1.1 *The* Operator Congruence *is defined as the relation on Q*

$$q_1 \equiv_{Op} q_2 \text{ if and only if } \hat{q}_1 = \hat{q}_2 \text{ in } Op(Q).$$

The Associate *Congruence is defined as the relation on* Q

$$q_1 \equiv_{As} q_2 \text{ if and only if } \hat{q}_1 = \hat{q}_2 \text{ in } A(Q).$$

The congruence classes containing some element q *will be denoted* $[q]_{Op}$ *and* $[q]_{As}$ *respectively. A quandle is* reduced associate *if and only if it injects into its associated group, and* reduced operator *if and only if it injects into the operator group.*

The operator congruence is called 'behavioural equivalence' in [12]. Note that $q_1 \equiv_{As} q_2 \Rightarrow q_1 \equiv_{Op} q_2$ that is, $\equiv_{As} < \equiv_{Op}$, and so Q is reduced operator $\Rightarrow Q$ is reduced associate. These are necessarily congruences since equivalence classes are the preimages of elements in the operator and associated groups respectively, considers as conjugacy quandles. These have been defined and studied before. See for example Joyce in [12] and Ryder in [22] and [23].

Example - Let Q be a group with the conjugacy quandle structure on it, then $Q/\equiv_{Op}$ is the group $Q/Z(Q)$ with the conjugacy quandle structure on it, where $Z(Q)$ is the group-theoretic centre of Q.

Joyce observes in [12] that a quandle injects into some group with the standard quandle structure on it, if and only if it is reduced associate. The associate congruence is particularly interesting for knot quandles: a theorem of Ryder [22] states that a knot quandle is reduced associate if and only if the knot is prime. We next state and prove some basic facts about these congruences, starting with one originally stated and proved in [22] as Theorem 2.18.

Lemma 6.1.2 $A(Q) \cong A(Q/\equiv_{As})$ *hence* $Op(Q) \cong Op(Q/\equiv_{As})$*.*

Proof.

We will denote the congruence classes of the associate congruence by $[q]$, suppressing the subscript As.

Let ϕ be the canonical map from Q to $Q/\equiv_{As}$. A is functorial, so let $\hat{\phi} = A(\phi) : A(Q) \to A(Q/\equiv_{As})$, that is $(\hat{q})\hat{\phi} = \widehat{(q)\phi} = \widehat{[q]}$. I claim that this is an isomorphism, and prove it by providing an inverse. Define

$$(\widehat{[q]})\hat{\phi}' = \hat{q}$$

on the generators of $A(Q/\equiv_{As})$, and extend linearly. This is well defined on the generators since $q_1 \equiv_{As} q_2$ precisely when $\hat{q}_1 = \hat{q}_2$. To check that it is well defined on the whole of $A(Q/\equiv_{As})$, first note that the relations to be checked are

$$\widehat{[q_1] \rhd [q_2]} = \widehat{[q_1 \rhd q_2]} = \widehat{[q_2]}^{-1}\widehat{[q_1]}\widehat{[q_1]}$$

so we require

$$\widehat{q_1 \triangleright q_2} = ([\widehat{q_1] \triangleright [q_2}])\phi' = (\widehat{[q_2]}^{-1})\phi'(\widehat{[q_1]})\phi'(\widehat{[q_1]})\phi' = \widehat{q_2}^{-1}\widehat{q_1}\widehat{q_1}$$

but these are just the defining relations of $A(Q)$.
It is easy to see that ϕ' is an inverse to $\hat{\phi}$. $\square$

Lemma 6.1.3 *Congruence classes of the associated and operator congruences are trivial.*

Proof.
Since $\equiv_{As} < \equiv_{Op}$ we need only show that $q_1 \equiv_{Op} q_2 \Rightarrow q_1 \triangleright q_2 = q_1$. But $q_1 \equiv_{Op} q_2 \Rightarrow q_1 \triangleright q_2 = q_1\hat{q}_2 = q_1\hat{q}_1 = q_1$ $\square$

Lemma 6.1.4 *The congruence classes of $\equiv_{As}$ are $Op(Q)$-connected.*

Proof.
Recall that A is functorial and consider the following commuting diagram.

$$\begin{tikzcd}
Q \arrow[r, "\bigcirc"] \arrow[d, "A"'] & \bigcirc(Q) \arrow[d, "A"] \\
A(Q) \arrow[r, "\hat{\bigcirc}"] & A(\bigcirc(Q))
\end{tikzcd}$$

Where $\bigcirc$ is the orbit congruence. From Lemma 1.1.12, $\bigcirc(Q)$ is trivial and, as noted in the comments to section 1.1.4, page 9, must therefore inject into $A(\bigcirc(Q))$. Hence for $q_1, q_2 \in Q$ we have

$$\hat{q}_1 = \hat{q}_2 \Rightarrow \widehat{\bigcirc(q_1)} = \widehat{\bigcirc(q_2)} \Rightarrow \bigcirc(q_1) = \bigcirc(q_2)$$

That is any two elements in the same congruence class of A are connected. $\square$

Lemma 6.1.5 *The associate and operator congruences are respected by automorphisms.*

Proof.
First the associate congruence. Let g be an automorphism of Q. Then

$$(\widehat{q_1 \triangleright q_2})g = (\widehat{q_1g) \triangleright (q_2g}) = \widehat{q_2g}^{-1}\widehat{q_1g}\,\widehat{q_2g}$$

Since these are the defining relations of $A(Q)$, all relations that hold between the $\hat{q}_i$ also hold between $\widehat{q_i g}$, in particular $\hat{q}_1 = \hat{q}_2 \Rightarrow \widehat{q_1 g} = \widehat{q_2 g}$.

Second, the operator congruence. Let g be an arbitrary automorphism.

$$
\begin{aligned}
\hat{q}_1 = \hat{q}_2 &\Rightarrow q\hat{q}_1 = q\hat{q}_2 \ \forall q \in Q \\
&\Rightarrow ((qg^{-1}) \rhd q_1)g = ((qg^{-1}) \rhd q_2)g \ \forall q \in Q \\
&\Rightarrow q \rhd (q_1 g) = q \rhd (q_2 g) \ \forall q \in Q \\
&\Rightarrow \overline{q_1 g} = \overline{q_2 g} \quad \Box
\end{aligned}
$$

Lemma 6.1.6 *If Q is reduced associate, and $Q' < Q$ then Q' is reduced associate.*

Proof.
Consider the following commuting diagram.

$$
\begin{tikzcd}
Q' \arrow[r, "\iota"] \arrow[d, "A_{Q'}"'] & Q \arrow[d, "A_Q"] \\
A(Q') \arrow[r, "\hat{\iota}"] & A(Q)
\end{tikzcd}
$$

$\hat{\iota} A_{Q'} = A_Q \iota$ injects, so $A_{Q'}$ injects. $\Box$

Lemma 6.1.7 *If Q is reduced associate, then for two elements q and p, q acts trivially on p if and only if p acts trivially on q.*

Proof.
If Q is reduced associate, then

$$
\begin{aligned}
q \rhd p = q &\Leftrightarrow \widehat{q \rhd p} = \hat{q} \\
&\Leftrightarrow \hat{p}^{-1}\hat{q}\hat{p} = \hat{q} \\
&\Leftrightarrow \hat{q}^{-1}\hat{p}\hat{q} = \hat{p} \\
&\Leftrightarrow \widehat{p \rhd q} = \hat{p} \\
&\Leftrightarrow p \rhd q = p \quad \Box
\end{aligned}
$$

We next look at whether a coset quandle is reduced operator or not, or more generally, when two elements of the coset quandle map to the same element in the operator group. For any coset quandle we will define a subgroup of the base group G of the quandle, the Corner group, and then show that this consists of all elements of the quandle which map to the same operator element as the defining group H. Then the preimage of any element in the operator group will be a right coset of the Corner group. Finally we will give a worked example of the Corner group in action.

Definition 6.1.8 *For some Q-data set* $P = \{G, H, g_0\}$*, the* corner group*, $C(P)$ is the subset of* $G, C(P) = \{g \in G | g_0 g g_0^{-1} g^{-1} \in core(P)\}$.

Lemma 6.1.9 *The corner group of a Q-data set is a group. Furthermore* $H < C(P)$ *and* $Z_G(g_0) < C(P)$.

Proof.

That the identity of G is in $C(P)$ is obvious. First we show that if $g \in C(P)$ then $g^{-1} \in C(P)$.

$$\begin{array}{rcll} & g & \in & C(P) \\ \Rightarrow & (g_0 g g_0^{-1} g^{-1})^{-1} & \in & core(P) \\ \Rightarrow & g_0 g^{-1} g_0^{-1} g & \in & g^{-1}\, core(P) g = core(P) \\ \Rightarrow & g^{-1} & \in & C(P) \end{array}$$

Next, if g_1 and g_2 are in $C(P)$, then so is $g_1 g_2$.

$$g_0(g_1 g_2) g_0^{-1}(g_2^{-1} g_1^{-1}) = (g_0 g_1 g_0^{-1} g_1^{-1}) g_1 (g_0 g_2 g_0^{-1} g_2^{-1}) g_1^{-1}$$

Now by hypothesis the bracketed terms are in $core(P)$ which is normal, and so after conjugation by g_1 remain in $core(P)$. Hence the whole expression is in $core(P)$, and $C(P)$ is a group.
That $H < C(P)$ is precisely the condition given in the construction of coset quandles. The last statement about $Z_G(q_0)$ are obvious. $\square$

The Corner group is, in some sense, the generalisation of the centraliser of g_0 in G; if the coset quandle $Qd\{G, H, g_0\}$ is corefree then the Corner group is just the centralizer of g_0 in G. If H is the whole of G, then the Corner group is also the whole of G.

Lemma 6.1.10 *Let* $P = \{G, H, g_0\}$ *be a Q-data set and let* $Q = Qd(P)$*. Then Q is reduced operator precisely when* $C(P) = H$.
Furthermore, if Q is corefree then Q is reduced operator precisely when $H = Z_G(g_0)$*, the centraliser of* $\hat{g}$ *in* G.

Proof.

$$\begin{array}{rl} & \overline{Hg_1} = \overline{Hg_2} \\ \text{if and only if} & Hg \rhd Hg_1 = Hg \rhd Hg_2 \ \ \forall g \in G \\ \text{if and only if} & g g_1^{-1} g_0 g_1 g_2^{-1} g_0^{-1} g_2 g^{-1} \in H \ \ \forall g \in G \\ \text{if and only if} & g_0 g_1 g_2^{-1} g_0^{-1} g_2 g_1^{-1} \in core(H) \\ \text{if and only if} & g_1 g_2^{-1} \in C(P) \end{array}$$

So for Q to be reduced operator requires that $g \in C(P)$ if and only if $g \in H$, or to put it more conventionally, $C(P) = H$. If Q is a corefree coset quandle, then $C(P) = Z_G(g_0)$ and so Q is reduced operator precisely when $H = Z_G(g_0)$. $\square$

Now for a worked example of the Corner group. Let G be the dihedral group D_{16}

$$G = D_{16} = \langle a, b | a^8 = b^2 = id, b^{-1}ab = a^{-1} \rangle$$

Let H be the subgroup generated by a^4 and b.

$$H = \{id, a^4, b, a^4b\}$$

The core of H in G is $\{id, a^4\}$. This is easily seen to be normal in G since $b^{-1}a^4b = a^4$, on the other hand any normal subgroup of G containing b must also contain $a^{-1}ba = a^6b$ and so cannot be contained in H. Let $g_0 = b$, then $ba^4b^{-1}a^{-4} = id$ and so $P = \{D_{16}, H, b\}$ satisfies the conditions for a Q-data set. Let $Q = Qd\{D_{16}, H, b\}$. Now the corner group for P is

$$C(P) = \{g \in D_{16} | bgb^{-1}g^{-1} \in core(P)\}.$$

Since b is already known to be in $C(P)$, we only need check for which values of n we have that a^n in $C(P)$. Since $ba^nb^{-1}a^{-n} = a^{-2n}$, this will happen precisely when $2n = 0$ or $4 \bmod 8$. Hence we can list the elements of the Corner group for this Q-data set.

$$C(P) = \{id, b, a^2, a^2b, a^4, a^4b, a^6, a^6b\}$$

We mention another use for the Corner group. In Corollary 5.2.2 it is shown that for g some element of $N(H)$, the normaliser of H in G, then for fixed g, the conjugation function that takes $[g'] \mapsto [g^{-1}g'g]$ is a Q-isomorphism from $Qd\{G, H, g_0\}$ to $Qd\{G, H, g^{-1}g_0g\}$. Lemma 5.2.5 shows that if $g_0 g_0'^{-1} \in core(P)$ then $Qd\{G, H, g_0\}$ is identical to $Qd\{G, H, g_0'\}$. Putting $g_0' = g^{-1}g_0g$ we see the following lemma.

Lemma 6.1.11 *Let g be some element of $N(H) \cap C(H)$. Then conjugation by g is a Q-automorphism of $Q = Qd\{G, H, g_0\}$.*

6.2 Stabilizers And Centralisers

For S some subset of a quandle Q and some group G acting on Q, we mean by the stabilizer of S, the setwise stabilizer.

$$St_G(S) = \{\hat{g} \in G | q\hat{g} \in S \ \forall q \in S\}$$

In the case $G = Op(Q), Aut(Q)$ or $A(Q)$ we refer to $St_{Op}(\cdot), St_{Aut}(\cdot)$ or $St_A(\cdot)$.

Lemma 6.2.1 *Let G be the operator (resp. associated group) of some quandle Q. Then -*

$$Z_G(\hat{q}_0) = St_G([q_0]_{\equiv_G})$$

If furthermore Q is connected then

$$|[q_0]_{\equiv_G}| = [St_G(q_0) : Z_G(\hat{q}_0)]$$

where $[St_G(q_0) : Z_G(\hat{q}_0)]$ is the index of $St_G(q_0)$ in $Z_G(\hat{q}_0)$.

$St_G(q_0) = Z_G(\hat{q}_0)$ if and only if Q is reduced operator (resp. if and only if Q is reduced associate).

Proof.

First statement.

$$Z_G(\hat{q}_0) = \{g \in G | g^{-1}\hat{q}_0 g = \hat{q}_0\} = \{g \in G | \overline{\hat{q}_0 g} = \hat{q}_0\} = St_G([q_0]_{\equiv_G})$$

Second statement.

Recall from 5.1.5 that ρ is the canonical isomorphism from Q to one of its representations.

$$\begin{aligned} Z_G(\hat{q}_0) &= \{g \in G | \overline{q_0 g} = \hat{q}_0\} \\ &= \textstyle\bigcup_{q \in [q_0]_{\equiv_G}} \{g \in G | q_0 g = q\} \\ &= \textstyle\bigcup_{q \in [q_0]_{\equiv_G}} (q)\rho \end{aligned}$$

Since $(q)\rho$ is the right coset of $St_G(\hat{q}_0)$ corresponding to q, we have that $|[q_0]_{\equiv_G}| = [St_G(q_0) : Z_G(\hat{q}_0)]$.

Third statement.

$Z_G(\hat{q}_0) = St_G(q_0)$ if and only if $[q_0]_G = q_0$, that is if and only if Q is reduced operator (resp. if and only if Q is reduced associate). $\square$

With this result we can obtain the following.

Lemma 6.2.2 *If Q and Q' are connected quandles that are both reduced associate, with $q_0 \in Q$ and $q'_0 \in Q'$ then $Q \cong Q'$ if and only if there exists a group isomorphism $\theta : A(Q) \to A(Q')$ s.t. $\theta(\hat{q}_0) = \hat{q}'_0$.*

Proof.

From comments in section 5.1 of chapter 5, page 55 onwards, two connected quandles Q and Q' are Q-isomorphic if and only if their associated representations are C-isomorphic. That is, if and only if for $q_0 \in Q$ and $q'_0 \in Q'$ there is an isomorphism $\theta : A(Q) \to A(Q')$ with the property that $\theta(St_{A(Q)}(q_0)) = St_{A(Q')}(q'_0)$ and $\theta(q_0) = q'_0$. Since the hypotheses of the lemma state that Q and Q' are both reduced associate, we have that $St_{A(Q)}(q_0) = Z_{A(Q)}(\hat{q}_0)$ and similarly for Q'. We need only note that any isomorphism obeying the above conditions automatically takes the centraliser of q_0 in Q to the centraliser of q'_0 in Q' and we are done. $\square$

A result of Ryder (see 3.2.4) states that a knot quandle is reduced associate if and only if the knot is prime. For some knot quandle Q then, where $A(Q)$ is the fundamental group of the knot, and $\hat{q}_0$ the image of q_0 in $A(Q)$ is a meridian in the fundamental group of the knot, and the above can be interpreted as a well known result of C.M.Gordon (see [10]) that two prime knots are equivalent if and only if there is an isomorphism between their fundamental groups taking a meridian to a meridian.

We can now contribute towards an answer to the question asked on page 9: if $\pi : A(Q) \to Op(Q)$, then what is the kernel of π?

Lemma 6.2.3 *If Q is reduced associate then $ker(\pi) = Z(A(Q))$, the centre of $A(Q)$.*

Proof.
Suppose that Q is a reduced associate quandle. Let $\hat{g} \in A(Q)$. Then

$$\begin{aligned} \hat{g} \in ker(\pi) &\Leftrightarrow q = q\hat{g} \ \forall q \in Q \\ &\Leftrightarrow \hat{q} = \widehat{q\hat{g}} = \hat{g}^{-1}\hat{q}\hat{g} \ \forall q \in Q \end{aligned}$$

Since $\{\hat{q}|q \in Q\}$ generates $A(Q)$, we have that $\hat{g} \in ker(\pi)$ if and only if $\hat{g}$ is in the centre of $A(Q)$. $\square$

Chapter 7

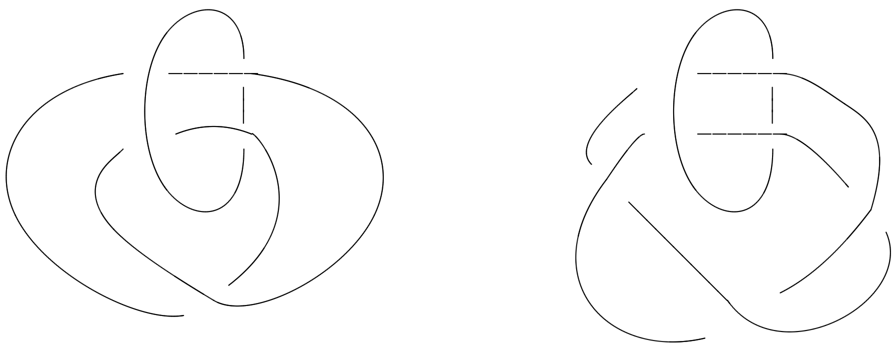
Information Loss In The Passage From Fundamental Quandles To Fundamental Groups

The fundamental quandle is an (almost) complete invariant of non-split links, the only fly in the ointment being two links which are mirror images of each other, with the orientations of each component reversed. The fundamental group of a link is the associated group of the fundamental quandle, and computing a presentation of the fundamental group of a link given the fundamental quandle is easy, yet the fundamental group is not a complete invariant.

It is known that in the case of knots the fundamental group determines the prime constituents of a knot up to a possible change of orientation, see [9] and also [25]. This is connected to Gordon and Luecke's result that knots are determined by their complements, see [**?**], and also to Ryder's theorem that the fundamental quandle of a knot injects into the fundamental group of the knot if and only if the knot is prime (see theorem 3.2.4 in this thesis.) As a consequence of this, a knot with n prime factors shares its fundamental group with at most $(2^{n-1} - 1)$ other knots.

It has also been shown that the generalised knot groups of a knot do tell (unoriented) knots apart, up to mirror image, as has been shown by S.Nelson in [20]. For a specific example see C.Tuffley in [26]. These generalised knot groups are just the second and higher associated groups of the fundamental quandle of a knot, see A.J. Kelly [15], although they were also independently defined in a topological way by M.Wada in [28].

In the more general case of links, the fundamental group holds less information. There are non-split links whose components are quite different, yet they have homeomorphic exteriors and so homomorphic fundamental groups. The following example of two links with homeomorphic exteriors is taken from Dale Rolfsen's book [21, p.49].



This cannot happen with the fundamental quandle, indeed as shown in chapter 4, it is quite easy to find the fundamental quandles of the individual components given the fundamental quandle of the entire link.

Where has information been lost in the passage from quandle to group? We will look at this question from two different perspectives. First of all the image of the fundamental quandle in the fundamental group is examined, and then, in the case of knots, the associated coset quandle will be used to gain insight into this question. Of course the fundamental group of the link is just the associated group of the fundamental quandle.

The quandle homomorphism from the fundamental quandle to the fundamental group.

Lack of Surjectivity.
The image of a quandle in its associated group is never the whole group, indeed it never includes the identity of the group (see 1.1.4). In the case of link quandles, the image is the set of 'meridian paths' in the fundamental group. The use of this extra information is underlined by the theorem of Gordon that two prime knots are equivalent if and only if there is an isomorphism of their fundamental groups which take a meridian to a meridian. For knots, the set of meridians is a single conjugacy class, and so knowing one meridian is equivalent to knowing all meridians. However in the more general case of links the image is a union of conjugacy classes, making life potentially more difficult.

Lack of Injectivity
In the case of knots, by Ryder's theorem (theorem 3.2.4) it is precisely when

a knot is prime that its quandle injects to the group, and so is isomorphic to its image. This is one reason why Gordon's theorem only works for prime knots: If we know that the images of the quandles of two prime knots in their respective fundamental groups are isomorphic, then the two knots are equivalent. Since the image of the fundamental quandle is the conjugacy class of meridians, this is precisely the content of Gordon's theorem.

Reconstructing the Quandle.
Suppose for some knot that the set of meridians, that is the image of the fundamental quandle in the fundamental group, is known. Suppose also that the preimage of any meridian in the fundamental quandle is known. Since this is a trivial quandle, (see Lemma 6.1.3 on page 77) this amounts to knowing the number of prime summands of the knot. What else is needed to reconstruct the original quandle? Quandle cohomology is being studied, notably by J.Scott Carter et.al. see for example [6]. This may help provide answers.

The associated coset quandle representation.

In the case of knots, the question may be looked at from a different perspective, coset quandles. For some knot K with fundamental quandle $Q(K)$ and fundamental group $F(K)$ we have that $Q(K) \cong Qd\{F(K), St(q_0), \hat{q}_0\}$ and so two such quandles are isomorphic if and only if there is an isomorphism between the fundamental groups which takes the stabilizer subgroup of one to the stabilizer subgroup of the other and a meridian of one to a meridian of the other. Hence these three pieces of data, $F(K), St(q_0)$ and $\hat{q}_0$ provide a complete invariant of K. The last datum is, as before, knowledge of the meridians in the fundamental group. So the middle datum, $St(q_0)$ holds the extra information about injectivity of the quandle into the group, and also the 'quandle reconstruction' information. The 'injectivity information' is kept in the stabilizer, since it is the inverse image of the centraliser, not the stabilizer, which maps to the same element of the fundamental group, as noted in the above paragraph on injectivity. Hence the 'reconstruction' information is kept in the nature of the stabilizer as a subgroup of the centraliser.

One last comment. The stabilizer is a key piece of information. In Theorem 2.3.2 there is for any finitely presented quandle Q, and any given element q of Q, an algorithm to give a generating set for the stabilizer Of q in Q. This generating set consists of $\hat{q}$ and certain elements $g \in A(Q)$ where $qg = q$ are defining relators in the quandle presentation. The algorithm mentioned takes one relator for each generator after the first, and any relators left over contribute to the generating set. In the case of knot quandles there will be exactly as many relators as crossings, as there are in a Wirtinger knot presentation, and so there is precisely one generator of the stabilizer arising from a relator in Q. This is unsurprising, the stabilizer is the peripheral subgroup of q, which in turn is essentially the fundamental group of a regular neighbourhood of the knot, which is a torus. Hence the peripheral subgroup of q is the free abelian group on two generators, one of which is the meridian, $\hat{q}$ and the other, a longitude, which arises from the relator. Now, when the quandle is taken to the fundamental group, each relator $q_1 = q_2 g$ maps to a relator $\hat{q}_1 = g^{-1}\hat{q}_2 g$ in a Wirtinger presentation of the fundamental group. This can be interpreted as the unsurprising result that longitudes commute with meridians in the fundamental group. However, as is well known, once a Wirtinger presentation for the fundamental group of a knot has been written down, any one relator can be deleted as a consequence of the rest. It seems to be in this operation that information is being lost.

Chapter 8

Homomorphism Invariants

The idea of finding invariants of quandles by looking at homomorphisms onto other, simpler, quandles is not new, in fact the well known three colouring invariant can be interpreted as counting homomorphisms onto the quandle made from the group C_3 with the structure $q_1 \triangleright q_2 = 2q_2 - q_1 \ mod\ 3$. In [11] by Benita Ho and Sam Nelson, and [19] by Sam Nelson, work has been done towards listing finite quandles, and counting homomorphisms from finitely presented quandles to these. See also [17] by Pedro Lopes and Dennis Roseman. Here we will look at homomorphisms to a family of coset quandles constructed from lower triangular matrix groups.

8.1 Basic Ideas

Let Q_n^T be the coset quandle

$$Q_n^T = Qd\{M_n^T, D_n, \hat{m}_0\}$$

where M_n^T is the group of (invertible) $n \times n$ lower triangular matrices with entries in some ring R, which will usually be $\mathbb{C}$ or some ring of polynomials with coefficients in $\mathbb{C}$, and D_n is the subgroup of diagonal matrices. Notice that since D_n is abelian, the condition

$$\hat{m}\hat{m}_0\hat{m}^{-1}\hat{m}_0^{-1} \in core(D_n) \ \forall \ \hat{m} \in D_n$$

is satisfied for all $\hat{m}_0 \in D_n$, and so any such $\hat{m}_0$ suffices to form a quandle. As usual, for $\hat{m} \in M_n^T$ denote the right coset of D_n containing $\hat{m}$ by $[\hat{m}]$.

There are two advantages to using this coset quandle, first D_n is easy to parametrise. Secondly it is easily seen that each right coset of D_n in M^T, that is, each element of the quandle, has a unique representative with ones on the diagonal, making comparisons of elements easy.

Definition 8.1.1 *For some lower triangular matrix $\hat{m}$, the* standard representative*, $(\hat{m})R$ of $[\hat{m}]$ is*

$$(\hat{m})R_{i,j} = \frac{\hat{m}_{i,j}}{\hat{m}_{i,i}}$$

Note that $[\hat{m}_1] = [\hat{m}_2]$, if and only if $(\hat{m}_1)R = (\hat{m}_2)R$ and that $((\hat{m})R)R = (\hat{m})R$. Sometimes rather than specifying a homomorphism $\theta : Q \to Q_n^T$, it is easier to specify the function $\tilde{\theta} : Q \to M_n^T$, $(q)\tilde{\theta} = ((q)\theta)R$ so $(q)\theta = [(q)\tilde{\theta}]$. Of course the $\hat{m}_0$ is also important to the structure of the target quandle, however this will not be specified, instead all possible $\hat{m}_0$ will be analysed simultaneously.

Definition 8.1.2 *A homomorphism is* trivial *if and only if the image of Q under the homomorphism is a trivial quandle.*

Trivial homomorphisms are always possible, map everything in Q onto (any) single element, and at best only reflect the number of orbits in Q, so we will only count non-trivial homomorphisms. Since right multiplication by any element of M_n^T induces an automorphism of Q_n^T, if there are any non-trivial solutions, then there are infinitely many.

Recall from chapter 5 that the core of Q_n^T is of interest.

Lemma 8.1.3 *Let P_n^T be the Q-data set $P_n^T = \{M_n^T, D_n, \hat{m}_0\}$. Then $Core(P^T) = \{\lambda I | \lambda \in \mathbb{C}\}$ where I is the identity matrix.*

Proof.

That the given group of matrices is normal in M^T and contained in D_n is obvious and so must be contained in the core of P_n. Conversely suppose $\hat{m}$ is a diagonal matrix with $m_{i,i} \neq m_{j,j}$ for some $i > j$. Let $\hat{m}'$ be the matrix that has ones on the diagonal and in the (i,j)'th position with zeros everywhere else. Then $\hat{m'}^{-1}\hat{m}\hat{m}'$ is the matrix with ones on the diagonal and $m_{j,j} - m_{i,i}$ in the (i,j)'th position. So $\hat{m} \notin core(P^T)$. $\square$

From Lemma 5.2.5 we have that values of $\hat{m}$ that differ by left multiplication by an element of $core(D_n)$, that is by multiplication of a non-zero scalar, will give the same target quandle. Hence we can simplify calculations by fixing (any) one of the entries of $\hat{m}$ to be 1.

Given some homomorphism θ from some quandle Q to Q_n^T it is useful in calculations to define the function $\hat{\theta} : Q \to M^T, (q)\hat{\theta} = \hat{m}^{-1}\hat{m}_0\hat{m}$ where $\hat{m} = ((q)\theta)R$ is the standard representative of $(q)\theta$ so that we have

$$(q_1\hat{q}_2)\theta = [((q_1)\tilde{\theta})(q_2)\hat{\theta}]$$

Of course $\hat{\theta}$ is just a function, not a homomorphism.

8.2 Examples

8.2.1 4_1, The Figure 8 Knot.

We calculate homomorphisms from the fundamental quandle of the figure eight knot to Q_2^T. Presentations for the fundamental quandles of various knots are given in an appendix, the fundamental quandle of the figure eight knot has a presentation

$$Q(4_1) \cong \langle q_1, q_3 | q_3 = q_1 \hat{q}_3 \hat{q}_1^{-1} \hat{q}_3^{-1} \hat{q}_1, q_1 = q_3 \hat{q}_1 \hat{q}_3^{-1} \hat{q}_1^{-1} \hat{q}_3 \rangle$$

So choose arbitrary matrices

$$\hat{m}_0 = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, (q_1)\theta = \left[\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\right], (q_3)\theta = \left[\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\right]$$

Then we have

$$(q_1)\hat{\theta} = \begin{pmatrix} 1 & 0 \\ (m-1)a & m \end{pmatrix}, (q_3)\hat{\theta} = \begin{pmatrix} 1 & 0 \\ (m-1)b & m \end{pmatrix}$$

The relations become

$$\begin{aligned}
(q_1 \hat{q}_3 \hat{q}_1^{-1} \hat{q}_3^{-1} \hat{q}_1)\theta &= \left[\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ (m-1)b & m \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -m^{-1}(m-1)a & m^{-1} \end{pmatrix}\right. \\
&\qquad \left.\begin{pmatrix} 1 & 0 \\ -m^{-1}(m-1)b & m^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ (m-1)a & m \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 1 & 0 \\ \frac{(m^2-2m+1)(b-a)+am}{m} & 1 \end{pmatrix}\right] \\
&:= \left[\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\right] \\
&= (q_3)\theta
\end{aligned}$$

and

$$\begin{aligned}
(q_3 \hat{q}_1 \hat{q}_3^{-1} \hat{q}_1^{-1} \hat{q}_3)\theta &= \left[\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ (m-1)a & m \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -m^{-1}(m-1)b & m^{-1} \end{pmatrix}\right. \\
&\qquad \left.\begin{pmatrix} 1 & 0 \\ -m^{-1}(m-1)a & m^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ (m-1)b & m \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 1 & 0 \\ \frac{(m^2-2m+1)(a-b)+bm}{m} & 1 \end{pmatrix}\right] \\
&:= \left[\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\right] \\
&= (q_3)\theta
\end{aligned}$$

So for θ to define a proper homomorphism we must have that

$$\frac{(m^2 - 2m + 1)(b - a) + am}{m} = b$$

and

$$\frac{(m^2 - 2m + 1)(a - b) + bm}{m} = a$$

Both relations are equivalent to the equation

$$\frac{(m^2 - 3m + 1)(a - b)}{m} = 0$$

This is clearly satisfied when either $a = b$, the trivial case, or when $m = \frac{3 \pm \sqrt{5}}{2}$. In the latter case, the two generators can map to *any* pair of cosets, to define a homomorphism.

8.2.2 Braid Index Two Knots

In this section we calculate homomorphisms from knots which are the closures of braids on two strands, and show the following.

Proposition 8.2.1 *Let K_{2n+1}, for n an integer greater than or equal to one, be the braid index 2 knot that is the closure of the braid σ^{2n+1} where σ is the standard generator for the braid group on two strings. Then there exist non-trivial homomorphisms of $Q(K_{2n+1})$ into $Q^T_{\hat{m}_0,2}$ if and only if $\hat{m}_0 = \begin{pmatrix} x & 0 \\ 0 & m_2 x \end{pmatrix}$ where m_2 is a $(2n+1)$'th root of -1, $m_2 \neq -1$.*

Corollary 8.2.2 *For distinct n, the knots K_{2n+1} are distinct from each other, and from the figure 8 knot.*

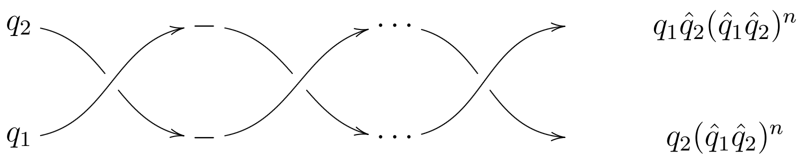
First we state

Lemma 8.2.3 *For $n \in \mathbb{N}$ we have*

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ b\sum_{i=0}^{n-1}(a^i c^{n-i-1}) & c^n \end{pmatrix}$$

Proof.
Use induction on n. $\square$
Next, write down a quandle presentation for K_{2n+1}.

Lemma 8.2.4 *Colour a diagram of σ^{2n+1} s.t. at the left hand the strands are labeled q_1 and q_2. Then at the right hand side, the strands are labeled by $q_2(\hat{q}_1\hat{q}_2)^n$ and $q_1\hat{q}_2(\hat{q}_1\hat{q}_2)^n$.*



Recall from identity 1.1 that for any quandle Q, $q_1, q_2 \in Q$ and $\hat{g} \in Op(Q)$ we have $q_1 \triangleright (q_2\hat{g}) = q_1\hat{g}^{-1}\hat{q}_2\hat{g}$.

Proof.

Use induction on n. If $n = 0$ then the output strands are labeled q_2 and $q_1\hat{q}_2$, as claimed. If after $2(n-1)+1 = 2n-1$ 'twists' strands are labeled $q_2(\hat{q}_1\hat{q}_2)^{n-1}$ and $q_1\hat{q}_2(\hat{q}_1\hat{q}_2)^{n-1}$ then after the next application of σ they will be labeled $q_1\hat{q}_2(\hat{q}_1\hat{q}_2)^{n-1}$ and

$$\begin{aligned} &(q_2(\hat{q}_1\hat{q}_2)^{n-1}) \triangleright (q_1\hat{q}_2(\hat{q}_1\hat{q}_2)^{n-1}) \\ &= q_2(\hat{q}_1\hat{q}_2)^{n-1}(\hat{q}_1\hat{q}_2)^{-(n-1)}\hat{q}_2^{-1}\hat{q}_1\hat{q}_2(\hat{q}_1\hat{q}_2)^{n-1} \\ &= q_2(\hat{q}_1\hat{q}_2)^n \end{aligned}$$

Then after one further 'twist', this becomes $q_2(\hat{q}_1\hat{q}_2)^n$, and

$$\begin{aligned} &(q_1\hat{q}_2(\hat{q}_1\hat{q}_2)^{n-1}) \triangleright (q_2(\hat{q}_1\hat{q}_2)^n) \\ &= q_1\hat{q}_2(\hat{q}_1\hat{q}_2)^{n-1}(\hat{q}_1\hat{q}_2)^{-n}\hat{q}_2(\hat{q}_1\hat{q}_2)^n \\ &= q_1\hat{q}_2(\hat{q}_1\hat{q}_2)^n \end{aligned}$$

and the lemma is proved. $\square$

Corollary 8.2.5

$$Q(K_{2n+1}) \cong \langle q_1, q_2 | q_1 = q_2(\hat{q}_1\hat{q}_2)^n, q_2 = q_1\hat{q}_2(\hat{q}_1\hat{q}_2)^n\rangle \quad \square$$

Now to prove Proposition 8.2.1, proceed as in the above section. Let

$$\hat{m}_0 = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, (q_1)\theta = \left[\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\right], (q_2)\theta = \left[\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\right]$$

So

$$(q_1)\hat{\theta}(q_2)\hat{\theta} = \begin{pmatrix} 1 & 0 \\ (m-1)(a+mb) & m^2 \end{pmatrix}$$

and, using Lemma 8.2.3 above,

$$((q_1)\hat{\theta}(q_2)\hat{\theta})^n \quad = \quad \begin{pmatrix} 1 & 0 \\ (m-1)(a+mb)\sum_{i=0}^{n-1} m^{2i} & m^{2n} \end{pmatrix}$$

So the relations we need to satisfy become

$$\left[\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & 0 \\ b+(m-1)(a+mb)\sum_{i=0}^{n-1} m^{2i} & m^{2n} \end{pmatrix}\right]$$

or

$$a = \frac{b+(m-1)(a+mb)\sum_{i=0}^{n-1} m^{2i}}{m^{2n}}$$

and

$$\left[\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & 0 \\ a+(m-1)b+m(m-1)(a+mb)\sum_{i=0}^{n-1} m^{2i} & m^{2n+1} \end{pmatrix}\right]$$

or

$$b = \frac{a+(m-1)b+m(m-1)(a+mb)\sum_{i=0}^{n-1} m^{2i}}{m^{2n+1}}$$

We look at two cases.
<u>Case1: $m=-1$</u>
In this case both equations reduce to $(2n+1)(b-a)=0$. $2n+1=0$ is clearly impossible, while $a=b$ gives the trivial solutions.
<u>Case2: $m \neq -1$</u>
In this case, the two equations to be solved can be rewritten

$$(m+1)m^{2n}a = (m+1)b + (a+mb)(m^2-1)\sum_{i=0}^{n-1} m^{2i}$$

and

$$(m+1)m^{2n+1}b = (m+1)a + (m^2-1)b + m(a+mb)(m^2-1)\sum_{i=0}^{n-1} m^{2i}$$

Recall the identity

$$(x-1)\sum_{i=0}^{n} x^i = x^{n+1}-1$$

from which we obtain

$$(m+1)m^{2n}a = (m+1)b + (a+mb)(m^{2n}-1)$$

and

$$(m+1)m^{2n+1}b = (m+1)a + (m^2-1)b + m(a+mb)(m^{2n}-1).$$

These both reduce to

$$(m^{2n+1}+1)(a-b) = 0$$

The solutions $a = b$ are the trivial solutions, and so we have non-trivial homomorphisms precisely when m is a $(2n+1)$th root of -1, $(m \neq -1)$ and in this case we again have that *any* function from $\{q_1, q_2\}$ to Q_2^T defines a homomorphism. This finishes the proof of the proposition. $\square$

8.3 Comments And Questions

The whole procedure of finding the polynomials to be satisfied for a given quandle presentation to have a homomorphism onto Q_n^T can be automated in a computer program. This has been done, and the program is given in appendix B. However the list of polynomials tends to be rather too big to be manageable. Even in the case of homomorphisms of the trefoil quandle to 3×3 lower triangular matrices, the end result was a couple of pages of polynomials to be satisfied. Trying to tell apart large knots using this method is likely to be prohibitively long in practice.

8.3.1 Transitivity Of The Automorphism Group.

In both of the above examples, whenever there are any non-trivial homomorphisms, it is possible to map the generators of the quandle to any set of two elements of $Q^T_{\hat{m}_0,2}$. This is less surprising than it might seem at first sight. Recall that for fixed $\hat{m} \in M_n^T$ the function $\eta_{\hat{m}} : [\hat{m}'] \mapsto [\hat{m}'\hat{m}]$ is an automorphism of $Q^T_{\hat{m}_0,n}$. Taking $n = 2$ and $\hat{m} = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$ we see that

$$\left[\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right]\eta_{\hat{m}} = \left[\begin{pmatrix} a & 0 \\ ax+b & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & 0 \\ ax+b & 1 \end{pmatrix}\right]$$

Hence the automorphism group of $Q^T_{\hat{m}_0,2}$ is 2-transitive, that is, given any two pairs of distinct quandle elements, $q_1 \neq q_2$ and $q_1' \neq q_2'$, there is an automorphism which takes q_i to q_i'. Since the examples above both have two generators, if there is any non-trivial homomorphism at all, that is a homomorphism which takes the two generators to distinct elements, then there is an automorphism of $Q^T_{\hat{m}_0,n}$ which takes the images of the generators to any pair of distinct elements.

This leads to two questions. Does $Q^T_{\hat{m}_0,n}$ have a 2-transitive automorphism group for $n > 2$? Does $Q^T_{\hat{m}_0,n}$ have a k-transitive automorphism group for $k > 2$? The answer to the second question is easy.

Lemma 8.3.1 *Let Q be a non-trivial quandle. Then $Aut(Q)$ is at most 2-transitive.*

Proof.

Q is non-trivial, so there exist $q_1, q_2 \in Q$ s.t. $q_1\hat{q}_2 \neq \underline{q_1.}$ Also $q_1\hat{q}_2 \neq q_2$ else $q_1 = q_2$. For any automorphism ϕ, $\phi(q_1\hat{q}_2) = \phi(q_1)\overline{\phi(q_2)}$ is determined by $\phi(q_1)$ and $\phi(q_2)$, hence cannot be mapped to an arbitrary element of Q. $\Box$.

Of course if Q is trivial, then any permutation is an automorphism, and so the automorphism group is as transitive as could be asked for. This leads to a further simplification of the problem of finding homomorphisms from a quandle Q to Q^T_n. Since the automorphism group of Q^T_n is transitive, if there are any non-trivial homomorphisms at all, then for any $q_1 \in Q$ we can find a non-trivial homomorphism that takes q to $[id]$. Further, for a second $q_2 \in Q$, since the automorphism group is 2-transitive, if there is any non-trivial homomorphism θ such that $(q_2)\theta \neq (q_1)\theta$ then there is a homomorphism θ' such that $(q_1)\theta' = [id]$ and $(q_2)\theta' = [\hat{m}]$ where $\hat{m}$ is the identity matrix with just one more 1 added below the leading diagonal. Of course there may be non-trivial homomorphisms which take q_1 and q_2 to the same element of M^T_n, indeed for a given presentation, it is not usually immediately obvious that two given elements are in fact distinct, so it is usually a good idea to test for homomorphisms $(q_2)\theta' = [id]$ also.

8.3.2 The $n = 2$ Case

In the $n = 2$ case we are looking for a function $\tilde{\theta} : Q \longrightarrow M^T_2$. Since 2×2 triangular matrices with ones on the diagonal have only one free entry to choose, this is equivalent to finding a function θ' from Q to the underlying ring R with the property that

$$\left[\begin{pmatrix} 1 & 0 \\ (q_1)\theta' & 1 \end{pmatrix}\right] \rhd \left[\begin{pmatrix} 1 & 0 \\ (q_2)\theta' & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & 0 \\ (q_1 \rhd q_2)\theta' & 1 \end{pmatrix}\right]$$

but for $\hat{m} = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ this is just the condition

$$\left[\begin{pmatrix} 1 & 0 \\ m(q_1\theta') + (1-m)(q_2\theta') & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & 0 \\ (q_1 \rhd q_2)\theta' & 1 \end{pmatrix}\right]$$

For fixed m then, this is essentially about finding a module homomorphism ϕ from the universal Alexander quandle for Q to R, where the underlying group of R is regarded as a $\mathbb{Z}[t^{\pm 1}]$ module with t acting as multiplication by m.

Notice that in the above examples, the values of m which provide non-trivial homomorphisms, are precisely the roots of the Alexander polynomial of the relevant knot.

8.3.3 The General Case

In the general case, note that if $\hat{m}_1$ and $\hat{m}_2$ are standard representatives, then

$$([m_1] \rhd [m_2])R = (\hat{m}_1\hat{m}_2^{-1}\hat{m}_0\hat{m}_2)R = (\hat{m}_0^{-1}\hat{m}_1\hat{m}_2^{-1}\hat{m}_0\hat{m}_2)$$

and so attention can be restricted to the subgroup M_n^{TU} of lower triangular matrices with unit diagonal. Conjugation by $\hat{m}_0$ is an automorphism of this group, call this automorphism γ, and the quandle Q_n^T is isomorphic to the group quandle constructed from M_n^{TU} by endowing it with the quandle structure $\hat{m}_1 \rhd \hat{m}_2 = \gamma(\hat{m}_1\hat{m}_2^{-1})\hat{m}_2$. This will be an Alexander quandle whenever M_n^{TU} commutes, that is, only in the $n = 2$ case.

8.4 Matters Geometric

Finding homomorphisms from a given finitely presented quandle to Q_n^T reduces to finding solutions to a set of polynomial equations. This in turn can be viewed from then perspective of algebraic geometry. This idea has already received considerable study when applied to finitely presented groups, see for example [3]. We briefly look at this approach in this section.

We will restrict our attention to the case where the ground ring R is $\mathbb{C}$, the complex numbers. We also fix n and $\hat{m}_0$ throughout. In this case, the problem of finding homomorphisms from some finitely presented quandle Q to $Q^T_{\hat{m}_0,n}$ reduces to finding the solution set for a collection of polynomials in $a\frac{n(n-1)}{2}$ variables where a is the number of generators in the presentation. This is essentially a problem in algebraic geometry.

We will show that the varieties defined by the sets of polynomials of isomorphic presentations are themselves isomorphic, and so we can use invariants from algebraic geometry to define invariants of finitely presented quandles rather more powerful than simply counting how many non-trivial homomorphisms there are, which as noted above, is always none or infinitely many.

First we quote some standard definitions and results from algebraic geometry, see for example [16].

Definition 8.4.1 *Let $f_1, \cdots, f_s$ be polynomials in $\mathbb{C}[x_1, \cdots, x_m]$. Then let*

$$\boldsymbol{V}(f_1, \cdots, f_s) = \{(a_1, \cdots, a_m) \in \mathbb{C}^m | f_i(a_1, \cdots, a_m) = 0 \;\; \forall\, 1 \leq i \leq s\}.$$

$\boldsymbol{V}(f_1, \cdots, f_s)$ is called the variety *defined by $f_1, \cdots, f_s$.*
Conversely, given a variety $V \subset \mathbb{C}^m$, then let

$$J(V) = \{f \in \mathbb{C}[x_1, \cdots, x_m] | f(v) = 0 \; \forall v \in V\}$$

$J(V)$ is called the ideal defined by V. *It is clearly an ideal of* $\mathbb{C}[x_1, \cdots x_m]$.
Let $V \subset \mathbb{C}^m, W \subset \mathbb{C}^k$ be varieties. A function $\phi : V \to W$ is a polynomial mapping *if there exist polynomials $f_1, \cdots f_k \in \mathbb{C}[x_1, \cdots x_m]$ such that*

$$\phi(a_1, \cdots, a_m) = (f_1(a_1, \cdots, a_m), \cdots, f_k(a_1, \cdots, a_m)) \; \forall (a_1, \cdots, a_m) \in V.$$

Let $V \subset \mathbb{C}^m, W \subset \mathbb{C}^k$ be varieties. Then V is isomorphic *to W ($V \cong W$) if and only if there exist polynomial mappings $\phi : V \to W$ and $\phi' : W \to V$ such that $\phi'\phi = Id_V$ and $\phi\phi' = Id_W$.*
For any variety $V \subset \mathbb{C}^m$, the coordinate ring *of V, denoted R_V, is the ring $\mathbb{C}[x_1 \cdots x_m]/J(V)$.*

Proposition 8.4.2 *Let $V \subset \mathbb{C}^m$ and $W \subset \mathbb{C}^k$ be varieties. Then $V \cong W \Leftrightarrow R_V \cong R_W$.*

A finitely presented quandle Q is a free quandle factored out by the congruence generated by the relations, and so we will start by looking at homomorphisms from FQ_a, the free quandle on a generators, $FQ_a = \langle q_1, \cdots, q_a | \rangle$. Let $X = \{x_{1,2,1}, \cdots, x_{a,n,n-1}\}$ be the *variable set* of Q. Take M_n^X to be the group of invertible $(n \times n)$ lower triangular matrices with entries in $\mathbb{C}[X]$ and constants on the leading diagonal. Let $Q_n^X = Qd\{M_n^X, D_n^X, \hat{m}_0\}$, where D_n^X is the group of diagonal matrices with entries in $\mathbb{C}$.

Lemma 8.4.3 *Let $\hat{m}$ be some element of M_n^X. Then $\hat{m}^{-1}$ will have entries which are polynomials in the entries of $\hat{m}$.*

Proof.

The standard algorithm for calculating inverses of matrices shows that each entry is of the form $\pm\frac{c}{d}$ where c is the determinant of some minor of $\hat{m}$, which is some polynomial in the entries of $\hat{m}$, and d is the determinant of $\hat{m}$, which is some non-zero complex number. $\square$

Define a function ψ' taking the generators of FQ_a to M^X by

$$(\psi'(q_i))_{j,k} = \begin{cases} x_{i,j,k} & : \ j > k \\ 1 & : \ j = k \\ 0 & : \ j < k \end{cases}$$

For each generator q_i define the function $\psi_i : q_i \mapsto [\psi'(q_i)]$. We can regard each generator q_i as a unit quandle in its own right, and so each ψ_i can be regarded as a Q-homomorphism. By the definition of free quandle and free sum there is a unique homomorphism $\psi : FQ_a \to Q_n^X$ s.t. $\psi(q_i) = \psi_i(q_i)$. Explicitly, for $p = p_1\hat{p}_2^{\pm 1} \cdots \hat{p}_b^{\pm 1}$, where the $p_j \in \{q_i\}$

$$\psi(p) = [(\psi'(p_1)\psi'(p_2)^{-1}\hat{m}_0^{\pm 1}\psi'(p_2) \cdots \psi'(p_b)^{-1}\hat{m}_0^{\pm 1}\psi'(p_b))] \tag{8.1}$$

Define $\tilde{\psi}(p)$ to be the standard representative of $\psi(p)$.

Then a homomorphism $\theta : FQ_a \to Q^T_{n,\hat{m}_0}$ is simply an evaluation of $\mathbb{C}[X]$, which takes the variables $x_{i,j,k}$ to arbitrary complex numbers.

Next consider a general quandle presentation P. Each relator R_i in P is a pair $p_{i,1} = p_{i,2}$ of elements of FQ_a. For each such relator, let $r_i = \tilde{\psi}(p_{i,1}) - \tilde{\psi}(p_{i,2})$, let $S_i = \{f | f \text{ is an entry in } r_i\}$, let $S = \bigcup S_i$, and let $I = I_P = I_{P,n,\hat{m}_0}$ be the ideal of $\mathbb{C}[X]$ generated by S. Then a homomorphism $\theta : Qd(P) \to Q^T_{n,\hat{m}_0}$ is an evaluation of X such that I maps to zero. That is, it is a zero set for the set of polynomials S.

So far, we are only counting homomorphisms from Q to $Q^T_{n,\hat{m}_0}$, which as noted above will always give an answer of 0 or ∞ for any given $\hat{m}_0$; we are only finding which values of $\hat{m}_0$ yield non-trivial homomorphisms.

Definition 8.4.4 *For a fixed quandle presentation P, a fixed integer n, and a fixed $(n \times n)$ diagonal matrix $\hat{m}_0$ the ideal $I = I_{P,n,\hat{m}_0}$ of R defined above is called the* presentation ideal *of P at $n, \hat{m}_0$. The variety $V(I)$, that is the set $\{x \in \mathbb{C}^{a\frac{n(n-1)}{2}} | f(x) = 0 \ \ \forall f \in I_P\}$, is called the* presentation variety *of P at $n, \hat{m}_0$.*

Proposition 8.4.5 *For isomorphic quandle presentations, P, P' and fixed $n, \hat{m}_0$, the presentation varieties $V(I_P)$ and $V(I_{P'})$ are isomorphic.*

Proof.

We go through each of the Tietze moves, see Theorem 2.1.7, and show that these leave presentation varieties polynomially equivalent.

The first two moves, adding or deleting a consequence of the remaining relations.

Since the new relation $q_1 = q_2$ is a consequence of the remaining relations, the new polynomials added to S from $\tilde{\psi}(q_1) - \tilde{\psi}(q_2)$ are consequences of the remaining polynomials in S. Hence $I_P = I_{P'}$ and the proposition is trivially true.

The last move. Introducing a new generator together with a defining relator, or the converse of this.

Let P' be P together with the new generator and relator. Then X' is X together with the $\frac{n(n-1)}{2}$ variables contributed by the new generator. The original variables are clearly polynomially equivalent, via the identity function. The only work involves showing that the zeros of the defining equations in all the new variables are polynomial functions of the old variables. But the new polynomials in I' are all of the form $x' = f(X)$ where x' is a new variable. This provides the polynomial function. $\square$

This opens the way to using any property of varieties that remains invariant under isomorphism as invariants of finitely presented quandles. Examples include dimension and number of irreducible components.

Chapter 9

Where Next?

There is more work to be done. Firstly, when looking at homomorphisms between coset quandles, to what extent is there a match between the Q-homomorphisms and C-homomorphisms? Can the search for one be replaced with the other? If this is the case, then G-homomorphisms are being used which are well known. For example it would be possible to study Q-homomorphisms onto coset quandles constructed from matrix groups by studying representations of the associated group instead. If this is the case, is it possible to tell quandles apart by their group representations? How useful is this in practice?

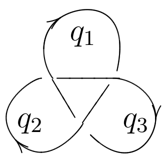
In the section on homomorphism invariants, homomorphisms onto lower triangular coset quandles were looked at. Other groups could be used to find invariants, in particular groups like $GL(n, R)$ for some ring R. How useful are these invariants?

The section on homomorphism invariants was concluded by defining presentation varieties and showing that isomorphic presentations give isomorphic varieties. How useful is this as an invariant?

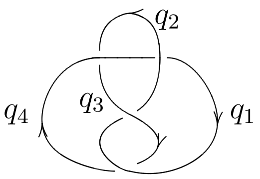
In the case of knots and (non-split) links, quandles theoretically provide complete information. How can the various invariants be extracted from this information? As an example, given the fundamental quandle of some knot, how can we calculate its Jones Polynomial?

Appendix A

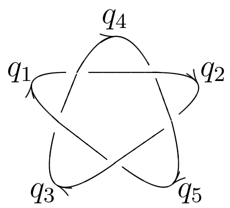
Table Of Knot Quandles



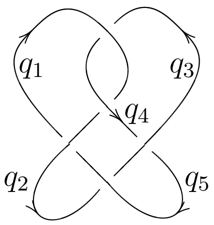
$$\begin{aligned} & \langle q_1, q_2, q_3 | q_1 = q_3\hat{q}_2, q_2 = q_1\hat{q}_3, q_3 = q_2\hat{q}_1\rangle \\ = \; & \langle q_1, q_2 | q_1 = q_2\hat{q}_1\hat{q}_2, q_2 = q_1\hat{q}_2\hat{q}_1\rangle \end{aligned}$$



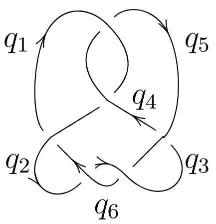
$$\begin{aligned} & \langle q_1, q_2, q_3, q_4 | q_1 = q_4\hat{q}_2, q_2 = q_1\hat{q}_3^{-1}, q_3 = q_2\hat{q}_4, q_4 = q_3\hat{q}_1^{-1}\rangle \\ = \; & \langle q_1, q_3 | q_1 = q_3\hat{q}_1^{-1}\hat{q}_3\hat{q}_1\hat{q}_3^{-1}, q_3 = q_1\hat{q}_3^{-1}\hat{q}_1\hat{q}_3\hat{q}_1^{-1}\rangle \end{aligned}$$



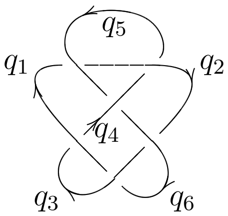
$$\begin{aligned} & \langle q_1, q_2, q_3, q_4, q_5 | q_1 = q_5\hat{q}_3^{-1}, q_2 = q_1\hat{q}_4^{-1}, q_3 = q_2\hat{q}_5^{-1}, \\ & \quad q_4 = q_3\hat{q}_1^{-1}, q_5 = q_4\hat{q}_2^{-1}\rangle \\ = \; & \langle q_1, q_3 | q_1 = q_3\hat{q}_1\hat{q}_3\hat{q}_1\hat{q}_3, q_3 = q_1\hat{q}_3\hat{q}_1\hat{q}_3\hat{q}_1\rangle \end{aligned}$$



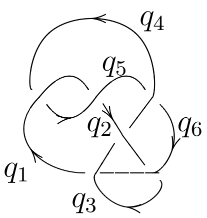
$$\begin{aligned}
&\langle q_1, q_2, q_3, q_4, q_5 | q_1 = q_5\hat{q}_2^{-1}, q_2 = q_1\hat{q}_4^{-1}, q_3 = q_2\hat{q}_5^{-1}, q_4 = q_3\hat{q}^{-1}, \\
&\qquad q_5 = q_4\hat{q}_3^{-1}\rangle \\
= \ &\langle q_1, q_3 | q_3 = q_1\hat{q}_3^{-1}\hat{q}_1^{-1}\hat{q}_3\hat{q}_1\hat{q}_3^{-1}\hat{q}_1^{-1} q_1 = q_3\hat{q}_1^{-1}\hat{q}_3^{-1}\hat{q}_1\hat{q}_3\hat{q}_1^{-1}\hat{q}_3^{-1}\rangle
\end{aligned}$$



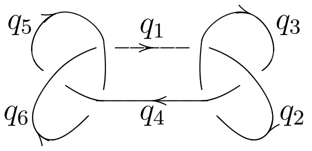
$$\begin{aligned}
&\langle q_1, q_2, q_3, q_4, q_5, q_6 | q_1 = q_6\hat{q}_2^{-1}, q_2 = q_1\hat{q}_4, q_3 = q_2\hat{q}_6^{-1}, q_4 = q_3\hat{q}_5^{-1}, \\
&\qquad q_5 = q_4\hat{q}_1^{-1}, q_6 = q_5\hat{q}_3^{-1}\rangle \\
= \ &\langle q_1, q_4 | q_4 = q_1\hat{q}_4\hat{q}_1\hat{q}_4^{-1}\hat{q}_1^{-1}\hat{q}_4\hat{q}_1\hat{q}_4^{-1}\hat{q}_1^{-1}, q_1 = q_4\hat{q}_1^{-1}\hat{q}_4^{-1}\hat{q}_1\hat{q}_4\hat{q}_1^{-1}\hat{q}_4^{-1}\hat{q}_1^{-1}\hat{q}_4\rangle
\end{aligned}$$



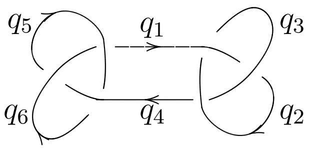
$$\begin{aligned}
&\langle q_1, q_2, q_3, q_4, q_5, q_6 | q_1 = q_6\hat{q}_3^{-1}, q_2 = q_1\hat{q}_5, q_3 = q_2\hat{q}_6^{-1}, q_4 = q_3\hat{q}_1^{-1}, \\
&\qquad q_5 = q_4\hat{q}_2, q_6 = q_5\hat{q}_4^{-1}\rangle \\
= \ &\langle q_1, q_2, q_3 | q_1 = q_3\hat{q}_1^{-1}\hat{q}_2\hat{q}_1\hat{q}_3^{-1}\hat{q}_1^{-1}\hat{q}_3^{-1}, q_2 = q_1\hat{q}_2^{-1}\hat{q}_1\hat{q}_3\hat{q}_1^{-1}, \\
&\qquad q_3 = q_2\hat{q}_1\hat{q}_3\hat{q}_1^{-1}\hat{q}_2^{-1}\hat{q}_1\hat{q}_3^{-1}\hat{q}_1^{-1}\hat{q}_2\hat{q}_1\hat{q}_3\hat{q}_1^{-1}\rangle
\end{aligned}$$



$$\begin{aligned}
&\langle q_1, q_2, q_3, q_4, q_5, q_6 | q_1 = q_6\hat{q}_3, q_2 = q_1\hat{q}_5^{-1}, q_3 = q_2\hat{q}_6, q_4 = q_3\hat{q}_2, \\
&\qquad q_5 = q_4\hat{q}_1^{-1}, q_6 = q_5\hat{q}_4^{-1}\rangle \\
= \ &\langle q_1, q_2, q_3 | q_1 = q_3\hat{q}_2\hat{q}_1^{-1}\hat{q}_2^{-1}\hat{q}_3^{-1}\hat{q}_2\hat{q}_3, q_2 = q_1\hat{q}_2^{-1}\hat{q}_3^{-1}\hat{q}_2\hat{q}_1, \\
&\qquad q_3 = q_2\hat{q}_3\hat{q}_2\hat{q}_1\hat{q}_2^{-1}\hat{q}_3\hat{q}_2\hat{q}_1^{-1}\hat{q}_2^{-1}\hat{q}_3^{-1}\hat{q}_2\rangle
\end{aligned}$$



$$
\begin{aligned}
&\langle q_1, q_2, q_3, q_4, q_5, q_6 | q_2 = q_1\hat{q}_3, q_3 = q_2\hat{q}_4, q_4 = q_3\hat{q}_2, \\
&\qquad q_5 = q_4\hat{q}_6^{-1}, q_6 = q_5\hat{q}_4^{-1}, q_1 = q_6\hat{q}_5^{-1}\rangle \\
= &\langle q_1, q_2, q_5 | q_2 = q_1\hat{q}_2\hat{q}_1, q_5\hat{q}_1\hat{q}_5 = q_2\hat{q}_1\hat{q}_2, q_5 = q_1\hat{q}_5\hat{q}_1\rangle
\end{aligned}
$$



$$
\begin{aligned}
&\langle q_1, q_2, q_3, q_4, q_5, q_6 | q_1 = q_6\hat{q}_5, q_2 = q_1\hat{q}_3, q_3 = q_2\hat{q}_4, \\
&\qquad q_4 = q_3\hat{q}_2, q_5 = q_4\hat{q}_6, q_6 = q_5\hat{q}_1\rangle \\
= &\langle q_1, q_3, q_6 | q_1 = q_6\hat{q}_1\hat{q}_6, q_3 = q_1\hat{q}_3\hat{q}_1, q_6 = q_3\hat{q}_1\hat{q}_3\hat{q}_6\hat{q}_1\rangle
\end{aligned}
$$

Appendix B

A Maple Program For Computing Homomorphism Invariants

```
> with(LinearAlgebra) :
> with(Groebner) :
```

This is a worksheet for calculating homomorphisms of quandles onto Qd{T,D,m} where T is the group of lower triangular matrices for some dimension dim, and D is the subgroup of diagonal matrices. The field from which the entries in the matrix are taken is not specified. The output is a set of rational polynomials which must be satisfied in the chosen field.

User Instructions

You should input:-
a) The dimension of the matrices to which you are looking for a homomorphism.
b) The quandle presentation. This should look like -
[[*List of generators.*] , [[*Relation1*] , [*Relation2*] , ... , [*Relation n*]]]
Each [relation] should look something like -
[[q1, q2, q1!], [q2, q1!, q2, q1]]
Where the q1 and q2 are generators, and q1! = q1^{-1} etc.
c) The general matrix M to be used to define the target quandle. This will usually be the arbitrary diagonal matrix given
below, but a more specific one may be used if desired.

The output is a list of equations to be solved in order to find homomorphisms. This can be fed into some Groebner basis function
to be simplified, if so desired. VarSet gives a list of the variables. It returns a list of two elements, the first is the set of variables
used in the matrices representing the generators of the quandle, and the second is the variables in the defining matrix M.

Note:- There is no error trapping, so relations must make sense, or something strange will happen.

After the program, a list of presentations of the fundamental quandles of some knots is supplied.

First we have the procedures.

```
>
>
> Canon := proc(M) local Fcan:

    # The input M is a lower triangular non-singular matrix, and the output is a
    function defining a matrix which gives

    # A 'canonical form' that is the matrix in the same coset of the diagonal
    matrices with ones on the diagonal.
  Fcan := (i, j) -> M[i, j]/M[i, i];
  end proc:

> RepBasis := proc(Q, Dim) local k:
  # Q is a quandle presentation. This returns an arbitrary (Dim x Dim)
  # unit lower diagonal matrix for each generator of Q.
  # For q a generator of Q, the arbitrary matrix is called mq and the entries
  # are eq_{i,j}.
  for k in Q[1] do
  m||k := Matrix(Dim, symbol = e||k, shape = triangular[unit, lower]) :
  end do:
  end proc:
> CalculateRelations := proc(Q, Dim, M)local i, j, k, x, y, x1, x2, y1, y2 global B:

    # This calculates the polynomials that have to be satisfied for the 'quandle
    generators to matrices'

    # mapping in RepBasis to be a homomorphism. So for each relation in Q,
    calculate a standard
  # representative for each side, then compare.
  # The resulting relations are stored in a list in B.
  # First assign each generator to a matrix of indeterminants
  # using the procedure 'RepBasis' above.
  RepBasis(Q, Dim) :
  # Q[2] is the set of relations.
  for i from 1 to nops(Q[2]) do
  # Q[2,i] is the i'th relation.
  # Q[2,i,1] is the left hand side of the i'th relation.
  # The left hand side translated into matrices is stored in rL||i.
```

```
x := Q[2, i, 1, 1]:
rL||i := m||x:
# Q[2,i,1,1] is the primary element in the left hand side of the i'th relation.
   for j from 2 to nops(Q[2, i, 1]) do
# Q[2,i,1,j] is the j'th element in the left hand side of the i'th relation.
    x := Q[2, i, 1, j]:
    x1 := convert(x, string) :
# Check to see if Q[2,i,1,j] is 'inverted', and act accordingly.
# It will be an inverse element if it ends with a '!'
      if MmaTranslator[Mma][StringPosition](x1, "!") ≠ [ ] then
       x2 := eval(StringTools[Chop](x1)) :
       rL||i := rL||i.MatrixInverse(m||x2).MatrixInverse(M).m||x2:
      else
       rL||i := rL||i.MatrixInverse(m||x).M.m||x:
     end if:
   end do:
# Now take the canonical form.
 rL||i := Matrix(Dim, Canon(rL||i)):
# And the same all over again for the right hand side.

   # This time the right hand side, Q[2,i,1], is used, and the output is stored in
   rR||i.
 y := Q[2, i, 2, 1]:
 rR||i := m||y:
   for j from 2 to nops(Q[2, i, 2]) do
    y := Q[2, i, 2, j]:
    y1 := convert(y, string) :
      if MmaTranslator[Mma][StringPosition](y1, "!") ≠ [ ] then
       y2 := eval(StringTools[Chop](y1)) :
       rR||i := rR||i . MatrixInverse(m||y2) . MatrixInverse(M) . m||y2:
     else
    rR||i := rR||i.MatrixInverse(m||y).M.m||y:
     end if:
   end do:
 rR||i := Matrix(Dim, Canon(rR||i)) :

   # To be a homomorphism, both sides of the relators must map to the same
   value.
# rR||i should equal rL||i  for all i, so the following should be 0.
r||i := rL||i - rR||i:
# Now normalize.
normal(r||i):
end do:
# Now extract the relations. Keep them in the list B.
B := [ ]:
 for i from 1 to nops(Q[2]) do

   # Each entry in the lower triangle must be zero. Note that we only need the
```

```
    numerators, as the denominators
  # are always powers of the diagonal entries in M which are non-zero.
   for j from 2 to Dim do:
    for k from 1 to (j - 1) do:
     B := [op(B), factor(numer(r||i[j, k]))];
    end do
   end do:
  end do:
  end proc:
> #
  # The above procedures compute the equations that need to be satisfied.
  # The following procedures are here to help simplify these by removing
  # repeated roots and finding Groebner Basis'.
  #
> VarSet := proc(Q, Dim, M) local i, j, k, x, V1, V2, V :
  # This gives the set of variables used in the relations given by the above
      procedure,
  # which is useful for calculating Groebner basis with.
  # We store the variables in the representatives of the generators of Q in V1.
  V1 := [ ]:
  for i from 1 to nops(Q[1]) do
   for j from 2 to Dim do
    for k from 1 to j - 1 do
      x := Q[1, i]:
      V1 := [op(V1), e||x_{j, k}]:
    end do:
   end do:
  end do:
  # We store the variables in the matrix M in V2.
  V2 := [ ]:
  for i from 1 to Dim do
    V2 := [op(V2), M[i, i]]:
  end do:
  # Now put them together in V.
  V := [V1, V2]:
  end proc:
>
> RemoveFactor := proc(B, V2) local x, y, i, p, m, B1 :

      # B is a list of polynomials to which we wish to find solutions. V2 is a list of
      indeterminants
  # which cannot be zero.

      # If any polynomials have a multiple factor, then we can dispose of it, since
      this will not affect
  # the common zeros.

      # Since we know that all the variables in V2 are non-zero, we can remove
```

```
    multiples of them from
# the polynomials in B.
B1 := [ ]:
for x in B do
p := factors(x) :
m := p[1]:
 for i from 1 to nops(p[2]) do
 m := m·p[2, i, 1]:
 end do:
 x := expand(m) :
  for y in V2 do
    while numer(normal(x/y)) ≠ numer(normal(x)) do
      x := normal(x/y) :
    end do:
  end do:
 B1 := [op(B1), x]:
end do:
end proc:
> Specify := proc(B, Q, M, Dim)local L, i, j, x:
# B is the set of equations to be simplified.
# Q is the original quandle.
# M is the defining matrix for the target quandle.
# Dim is the dimension of the target quandle.
#

    # Since the quandle we are mapping to has a transitive automorphism
    group, we can fix one of the

    # quandle elements to be the identity. Also we can fix the first element of M
    to be 1.
L := [M_{1, 1} = 1]:
x := Q[1, 1]:
for i from 2 to Dim do
 for j from 1 to i − 1 do
  L := [op(L), e || x[i, j] = 0]
 end do:
end do:
 eval(B, L) :
end proc:
> Simp := proc(B, V2, Ord)local B1, B2:
# Repeatedly applies 'Basis' and 'RemoveFactor'
# B is the list of equations to be simplified.
# V2 is a list of indeterminants which cannot be zero, and so factors
# consisting of these elements can be thrown away.
# Ord is a term order to be used in computing a Groebner Basis.
B1 := B:
```

```
B2 := [ ]:
while B1 ≠ B2   do
  B2 := RemoveFactor(B1, V2) :
  B1 := Basis(B2, Ord) :
end do:
factor(B1) :
end proc:
> # For conveniance here is a list of some presentations of knot quandles.
  # The knots are numbered according to the list in Ray Lickorish's book 'Knot
      Theory'.
>
> # Two different presentations of the Trefoil knot.
> QTref := [[q1, q2], [[[q1, q2, q1, q1, q2], [q1]], [[q1], [q2, q1, q2]]]]:
>
> #
  #  Now we will show the program in action by computing the polynomials
      needing to be
  #  satisfied  for a non-trivial homomorphism from the fundamental quandle of
      the trefoil
  #  knot to the coset quandle on 3x3 lower triangular matrices to be defined.
  #
> K1 := QTref:
> Dim := 3 :
> M := Matrix(Dim, symbol = m, shape = diagonal) :
      # This is the defining matrix of the coset quandle.
> Rel1 := CalculateRelations(K1, Dim, M);
```

$$
\begin{aligned}
Rel1 := \Big[& \left(-m_{2,2}+m_{1,1}\right)\left(m_{2,2}+m_{1,1}\right)\left(m_{2,2}^{2}-m_{1,1}\,m_{2,2}+m_{1,1}^{2}\right)\left(eq1_{2,1}\right.\\
& \left.-eq2_{2,1}\right),\ m_{3,3}^{4}\,eq2_{3,1}+m_{2,2}^{4}\,eq2_{2,1}\,eq1_{3,2}-m_{2,2}^{4}\,eq2_{2,1}\,eq2_{3,2}-\\
& m_{1,1}^{4}\,eq1_{3,2}\,eq2_{2,1}+m_{1,1}^{4}\,eq2_{2,1}\,eq2_{3,2}+m_{1,1}^{3}\,m_{3,3}\,eq2_{3,1}-m_{1,1}^{3}\,m_{3,3}\,eq1_{3,1}\\
& -m_{1,1}\,m_{3,3}^{3}\,eq2_{3,1}+m_{1,1}^{4}\,eq1_{3,1}-m_{1,1}^{4}\,eq2_{3,1}-eq1_{3,1}\,m_{3,3}^{4}+\\
& m_{1,1}^{3}\,m_{2,2}\,eq2_{2,1}\,eq1_{3,2}-m_{1,1}^{3}\,m_{2,2}\,eq2_{2,1}\,eq2_{3,2}-m_{1,1}^{3}\,eq1_{2,1}\,m_{2,2}\,eq1_{3,2}+\\
& m_{1,1}^{3}\,eq1_{2,1}\,m_{2,2}\,eq2_{3,2}-m_{1,1}^{3}\,eq1_{2,1}\,m_{3,3}\,eq2_{3,2}+m_{1,1}^{3}\,m_{3,3}\,eq1_{2,1}\,eq1_{3,2}\\
& +m_{1,1}\,m_{2,2}^{3}\,eq1_{2,1}\,eq1_{3,2}-m_{1,1}\,m_{2,2}^{3}\,eq1_{2,1}\,eq2_{3,2}-m_{1,1}\,eq2_{2,1}\,m_{2,2}^{3}\,eq1_{3,2}\\
& +m_{1,1}\,eq2_{2,1}\,m_{2,2}^{3}\,eq2_{3,2}+m_{1,1}\,m_{2,2}^{2}\,eq1_{2,1}\,m_{3,3}\,eq2_{3,2}-m_{1,1}\\
& m_{2,2}^{2}\,eq1_{2,1}\,m_{3,3}\,eq1_{3,2}+m_{1,1}\,m_{3,3}^{3}\,eq1_{3,1}-m_{1,1}\,eq2_{2,1}\,m_{2,2}^{2}\,m_{3,3}\,eq2_{3,2}\\
& +m_{1,1}\,eq2_{2,1}\,m_{2,2}^{2}\,m_{3,3}\,eq1_{3,2}-m_{1,1}\,eq2_{2,1}\,m_{3,3}^{3}\,eq1_{3,2}+\\
& m_{2,2}^{3}\,eq2_{2,1}\,m_{3,3}\,eq2_{3,2}-m_{2,2}^{3}\,eq2_{2,1}\,m_{3,3}\,eq1_{3,2}+m_{2,2}\,eq2_{2,1}\,m_{3,3}^{3}\,eq1_{3,2}\\
& -m_{2,2}\,eq2_{2,1}\,m_{3,3}^{3}\,eq2_{3,2}+m_{1,1}\,m_{3,3}^{3}\,eq2_{2,1}\,eq2_{3,2},\ \left(-m_{3,3}+m_{2,2}\right)\left(m_{3,3}\right.\\
& \left.+m_{2,2}\right)\left(m_{3,3}^{2}-m_{2,2}\,m_{3,3}+m_{2,2}^{2}\right)\left(eq1_{3,2}-eq2_{3,2}\right),\ \left(m_{2,2}^{2}-m_{1,1}\,m_{2,2}+\right.
\end{aligned}
\tag{1}
$$

$m_{1,1}^2)\ (eq1_{2,1} - eq2_{2,1}),\ m_{3,3}^2\, eq1_{3,1} - m_{1,1}^2\, eq2_{3,1} + m_{1,1}^2\, eq2_{3,2}\, eq1_{2,1} -$
$m_{1,1}^2\, eq1_{2,1}\, eq1_{3,2} + m_{1,1}^2\, eq1_{3,1} - m_{1,1}\, eq1_{2,1}\, m_{2,2}\, eq2_{3,2}$
$+ m_{1,1}\, eq1_{2,1}\, m_{2,2}\, eq1_{3,2} - m_{1,1}\, m_{3,3}\, eq1_{3,1} + m_{1,1}\, m_{2,2}\, eq2_{2,1}\, eq2_{3,2}$
$- m_{1,1}\, m_{2,2}\, eq2_{2,1}\, eq1_{3,2} + m_{1,1}\, eq2_{2,1}\, m_{3,3}\, eq1_{3,2} - m_{1,1}\, m_{3,3}\, eq2_{2,1}\, eq2_{3,2}$
$+ m_{1,1}\, m_{3,3}\, eq2_{3,1} - m_{2,2}^2\, eq2_{2,1}\, eq2_{3,2} + m_{2,2}^2\, eq2_{2,1}\, eq1_{3,2}$
$- m_{2,2}\, eq2_{2,1}\, m_{3,3}\, eq1_{3,2} + m_{2,2}\, eq2_{2,1}\, m_{3,3}\, eq2_{3,2} - m_{3,3}^2\, eq2_{3,1},\ (m_{3,3}^2$
$- m_{2,2}\, m_{3,3} + m_{2,2}^2)\ (eq1_{3,2} - eq2_{3,2})]$

```
> #` `
  # And Simplify.
  #
  # First specify the values of certain indeterminants. By transitivity of
  # the automorphism group of a coset quandle, we can assume that
  # the first generator in the quandle presentation maps to [Id]. We can
  # also assume that the first entry  in the defining matrix M is 1.
  #
> Rel2 := Specify(Rel1, K1, M, Dim);
```

$$Rel2 := [-(-m_{2,2}+1)\,(m_{2,2}+1)\,(m_{2,2}^2 - m_{2,2}+1)\, eq2_{2,1},\ -m_{3,3}^3\, eq2_{3,1} + eq2_{2,1}\, m_{2,2}^3\, eq2_{3,2} + m_{3,3}^3\, eq2_{2,1}\, eq2_{3,2} + m_{3,3}^4\, eq2_{3,1} + eq2_{2,1}\, eq2_{3,2} + m_{3,3}\, eq2_{3,1} - m_{2,2}^4\, eq2_{2,1}\, eq2_{3,2} - eq2_{3,1} - eq2_{2,1}\, m_{2,2}^2\, m_{3,3}\, eq2_{3,2} - m_{2,2}\, eq2_{2,1}\, eq2_{3,2} + m_{2,2}^3\, eq2_{2,1}\, m_{3,3}\, eq2_{3,2} - m_{2,2}\, eq2_{2,1}\, m_{3,3}^3\, eq2_{3,2},\ -(-m_{3,3}+m_{2,2})\,(m_{3,3}+m_{2,2})\,(m_{3,3}^2 - m_{2,2}\, m_{3,3} + m_{2,2}^2)\, eq2_{3,2},\ -(m_{2,2}^2 - m_{2,2}+1)\, eq2_{2,1},\ -eq2_{3,1} + m_{2,2}\, eq2_{2,1}\, eq2_{3,2} - m_{3,3}\, eq2_{2,1}\, eq2_{3,2} + m_{3,3}\, eq2_{3,1} - m_{2,2}^2\, eq2_{2,1}\, eq2_{3,2} + m_{2,2}\, eq2_{2,1}\, m_{3,3}\, eq2_{3,2} - m_{3,3}^2\, eq2_{3,1},\ -(m_{3,3}^2 - m_{2,2}\, m_{3,3} + m_{2,2}^2)\, eq2_{3,2}] \quad \textbf{(2)}$$

```
> # Next dispose of multiple factors, and apply Groebner basis techniques.
> V := VarSet(K1, Dim, M) :
> Ord := prod(plex(V[1][ ]), plex(V[2][ ])) :
> Simp(Rel2, V[2], Ord);
```

$$[(m_{3,3}^2 - m_{2,2}\, m_{3,3} + m_{2,2}^2)\, eq2_{3,2},\ eq2_{3,1}\,(m_{3,3}^2 - m_{3,3}+1)\,(m_{3,3}^2+1+m_{3,3}),\ eq2_{3,1}\,(m_{3,3}^2 - m_{3,3}+1)\,(-m_{3,3}-1+m_{2,2}),\ (m_{2,2}^2 - m_{2,2}+1)\, eq2_{2,1},\ -eq2_{3,1} + eq2_{2,1}\, eq2_{3,2} - m_{3,3}^3\, eq2_{3,1}] \quad \textbf{(3)}$$

```
>
```

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