Gevrey Regularity of Solutions of Evolution Equations and Boundary Controllability

A THESIS

SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL

OF THE UNIVERSITY OF MINNESOTA

BY

STEPHEN WILLIAM TAYLOR

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

1989

Acknowledgements

I would like to thank Professors Lawrence Markus and Yasutaka Sibuya for their helpful suggestions, and Dr. George McIvor for his proof reading of the manuscript. I especially wish to thank Professor Walter Littman for his excellent advice while I was doing this research.

I am also grateful to the Graduate School for its support in the form of a University of Minnesota Dissertation Fellowship.

Stephen W. Taylor.

For Janice.

Preface

Chapters 1 to 4 of this work generalize the work of W. Littman and L. Markus [17] on the 'Exact Boundary Controllability of a Hybrid System of Elasticity'. The remaining chapter deals with the Gevrey regularity of strongly continuous semigroups.

The work of [17] treats the exact boundary controllability of a simplified version of the SCOLE model. The SCOLE [21] (Spacecraft Control Laboratory Experiment) model consists of a long flexible mast clamped at one end to a space shuttle and at the other end to an antenna (see figure 8). The authors of [17] aim to capture some of the mathematical features of the problem by modelling the mast as an Euler-Bernoulli beam vibrating in a fixed plane. The transverse deflections of the beam satisfy the equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0. \tag{1}$$

The requirement that the beam is clamped to the shuttle (which is assumed to be inertial) at $x = 0$ yields the 'clamped end conditions' $w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0$. The ordinary differential equation of motion of the antenna at the other end of the beam yields unusual boundary conditions for the partial differential equation. These may be written as $B_1 w(d,t) = f_1(t)$, $B_2 w(d,t) = f_2(t)$, where the functions f_1 and f_2 are a controlling force and a controlling torque respectively, both applied only to the antenna. The control problem solved in [17] is the following: Given an initial disturbance (satisfying certain compatibility and smoothness assumptions) $w(x,0) = w_0(x)$, $\frac{\partial w}{\partial t}(x,0) = v_0(x)$, and given $T > 0$, find appropriate functions f_1 and f_2 that will drive the whole system to rest for all times $t \geq T$.

The method used in [17] to prove the controllability is also described by W. Littman in [16]. We summarize it below:

(1) Extend the domains of the initial data w_0, v_0 so that these functions are defined on $[0,\infty)$ and have compact support.

(2) Taking the clamped end conditions into account, solve the beam equation for $x \in [0,\infty)$ with the modified initial data.

(3) Let ψ be a cut-off function satisfying $\psi(t) = 1$ for $t \le T/2$, $\psi(t) = 0$ for $t \ge T$. Set $F(x,t) = (\partial^2/\partial t^2 + \partial^4/\partial x^4)\{\psi(t)\, w(x,t)\} = 2\psi'(t)\, \frac{\partial w}{\partial t}(x,t) + \psi''(t)\, w(x,t)$, where w is the solution of Step 2.

(4) Solve the problem

$$\frac{\partial^2 W}{\partial t^2} + \frac{\partial^4 W}{\partial x^4} = F, \quad W(0,t) = \frac{\partial W}{\partial x}(0,t) = \frac{\partial^2 W}{\partial x^2}(0,t) = \frac{\partial^3 W}{\partial x^3}(0,t) = 0,$$

to get a solution which vanishes for $t \le T/2$ and $t \ge T$.

(5) Put $u(x,t) = \psi(t)\, w(x,t) - W(x,t)$. This function satisfies the beam equation, the clamped end conditions and the initial conditions. Further, it vanishes for $t > T$. The control functions are then obtained from the equations $f_1(t) = B_1 u(d,t)$, $f_2(t) = B_2 u(d,t)$.

Step 4 is difficult, for the beam equation must be solved in the *x-direction*. This leads to a problem which is not well-posed. However, it is shown in [16] and [17] that this can be solved if F is of Gevrey class 2 in the time variable for $t > 0$. What this means (c.f. Hörmander [8]) is that for any given $\theta > 0$ and any given compact subset K of $(0,\infty)$, there exists a constant $C \ge 0$ such that

$$\left| \frac{\partial^n F}{\partial t^n}(x,t) \right| \le C\, \theta^n\, (n!)^2, \qquad \forall\, t \in K, \quad n \ge 0.$$

These Gevrey functions form an algebra which is closed under differentiation with respect to t. Further, it is possible to choose ψ in Step 3 to be a Gevrey 2 function (such functions can be explicitly written - see [8]). So F may be obtained as a Gevrey 2 function, as long as the solution of Step 2 is a Gevrey 2 function.

In [17], an *explicit formula* is given for the solution of Step 2. The formula shows that w(x,t) is actually analytic in t for $t > 0$ (and is thus Gevrey 2 in t for $t > 0$). Thus, Steps 1-5 are easily carried out to solve the control problem.

In the present work, we generalize the work of [17] to obtain results applicable to equations of the form

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \sum_{i=0}^{4} a_i(x)\, \frac{\partial^i w}{\partial x^i}(x,t) + \sum_{i=0}^{2} b_i(x)\, \frac{\partial^{i+1} w}{\partial x^i \partial t}(x,t) = 0, \tag{2}$$

where $a_4(x) > 0$ and $b_2(x) \leq 0$. Obviously, one cannot present an explicit fundamental solution for Step 2, so a considerable amount of work is required to show that such fundamental solutions exist and that they have the required properties. It is proved in Chapter 1 that when the coefficients are variable (and satisfy mild assumptions), such fundamental solutions exist and are of Gevrey class 2 in t for $t > 0$. It appears that the analyticity in t for $t > 0$ seen in [17] does not always occur. However, in Chapter 2 we show that if the coefficients of equation (2) are constants, then the fundamental solution is analytic in t for $t > 0$.

We carry out Step 4 in Chapter 3. In [17] an explicit series solution is presented for the corresponding step, and the convergence of the series is based on Gevrey estimates. Here, we prefer to use a different approach, based on the Ovcyannikov Theorem (see Treves [22], [23]). This allows us to treat partial differential equations like (2) in which we allow also a dependence of the coefficients on time.

The theory of Chapters 1-3 is applied in Chapter 4 to solve problems similar to the one solved in [17]. However, here we have an Euler-Bernoulli beam with *variable* physical characteristics. We also treat the case in which the shuttle is rotating with a varying angular velocity.

It should be noted that there are other methods for proving boundary controllability results. One method which has achieved considerable success is the HUM (Hilbert Uniqueness) Method (see Lions [11], [12]). However, it is not clear that this method can be used to get similar results for the type of problems we consider. Even for the simple beam equation considered in [17], an application of the method yields controllability results in spaces with very poor regularity. Indeed, the spaces are occasionally spaces of distributions. It is possible that the process of 'weakening the norm' (Lagnese, Lions [10]) can overcome this problem, but even then the HUM method in its present form does not apply to equations of the form (2).

Chapter 5 of this work can be read independently of the preceding chapters and deals with the Gevrey regularity of strongly continuous semigroups. We see that this property falls naturally between those of differentiability and analyticity of semigroups. Theorems analogous to those already known for differentiable and analytic semigroups are given.

Finally, we note that the definition of Gevrey functions which we use here differs little from that of the functions introduced by Maurice Gevrey [6], in 1918. His name for these functions is *functions of class* α. We use the following definition:

Definition: Let Ω be an open subset of $\mathbf{R}^n$, let $\delta > 0$, and let $(B, \|\ \|)$ be a Banach space. If $f : \Omega \to B$ is infinitely differentiable then we say that f is of **Gevrey class δ** (i.e. $f \in \gamma^\delta(\Omega; B)$) if for each compact subset K of Ω and each $\theta > 0$, there exists a constant $C \geq 0$ such that†

$$|D^\alpha f(x)| \leq C\, \theta^{|\alpha|}\, (|\alpha|!)^\delta, \quad \text{for all multi-indices } \alpha, \text{ and all } t \in K.$$

If $B = \mathbf{R}$, we just write $\gamma^\delta(\Omega; \mathbf{R}) = \gamma^\delta(\Omega)$.

Occasionally we need to use this concept in a locally convex space X topologized by a family of separating seminorms P. In this case, we say that an infinitely differentiable function $f : \Omega \to X$ is of **Gevrey class δ with respect to the seminorms P** (i.e $f \in \gamma^\delta(\Omega; X, P)$) if for each compact subset K of Ω, each $p \in P$ and each $\theta > 0$, there exists a constant $C \geq 0$ such that

$$p(D^\alpha f(x)) \leq C\, \theta^{|\alpha|}\, (|\alpha|!)^\delta, \quad \text{for all multi-indices } \alpha, \text{ and all } t \in K.$$

Stephen W. Taylor.

---

†*The notation here is that of Laurent Schwartz [9]. If* $\alpha = (i,j)$ *then* $D^\alpha u(x,a) = \frac{\partial^{i+j} u}{\partial x^i\, \partial a^j}(x,a)$, *and* $|\alpha| = i+j$.

Table of Contents

Abstract

Chapters 1 to 4 of this work generalize the work of Littman and Markus [17] on the 'Exact Boundary Controllability of a Hybrid System of Elasticity'. Chapter 5 is independent of the earlier chapters and it deals with the Gevrey regularity of strongly continuous semigroups.

In Chapter 1, we consider equations of the form

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \sum_{i=0}^{4} a_i(x)\,\frac{\partial^i w}{\partial x^i}(x,t) + \sum_{i=0}^{2} b_i(x)\,\frac{\partial^{i+1} w}{\partial x^i \partial t}(x,t) = 0,$$

where $b_2(x) \le 0$, $a_4(x) > 0$ and the coefficients satisfy other mild conditions. We investigate solutions belonging to certain spaces of functions and show that if the initial values of w and $\frac{\partial w}{\partial t}$ have compact support, then the solutions are infinitely differentiable with respect to t for $t > 0$. Moreover, the derivatives are shown to satisfy certain Gevrey estimates.

In Chapter 2, we restrict the coefficients to be constants and show that in this case we obtain solutions which are analytic functions of t for $t > 0$.

In Chapter 3, we use the Gevrey regularity established in Chapter 1 to obtain solutions which vanish for $t > T$ (T being a given positive constant). In fact, we consider more general equations which are assumed to satisfy results similar to those proved in Chapter 1 and which have coefficients possessing a 'Gevrey' dependence on time. We need the Gevrey regularity to solve the equations in the 'x-direction'.

We apply the theory of the previous chapters in Chapter 4. Here, we consider the boundary controllability of a hybrid system consisting of an elastic beam clamped at one end to a space shuttle and at the other end to an antenna. The beam equation has variable coefficients and the ordinary differential equation of motion for the antenna yields unusual boundary conditions. We prove that the system is exactly controllable with two open loop controllers applied only to the antenna and that the rest state may be reached during an arbitrarily short time duration. The case in which the shuttle is rotating is also considered. This generalizes the work in [17].

The Gevrey regularity of semigroups is dealt with in Chapter 5. We develop theorems yielding both necessary and sufficient conditions for this regularity, which is a property falling naturally between those of differentiability and analyticity of semigroups.

Chapter 1

Gevrey Solutions for Equations of Euler-Bernoulli Type

1.0 Introduction

In this chapter, we consider the following partial differential equation

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \sum_{i=0}^{4} a_i(x)\,\frac{\partial^i w}{\partial x^i}(x,t) + \sum_{i=0}^{2} b_i(x)\,\frac{\partial^{i+1} w}{\partial x^i \partial t}(x,t) = 0, \tag{1}$$

for $t > 0$ and x in an interval $(0,r)$.

What we have in mind are boundary control problems of the following form:

> Find functions $f_1(t)$, $f_2(t)$ so that the solution $w(x,t)$ of (1), with the following boundary and initial conditions, vanishes for $t \geq T$, where $T > 0$ is given.
>
> $$w(x,0) = w_0(x), \qquad \frac{\partial w}{\partial t}(x,0) = v_0(x) \qquad 0 \leq x \leq r, \tag{2}$$
>
> $$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0 \qquad \text{for } t \geq 0, \tag{3}$$
>
> $$B_1 w(r,t) = f_1(t), \qquad B_2 w(r,t) = f_2(t) \qquad \text{for } t \geq 0, \tag{4}$$
>
> where B_1 and B_2 are linear differential operators defined at the boundary.

There are many examples of such control problems when equation (1) is, for instance, the Euler-Bernoulli beam equation,

$$\rho(x)\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left[E(x)\,I(x)\,\frac{\partial^2 w}{\partial x^2}(x,t)\right] = 0.$$

which will be discussed in Chapter 4.

As was mentioned in the Preface, a first step in solving such problems is to generate solutions of (1), (2) and (3) which are of Gevrey class in the time variable, t, for $t > 0$. This first step is the purpose of Chapter 1.

We achieve this goal by first solving equations (1), (2) and (3) for $x \in (0,\infty)$, after extending the domains of the coefficients of (1) and of the initial data.

In Sections 1.1 to 1.7, we make the restrictions

$$a_4(x) \equiv 1, \qquad a_3(x) \equiv 0. \tag{5}$$

for this can be achieved by means of an elementary transformation of variables. This restriction is removed in Section 1.8.

We note that the reader will quickly realize that the methods developed in this chapter are applicable when equation (3) is replaced by certain other types of homogeneous boundary conditions. However for the sake of clarity of exposition, we consider only the boundary conditions (3).

Our main result is that there is a fundamental solution K(x,a,t), of equation (1), which is of Gevrey class 2 in t, and is such that solutions of (1) are given by

$$w(x,t) = \int_0^\infty K(x,a,t)\, v_0(a) + L_2\, w_0(a)\, da \;+\; \frac{\partial}{\partial t}\int_0^\infty K(x,a,t)\, w_0(a)\, da, \tag{6}$$

provided that w_0 and v_0 are in certain spaces of functions† with compact support. In this expression, $L_2=b_2(x)d^2/dx^2+b_1(x)d/dx+b_0(x)$.

The chapter is set out as follows:

Section 1.1 deals with the standard semigroup approach for solving equation (1).

In Sections 1.2 to 1.4, we use an asymptotic analysis approach to construct a fundamental solution (Green's function) for the resolvent ordinary differential equation. Standard

---

†*For a precise statement of this, see section (1.8).*

analyses of this type impose the condition that the spectral parameter be confined to a certain sector of the complex plane. However, we require an analytic continuation into a larger set, and our assumptions allow us to achieve this.

In Sections 1.5 to 1.7, we use the spectral properties found in the preceding sections to construct the fundamental solution of (1) and show that it is indeed of Gevrey class 2.

Finally, in Section 1.8, we remove the restrictions (5) and present the final results of the chapter.

1.1 The Semigroup Formulation For The Solution of Equation (1) for $x \in (0,\infty)$

Let L_1 and L_2 denote the differential expressions:

$$L_1=a_2(x)d^2/dx^2+a_1(x)d/dx+a_0(x)$$

$$L_2=b_2(x)d^2/dx^2+b_1(x)d/dx+b_0(x).$$

Here we have assumed that the domains of the coefficient functions $a_i(x)$, $b_i(x)$ have been extended so that these functions are continuous and bounded on $[0,\infty)$. Later, in Section 1.2, we will restrict attention to the case in which these functions have compact support in $[0,\infty)$, but this assumption is not required in the present section.

Equation (1) may be written as a system:

$$\frac{\partial}{\partial t}\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\frac{\partial^4 w}{\partial x^4} - L_1 w - L_2 v \end{bmatrix} . \tag{7}$$

We shall solve equation (7) by considering the corresponding abstract ordinary differential equation in a Hilbert space. We let Hdenote the Hilbert space:

$$\mathsf{H} = H_0^2(\mathbf{R}^+) \times L^2(\mathbf{R}^+)$$

endowed with the inner product (,), where:

$$\left(\begin{pmatrix} w_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ v_2 \end{pmatrix}\right) = \int_0^\infty v_1(x)v_2(x) + w_1(x)w_2(x) + w_1'(x)w_2'(x) + w_1''(x)w_2''(x)\, dx \qquad (8)$$

We use the notation $(u,u) = \|u\|^2$.

We may define an unbounded operator A; on H as follows:

The domain of A, $D_A = (H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+)) \times H_0^2(\mathbf{R}^+)$,

$$A \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\dfrac{\partial^4 w}{\partial x^4} - L_1 w - L_2 v \end{bmatrix} . \qquad (9)$$

It is clear that A is a closed, densely defined operator. We shall be interested in solving the abstract ordinary differential equation initial value problem:

$$\frac{du}{dt} = Au, \qquad u(0) = \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \in D_A . \qquad (10)$$

The following lemmas contain the properties of A which we will need to solve equation (10).

Lemma 1

Suppose that a_i, b_i, b_i', b_i'', for $i \in \{0,1,2\}$ are all continuous and bounded on $[0,\infty)$ and that $b_2(x) \le 0$ for all $x \in [0,\infty)$.
Then there exists a constant

$$\Lambda = \Lambda(\|a_0\|_{L^\infty}, \|a_1\|_{L^\infty}, \|a_2\|_{L^\infty}, \|b_2''\|_{L^\infty}, \|b_1'\|_{L^\infty}, \|b_0\|_{L^\infty})$$

such that $(Au,u) \le \Lambda\|u\|^2$ for all $u \in D_A$. (11)

Proof

Let $u = \begin{bmatrix} w \\ v \end{bmatrix} \in D_A$. Then:

$$(Au, u) = \left(\begin{bmatrix} v \\ -\frac{\partial^4 w}{\partial x^4} - L_1 w - L_2 v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right)$$

$$= \int_0^\infty vw + v'w' + v''w'' - vw^{(4)} - vL_1 w - vL_2 v \, dx$$

$$= \int_0^\infty vw + v'w' - vL_1 w - vL_2 v \, dx$$

$$= \int_0^\infty vw + v'w' - v\sum_{i=0}^{2} a_i w^{(i)} - v\sum_{i=0}^{2} b_i v^{(i)} \, dx$$

$$= \int_0^\infty vw + v'w' - v\sum_{i=0}^{2} a_i w^{(i)} + b_2(v')^2 + (b_2' - b_1)vv' - b_0 v^2 \, dx$$

$$= \int_0^\infty vw - vw'' - v\sum_{i=0}^{2} a_i w^{(i)} + b_2(v')^2 + \frac{1}{2}(b_1' - b_2'' - 2b_0)v^2 \, dx$$

$$\leq \int_0^\infty vw - vw'' - v\sum_{i=0}^{2} a_i w^{(i)} + \frac{1}{2}(b_1' - b_2'' - 2b_0)v^2 \, dx$$

$$\leq \Lambda \|u\|^2$$

with $\Lambda = 2 \max (\|a_0\|_{L^\infty} + 1, \ \|a_1\|_{L^\infty}, \ \|a_2\|_{L^\infty} + 1, \ 2\|b_0\|_{L^\infty} + \|b_1'\|_{L^\infty} + \|b_2''\|_{L^\infty})$.

□

Remark: Note that equation (11) simply states that $A - \Lambda I$ is a dissipative operator, i.e. that $((A - \Lambda I)u,u) \leq 0$ for all u in D_A. If the function b_2 were to vanish, a slight modification of the proof of Lemma 1 shows that $-A - \Lambda I$ would also be dissipative.

We shall denote by $R(\lambda)$ the resolvent of A; i.e. $R(\lambda) = (\lambda I - A)^{-1}$, where I is the identity mapping on H.

Observe that:

$$(\lambda I - A)\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathsf{H}$$

$$\Leftrightarrow \quad \lambda\begin{bmatrix} w \\ v \end{bmatrix} + \begin{bmatrix} -v \\ w^{(4)} + L_1 w + L_2 v \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{cases} v = \lambda w - f_1 \\ \lambda^2 w + w^{(4)} + \lambda L_2 w + L_1 w = f_2 + \lambda f_1 + L_2 f_1 \end{cases} . \tag{13}$$

Thus a study of the spectral properties of A amounts to investigating the solutions $w \in H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+)$ of equation (13) with $f_1 \in H_0^2(\mathbf{R}^+)$ and $f_2 \in L^2(\mathbf{R}^+)$.

Lemma 2

For each $k > 0$ there exists a mapping

$$G_k \in \mathsf{b}\,(L^2(\mathbf{R}^+),\ H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+))$$

with the following properties:

$$k^4 w + w^{(4)} = f\,, \tag{14}$$

$$\|w^{(i)}\|_{L^2} \leq C|k|^{i-4}\|f\|_{L^2} \ ,\ \ (\ i = 0, 1, 2, 3, 4\ ), \tag{15}$$

where $w = G_k f$, and C is a constant independent of k. Further, $w = G_k f$ is the unique member of $H_0^2(\mathbf{R}^+)$ such that

$$g_k(w\,,\ u) = \int_0^\infty w''u'' + k^4 wu\,dx = \int_0^\infty fu\,dx \qquad \forall\ u \in H_0^2(\mathbf{R}^+). \tag{16}$$

Proof

Consider the functions

$$\mathsf{b}(x,k) = \frac{\sqrt{2}}{8k^3}\{(1+i)\exp[-k\frac{(1+i)}{\sqrt{2}}|x|] + (1-i)\exp[-k\frac{(1-i)}{\sqrt{2}}|x|]\}, \tag{17}$$

$$\mathsf{c}(x,a,k) = \frac{\sqrt{2}}{8k^3}\{(1+i)\exp[-k\frac{(1-i)}{\sqrt{2}}(x+a)] + (1-i)\exp[-k\frac{(1+i)}{\sqrt{2}}(x+a)]$$

$$-2\exp[-\frac{k}{\sqrt{2}}((x+a)-i|x-a|)] - 2\exp[-\frac{k}{\sqrt{2}}((x+a)+i|x-a|)]\}, \tag{18}$$

$$\mathsf{g}(x,a,k) = \mathsf{b}(x-a,k) + \mathsf{c}(x,a,k). \tag{19}$$

We define mappings B_k and C_k on $C_0^\infty(R^+)$ by:

$$\mathsf{B}_k f(x) = \int_0^\infty \mathsf{b}(x-a,k)f(a)\,da\,, \qquad \mathsf{C}_k f(x) = \int_0^\infty \mathsf{c}(x,a,k)f(a)\,da\,. \tag{20}$$

Now it is clear that

$$\left|\frac{\partial^j \mathsf{c}}{\partial x^j}(x,a,k)\right| \le 2\,|k|^{j-3}\exp[-\frac{k}{\sqrt{2}}(x+a)]\,, \quad \text{for } j = 0,\ 1,\ 2,\ 3,\ 4. \tag{21}$$

Hence by the Cauchy - Schwarz inequality and from the fact that

$$\left(\int_0^\infty\int_0^\infty e^{-\sqrt{2}\,k(x+a)}\,dx\,da\right)^{\frac{1}{2}} = \int_0^\infty e^{-\sqrt{2}\,kx}\,dx = \frac{1}{\sqrt{2}\,k}\,,$$

estimates like those of (15) hold for the mapping C_k.

B_k can be conveniently analysed by redefining it to act on $C_0^\infty(R)$ as follows:

$$\mathsf{B}_k f(x) = \int_{-\infty}^{\infty} \mathsf{b}(x-a, k) f(a)\, da = (\, \mathsf{b}(.,k) * f\,)(x). \tag{22}$$

An elementary calculation shows that the Fourier transform of $\mathsf{b}(.,k)$ is

$$\Psi_k(\xi) = \frac{1}{\sqrt{2\pi}\,(k^4 + \xi^4)}. \tag{23}$$

Thus we have:

$$(D^n \mathsf{B}_k f)^\wedge\,(\xi) = \Psi_k(\xi)\,(i\xi)^n\,\hat{f}(\xi)\,, \qquad \text{for } n = 0,\ 1,\ 2,\ 3,\ 4,\ \ldots\ldots \tag{24}$$

where D denotes the first derivative operator. A further elementary calculation shows that the suprema of the expressions $|\xi^n(k^4 + \xi^4)^{-1}|$ are:

$$\begin{cases} \frac{1}{4}\, n^{n/4}(4-n)^{1-n/4} k^{n-4} & \text{if } n = 0,\ 1,\ 2,\ 3 \\ 1 & \text{if } n = 4. \end{cases} \tag{25}$$

Since $f \in C_0^\infty(\mathbf{R})$, it follows that $\hat{f}$ is a member of the Schwarz class of functions. In particular, $\hat{f} \in L^2(\mathbf{R})$. Hence, we see that

$$\|(D^n \mathsf{B}_k f)^\wedge\|_{L^2} \le C k^{n-4} \|\hat{f}\|_{L^2}. \qquad n = 0,\ 1,\ 2,\ 3,\ 4. \tag{26}$$

But the Fourier transform preserves L^2 norms, so estimates like those of (15) also hold for the mapping B_k, where $f \in C_0^\infty(\mathbf{R}^+)$. Hence, estimates (15) hold for the mapping $G_k = \mathsf{B}_k + \mathsf{C}_k$, provided that $f \in C_0^\infty(\mathbf{R}^+)$. Since $C_0^\infty(\mathbf{R}^+)$ is dense in $L^2(\mathbf{R}^+)$, G_k extends uniquely to a mapping from $L^2(\mathbf{R}^+)$ into $H^4(\mathbf{R}^+)$ having the properties listed in (15).

The function g was constructed to have the following properties:

$$\mathsf{g}(.,.,k) \in C^2(\mathbf{R}^2) \cap C^\infty(\ \{(x, a) : x \ne a\}\ )\,, \tag{27}$$

$$\frac{\partial^3 g}{\partial x^3}(a, a^-, k) - \frac{\partial^3 g}{\partial x^3}(a, a^+, k) = 1, \tag{28}$$

$$\frac{\partial^4 g}{\partial x^4}(x, a, k) + k^4 g(x, a, k) = 0 \qquad \text{for } x \neq a, \tag{29}$$

$$g(0, a, k) \equiv \frac{\partial g}{\partial x}(0, a, k) \equiv 0. \tag{30}$$

Equation (30) implies that G_k maps $C_0^\infty(\mathbf{R}^+)$ into $H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+)$. Since this is a closed subspace of $H^4(\mathbf{R}^+)$, it follows that G_k maps $L^2(\mathbf{R}^+)$ into $H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+)$. Properties (27), (28) and (29) imply that $g_k(.,.,k)$ is a fundamental solution for the operator $\partial^4/\partial x^4 + k^4$, so equation (14) is satisfied as stated.

Equation (16) follows from equation (14) and integration by parts. The uniqueness assertion follows from the fact that the form g_k is positive definite.

This completes the proof of Lemma 2.

□

Lemma 3

Suppose that all the coefficients of L_1 and L_2 are continuous and bounded, that $b_2(x) \leq 0$ for all x in $[0,\infty)$, and that b_2' is continuous and bounded.

Then there exist constants $k_0 > 0$, $C > 0$ such that for each $k > k_0$ there is a unique mapping

$$H_k \in b\,(L^2(\mathbf{R}^+),\ H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+))$$

with the following properties:

$$k^4 w + w^{(4)} + k^2 L_2 w + L_1 w = f, \tag{31}$$

$$\|w^{(i)}\|_{L^2} \leq C|k|^{i-4}\|f\|_{L^2}, \quad (\ i = 0, 1, 2, 3, 4\ ), \tag{32}$$

where $w = H_k f$.

Proof

Consider the bilinear form h_k on $H_0^2(\mathbf{R}^+)$:

$$h_k(w, u) = \int_0^\infty w''u'' + k^4 wu + k^2(b_2 w''u + b_1 w'u + b_0 wu) + a_2 w''u + a_1 w'u + a_0 wu \, dx. \tag{33}$$

This bilinear form is clearly continuous. We shall see below that there exists a constant $k_0 > 0$ such that if $k > k_0$ then

$$h_k(w, w) \geq \frac{1}{9}(\|w''\|^2_{L^2} + k^2\|w'\|^2_{L^2} + k^4\|w\|^2_{L^2}). \tag{34}$$

To prove inequality (34) we make use of the standard interpolation inequality (c.f. Gilbarg and Trudinger [7]), which states that if w" and w are in $L^2(\mathbf{R}^+)$ then w' is in $L^2(\mathbf{R}^+)$ and for $\varepsilon > 0$,

$$\|w'\|^2_{L^2} \leq 6(\varepsilon\|w''\|^2_{L^2} + \frac{1}{\varepsilon}\|w\|^2_{L^2}). \tag{35}$$

This of course implies the following inequality:

$$\|w'\|_{L^2} \leq 3(\varepsilon\|w''\|_{L^2} + \frac{1}{\varepsilon}\|w\|_{L^2}). \tag{36}$$

Now,

$$h_k(w, w) = \int_0^\infty (w'')^2 + k^4 w^2 + k^2((b_1 - b_2')ww' + b_0 w^2)$$
$$- k^2 b_2 (w')^2 + (a_2 w''w + a_1 w'w + a_0 w^2) \, dx$$

$$\geq \|w''\|^2_{L^2} + k^4\|w\|^2_{L^2} - k^2C_1(\|w\|_{L^2}\|w'\|_{L^2} + \|w\|^2_{L^2}) - C_2(\|w''\|_{L^2}\|w\|_{L^2} + \|w'\|_{L^2}\|w\|_{L^2} + \|w\|^2_{L^2}),$$

where $C_1 = \max(\|b_1 - b_2'\|_{L^\infty}, \|b_0\|_{L^\infty})$ and $C_2 = \max(\|a_2\|_{L^\infty}, \|a_1\|_{L^\infty}, \|a_0\|_{L^\infty})$.
But by equations (35) and (36),

$$\|w'\|^2_{L^2} \leq 6(\frac{1}{k^2}\|w''\|^2_{L^2} + k^2\|w\|^2_{L^2}) \quad \text{and} \quad \|w'\|_{L^2} \leq 3(\frac{1}{k}\|w''\|_{L^2} + k\|w\|_{L^2}).$$

Hence,

$$h_k(w, w) \geq \|w''\|^2_{L^2} + k^4\|w\|^2_{L^2} - k^2C_1(\frac{3}{k}\|w''\|_{L^2}\|w\|_{L^2} + (3k+1)\|w\|^2_{L^2})$$
$$- C_2((1+\frac{3}{k})\|w''\|_{L^2}\|w\|_{L^2} + (3k+1)\|w\|^2_{L^2})$$

$$\geq \|w''\|^2_{L^2} + k^4\|w\|^2_{L^2} - k^2C_1\left(\frac{3}{k}\left[\frac{1}{2k^2}\|w''\|^2_{L^2} + \frac{k^2}{2}\|w\|^2_{L^2}\right] + (3k+1)\|w\|^2_{L^2}\right)$$
$$- C_2\left(\left(1 + \frac{3}{k}\right)\left[\frac{1}{2k^2}\|w''\|^2_{L^2} + \frac{k^2}{2}\|w\|^2_{L^2}\right] + (3k+1)\|w\|^2_{L^2}\right)$$

$$= \left(1 - \frac{3C_1}{2k} - \frac{C_2}{2k^2}\left[1 + \frac{3}{k}\right]\right)\|w''\|^2_{L^2} +$$
$$\left(1 - \frac{C_1}{k}\left[\frac{9}{2} + \frac{1}{k}\right] - \frac{C_2}{k^2}\left[\frac{1}{2} + \frac{9}{2k} + \frac{1}{k^2}\right]\right)k^4\|w\|^2_{L^2},$$

So there exists $k_0 > 0$ such that for $k > k_0$,

$$h_k(w\,,\ w) \geq \frac{7}{9}(\|w''\|^2_{L^2} + k^4\|w\|^2_{L^2})$$

$$\geq \frac{7}{9}(\|w''\|^2_{L^2} + k^4\|w\|^2_{L^2}) + \frac{1}{9}k^2\|w'\|^2_{L^2} - \frac{6}{9}k^2(\frac{1}{k^2}\|w''\|^2_{L^2} + k^2\|w\|^2_{L^2})$$

$$\geq \frac{1}{9}(\|w''\|^2_{L^2} + k^2\|w'\|^2_{L^2} + k^4\|w\|^2_{L^2}).$$

This proves inequality (34).

By the Lax-Milgram lemma [13], for each f in L^2 there exists a unique w in H^2_0 (defining the mapping $w = H_k f$) such that:

$$h_k(w\,,\ u) = \int_0^\infty fu\,dx \quad \forall\ u \in H^2_0(\mathbf{R}^+). \tag{37}$$

By equation (37) and inequality (34) we immediately obtain:

$$\|w\|_{L^2} \leq \frac{9}{k^4}\|f\|_{L^2}, \qquad \|w'\|_{L^2} \leq \frac{27}{k^3}\|f\|_{L^2}, \qquad \|w''\|_{L^2} \leq \frac{27}{k^2}\|f\|_{L^2}.$$

This establishes part of (32). For the rest of the proof we use Lemma 2 and note that equation (37) may be rewritten as follows:

$$g_k(H_k f\,,\ u) = \int_0^\infty \{- k^2(b_2[H_k f]'' + b_1[H_k f]' + b_0[H_k f])$$

$$- (a_2[H_k f]'' + a_1[H_k f]' + a_0[H_k f] + f)\}dx \quad \forall\ u \in H^2_0(R^+). \tag{38}$$

From equation (38) and Lemma 2 we obtain:

$$H_k = G_k\ \{I - k^2(b_2D^2 + b_1D + b_0)H_k - (a_2D^2 + a_1D + a_0)H_k\}, \tag{39}$$

where D denotes the first derivative operator.

From equation (39) it follows that the range of H_k is contained in $H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+)$. Further, the estimates in (32) which are yet to be proved (i.e. for i = 3,4) follow from those already found above, and from the fact that G_k satisfies (32).

Finally, the uniqueness assertion follows from the fact that if $w \in H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+)$ satisfies (31), then it also satisfies (37).This completes the proof of Lemma 3.

□

Theorem 4

Under the conditions of Lemma 1, A is the infinitesimal generator of a strongly continuous semigroup U on $\mathcal{H}$, and $\|U(t)\| \le e^{\Lambda t}$ for $t \ge 0$.

Proof

An immediate deduction from the result of lemma 3 is that for $\lambda > k_0^2$, R_λ exists.
In fact,

$$R_\lambda = \begin{bmatrix} H_{\sqrt{\lambda}}(\lambda+L_2) & H_{\sqrt{\lambda}} \\ H_{\sqrt{\lambda}}(\lambda^2+\lambda L_2) - 1 & \lambda H_{\sqrt{\lambda}} \end{bmatrix}. \tag{40}$$

Thus the resolvent of $A - \Lambda I$ exists for $\lambda > k_0^2 + \Lambda$. Equation (11) of lemma 1 states that $A - \Lambda I$ is dissipative. We have already observed that $A - \Lambda I$ is closed and densely defined. Thus it follows from the Lumer - Phillips theorem [18] that $A - \Lambda I$ is the infinitesimal generator of a strongly continuous semigroup of contractions. Consequently, A is the infinitesimal generator of a strongly continuous semigroup U satisfying $\|U(t)\| \le e^{\Lambda t}$ for $t \ge 0$. This proves Theorem 4.

□

Remark: If the function b_2 were identically zero, a slight modification of Lemma 3 shows that the resolvent of $-A - \Lambda I$ would also exist for λ sufficiently large. By the remark following Lemma 1 and by the proof of Theorem 3 it follows that -A would in this case also be the infinitesimal generator of a strongly continuous semigroup. This implies that the semigroup U of Theorem 3 could be continued in the negative t direction so as to obtain a strongly continuous group satisfying $\|U(t)\| \le e^{|\Lambda| t}$.

1.2 Approximate Solutions of $L_k u = 0$

We now restrict attention to the case in which the coefficient functions a_i, b_i (for $i \in \{0,1,2\}$) have compact support in $[0,\infty)$. In addition to this,we assume that A satisfies a certain "light damping" assumption.

Definition: We say that A satisfies the ***light damping assumption*** if $-2 < b_2(x) \le 0$ for all x in $\mathbf{R}^+$. Since b_2 is continuous with compact support, we can use the representation

$$b_2(x) = -2\cos\theta(x), \tag{41}$$

where we can find $\theta_1 > 0$ such that

$$\theta_1 < \theta(x) \le \frac{\pi}{2} \qquad \forall\, x \in [0,\infty). \tag{42}$$

The function θ is just as smooth as the function b_2 and is identically equal to $\pi/2$ outside a compact set.

We shall now investigate approximate solutions of equation (13). For this it is convenient to set $\lambda = k^2$ and consider the differential expression:

$$L_k = \frac{d^4}{dx^4} + k^2\left(2\cos\theta\frac{d^2}{dx^2} + b_1\frac{d}{dx} + b_0\right) + a_2\frac{d^2}{dx^2} + a_1\frac{d}{dx} + a_0 + k^4, \tag{43}$$

for $k \in \mathbf{C}$.

Lemma 5

In addition to the assumptions already stated in Lemma 1 and those stated above, let $\theta \in C^4[0,\infty)$ and $b_1 \in C^3[0,\infty)$. Then there exist functions r_j of the form:

$$r_j(x,k) = k\nu_j(x) + \mu_j(x), \qquad j = 1, 2, 3, 4, \tag{44}$$

such that:

(i) $\mu_j, \nu_j \in C^4[0,\infty)$, j = 1, 2, 3, 4;

(ii) The functions $u_j(x,k) = \exp(r_j(x,k))$, j = 1, 2, 3, 4, are such that $\frac{1}{u_j(x,k)} L_k u_j(x,k)$ is a polynomial in k of degree no greater than two with coefficients which are continuous functions of x with compact support;

(iii) The Wronskian $W(x,k) = \det\left(\frac{\partial^j u_i(x,k)}{\partial x^j}\right)$ is a polynomial in k of degree six. The coefficient of k^6 is $16 \sin^2\theta(0)$, while the coefficients of lower powers of k are continuous functions of x with compact support.

Proof

Consider $u(x,k) = \exp(k\nu(x) + \mu(x))$. The expansion of $\frac{1}{u(x,k)} L_k u(x,k)$ is a polynomial in k of degree no greater than four. Setting the third and fourth order terms equal to zero yields the two equations:

$$(\nu')^4 - 2\cos\theta(\nu')^2 + 1 = 0, \tag{45}$$

$$4(\nu')^3\mu' + 6(\nu')^2\nu'' - 2\cos(\theta)(2\nu'\mu' + \nu'') + b_1\nu' = 0. \tag{46}$$

Equation (45) yields $(\nu')^2 = e^{\pm i\theta}$. From this we take the four solutions:

$$\nu_1(x) = \int_0^x \exp[\tfrac{1}{2} i\theta(s)]\, ds, \tag{47a}$$

$$\nu_2(x) = \int_0^x \exp[-\tfrac{1}{2} i\theta(s)]\, ds, \tag{47b}$$

$$\nu_3(x) = \int_0^x -\exp[\tfrac{1}{2} i\theta(s)]\, ds, \tag{47c}$$

$$\nu_4(x) = \int_0^x \exp[-\frac{1}{2} i\theta(s)]\, ds. \tag{47d}$$

Substituting ν_j into equation (46) (for j = 1, 2, 3, 4) yields the following solutions μ_j :

$$\mu_1(x) = -\frac{1}{2}\ln\frac{\sin(\theta(x))}{\sin(\theta(0))} + \frac{3i}{4}(\theta(x) - \theta(0)) + \frac{i}{4}\int_0^x \frac{b_1(s)}{\sin(\theta(s))}\, ds, \tag{48a}$$

$$\mu_2(x) = -\frac{1}{2}\ln\frac{\sin(\theta(x))}{\sin(\theta(0))} - \frac{3i}{4}(\theta(x) - \theta(0)) - \frac{i}{4}\int_0^x \frac{b_1(s)}{\sin(\theta(s))}\, ds, \tag{48b}$$

$$\mu_3(x) = -\frac{1}{2}\ln\frac{\sin(\theta(x))}{\sin(\theta(0))} + \frac{3i}{4}(\theta(x) - \theta(0)) + \frac{i}{4}\int_0^x \frac{b_1(s)}{\sin(\theta(s))}\, ds, \tag{48c}$$

$$\mu_4(x) = -\frac{1}{2}\ln\frac{\sin(\theta(x))}{\sin(\theta(0))} - \frac{3i}{4}(\theta(x) - \theta(0)) - \frac{i}{4}\int_0^x \frac{b_1(s)}{\sin(\theta(s))}\, ds. \tag{48d}$$

Now we define r_j by equation (44) and u_j by equation (ii). Clearly from equations (47) and (48) we have $\nu_j, \mu_j \in C^4[0,\infty)$. The expressions

$$\frac{1}{u_j(x,k)} L_k u_j(x,k), \qquad j = 1, 2, 3, 4,$$

have been constructed to be polynomials in k of degree no greater than two, the coefficients of which are continuous functions of x with compact support. This establishes (ii).

By equations (47) and (48) we have:

$$W(x,k) = \left(\frac{\sin\theta(0)}{\sin\theta(x)}\right)^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ r_1' & r_2' & r_3' & r_4' \\ (r_1')^2 + r_1'' & (r_2')^2 + r_2'' & (r_3')^2 + r_3'' & (r_4')^2 + r_4'' \\ (r_1')^3 + 3r_1'r_1'' + r_1^{(3)} & (r_2')^3 + 3r_2'r_2'' + r_2^{(3)} & (r_3')^3 + 3r_3'r_3'' + r_3^{(3)} & (r_4')^3 + 3r_4'r_4'' + r_4^{(3)} \end{vmatrix}.$$

Since each r_j' is a first degree polynomial in k, it is apparent that the determinant above is a polynomial in k of degree no larger than six. The coefficient of k^6 is given by the usual Vandermonde expression:

$$W(x,k) = \left(\frac{\sin\theta(0)}{\sin\theta(x)}\right)^2 [(r_2'-r_1')(r_3'-r_2')(r_3'-r_1')(r_4'-r_3')(r_4'-r_2')(r_4'-r_1')]k^6 + p_1(x,k)$$

$$= \left(\frac{\sin\theta(0)}{\sin\theta(x)}\right)^2 (e^{-i\theta/2} - e^{i\theta/2})(-e^{i\theta/2} - e^{-i\theta/2})(-e^{i\theta/2} - e^{i\theta/2})(-e^{-i\theta/2} + e^{i\theta/2})$$

$$. (-e^{-i\theta/2} - e^{-i\theta/2})(-e^{-i\theta/2} - e^{i\theta/2})k^6 + p_2(x,k)$$

$$= 16 \sin^2\theta(0)\, k^6 + p_3(x,k). \tag{49}$$

Here $p_1(x,k)$, $p_2(x,k)$ and $p_3(x,k)$ are polynomials in k of degree less than or equal to five. Since b_1 has compact support and $\theta(x) = \pi/2$ outside a compact set, it is easy to see that the coefficients of order less than or equal to five in the expression for W(x,k) vanish outside a compact set. Thus $W(x,k) = 16 \sin^2\theta(0)\, k^6$ for large values of x. This proves (iii).

□

The functions u_1, u_2, u_3, u_4 have been constructed as approximate solutions of the ordinary differential equation $L_k u = 0$. However it is more useful to consider them as actual solutions of an equation which approximates the equation $L_k u = 0$. The following lemma shows how we can do this.

Lemma 6

Under the assumptions of Lemma 5, there exists a constant $C_1 > 0$ such that for $|k| > C_1$ the functions u_1, u_2, u_3, u_4 form a fundamental set of solutions for a homogeneous equation:

$$\tilde{L}_k y = y^{(4)} + k^2(2\cos\theta y'' + b_1 y') + k^4 y + c_3 y^{(3)} + c_2 y'' + c_1 y' + c_0 y = 0 . \tag{50}$$

The functions c_3, c_2, c_1, c_0 are all rational functions of k with coefficients which are continuous functions of x. There exists a constant M_1 such that

$$|c_j(x,k)| \le M_1 |k|^{2-j}, \qquad j = 0, 1, 2, 3, \qquad |k| > C_1. \tag{51}$$

Further, the functions $c_j(x,k)$ vanish for x outside a compact set. If $\theta \in C^5[0,\infty)$, $b_1 \in C^4[0,\infty)$, then the functions $c_j(x,k)$ are each $C^1[0,\infty)$ functions of x and an estimate of the same form as (51) holds with c_j' replacing c_j.

Proof

By (42) and part (iii) of Lemma 5, we can find $C_1 > 0$ such that

$$\left| \frac{W(x,k)}{16k^6 \sin^2\theta(0)} \right| \ge \frac{1}{2} \qquad \text{for } |k| > C_1.$$

Since u_1, u_2, u_3, u_4 are all $C^4[0,\infty)$ functions of x which have a Wronskian which does not vanish, an elementary result from the theory of ordinary differential equations (see Coddington and Levinson [4]) yields the fact that these functions form a fundamental solution set for a linear, homogeneous, fourth order equation. In fact, the equation can readily be written down in terms of determinants:

Wronskian of $(y, u_1, u_2, u_3, u_4) = 0$,

$$\text{i.e.,} \quad \begin{vmatrix} y & u_1 & u_2 & u_3 & u_4 \\ y' & u_1' & u_2' & u_3' & u_4' \\ y'' & u_1'' & u_2'' & u_3'' & u_4'' \\ y^{(3)} & u_1^{(3)} & u_2^{(3)} & u_3^{(3)} & u_4^{(3)} \\ y^{(4)} & u_1^{(4)} & u_2^{(4)} & u_3^{(4)} & u_4^{(4)} \end{vmatrix} = 0. \tag{52}$$

After factoring out an exponential term, we may write equation (52) as

$$\det(\vec{Y}, \vec{R}_1, \vec{R}_2, \vec{R}_3, \vec{R}_4) = 0, \tag{53}$$

where $\vec{Y} = \begin{bmatrix} y \\ y' \\ y'' \\ y^{(3)} \\ y^{(4)} \end{bmatrix}$, and $\vec{R}_j = \begin{bmatrix} 1 \\ r_j' \\ (r_j')^2 + r_j'' \\ (r_j')^3 + 3r_j'r_j'' + r_j^{(3)} \\ (r_j')^4 + 6(r_j')^2 r_j'' + 4r_j'r_j^{(3)} + 3(r_j'')^2 + r_j^{(4)} \end{bmatrix}$ for j = 1, 2, 3, 4.

Equation (53) is equivalent to the equation

$$\det(\vec{Z}, \vec{S}_1, \vec{S}_2, \vec{S}_3, \vec{S}_4) = 0, \tag{54}$$

$$\text{where } \vec{Z} = \vec{Y} + \{k^2(2\cos\theta\ y'' + b_1 y') + k^4 y\}\vec{e}_5$$

$$\text{and } \vec{S}_j = \vec{R}_j + \{k^2(2\cos\theta[(r_j')^2 + r_j''] + b_1 r_j') + k^4\}\vec{e}_5 .$$

It is clear that for i = 1, 2, 3, 4, the i^{th} component in each of the vectors $\vec{S}_j$ is a polynomial in k of degree no more than i-1. However, by the construction of the functions r_j in Lemma 5, it follows that the fifth component of each of the vectors $\vec{S}_j$ is a polynomial in k of degree no greater than two.

Thus in the cofactor expansion of the left side of equation (54), down its first column, we see that the coefficient of $y^{(4)} + k^2(2\cos\theta\ y'' + b_1 y') + k^4 y$ is a polynomial of degree six, while the coefficients of $y^{(3)}$, y", y' and y are polynomials of degree no more than five, six, seven and eight respectively.

The coefficient of $y^{(4)} + k^2(2\cos\theta\ y'' + b_1 y') + k^4 y$ has as its highest order (k^6) term the same Vandermonde determinant that was encountered in the proof of Lemma 5. Thus, for $|k| > C_1$, we can divide by this coefficient to get equation (50), where the functions c_j are rational functions of k which satisfy estimates (51). That the functions c_j are continuous functions of x is an immediate consequence of the fact that the functions r_j are $C^4[0,\infty)$ in x.

The last statement in the lemma follows from the fact that the r_j are $C^5[0,\infty)$ functions of x if $\theta \in C^5[0,\infty)$ and $b_1 \in C^4[0,\infty)$.

□

1.3 A Fundamental Solution for $\tilde{L}_k \tilde{g} = \delta(x-a)$

Theorem 7

Let $\tilde{L}_k$ be as in Lemma 6. There is a constant $C_2 > 0$ such that for $|k| > C_2$ there exists a fundamental solution $\tilde{g}(x,a,k)$ of

$$\tilde{L}_k y = \delta(x-a)$$

satisfying:

$\tilde{g}$, $\tilde{g}_x$, $\tilde{g}_{xx}$ $\tilde{g}_a$, $\tilde{g}_{ax}$ are all continuous functions of x and a, $\tilde{g}_{xxx}$, $\tilde{g}_{xxxx}$, $\tilde{g}_{xxa}$, $\tilde{g}_{xxxa}$, $\tilde{g}_{xxxxa}$ are all continuous functions of x and a for x≠a, and all mixed derivatives, of the same order with respect to a and of the same order with respect to x as one listed here, are equal. Further, the restrictions of these functions to either of the domains $\{(x,a) : 0 \le x < a\}$ and $\{(x,a) : 0 \le a < x\}$ can be extended to be continuous functions on the closure of that domain. (55)

$$\frac{\partial^3 \tilde{g}}{\partial x^3}(a, a^-, k) - \frac{\partial^3 \tilde{g}}{\partial x^3}(a, a^+, k) = 1, \tag{56}$$

$$\tilde{L}_k \tilde{g}(x,a,k) = 0 \quad \text{provided } x \neq a, \tag{57}$$

$$\tilde{g}(0, a, k) \equiv \frac{\partial \tilde{g}}{\partial x}(0, a, k) \equiv 0. \tag{58}$$

Further, if $f \in C_0^\infty(\mathbf{R}^+)$, $w_k(x) = \int_0^\infty \tilde{g}(x, a, k) f(a)da$, and $|\arg k| < \frac{\pi}{4}$,

$w_k(x)$, $w_k'(x)$, $w_k''(x)$, $w_k^{(3)}(x)$, $w_k^{(4)}(x)$ all $\to 0$ as $x \to \infty$. (59)

Proof

It is convenient to define two functions:

$$\alpha(k) = -\frac{1}{2}(e^{-i\theta(0)} + 1) + \frac{1}{4k}e^{-i\theta(0)/2}(3i\theta'(0) + \frac{ib_1(0)}{\sin\theta(0)}), \tag{60}$$

$$\bar{\alpha}(k) = -\frac{1}{2}(e^{i\theta(0)} + 1) - \frac{1}{4k}e^{i\theta(0)/2}(3i\theta'(0) + \frac{ib_1(0)}{\sin\theta(0)}). \tag{61}$$

Note that $\overline{\alpha(k)} = \bar{\alpha}(\bar{k})$ and that

$$\alpha(k)\bar{\alpha}(k) = \frac{1}{2}(1 + \cos\theta(0)) + \frac{1}{16k^2}\left(3\theta'(0) + \frac{b_1(0)}{\sin\theta(0)}\right)^2. \tag{62}$$

We now set

$$y_1 = u_1 + \bar{\alpha}u_2 - (1 + \bar{\alpha})u_4, \qquad y_2 = u_2 + \alpha u_1 - (1 + \alpha)u_3.$$

Clearly $y_1(0,k) = y_2(0,k) = 0$, and it is easy to check that $y_1'(0,k) = y_2'(0,k) = 0$. Thus $\{y_1,y_2\}$ form a basis for the set of solutions y of $\tilde{L}_k y = 0$, $y(0) = y'(0) = 0$.

Next, we define for $|k| > C_1$ (C_1 is the constant of Lemma 6)

$$P(x,a,k) = \begin{cases} \dfrac{1}{W(a,k)}\begin{vmatrix} u_1(a,k) & u_2(a,k) & u_3(a,k) & u_4(a,k) \\ u_1'(a,k) & u_2'(a,k) & u_3'(a,k) & u_4'(a,k) \\ u_1''(a,k) & u_2''(a,k) & u_3''(a,k) & u_4''(a,k) \\ u_1(x,k) & u_2(x,k) & u_3(x,k) & u_4(x,k) \end{vmatrix}, & x > a \\ 0, \quad x \le a. \end{cases}$$

The function P satisfies (55), (56), (57) and (58) and is therefore a fundamental solution for $\tilde{L}_k$. However, (59) is not satisfied because in the definition of P the cofactors of the increasing exponential terms $u_1(x,k)$ and $u_2(x,k)$ are not necessarily equal to zero. We show below that we can uniquely find functions $m_1(a,k)$ and $m_2(a,k)$, both C^1 functions of 'a' such that

$$\tilde{g}(x,a,k) = P(x,a,k) + m_1(a,k)y_1(x,k) + m_2(a,k)y_2(x,k) \tag{63}$$

has all of the required properties.

Clearly (55), (56), (57) and (58) are automatically satisfied. To obtain (59) we need only choose m_1 and m_2 so that the coefficients of $u_1(x,k)$ and $u_2(x,k)$ vanish for $x > a$. This requires:

$$\begin{bmatrix} 1 & \alpha(k) \\ \bar{\alpha}(k) & 1 \end{bmatrix} \begin{bmatrix} m_1(a,k) \\ m_2(a,k) \end{bmatrix} = \frac{1}{W(a,k)} \begin{bmatrix} \beta(a,k) \\ \bar{\beta}(a,k) \end{bmatrix}, \tag{64}$$

$$\text{where } \beta(a,k) = \begin{vmatrix} u_2(a,k) & u_3(a,k) & u_4(a,k) \\ u_2'(a,k) & u_3'(a,k) & u_4'(a,k) \\ u_2''(a,k) & u_3''(a,k) & u_4''(a,k) \end{vmatrix}, \tag{65}$$

$$\text{and } \bar{\beta}(a,k) = \begin{vmatrix} u_1(a,k) & u_4(a,k) & u_3(a,k) \\ u_1'(a,k) & u_4'(a,k) & u_3'(a,k) \\ u_1''(a,k) & u_4''(a,k) & u_3''(a,k) \end{vmatrix}. \tag{66}$$

By the light damping assumption ((41) and (42)) and by equation (62) it follows that there is a constant $C_2 > C_1$ such that the determinant of the system (64) is non-zero for $|k|>C_2$.

This establishes the existence and uniqueness of the functions m_1 and m_2, and completes the proof of the theorem.

□

Remark: For the particular case $\theta = \pi/2$, $b_1 = 0$, the fundamental solution $\tilde{g}$ constructed above is precisely the function g of Lemma 2.

Our aim is to construct a fundamental solution for the differential expression L_k with properties similar to those of the function $\tilde{g}$ of Theorem 7. In the process of doing this we shall encounter an integro-differential equation, the solution of which requires the estimates of the following lemma.

Lemma 8

$$\text{Let } B(d,k) = \begin{cases} 1, & \text{if } |\arg k| < \frac{\pi}{4} \\ \exp[\frac{d}{\sqrt{2}}(|\text{Im } k| - \text{Re } k)], & \text{if } \frac{\pi}{4} \le |\arg k| \le \frac{\pi}{2} \end{cases} \tag{67}$$

There is a constant M_2 such that for all k satisfying $|k| > C_2$ (C_2 being the constant of Theorem 7), the following estimates hold:

$$\left| \frac{\partial^{m+n}\tilde{g}}{\partial x^n \partial a^m}(x,a,k) \right| \le M_2 B(x+a,k)|k|^{n+m-3} \quad n = 0, 1, 2, 3, 4; \; m = 0, 1. \tag{68}$$

These estimates hold for all $x > 0$, $a > 0$ if $m + n < 3$, otherwise they hold only for $x \ne a$.

Proof

It is easily verified that $\tilde{g}(x,a,k)$ equals

$$\begin{cases} \frac{1}{W(a,k)}\{\beta(a,k)u_1(x,k)+\bar{\beta}(a,k)u_2(x,k) \\ \quad +(1-\alpha(k)\bar{\alpha}(k))^{-1}[(\bar{\alpha}(k)+1)(\alpha(k)\bar{\beta}(a,k)-\beta(a,k))u_4(x,k) \\ \quad +(\alpha(k)+1)(\bar{\alpha}(k)\beta(a,k)-\bar{\beta}(a,k))u_3(x,k)]\}, & \text{if } x \le a, \\ \frac{1}{W(a,k)}\{[-\bar{\gamma}(a,k) + (1-\alpha(k)\bar{\alpha}(k))^{-1}(\bar{\alpha}(k)+1)(\alpha(k)\bar{\beta}(a,k)-\beta(a,k))]u_4(x,k) \\ \quad +[-\gamma(a,k) + (1-\alpha(k)\bar{\alpha}(k))^{-1}(\alpha(k)+1)(\bar{\alpha}(k)\beta(a,k)-\bar{\beta}(a,k))]u_3(x,k)\}, & \text{if } x \ge a. \end{cases} \tag{69}$$

$$\text{where } \gamma(a,k) = \begin{vmatrix} u_1(a,k) & u_2(a,k) & u_4(a,k) \\ u_1'(a,k) & u_2'(a,k) & u_4'(a,k) \\ u_1''(a,k) & u_2''(a,k) & u_4''(a,k) \end{vmatrix}, \tag{70}$$

$$\text{and } \bar{\gamma}(a,k) = \begin{vmatrix} u_2(a,k) & u_1(a,k) & u_3(a,k) \\ u_2'(a,k) & u_1'(a,k) & u_3'(a,k) \\ u_2''(a,k) & u_1''(a,k) & u_3''(a,k) \end{vmatrix}. \tag{71}$$

By equations (47), (48) and (65) we obtain:

$$\beta(a,k) = \exp\left[k\left(\int_0^a -e^{i\theta(s)/2}\,ds\right) - \frac{3}{2}\ln\left(\frac{\sin\theta(a)}{\sin\theta(0)}\right) - \frac{3i}{4}(\theta(a) - \theta(0)) - \frac{i}{4}\int_0^a \frac{b_1(s)}{\sin\theta(s)}\,ds\right]$$

$$\bullet \begin{vmatrix} 1 & 1 & 1 \\ r_2' & r_3' & r_4' \\ r_2''+(r_2')^2 & r_3''+(r_3')^2 & r_4''+(r_4')^2 \end{vmatrix}.$$

But the determinant above is a polynomial in k, of degree three, with bounded coefficients. Hence for $|k| > C_2$ we can find a constant C such that

$$|\beta(a,k)| \le C|k|^3 \exp\left[-\mathrm{Re}\left(\int_0^a e^{i\theta(s)/2}\,ds\right)\right]. \tag{72}$$

$$\text{Now } \beta'(a,k) = \begin{vmatrix} u_2(a,k) & u_3(a,k) & u_4(a,k) \\ u_2'(a,k) & u_3'(a,k) & u_4'(a,k) \\ u_2^{(3)}(a,k) & u_3^{(3)}(a,k) & u_4^{(3)}(a,k) \end{vmatrix},$$

and it is easy to see that C can be chosen so that also:

$$|\beta'(a,k)| \le C|k|^4 \exp\left[-\mathrm{Re}\left(\int_0^a e^{i\theta(s)/2}\,ds\right)\right]. \tag{73}$$

We may treat $\bar{\beta}, \gamma, \bar{\gamma}$ and these functions' first derivatives similarly to obtain:

$$|\bar{\beta}(a,k)| \leq C|k|^3 \exp\left[-\mathrm{Re}\left(\int_0^a e^{-i\theta(s)/2}\, ds\right)\right], \tag{74}$$

$$|\bar{\beta}'(a,k)| \leq C|k|^4 \exp\left[-\mathrm{Re}\left(\int_0^a e^{-i\theta(s)/2}\, ds\right)\right], \tag{75}$$

$$|\gamma(a,k)| \leq C|k|^3 \exp\left[\mathrm{Re}\left(\int_0^a e^{i\theta(s)/2}\, ds\right)\right], \tag{76}$$

$$|\gamma'(a,k)| \leq C|k|^4 \exp\left[\mathrm{Re}\left(\int_0^a e^{i\theta(s)/2}\, ds\right)\right], \tag{77}$$

$$|\bar{\gamma}(a,k)| \leq C|k|^3 \exp\left[\mathrm{Re}\left(\int_0^a e^{-i\theta(s)/2}\, ds\right)\right], \tag{78}$$

$$|\bar{\gamma}'(a,k)| \leq C|k|^4 \exp\left[\mathrm{Re}\left(\int_0^a e^{-i\theta(s)/2}\, ds\right)\right]. \tag{79}$$

From equations (47) and (48),

$$u_1(x,k) = \exp\left[k\left(\int_0^x e^{i\theta(s)/2}\, ds\right) - \frac{1}{2}\ln\left(\frac{\sin\theta(x)}{\sin\theta(0)}\right) + \frac{3i}{4}(\theta(x)-\theta(0)) + \frac{i}{4}\int_0^x \frac{b_1(s)}{\sin\theta(s)}\, ds\right].$$

From this we obtain:

$$|u_1^{(n)}(x,k)| \leq C|k|^n \exp\left[\mathrm{Re}\left(k\int_0^x e^{i\theta(s)/2}\, ds\right)\right], \qquad n = 0, 1, 2, 3, 4. \tag{80}$$

Similarly, for $n = 0, 1, 2, 3, 4$:

$$|u_2^{(n)}(x,k)| \leq C|k|^n \exp\left[\mathrm{Re}\left(k\int_0^x e^{-i\theta(s)/2}\, ds\right)\right], \tag{81}$$

$$|u_3^{(n)}(x,k)| \le C|k|^n \exp\left[-\mathrm{Re}\left(k\int_0^x e^{i\theta(s)/2}\,ds\right)\right], \tag{82}$$

$$|u_4^{(n)}(x,k)| \le C|k|^n \exp\left[-\mathrm{Re}\left(k\int_0^x e^{-i\theta(s)/2}\,ds\right)\right]. \tag{83}$$

Furthermore, from Lemma 6 we have for $|k| > C_2 > C_1$,

$$W(a,k) \ge \text{constant} \cdot |k|^6 . \tag{84}$$

and from the proof of Lemma 5, it is clear that we have

$$W'(a,k) \le \text{constant} \cdot |k|^6 \qquad \text{for } |k| > C_2. \tag{85}$$

Hence from equation (69) and the estimates above, we see that for $x \le a$,

$$\begin{aligned}|\tilde{g}(x,a,k)| \le \frac{\text{constant}}{|k|^3}\Bigg\{ &\exp\left[-\mathrm{Re}\left(k\int_x^a e^{i\theta(s)/2}\,ds\right)\right] + \exp\left[-\mathrm{Re}\left(k\int_x^a e^{-i\theta(s)/2}\,ds\right)\right]\\
&+\exp\left[-\mathrm{Re}\left\{k\left(\int_0^a e^{-i\theta(s)/2}\,ds + \int_0^x e^{-i\theta(s)/2}\,ds\right)\right\}\right]\\
&+\exp\left[-\mathrm{Re}\left\{k\left(\int_0^a e^{i\theta(s)/2}\,ds + \int_0^x e^{-i\theta(s)/2}\,ds\right)\right\}\right]\\
&+\exp\left[-\mathrm{Re}\left\{k\left(\int_0^a e^{-i\theta(s)/2}\,ds + \int_0^x e^{i\theta(s)/2}\,ds\right)\right\}\right]\\
&+\exp\left[-\mathrm{Re}\left\{k\left(\int_0^a e^{i\theta(s)/2}\,ds + \int_0^x e^{i\theta(s)/2}\,ds\right)\right\}\right]\Bigg\}.\end{aligned}$$

For $x \ge a$, the same formula nearly applies, the only change necessary being that " $\int_x^a$ " should be replaced by " $\int_a^x$ ".

Recall that $0 < \theta_1 < \theta(x) \le \pi/2$, so if $|\arg k| \le \pi/4$ we obtain

$$\tilde{g}(x,a,k) \le \frac{\text{constant}}{|k|^3} .$$

If $\pi/4 \le |\arg k| \le \pi/2$, the dominant exponential term in the expression is

$$\text{either} \qquad \exp\left[-\operatorname{Re}\left\{k\left(\int_0^a e^{-i\theta(s)/2}\,ds + \int_0^x e^{-i\theta(s)/2}\,ds\right)\right\}\right],$$

$$\text{or} \qquad \exp\left[-\operatorname{Re}\left\{k\left(\int_0^a e^{i\theta(s)/2}\,ds + \int_0^x e^{i\theta(s)/2}\,ds\right)\right\}\right],$$

both of which are no greater than

$$\exp[-(\operatorname{Re} k)(\textstyle\int_0^a \cos(\theta(s)/2)\,ds + \int_0^x \cos(\theta(s)/2)\,ds)$$
$$+ |\operatorname{Im} k| \textstyle\int_0^a \sin(\theta(s)/2)\,ds + \int_0^x \sin(\theta(s)/2)\,ds)]$$

$$\le \exp[(|\operatorname{Im} k| - (\operatorname{Re} k))(x+a)/\sqrt{2}\,].$$

This establishes the first of inequalities (68) (for m = n = 0). The others follow in a similar fashion from the preceding estimates. This completes the proof of Lemma 8.

□

1.4 A Fundamental Solution for $L_k g = \delta(x\text{-}a)$

We are now in a position to construct a suitable fundamental solution for the differential expression L_k. For the sake of clarity of exposition, we first define some seminorms on the class of functions $f : [0,\infty)^2 \to \mathbf{C}$ which satisfy the regularity result (55) of Theorem 7. If $i \in \{0,1,2,3,4\}$ and $j \in \{0,1\}$, we define

$$|f|_{i,j}(d'', d') = \sup_{\substack{0\le x\le d'' \\ 0\le a\le d' \\ x\ne a}} \left| \frac{\partial^{i+j} f(x,a)}{\partial x^i \partial a^j} \right|. \tag{86}$$

Theorem 9

Suppose that the supports of the coefficient functions of the differential expressions L_1 and L_2 are contained in [0,d]. Then there exists a constant p such that for each k in the region

$$\Omega = \{k \in \mathbf{C} : B(2d,k) < p|k|\} \tag{87}$$

(i) there is a fundamental solution g(x,a,k) of

$$L_k\, g(x,a,k) = \frac{d^4}{dx^4}\, g(x,a,k) + L_1 g(x,a,k) + k^2 L_2 g(x,a,k) + k^4 g(x,a,k) = \delta(x-a).$$

(ii) g satisfies the regularity result (55) of Theorem 7 and there is a constant M_3 such that for all $d' \geq d$, $d'' \geq d$, $i \in \{0,1,2,3,4\}$ and $j \in \{0,1\}$,

$$\frac{|g|_{i,j}(d'', d')}{|k|^{i+j}} \leq M_3\, \frac{B(d''+d', k)}{|k|^3}\,. \tag{88}$$

(iii) g satisfies (56), (58), and (59) of Theorem 7. Equation (57) is of course replaced with

$$L_k g(x,a,k) = 0 \quad \text{for } x \neq a.$$

(iv) g(x,a,k) and its partial derivatives listed in (55) are analytic functions of k for $k \in \Omega$.

Proof

We start by solving the integro-differential equation

$$g(x, a, k) = \tilde{g}(x, a, k) + \int_0^d \tilde{g}(x, b, k)\, P_k g(b, a, k)\, db, \tag{89}$$

$$\text{where} \quad P_k g(b, a, k) = [c_3(b, k)\frac{\partial^3}{\partial b^3} + \{c_2(b, k) - a_2(b)\}\frac{\partial^2}{\partial b^2} + \{c_1(b, k) - a_1(b)\}\frac{\partial}{\partial b}$$
$$+\{c_0(b, k) - a_0(b) - k^2 b_0(b)\}]\, g(b, a). \tag{90}$$

We solve equation (89) by successive approximations, by defining

$$K_0(x, a, k) = \tilde{g}(x, a, k), \quad \text{and for } n \geq 1 \quad K_n(x, a, k) = \int_0^d \tilde{g}(x, b, k)\, P_k K_{n-1}(b, a, k)\, db. \tag{91}$$

By inequalities (51) of Lemma 6, we obtain for $d' \geq d$ and $i \in \{0,1,2,3\}$,

$$|K_n|_{i,0}(d,d') \leq d|\tilde{g}|_{i,0}(d,d)\,(M_1|k|^{-1}|K_{n-1}|_{3,0}(d,d') + [M_1 + \|a_2\|_{L^\infty}]\,|K_{n-1}|_{2,0}(d,d') + [M_1|k| + \|a_1\|_{L^\infty}]\,|K_{n-1}|_{1,0}(d,d') + [M_1|k|^2 + \|a_0\|_{L^\infty} + \|b_0\|_{L^\infty}|k|^2]\,|K_{n-1}|_{0,0}(d,d')).$$

But if we set

$$M = d\max\left(M_1 + \|a_2\|_{L^\infty},\ M_1 + \frac{\|a_1\|_{L^\infty}}{C_2},\ M_1 + \|b_0\|_{L^\infty} + \frac{\|a_0\|_{L^\infty}}{C_2^2}\right),$$

we get for $|k| > C_2$,

$$|K_n|_{i,0}(d,d') \leq |k|^2\,|\tilde{g}|_{i,0}(d,d)\,M\sum_{j=0}^{3}|k|^{-j}\,|K_{n-1}|_{j,0}(d,d') \qquad i \in \{0,1,2,3\}.$$

But by inequalities (68) of Lemma 8, we have

$$|\tilde{g}|_{i,0}(d,d) \leq M_2\,B(2d,k)\,|k|^{i-3}.$$

The last two inequalities imply that for $i \in \{0,1,2,3\}$,

$$|k|^{-i}\,|K_n|_{i,0}(d,d') \leq |k|^{-1}\,M\,M_2\,B(2d,k)\sum_{j=0}^{3}|k|^{-j}\,|K_{n-1}|_{j,0}(d,d'). \tag{92}$$

However, since $K_0 = \tilde{g}$, we have for $j \in \{0,1,2,3\}$,

$$|k|^{-j}\,|K_0|_{j,0}(d,d') \leq |k|^{-3}M_2\,B(d+d',k).$$

So iteration of inequality (92) yields

$$|k|^{-i}\,|K_n|_{i,0}(d,d') \leq |k|^{-3}M_2\,B(d+d',k)\left[\frac{4MM_2\,B(2d,k)}{|k|}\right]^n.$$

We choose the constant p in equation (87) by

$$p^{-1} = \max\,(8MM_2\,,\, C_2),$$

so we get:

$$|k|^{-i}\, |K_n|_{i,0}(d, d') \le |k|^{-3} M_2\; B(d+d',k) \left[\frac{1}{2}\right]^n .$$

Thus the series $K_0+K_1+K_2+K_3+\ldots$ together with the series of first and second partial derivatives with respect to x converge absolutely, uniformly on the compact set $[0,d] \times [0,d']$. The third partial derivative of each K_n with respect to x exists and is continuous when $x \neq a$. Thus we get convergence to a function g which, along with g_x and g_{xx} is continuous on $[0,d] \times [0,d']$. Also, g_{xxx} is continuous on $([0,d] \times [0,d']) \cap \{(x,a) : x \neq a\}$ and its restriction to either one of the components of this set can be continued to be a continuous function on that component. Furthermore, since the series converge uniformly, we see that they converge to analytic functions of k in Ω, and that g satisfies equation (89). Also,

$$|k|^{-i}\, |g|_{i,0}(d, d') \le |k|^{-3}\, M_2\; B(d+d',k) \sum_{n=0}^{\infty} \left[\frac{1}{2}\right]^n$$

$$= 2|k|^{-3}\, M_2\; B(d+d',k).$$

Next, note that for $a \le d$,

$$\frac{\partial K_n}{\partial a}(x,a,k) = \int_0^a \tilde{g}(x,b,k)\; P_k \frac{\partial K_{n-1}}{\partial a}(b,a,k)\; db + \int_a^d \tilde{g}(x,b,k)\; P_k \frac{\partial K_{n-1}}{\partial a}(b,a,k)\; db$$

$$+\; \tilde{g}(x,a,k)\; c_3(a,k) \left[\frac{\partial^3 K_{n-1}}{\partial x^3}(a^-,a,k) - \frac{\partial^3 K_{n-1}}{\partial x^3}(a^+,a,k) \right].$$

For $a \ge d$ the same formula applies, but the last term vanishes. From this we obtain:

$$|K_n|_{i,1}(d, d') \le |k|^2\, |\tilde{g}|_{i,0}(d, d)\; M \sum_{j=0}^{3} |k|^{-j}\, |K_{n-1}|_{j,1}(d, d')$$

$$+\; |\tilde{g}|_{i,0}(d, d)\; M_1 |k|^{-1}\, |K_{n-1}|_{3,0}(d, d).$$

Hence,

$$|k|^{-i}\, |K_n|_{i,1}(d, d') \le |k|^{-1}\, M\, M_2\, B(2d, k) \sum_{j=0}^{3} |k|^{-j}\, |K_{n-1}|_{j,1}(d, d')$$

$$+\; |k|^{-4}\, M_1\, M_2^2\, [B(2d, k)]^2 \left[\frac{1}{2}\right]^{n-1}$$

$$\le \frac{1}{8} \sum_{j=0}^{3} |k|^{-j}\, |K_{n-1}|_{j,1}(d, d') \;+\; \frac{M_2}{8|k|^3 d}\, B(2d, k) \left[\frac{1}{2}\right]^{n-1}.$$

But from inequality (68) of Lemma 8,

$$|k|^{-i}\, |K_0|_{i,1}(d, d') \le M_2\, |k|^{-2}\, B(d{+}d', k).$$

The last two inequalities imply that for $i \in \{0,1,2,3\}$,

$$|k|^{-i}\, |K_n|_{i,1}(d, d') \le \left[\frac{1}{2}\right]^{n} M_2\, B(d{+}d', k)\, |k|^{-2} + n\, M_2\, \frac{B(2d, k)}{8|k|^3 d} \left[\frac{1}{2}\right]^{n-1}.$$

This shows that the series

$$\sum_{n=0}^{\infty} \frac{\partial^{i+1} K_n}{\partial x^i \partial a} \qquad\qquad i \in \{0, 1\}$$

converge uniformly on $[0,d] \times [0,d']$, and that the series

$$\sum_{n=0}^{\infty} \frac{\partial^{i+1} K_n}{\partial x^i \partial a} \qquad i \in \{2,3\}$$

converge uniformly on each component of the set $([0,d] \times [0,d']) \cap \{(x,a) : x \neq a\}$. The restriction of the sums

$$\frac{\partial^{i+1} g}{\partial^i x \partial a} \qquad i \in \{2,3\}$$

to either one of the components can be continued to continuous functions on that component. The mixed partial derivatives of g may be taken in any order because the same is true for each K_n. Further, the partial derivatives of g are analytic in k because of the uniform convergence. Summing the last inequality yields for $i \in \{0,1,2,3\}$:

$$|k|^{-i} |g|_{i,1}(d,d') \leq 2|k|^{-2} M_2 B(d+d',k) + (2d)^{-1}|k|^{-3} M_2 B(2d,k)$$

$$\leq |k|^{-2} M_3 B(d+d',k), \qquad \text{where } M_3 = 2M_2 + \frac{M_2}{2C_2 d}. \tag{93}$$

Note that equation (89) serves as a definition for g(x,a,k) for $x > d$, since g is known for $x \leq d$. From this we see that if $d' \geq d$ and $d'' \geq d$ then for $i \in \{0,1,2,3\}$,

$$|k|^{-i} |g|_{i,0}(d'',d') \leq |k|^{-i} |\tilde{g}|_{i,0}(d'',d') + M|k|^{2-i} |\tilde{g}|_{i,0}(d'',d) \sum_{j=0}^{3} |k|^{-j} |g|_{j,0}(d,d')$$

$$\leq |k|^{-3} M_2 B(d''+d',k) + |k|^{-3} M_2 B(d''+d,k)\, |k|^{-1} 8M M_2 B(d+d',k).$$

But since $B(\mu_1,k)\, B(\mu_2,k) = B(\mu_1+\mu_2,k)$, we obtain:

$$|k|^{-i} |g|_{i,0}(d'',d') \leq |k|^{-3} M_2 B(d''+d',k) + |k|^{-3} M_2 B(d''+d',k)\, |k|^{-1} 8M M_2 B(d+d',k)$$

$$\leq 2|k|^{-3} M_2 B(d''+d',k).$$

We may similarly deduce inequalities (88) for $j = 1$ and $i \leq 3$ by using inequality (93).

From equation (89) we get (for $x \neq a$)

$$\tilde{L}_k g(x, a, k) = \tilde{L}_k \int_0^d \tilde{g}(x, b, k)\, P_k g(b, a, k)\, db$$

$$= P_k g(x, a, k).$$

i.e $$L_k g(x,a,k) = 0 \quad \text{for } x \neq a.$$

From this the remaining estimates (88) (i.e. for $i = 4$) follow from the previous ones.

Finally, that g satisfies (56), (58) and (59) follows immediately from equation (89) and the fact that $\tilde{g}$ satisfies these conditions. This completes the proof of Theorem 9.

□

1.5 'Spectral' Properties of g

We will now see that the conclusions of Theorem 9 yield very strong regularity results for the solution of equation (1).

Notice that the function $g(x, a, \sqrt{\lambda})$ is analytic in a domain containing

$$\{\lambda \in \mathbb{C} : \text{Re}\,\lambda > p^{-2}\}.$$

(Here the square root function is defined so that the square root of positive real numbers is positive, with its branch line coinciding with the negative real axis). For $\text{Re}\,\lambda > p^{-2}$ we see from (88) that:

$$\sup_{\substack{x\in[0,\infty)\\ a\in[0,\infty)}} |g(x, a, \sqrt{\lambda})| \leq M_3 |\lambda|^{-3/2}.$$

Thus we may define $J : [0,\infty)^3 \to \mathbb{C}$ by

$$J(x, a, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} g(x, a, \sqrt{\lambda})\, e^{\lambda t}\, d\lambda, \qquad \text{where } \gamma > p^{-2}. \tag{94}$$

Now we define for $r > 0$,

$$\Omega(r) = \{k \in \mathbf{C} :\ B(r,k) < p|k|\}. \tag{95}$$

Recall that $\Omega = \Omega(2d)$. It is obvious that if $0 < r_1 < r_2$, then $\Omega(r_2) \subset \Omega(r_1)$. We define $\Omega'(r)$ as the square of $\Omega(r)$, i.e. $\Omega'(r); = sq(\Omega(r))$ where $sq(z) = z^2$. We set $\Omega' = \Omega'(2d)$.

We will make use of the following important property of $\Omega'(r)$.

Lemma 10

Given $r > 0$ and $b > 0$, there exists $c = c(b,r) > 0$ such that the set

$$\Sigma_{b,r} = \{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda \geq -b\,|\mathrm{Im}\,\lambda|^{1/2} + c\} \tag{96}$$

is contained in $\Omega'(r)$.

Further, if $r = d' + d''$ where $d' \geq d$ and $d'' \geq d$, then for $\lambda \in \Sigma_{b,r}$ we have the estimates:

$$|g(.,.,\sqrt{\lambda}\,)|_{i,j}(d'', d') \leq \text{constant}\,.\,|\lambda|^{(i+j-3)/2} \qquad i \in \{0, 1, 2, 3, 4\},\ \ j \in \{0, 1\}. \tag{97}$$

Proof

We notice that $\Omega(r)$ contains the set

$$A = \{e^{\pi i/4}(\sigma+i\delta) :\ \sigma > p^{-1},\ \ 0 \leq \delta < r^{-1}\ln \sigma p\},$$

for $B(r, e^{\pi i/4}(\sigma+i\delta)) = e^{\delta r} < \sigma p \leq p|e^{\pi i/4}(\sigma+i\delta)|$.

Similarly, A^*, the reflection of A in the real axis is contained in $\Omega(r)$. Thus,

$$\Omega(r) \supset \{k : |k| > p^{-1},\ |\arg k| \leq \pi/4\} \cup A \cup A^*.$$

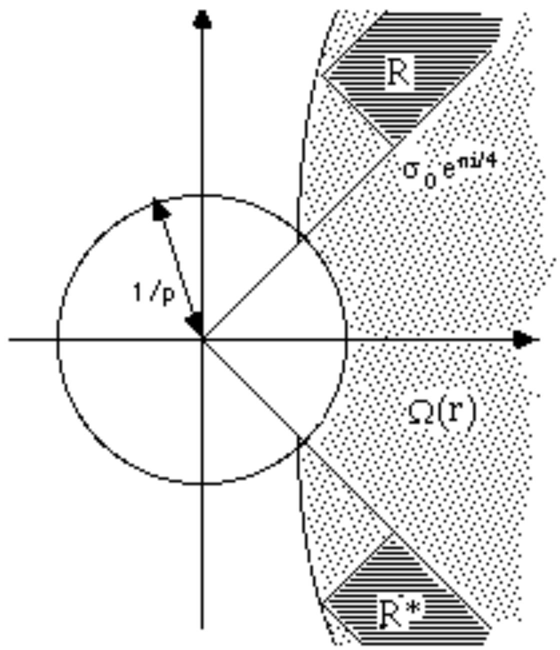

Figure 1

Given $b > 0$, choose $\sigma_0 = p^{-1}e^{br/2}$. We see that the infinite rectangle

$$R = \{e^{\pi i/4}(\sigma+i\delta) : \sigma > \sigma_0, 0 \le \delta < b/2\}$$

is contained in A, and is thus in Ω(r). Similarly R*, the reflection of R in the real axis is contained in Ω(r).

The left boundary of sq(R) (see figure 2) is parametrized by

$$\lambda = (e^{\pi i/4}[\sigma+ib/2])^2 = -b\sigma + i(\sigma^2 - b^2/4), \quad \text{for } \sigma > \sigma_0.$$

We see that such points satisfy

$$\text{Re } \lambda = -b\sigma \le -(\text{Im } \lambda)^{1/2}\, b.$$

Clearly points on the left boundary of R* satisfy

$$\text{Re } \lambda \le -|\text{Im } \lambda|^{1/2}\, b.$$

From this it is clear that we can pick $c > 0$ so that the entire curve

$$\mathrm{Re}\,\lambda = -\,|\mathrm{Im}\,\lambda|^{1/2}\, b + c$$

is to the right of the boundary of $\Omega'(r)$. This completes the proof of the first part of the lemma.

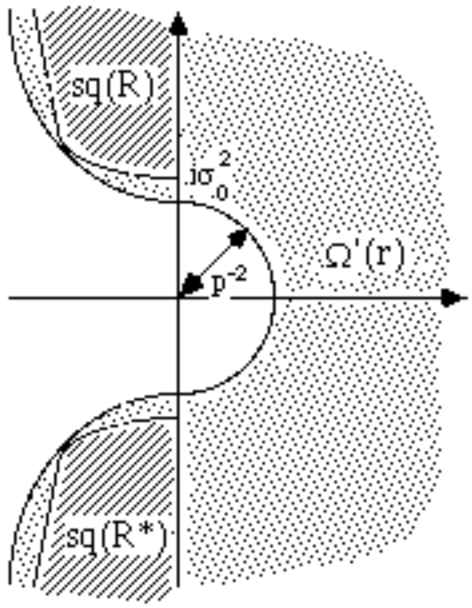

Figure 2

Notice that if $\mathrm{Re}\,\lambda > 0$ then estimates (97) follow immediately from estimates (88). If

$$\lambda = (e^{\pi i/4}[\sigma + i\delta])^2 \text{ where } \sigma > \sigma_0 \text{ and } 0 \le \delta < b/2,$$

we have

$$(d' + d'')(|\mathrm{Im}\,k| - \mathrm{Re}\,k)/\sqrt{2} = (d' + d'')\delta = r\delta < rb/2,$$

so that $B(d'' + d', k) \le e^{rb/2}$. With (88) this establishes estimates (97) in

$$\Sigma_{b,r} \cap \{\lambda : \mathrm{Im}\,\lambda > \rho\}$$

for ρ sufficiently large. Similarly, (97) holds in

$$\Sigma_{b,r} \cap \{\lambda : \operatorname{Im} \lambda < -\rho\}.$$

The only part of $\Sigma_{b,r}$ for which we have not verified (97) is a bounded part. But then the boundedness is enough to get (97) from (88). This completes the proof of Lemma 10.

$\square$

1.6 The Gevrey Smoothness of J

We now show that the function J, defined in equation (94), is of Gevrey class γ^2 in the time variable t for $t > 0$. For this we need to specify the locally convex space X and the family of seminorms $\mathcal{P}$ which topologize X.

We let X be the space of complex-valued functions on $[0,\infty)^2$ satisfying the regularity conditions (55) of Theorem 7. X may be topologized by the separating family of seminorms

$$\mathcal{P} = \{|\ \ |_{i,j}(d',d') : \ d' \geq d,\ i \in \{0,1,2,3,4\},\ j \in \{0,1\}\},$$

which makes X into a locally convex topological vector space.

To prove that $J \in \gamma^2((0,\infty),X,\mathcal{P})$, it is clearly enough to show that for any $d' \geq d$, the restriction of J to $(x,a) \in [0,d']^2$ is in $\gamma^2((0,\infty),X_{d'})$ where $X_{d'}$ is the Banach space of functions on $[0,d']^2$ having the regularity† specified in (55) and having the norm

$$\sum_{\substack{0 \leq i \leq 4 \\ 0 \leq j \leq 1}} |\ \ |_{i,j}(d',d').$$

Theorem 11

$J : (0,\infty) \to X_{d'}$ is in the Gevrey class $\gamma^2((0,\infty),X_{d'})$.

Proof

---

†*The partial derivatives listed in (55) are assumed to be continuable continuously to the boundaries of each of the sets* $\{(x,a) : 0 \leq a < x < d'\}$ *and* $\{(x,a) : 0 \leq x < a < d'\}$.

The integral (94) defining the function J converges absolutely, uniformly for $(x,a) \in$ compact subsets of $[0,\infty)^2$. We first show that the integral can be taken over a different contour which will allow us to make use of the estimates (88) of Theorem 9.

Let $b > 0$. Choosing c and $\Sigma_{b,2d'}$ as in Lemma 10, we let Γ be the boundary of $\Sigma_{b,2d'}$ with upwards orientation.

Let Γ_R be the contour

$$\Gamma_R = \{s+iR : -bR^{1/2} + c \leq s \leq \gamma\},$$

with orientation in the direction of increasing $\mathrm{Re}\,\lambda$ (the definition of Γ_R makes sense if R is sufficiently large).

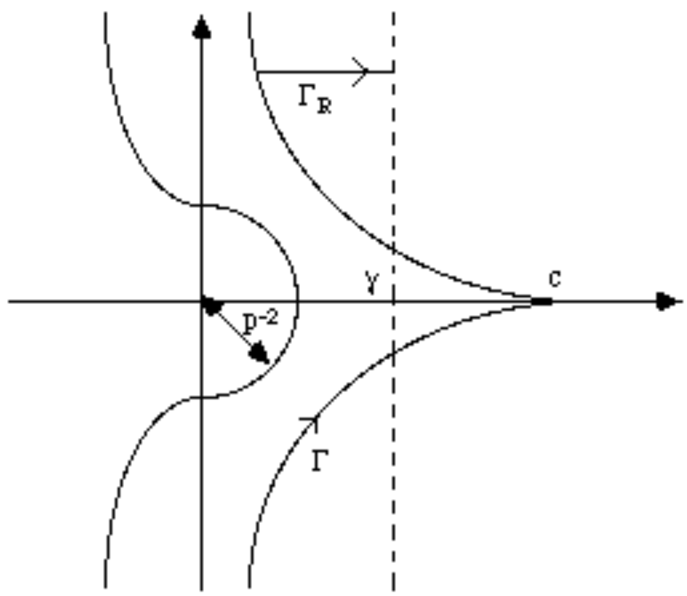

Figure 3

By estimates (88),

$$\left| \int_{\Gamma_R} g(.,.,\sqrt{\lambda}\,)\, e^{\lambda t}\, d\lambda \right|_{0,0} (d',d') \leq \text{constant}.\, R^{-3/2} \int_{-\infty}^{\gamma} e^{st}\, ds$$

$$= \text{constant}.\, R^{-3/2}\, e^{\gamma t}.$$

Thus $\int_{\Gamma_R} g(x, a, \sqrt{\lambda})\, e^{\lambda t}\, d\lambda \to 0$ as $R \to \infty$ uniformly for $(x, a) \in [0, d']^2$.

Similarly, with Γ_{-R} being the reflection of Γ_R in the real axis, we see that

$$\int_{\Gamma_{-R}} g(x, a, \sqrt{\lambda})\, e^{\lambda t}\, d\lambda \to 0 \text{ as } R \to \infty \text{ uniformly for } (x, a) \in [0, d']^2.$$

Thus we may replace the contour $\text{Re}\,\lambda = \gamma$ with the contour $\Gamma : \text{Re}\,\lambda = -b|\text{Im}\,\lambda|^{1/2} + c$, provided that the orientation is taken in the direction of increasing $\text{Im}\,\lambda$.

We now show that for $t > 0$ and $(x,a) \in [0,d']^2$,

$$\frac{\partial^n}{\partial t^n} J(x, a, t) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^n\, g(x, a, \sqrt{\lambda})\, e^{\lambda t}\, d\lambda, \quad n = 0,\ 1,\ 2,\ 3,\ 4,\ \ldots. \tag{98}$$

For now, we let $I_n(t)$ denote the integral on the right hand side of (98). If we can show that $I_n(t)$ is absolutely convergent in $X_{d'}$, uniformly in t for t on compact subsets of $(0,\infty)$, then (98) will follow, along with the fact that $J(.,.,t)$ is a C^∞ function of t in $X_{d'}$. To see why this is true, let $t_0 > 0$ be given and pick $\varepsilon < t_0$. For $n = 0, 1, 2, 3, 4, \ldots$, let B_n be a constant such that for $t \in [t_0-\varepsilon, t_0+\varepsilon]$,

$$\int_{\Gamma} |\lambda|^n\, \|g(.,.,\sqrt{\lambda})\|_{X_d}\, |e^{\lambda t}|\, |d\lambda| < B_n. \tag{99}$$

Then for $0 < |h| < \varepsilon$, we can find $\bar{t} \in [t_0-|h|, t_0+|h|]$, depending on λ, such that

$$\frac{\exp(\lambda(t_0+h)) - \exp(\lambda t_0)}{h} - \lambda\exp(\lambda t_0) = \lambda\exp(\lambda\bar{t}\,) - \lambda\exp(\lambda t_0)$$

$$= \lambda^2\, (\bar{t} - t_0)\, \exp(\lambda\tilde{t}\,), \quad \text{where } \tilde{t} \text{ is between } t_0 \text{ and } \bar{t}.$$

Thus,

$$\left|\frac{\exp(\lambda(t_0+h)) - \exp(\lambda t_0)}{h} - \lambda\exp(\lambda t_0)\right| \le |h|\,|\lambda|^2 \begin{cases} |\exp \lambda(t_0-\varepsilon)| & \text{if } \operatorname{Re}\lambda \le 0 \\ |\exp \lambda(t_0+\varepsilon)| & \text{if } \operatorname{Re}\lambda \ge 0 \end{cases}.$$

We get from this that

$$2\pi \left|\left|\frac{I_n(t_0+h)-I_n(t_0)}{h} - I_{n+1}(t_0)\right|\right|_{X_d} \le |h| \int_{\Gamma\cap\{\lambda:\ \operatorname{Re}\lambda<0\}} |\lambda|^{n+2}\, \|g(.,.,\sqrt{\lambda})\|_{X_d}\, |\exp \lambda(t_0-\varepsilon)|\,|d\lambda|$$

$$+ \ |h| \int_{\Gamma\cap\{\lambda:\ \operatorname{Re}\lambda>0\}} |\lambda|^{n+2}\, \|g(.,.,\sqrt{\lambda})\|_{X_d}\, |\exp \lambda(t_0+\varepsilon)|\,|d\lambda|$$

$$\le\ 2|h|\ B_{n+2}.$$

Thus it remains to prove (99). We will see that the estimation of B_n will also establish the Gevrey smoothness of J.

On Γ, set

$$\lambda = iR - b|R|^{1/2} + c. \quad \text{Then } d\lambda = idR \pm \tfrac{1}{2}\, b|R|^{-1/2} dR.$$

Let $R_0 > 1$ be such that for $|R| > R_0$ we have $|\lambda| \le 2|R|$ and $|d\lambda| \le 2|dR|$. Then for $i \in \{0,1,2,3,4\}$, $j \in \{0,1\}$ and $t \in [t_0-\varepsilon, t_0+\varepsilon]$, we get from Lemma 10 that

$$\int_{\Gamma\cap\{\lambda:\ \operatorname{Im}\lambda>R_0\}} |\lambda|^n\, |g(.,.,\sqrt{\lambda})|_{i,j}(d',d')\, |e^{\lambda t}|\, |d\lambda| \le \text{constant.} \int_{R_0}^{\infty} (2R)^n\, R^{(i+j-3)/2}\, e^{ct-bt\sqrt{R}}\, 2dR$$

$$\le \text{constant.}\ 2^{n+1}\, e^{ct} \int_0^{\infty} R^{n+1}\, e^{-bt\sqrt{R}}\, dR$$

$$= \text{constant}.\ 2^{n+2}\ e^{ct} \int_0^\infty u^{2n+3}\ e^{-btu}\ du$$

$$= \text{constant}. \left[\frac{2}{b^2 t^2}\right]^{n+2} e^{ct}\ (2n+3)!$$

But $(2n+3)! < 2^{2n+3}(n+2)!\,(n+1)!$, so the integral is bounded by

$$\tfrac{1}{2}\ \text{constant}. \left[\frac{8}{b^2(t_0-\varepsilon)^2}\right]^{n+2} \exp[c(t_0+\varepsilon)]\ (n+2)^3\ n!^2 \quad .$$

The constant $b > 0$ is arbitrary. Thus, given $\theta > 0$, we choose

$$b = \frac{4}{(t_0-\varepsilon)\sqrt{\theta}}$$

and note that since the sequence $\{(n+2)^3/2^{n+3}\}$ is bounded, there exists a constant C such that the integral is actually bounded by

$$C\,\theta^n\ n!^2.$$

The part of the integral taken over $\Gamma \cap \{\lambda : \mathrm{Im}\,\lambda < -R_0\}$ may be treated similarly. What remains is the integral over $\Gamma \cap \{\lambda : \mathrm{Im}\,\lambda < -R_0\}$, which is a compact set. If we set

$$D = \text{diameter}\ (\Gamma \cap \{\lambda : |\mathrm{Im}\,\lambda| \le R_0\}),$$

we see that

$$\int_{\Gamma\cap\{\lambda:|\mathrm{Im}\ \lambda|\le R_0\}} |\lambda|^n\ \|g(.,.,\sqrt{\lambda}\,)\|_{X_d}\ |e^{\lambda t}|\ |d\lambda| < \text{constant}.\ D^n \exp c(t_0+\varepsilon).$$

But the sequence $\{(D/\theta)^n / n!^2\}$ is bounded. Thus we have proved estimate (99) and equation (98). In so doing, we have found that for any $\theta > 0$ and compact sub-interval K of $(0,\infty)$, there exists a constant C' such that for all $t \in K$,

$$\left\| \frac{d^n}{dt^n} J(.,.,t) \right\|_{X_d} < C' \theta^n n!^2, \qquad n = 0, 1, 2, 3, 4, \ldots .$$

This completes the proof of the theorem.

□

Recall that some of the partial derivatives of the function g do not exist on the line $x = a$, while limits of these derivatives from either side of this line do exist. The following result shows that J does not inherit this discontinuous behavior.

Lemma 12

For $t > 0$, all of the partial derivatives of J(x,a,t) listed in (55) exist and are continuous at points on the line $x = a$. Further, for fixed $a \in [0,\infty)$, J(.,a,.) satisfies the partial differential equation

$$\frac{\partial^2 J}{\partial t^2}(x, a, t) + \frac{\partial^4 J}{\partial x^4}(x, a, t) + L_2 \frac{\partial J}{\partial t}(x, a, t) + L_1 J(x, a, t) = 0 \qquad \text{for } (x, t) \in (0, \infty)^2, \tag{100}$$

and for $t > 0$, J satisfies the boundary conditions

$$J(0, a, t) = \frac{\partial J}{\partial x}(0, a, t) = 0. \tag{101}$$

Proof

Pick any $d' \geq d$ and let the contour Γ† be as in the proof of Theorem 11. Since g satisfies

$$\frac{\partial^3 g}{\partial x^3}(a, a^-, k) - \frac{\partial^3 g}{\partial x^3}(a, a^+, k) = 1,$$

we get from the proof of Theorem 11 that for any $a < d'$,

†Γ, of course, depends on a prescribed constant b > 0. However, for the purposes of this proof, any b will do.

$$\frac{\partial^3 J}{\partial x^3}(a, a^-, t) - \frac{\partial^3 J}{\partial x^3}(a, a^+, t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\, d\lambda.$$

Recall that Γ is the contour parametrized by

$$\lambda = iR - b|R|^{1/2} + c, \qquad -\infty < R < \infty.$$

For a given $R_0 > 0$, we consider a contour γ_{R_0} consisting of the line segment

$$\gamma_{R_0} = \{is - bR_0^{1/2} + c : -R_0 \le s \le R_0\},$$

with orientation in the direction of increasing s.

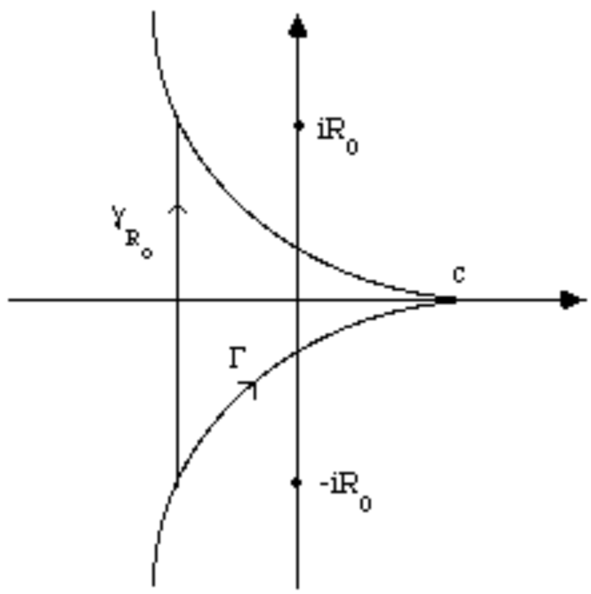

Figure 4

Now,

$$\left|\int_{\gamma_{R_0}} e^{\lambda t}\, d\lambda\right| \le \int_{-R_0}^{R_0} \exp[ct - bR_0^{1/2}t]\, ds = 2R_0 \exp[ct - bR_0^{1/2}t].$$

So we see that

$$\frac{\partial^3 J}{\partial x^3}(a, a^-, t) - \frac{\partial^3 J}{\partial x^3}(a, a^+, t) = \frac{1}{2\pi i} \lim_{R_0 \to \infty} \int_{\gamma_{R_0}} e^{\lambda t}\, d\lambda = 0.$$

But this together with

(i) $\frac{\partial^2 J}{\partial x^2}(x, a, t)$ exists and is continuous for all $(x, a) \in (0, \infty)^2$,

(ii) $\frac{\partial^3 J}{\partial x^3}(x, a, t)$ exists and is continuous on either side of the line $x = a$, and has

limiting values as (x,a) converges on either side of the line to a point on it,

imply that $\frac{\partial^3 J}{\partial x^3}(x, a, t)$ exists and is continuous at $x = a$.

Next, using equation (98) and part (iii) of Theorem 9, we see that equation (100) is satisfied provided that $x \neq a$. But every member of equation (100) except for the fourth order term is known to be a continuous function of (x,a), even on the line $x = a$. This implies that

$$\frac{\partial^4 J}{\partial x^4}(x, a, t)$$

exists and is continuous at points on the line.

Now we note that we have trivially:

$$\frac{\partial^2 J}{\partial x^2}(a, a^-, t) = \frac{\partial^2 J}{\partial x^2}(a, a^+, t),$$

which implies that

$$\frac{\partial^3 J}{\partial x^3}(a, a^-, t) + \frac{\partial^3 J}{\partial a \partial x^2}(a, a^-, t) = \frac{\partial^3 J}{\partial x^3}(a, a^+, t) + \frac{\partial^3 J}{\partial a \partial x^2}(a, a^+, t).$$

We have already seen that

$$\frac{\partial^3 J}{\partial x^3}(x, a, t)$$

exists and is continuous at points on the line $x = a$. Thus also

$$\frac{\partial^3 J}{\partial a \partial x^2}(x, a, t)$$

exists and is continuous at points on this line.

Similarly we may prove the existence and continuity of

$$\frac{\partial^4 J}{\partial a \partial x^3}(x, a, t),$$

while the existence and continuity of

$$\frac{\partial^5 J}{\partial a \partial x^4}(x, a, t)$$

can be proved simply by differentiating equation (100).

Equations (101) follow immediately from (98) and:

$$g(0, a, k) = \frac{\partial g}{\partial x}(0, a, k) = 0.$$

Finally we note that since the mixed partial derivatives may be taken in any order for $x \neq a$, the same is true at $x = a$ by continuity.

This completes the proof of Lemma 12.

□

Lemma 12 implies that for each $t > 0$, $J(.,.,t)$ belongs to the class Y of functions

$$f : [0,\infty)^2 \to \mathbf{C}$$

given by

$$Y = \{f : \text{for } i \in \{0,1,2,3,4\} \text{ and } j \in \{0,1\},\ \frac{\partial^{i+j} f}{\partial x^i \partial a^j}(x,a) \text{ is a continuous function of } (x,a),$$

$$\text{and all partial derivatives of order } j \text{ with respect to 'a' and } i \text{ with respect to } x \text{ are equal}\}.$$

If we topologize Y by the same seminorms P topologizing X, we see immediately that Y is a closed subspace of X. Further, we have the following result:

Corollary 13

$t \to J(.,.,t)$ is a member of $\gamma^2((0,\infty),Y,\mathsf{P})$.

□

1.7 Gevrey Solutions of Equation (1)

We now consider the following mappings on functions $f \in L^2(\mathbf{R}^+)$ with compact support:

$$\mathsf{G}_\lambda f(x) = \int_0^\infty g(x,a,\sqrt{\lambda})f(a)\,da \qquad (\lambda \in \Omega'), \tag{102}$$

$$\mathsf{J}(t)f(x) = \int_0^\infty J(x,a,t)f(a)\,da \qquad (t > 0). \tag{103}$$

Recall that A is the infinitesimal generator of a strongly continuous semigroup. This fact implies that there exists a real constant Λ_0 such that R_λ, the resolvent of A, exists and is analytic in

$$\Theta = \{\lambda : \text{Re}\,\lambda > \Lambda_0\}.$$

One easily verifies, using the results of Theorem 9, that if $(f_1,f_2) \in (C_0^\infty(\mathbf{R}^+))^2$ and

$\lambda \in \Omega' \cap \Theta$, then

$$R_\lambda \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} G_\lambda(\lambda + L_2) & G_\lambda \\ G_\lambda(\lambda^2 + \lambda L_2) - 1 & \lambda G_\lambda \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{104}$$

Now if $f_1 \in H_0^2(\mathbf{R}^+)$ and $f_2 \in L^2(\mathbf{R}^+)$ and these functions have support in [0,d'), one can find ϕ_1 and ϕ_2 in C_0^∞, with support in [0,d'], such that (ϕ_1,ϕ_2) is arbitrarily close to (f_1,f_2) in H.

Because of the continuity of the mapping R_λ, we see that equation (104) holds for all members of H with compact support. This forms the basis of the following theorem.

Theorem 14

Let $f_1 \in H_0^2(\mathbf{R}^+)$ and $f_2 \in L^2(\mathbf{R}^+)$ have compact support. Then for $t > 0$,

$$U(t) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} J'(t) + J(t)\, L_2 & J(t) \\ J''(t) + J'(t)\, L_2 & J'(t) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{105}$$

Proof

We first suppose that $(f_1,f_2) \in D_{A^2}$, the domain of A^2. In this case, a classical result (e.g. see Pazy [15]) is that if $\gamma > \Lambda_0$ then

$$U(t) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} R_\lambda \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} e^{\lambda t}\, d\lambda,$$

and the integral converges as an improper Riemann integral in H, uniformly for t on compact subsets of $(0,\infty)$.

If we let

$$\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = U(t) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

we get from the discussion preceding the statement of the theorem that, if γ is sufficiently large,

$$w(.,t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \int_0^{\infty} g(.,a,\sqrt{\lambda})(\lambda f_1(a) + L_2 f_1(a) + f_2(a))\, da\, d\lambda.$$

We pick an arbitrary $d' > d$ such that $[0,d']$ contains the supports of f_1 and f_2, and we show that for $x \in [0,d']$ and $t > 0$, $w(x,t)$ is given by the first component of the right side of equation (105).

Given $b^{\dagger} > 0$, we take $\Sigma_{b,2d'}$ as in the proof of Lemma 10, and let Γ be as in the proof of Theorem 11. By Lemma 10, we have

$$\max_{0\le x\le d'} |\lambda G_\lambda f_1(x) + G_\lambda f_2(x)| \le \text{constant}.\, (|\lambda|^{-1/2} \|f_1\|_{L^2} + |\lambda|^{-3/2}[\, \|f_2\|_{L^2} + \|f_1\|_{H^2}]).$$

This estimate allows us to deform contours as in the proof of Theorem 11, so that we get:

$$w(.,t) = \frac{1}{2\pi i} \int_{\Gamma} \int_0^{d'} g(.,a,\sqrt{\lambda})(\lambda f_1(a) + L_2 f_1(a) + f_2(a))\, da\, d\lambda.$$

one easily checks as in the proof of Theorem 11 that this integral is absolutely convergent for any $t > 0$, uniformly for $x \in [0,d']$. Hence, by Fubini's theorem, we can interchange the order of integration and arrive at

$$w(x,t) = \int_0^{\infty} J(x,a,t)\{f_2(a) + L_2 f_1(a)\}\, da \; + \; \frac{\partial}{\partial t} \int_0^{\infty} J(x,a,t) f_1(a)\, da, \tag{106}$$

---

†*Again, any fixed $b > 0$ will suffice for the proof of this theorem.*

for $t > 0$ and $x \in [0,d']$. But since d' can be arbitrarily large, it follows that equation (106) holds for all $x \in [0,\infty)$.

To show that equation (106) holds for all $(f_1,f_2) \in \mathsf{H}$ with compact support, we need to show that we can approximate (f_1,f_2) arbitrarily well in H by members of D_{A^2} with compact support. For then we can get (106) by using the boundedness of U(t). To see this, let $\varepsilon > 0$ be given. Since D_{A^2} is dense in H, we can find $(h_1,h_2) \in \mathsf{H}$ such that

$$\|(f_1,f_2) - (h_1,h_2)\| < \varepsilon.$$

However, it is not guaranteed that h_1 and h_2 have compact support. But we can overcome this problem. Recall that

$$D_{A^2} = \left\{ \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in D_A : \ A \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in D_A \right\}$$

$$= \{(g_1, g_2) : g_1 \in H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+), \ \ g_2 \in H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+),$$

$$g_1^{(4)} + \sum_{j=0}^{2} a_j\, g_1^{(j)} + \sum_{j=0}^{2} b_j\, g_2^{(j)} \in H_0^2(\mathbf{R}^+) \}.$$

But the supports of the functions a_j, b_j are contained in $[0,d]$. Thus if $\phi \in C_0^\infty$ and $\phi(x) = 1$ for $x \le d$, then $(\phi g_1, \phi g_2) \in D_{A^2}$ if $(g_1,g_2) \in D_{A^2}$.

We consider $\psi \in C_0^\infty$ satisfying

$$\psi(x) = \begin{cases} 1 & x \le 0 \\ 0 & x \ge 1 \end{cases}, \qquad \text{with } M = \max_{0 \le x \le 1} (|\psi(x)|, |\psi'(x)|, |\psi''(x)|).$$

We set (for $R > d$)

$$\psi_R(x) = \psi(x-R).$$

Since $\mathsf{H} = H_0^2(\mathbf{R}^+) \times L^2(\mathbf{R}^+)$, it is easy to see that by making R sufficiently large, we can get

$$\|(h_1,h_2) - (\psi_R h_1,\psi_R h_2)\| < \varepsilon \qquad \text{and thus} \qquad \|(f_1,f_2) - (\psi_R h_1,\psi_R h_2)\| < 2\varepsilon.$$

Hence equation (106) holds for arbitrary $(f_1,f_2) \in \mathsf{H}$ with compact support.

It remains for us to show that

$$v(.,t) = \frac{\partial}{\partial t}\int_0^\infty J(.,a,t)\{f_2(a) + L_2 f_1(a)\}\,da + \frac{\partial^2}{\partial t^2}\int_0^\infty J(.,a,t) f_2(a)\,da. \tag{107}$$

If $(f_1,f_2) \in D_A$ with compact support then this follows from equation (106), because the first component of the equation

$$\frac{d}{dt} U(t)\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = A\, U(t)\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

yields $v(.,t) = (d/dt)w(.,t)$ in $H_0^2(\mathbf{R}^+)$. Equation (107) now follows,by a continuity argument similar to the one used in proving (106), for all $(f_1,f_2) \in \mathsf{H}$ with compact support.

This completes the proof of Theorem 14.

□

We next consider the space $C^4[0,\infty)$ topologized by the seminorms $\mathsf{P}_1 = \{p_r : r > 0\}$, where

$$p_r(f) = \max_{0\le x\le r} \sum_{j=0}^{4} |f^{(j)}(x)|.$$

An immediate consequence of Corollary 13 and Theorem 14 is the following result.

Corollary 15

If $(f_1,f_2) \in \mathsf{H}$ with compact support and

$$\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = U(t) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

then both w and v are in $\gamma^2((0,\infty),C^4[0,\infty),\mathsf{P}_1)$.

□

1.8 Removing the Restrictions on Equation (1)

We now remove the restrictions $a_4(x) = 1$, $a_3(x) = 0$ in equation (1). We set

$$q(x) = \int_0^x (a_4(s))^{-1/4}\, ds,$$

and consider the transformation $Qw = \tilde{w}$, where

$$\tilde{w}(y,t) = w(q^{-1}(y),t)\ [a_4(q^{-1}(y))]^{-3/8} \exp\left[\frac{1}{4} \int_0^{q^{-1}(y)} \frac{a_3(s)}{a_4(s)}\, ds \right], \tag{108}$$

is the one referred to in Section 1.0 and it transforms equation (1) into

$$\frac{\partial^2 \tilde{w}}{\partial t^2} + \frac{\partial^4 \tilde{w}}{\partial y^4} + \sum_{j=0}^{2} \tilde{a}_j(y) \frac{\partial^j \tilde{w}}{\partial y^j} + \sum_{j=0}^{2} \tilde{b}_j(y) \frac{\partial^{j+1} \tilde{w}}{\partial y^j \partial t} = 0, \tag{109}$$

where $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{b}_0, \tilde{b}_1, \tilde{b}_2$ of course depend on $a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2$. The severest assumptions on the regularity of $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{b}_0, \tilde{b}_1, \tilde{b}_2$ arise in Lemma 5 and in the work following Lemma 5, where we require

$\tilde{b}_2 \in C^4[0,\infty)$, $\tilde{b}_1 \in C^3[0,\infty)$, and all other coefficients are in $C[0,\infty)$.

For these conditions to hold, it is sufficient for the original coefficients to satisfy the conditions

(I) $a_4 \in C^4[0,\infty)$, $a_3 \in C^3[0,\infty)$, $b_2 \in C^4[0,\infty)$, $b_1 \in C^3[0,\infty)$,
$a_j \in C[0,\infty)$ for $j \in \{0,1,2\}$, $b_0 \in C[0,\infty)$.

We shall also assume that the domains of the functions $a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2$ have been extended to $[0,\infty)$ in such a way that

(II) $a_0, a_1, a_2, a_3, b_0, b_1, b_2$ all have compact support, there exists $R > 0$ such that for $x > R$, $a_4(x) = 1$, and $a_4(x) > 0$ for all $x \in [0,\infty)$.

The requirement $-2 < \tilde{b}_2(x) \le 0$ is equivalent to what we call a ***light damping assumption***:

(III) $-2\sqrt{a_4(x)} < b_2(x) \le 0$ for all $x \in [0,\infty)$.

One easily verifies that, under these assumptions, the mapping† Q maps $\mathsf{H} = H_0^2(\mathbf{R}^+) \times L^2(\mathbf{R}^+)$ continuously into itself and that it has a continuous inverse.

If we define

$$S(t) = Q^{-1}\, U(t)\, Q, \tag{110}$$

it is obvious that S(t) is a strongly continuous semigroup on H. Further, one can easily compute the infinitesimal generator B of S(t) and find that

$$D_B = D_A = (H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+)) \times L^2(\mathbf{R}^+),$$

$$B\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\displaystyle\sum_{j=0}^{4} a_j(x)\,\frac{\partial^j w}{\partial x^j} - \sum_{j=0}^{2} b_j(x)\,\frac{\partial^j v}{\partial x^j} \end{bmatrix}. \tag{111}$$

We further define, for $(x,b,t) \in [0,\infty)^3$,

---

†*i.e. Q is applied component-wise.*

$$K(x,b,t) = [a_4(x)]^{3/8}[a_4(b)]^{-5/8}\exp\left[\frac{1}{4}\int_x^b \frac{a_3(s)}{a_4(s)}\,ds\right] J(q^{-1}(x), q(b), t). \tag{112}$$

For $t > 0$, we consider the following operator on functions $f \in L^2(\mathbf{R}^+)$ with compact support:

$$\mathsf{K}(t)f(x) = \int_0^\infty K(x,a,t)f(a)\,da. \tag{113}$$

The preceding theorems on U, J and J imply similar results about S, K and K. We summarize these results in the following theorem.

Theorem 16

Under conditions (I), (II) and (III), the following statements hold.

(i) $t \to K(.,.,t)$ is a member of $\gamma^2((0,\infty),Y,\mathsf{P})$.

(ii) For each fixed $a \in [0,\infty)$,

$$\frac{\partial^2 K}{\partial t^2}(x,a,t) + \sum_{j=0}^{4} a_j(x)\,\frac{\partial^j K}{\partial x^j}(x,a,t) + \sum_{j=0}^{2} b_j(x)\,\frac{\partial^{j+1} K}{\partial x^j \partial t}(x,a,t) = 0, \quad \text{for } (x,t) \in (0,\infty)^2.$$

(iii) Let $f_1 \in H_0^2(\mathbf{R}^+)$ and $f_2 \in L^2(\mathbf{R}^+)$ have compact support. Then for $t > 0$,

$$S(t)\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \mathsf{K}'(t) + \mathsf{K}(t)\,L_2 & \mathsf{K}(t) \\ \mathsf{K}''(t) + \mathsf{K}'(t)\,L_2 & \mathsf{K}'(t) \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

where $L_2 = b_2\, d^2/dx^2 + b_1\, d/dx + b_0$.

(iv) If $\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = S(t)\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, where f_1 and f_2 are as in (iii), then

$t \to w(.,t)$ and $t \to v(.,t)$ are both members of $\gamma^2((0,\infty),C^4[0,\infty),\mathsf{P}_1)$.

□

Chapter 2

Analyticity for Equations of Euler-Bernoulli Type With Constant Coefficients

2.0 Introduction

In this chapter, we once again investigate the the properties of solutions of the equation

$$\frac{\partial^2 w}{\partial t^2} + \sum_{i=0}^{4} a_i \frac{\partial^i w}{\partial x^i} + \sum_{i=0}^{2} b_i \frac{\partial^{i+1} w}{\partial x^i \partial t} = 0. \tag{1}$$

However, here we suppose that the coefficients a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2 are constants and we show that, under this assumption, we obtain even better regularity results for the fundamental solution K(x,a,t) than those obtained in Chapter 1. Indeed, we show that K may be continued analytically on to a certain subset of $\mathbf{C}^3$.

In this work, we assume that $a_4 > 0$ and that the light damping assumption,

$$-2\sqrt{a_4} < b_2 \leq 0, \tag{2}$$

is satisfied. As in Chapter 1, we begin by considering the case $a_4 = 1$, $a_3 = 0$, for this can be achieved by means of the transformation Q (see equation (108) of Chapter 1). In this case, we may pick $\theta \in (0,\pi/2]$ so that $b_2 = -2\cos\theta$.

We again investigate solutions of (1) satisfying the 'clamped end conditions'

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0, \qquad \text{for } t \geq 0.$$

However, it will become clear that the methods developed here are applicable to problems with certain other homogeneous boundary conditions at $x = 0$.

The semigroup formulation of Section 1.1 still, of course, applies for this case of constant coefficients. However, the construction of the function g in Chapter 1 is not valid here, because it was done under the assumption that certain coefficients of the partial differential equation have compact support. Thus, we start the investigation in Section 2.1 by constructing the function g. At the same time, we derive the properties of g which we later use to obtain regularity results for the function J.

In Section 2.2, we use changes in contours to demonstrate the analyticity of J in much the same way that we demonstrated that the function J of Chapter 1 is of Gevrey class in the time variable.

In Section 2.3, we remove the assumptions $a_4 = 1$, $a_3 = 0$ and present the final results of the chapter. The chapter ends with an example for which we can write down the fundamental solution of equation (1) explicitly. It exemplifies the results of the final theorem of the chapter.

2.1 The Construction and Properties of the Function g

In this section, we construct the function g corresponding to the case of constant coefficients with $a_4 = 1$ and $a_3 = 0$. Since g is a fundamental solution of an ordinary differential equation with constant coefficients, it may be expressed in terms of the roots of the characteristic polynomial of the differential equation. Thus we see that the properties of g depend largely on the behavior of the roots as functions of the spectral parameter. We start by considering these roots.

It is convenient to define for $r > 0$ the family of sets

$$S(r) = \{\omega \in \mathbf{C} : |\omega| > r\}. \tag{3}$$

Lemma 1

There exists a constant $d > 0$ depending on a_0, a_1, a_2, θ, b_0 and b_1 such that for all $k \in S(d)$, the roots of the equation

$$x^4 - 2\cos\theta\ k^2\ x^2 + k^2(b_1 x + b_0) + a_2 x^2 + a_1 x + a_0 + k^4 = 0 \tag{4}$$

are distinct and depend analytically on k. Also, each root is expressible as a Laurent series

$$m_1(k) = ke^{i\theta/2} + \sum_{j=0}^{\infty} c_{1j}\, k^{-j}, \qquad m_2(k) = ke^{-i\theta/2} + \sum_{j=0}^{\infty} c_{2j}\, k^{-j},$$

$$m_3(k) = -ke^{i\theta/2} + \sum_{j=0}^{\infty} c_{3j}\, k^{-j}, \qquad m_4(k) = -ke^{-i\theta/2} + \sum_{j=0}^{\infty} c_{4j}\, k^{-j}, \tag{5}$$

each of which converges absolutely, uniformly for $k \in \overline{S(d+\delta)}$ for all positive constants δ.

Proof

Setting $x = kz$ and $k = 1/\varepsilon$, we see that equation (4) is equivalent to

$$h(z,\varepsilon) = z^4 - 2\cos\theta\, z^2 + 1 + \varepsilon b_1 z + \varepsilon^2 a_2 z^2 + \varepsilon^3 a_1 z + \varepsilon^4 a_0 + \varepsilon^2 b_0 = 0.$$

Let $f(z) = z^4 - 2\cos\theta\, z^2 + 1$ and $\delta = \sin\theta/2$. Clearly if $z \in C_\delta$, where

$$C_\delta = \{z \in \mathbf{C} : |z - e^{i\theta/2}| = \delta\},$$

then

$$f(z) = (z - e^{i\theta/2})(z + e^{i\theta/2})(z - e^{-i\theta/2})(z + e^{-i\theta/2}) > \delta^4.$$

Further, if $z \in C_\delta$ then $|z| \le \delta+1 < 2$ and

$$|f(z) - h(z,\varepsilon)| \le 2|\varepsilon b_1| + 4|\varepsilon^2 a_2| + 2|\varepsilon^3 a_1| + |\varepsilon^4 a_0| + |\varepsilon^2 b_0|.$$

Hence if $|\varepsilon| < \rho = \frac{1}{5}\delta^4(1 + 2|b_1| + 2|a_2|^{1/2} + (2|a_1|)^{1/3} + |a_0|^{1/4} + |b_0|^{1/2})^{-1}$, then

$$|f(z) - h(z,\varepsilon)| < \delta^4 < |f(z)| \qquad \text{for all } z \in C_\delta.$$

Thus by Rouché's Theorem [20], we see that if $|\varepsilon| < \rho$ then f(z) and h(z,ε) have the same number of zeros inside the circle C_δ. But f(z) has only one zero, $e^{i\theta/2}$, in this set, so it follows that g has only one zero, $z_1(\varepsilon)$, in this set .

An elementary result† from complex analysis gives $z_1(\varepsilon)$ explicitly:

$$z_1(\varepsilon) = \frac{1}{2\pi i}\int_{C_\delta} \frac{zh'(z,\varepsilon)}{h(z,\varepsilon)}\,dz.$$

It was shown that $|f(z)| > \delta^4$ for $z \in C_\delta$. Thus we can find $\eta > 0$ so that $|f(z)| > \delta^4+\eta$ for $z \in C_\delta$. Hence,

$$|h(z,\varepsilon)| \geq |f(z)| - |h(z,\varepsilon) - f(z)| > \eta \qquad \text{for } z \in C_\delta \text{ and } |\varepsilon| \leq \rho.$$

It follows that $zh'(z,\varepsilon)/h(z,\varepsilon)$ is a uniformly continuous function of (z,ε) for $z \in C_\delta$ and $|\varepsilon| \leq \rho$. If C is any simple closed contour contained in $\{\omega \in \mathbf{C} : |\omega| < \rho\}$ then by Fubini's theorem

$$\int_C z_1(\varepsilon)\,d\varepsilon = \frac{1}{2\pi i}\int_{C_\delta}\int_C \frac{zh'(z,\varepsilon)}{h(z,\varepsilon)}\,d\varepsilon\,dz = 0.$$

This shows that $z_1(\varepsilon)$ is analytic for $|\varepsilon| < \rho$. Hence we can write $z_1(\varepsilon)$ as a power series* with radius of convergence no less than ρ:

$$z_1(\varepsilon) = e^{i\theta/2} + \sum_{j=0}^{\infty} c_{1j}\,\varepsilon^{j+1}.$$

If we set $m_1(k) = k\,z_1(1/k)$, then we obtain the first of equations (5). The convergence properties of the Laurent series for m_1 follow from those of the Taylor series for z_1.

The roots m_2, m_3 and m_4 may be treated similarly.

□

Remark: The constants c_{ij} of Lemma 1 satisfy the equations

$$c_{1j} = \bar{c}_{2j} \quad \text{and} \quad c_{3j} = \bar{c}_{4j} \qquad \text{for } j = 1, 2, 3, 4, \ldots\ldots$$

---

†*The expression yields the sum of the zeros of an analytic function inside a simple, closed contour. In this case, however, the sum is clearly $z_1(\varepsilon)$.*

**The constant coefficient is easily computed from the equation $h(z,\varepsilon) = 0$ to be $e^{i\theta/2}$.*

One easily sees this by recognizing that when k is real the roots appear as a pair of complex conjugates

$$m_1(k) = \overline{m}_2(k) \qquad \text{and} \qquad m_3(k) = \overline{m}_4(k).$$

With Lemma 1 in mind, we define functions g_1 and g_2 on $\mathbf{C}^2 \times S$:

$$g_1(x,a,k) = -\frac{\exp[m_1(x-a)]}{(m_1-m_2)(m_1-m_3)(m_1-m_4)} - \frac{\exp[m_2(x-a)]}{(m_2-m_1)(m_2-m_3)(m_2-m_4)}$$

$$+ \frac{\exp[m_3x-m_1a]}{(m_2-m_1)(m_3-m_1)(m_3-m_4)} + \frac{\exp[m_3x-m_2a]}{(m_2-m_1)(m_3-m_2)(m_4-m_3)}$$

$$+ \frac{\exp[m_4x-m_1a]}{(m_2-m_1)(m_4-m_3)(m_1-m_4)} + \frac{\exp[m_4x-m_2a]}{(m_1-m_2)(m_4-m_2)(m_4-m_3)}\ , \tag{6}$$

$$g_2(x,a,k) = -\frac{\exp[m_3(x-a)]}{(m_3-m_4)(m_3-m_2)(m_3-m_1)} - \frac{\exp[m_4(x-a)]}{(m_4-m_3)(m_4-m_2)(m_4-m_1)}$$

$$+ \frac{\exp[m_3x-m_1a]}{(m_2-m_1)(m_3-m_1)(m_3-m_4)} + \frac{\exp[m_3x-m_2a]}{(m_2-m_1)(m_3-m_2)(m_4-m_3)}$$

$$+ \frac{\exp[m_4x-m_1a]}{(m_2-m_1)(m_4-m_3)(m_1-m_4)} + \frac{\exp[m_4x-m_2a]}{(m_1-m_2)(m_4-m_2)(m_4-m_3)}\ . \tag{7}$$

These functions have the following properties:

(i) g_1 and g_2 are analytic in $\mathbf{C}^2 \times S$.

(ii) g_1-g_2 and its partial derivatives of order ≤ 2 with respect to (x,a) vanish at all points $(a,a,k) \in \mathbf{C}^2 \times S$.

(iii) If $|\arg k| < \pi/4$, $|k|$ is sufficiently large and $(x,a) \in [0,\infty)^2$, then $g_2(x,a,k)$ and all of its derivatives tend to zero as x tends to infinity. Further,

$$g_1(0,a,k) = \frac{\partial g_1}{\partial x}(0,a,k) = 0 \qquad \text{for all } (a,k) \in \mathbf{C} \times S.$$

(iv) $$\frac{\partial^3 g_2}{\partial x^3}(a,a,k) - \frac{\partial^3 g_1}{\partial x^3}(a,a,k) = 1 \qquad \text{for all } (a,k) \in \mathbf{C} \times S.$$

(v) $$L_k\, g_j(x,a,k) = 0 \qquad \text{for all } (x,a,k) \in \mathbf{C}^2 \times S,\ j \in \{1,2\}.$$

Further, we define $g : [0,\infty)^2 \times S \to \mathbf{C}$ by:

$$g(x,a,k) = \begin{cases} g_1(x,a,k) & \text{for } 0 \le x < a \\ g_2(x,a,k) & \text{for } 0 \le a < x \end{cases} . \tag{8}$$

It is easily seen that g has the following properties:

(i) Let $T_1 = \{(x,a) : 0 \le x < a\}$ and $T_2 = \{(x,a) : 0 \le a < x\}$. Then $g|_{T_1}$ and $g|_{T_2}$ have analytic continuations into $\mathbf{C}^2 \times S$ (the continuations are g_1 and g_2 respectively).

(ii) The function g and its partial derivatives of order ≤ 2 with respect to (x,a) are continuous.

(iii) If $|\arg k| < \pi/4$, $|k|$ is sufficiently large and $(x,a) \in [0,\infty)^2$, then $g(x,a,k)$ and all of its derivatives tend to zero as x tends to infinity. Further,

$$g(0,a,k) = \frac{\partial g}{\partial x}(0,a,k) = 0.$$

(iv) $$\frac{\partial^3 g}{\partial x^3}(a,a^-,k) - \frac{\partial^3 g}{\partial x^3}(a,a^+,k) = 1.$$

(v) If $x \neq a$ then $L_k\, g(x,a,k) = 0$. Further, g is a fundamental solution for $L_k u = f$.

(vi) There exist constants† $d' > d$ and $C > 0$ such that if $\arg k = \phi$ then* for $i = 1,2$,

---

†*The constant d is that of lemma 1.*

**The notation here is that of Laurent Schwartz [9]. If $\alpha = (i,j)$ then $D^\alpha u(x,a) = \frac{\partial^{i+j} u}{\partial x^i\, \partial a^j}(x,a)$, and $|\alpha| = i+j$.*

$$|D^{\alpha} g|_{T_i}(x, a, k)| \le \begin{cases} 2^{|\alpha|} C |k|^{|\alpha|-3} \exp[-|k| \cos\{(|\phi|+\theta/2)|x-a|\}] & \text{for } k \in S_1 \cap S(d') \\ 2^{|\alpha|} C |k|^{|\alpha|-3} \exp[-|k| \cos\{(|\phi|+\theta/2)(x+a)\}] & \text{for } k \in (S_2 \cup S_3) \cap S(d'). \end{cases}$$

Here S_1, S_2 and S_3 are the sectors

$$S_1 = \{\omega : \theta/2 - \pi/2 < \arg \omega < \pi/2 - \theta/2\},$$

$$S_2 = \{\omega : -\pi/2 \le \arg \omega \le \theta/2 - \pi/2\},$$

$$S_3 = \{\omega : \pi/2 - \theta/2 \le \arg \omega \le \pi/2\}.$$

By property (vi) with $\alpha = (0,0)$, we see that if $|\lambda| > (d')^2$ and $\operatorname{Re} \lambda > 0$ then

$$|g(x,a,(\lambda)^{1/2})| \le C\, |\lambda|^{-3/2}, \tag{9}$$

so we may define $J : [0,\infty)^3 \to \mathbf{C}$ as in Chapter 1 by

$$J(x, a, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} g(x, a, \sqrt{\lambda}\,)\, e^{\lambda t}\, d\lambda, \qquad \text{where } \gamma > (d')^2. \tag{10}$$

2.2 Analyticity of the Function J

It is easy to see from equation (10) that J is a continuous function of its arguments. However, as we see in the following theorems, much more is true.

Theorem 2

The functions $J|_{T_1 \times [0,\infty)}$ and $J|_{T_2 \times [0,\infty)}$ may be continued analytically on to the set $\mathbf{C}^2 \times \{t \in \mathbf{C} : \operatorname{Re} t > 0\}$.

Proof

We consider the contour Γ_R consisting of the line segment $\{s+iR : 0 \le s \le \gamma\}$, with orientation in the direction of increasing s. Inequality (9) shows that

$$\lim_{R\to\pm\infty} \int_{\Gamma_R} g(x, a, \sqrt{\lambda}\,)\, e^{\lambda t}\, d\lambda = 0.$$

This shows that we may replace the original contour $C_1 = \{\gamma + is : s \in R\}$ by the contour C_2 shown in figure 5.

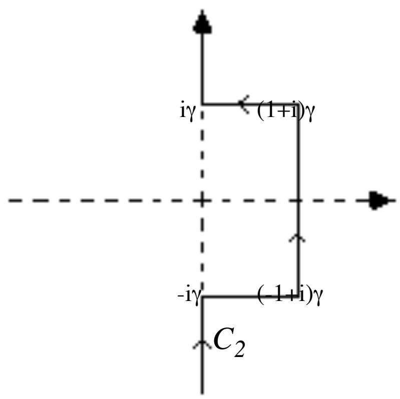

Figure 5

Thus we obtain

$$J(x,a,t) = \frac{1}{\pi i} \int_{C_3} g(x,a,k)\, k \exp[k^2 t]\, dk,$$

where C_3 is the image of C_2 under the transformation $\lambda \to \lambda^{1/2} = k$. C_3 coincides with the rays $\arg k = \pm\, \pi/4$ for large $|k|$.

Now consider the contour $\Gamma'_R = \{s+iR : 0 \le s \le R\}$, with orientation in the direction of increasing s (see figure 6).

From property (vi) of the function g, we see that for $k \in (S_2 \cup S_3) \cap S(d')$,

$$|g(x,a,k)| \le C\, |k|^{-3} \exp[|k|(x+a)(\sin|\phi|\, \sin\theta/2 - \cos\phi\, \cos\theta/2)]$$

$$\le C\, |k|^{-3} \exp[|k|(x+a)(\sin|\phi| - \cos\phi)/\sqrt{2}]$$

$$\le C\, |k|^{-3} \exp[(x+a)(|\mathrm{Im}\, k| - \mathrm{Re}\, k)/\sqrt{2}].$$

Inspection of (vi) shows that we even have

$$|g(x,a,k)| \le C\, |k|^{-3} \exp[(x+a)(|\mathrm{Im}\, k| - \mathrm{Re}\, k)/\sqrt{2}]$$

for all $k \in S(d')$ satisfying $\pi/4 \le |\arg k| \le \pi/2$. Thus if $k = s+iR$ on Γ'_R, with $R > \gamma^{1/2}$, then

$$|k\, g(x,a,k)| \le C\, |R|^{-2} \exp[(x+a)(R-s)/\sqrt{2}].$$

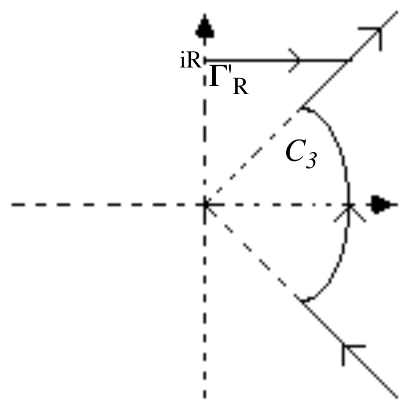

Figure 6

Thus,

$$\int_{\Gamma'_R} |g(x,a,k)\, k\, e^{k^2 t}|\, |dk| \le C\, R^{-2} \int_0^R \exp[(x+a)(R-s)/\sqrt{2}\,]\, \exp[(s^2-R^2)t]\, ds.$$

Consider the polynomial $p(s) = (x+a)(R-s)/\sqrt{2} + (s^2-R^2)t$. If $t > 0$ and $R > (x+a)/(\sqrt{2}\, t)$, then $p(s) \le 0$ for $s \in [0,R]$. Thus we see that for $t > 0$,

$$\lim_{R\to\infty} \int_{\Gamma'_R} g(x,a,k)\, k\, e^{k^2 t}\, dk = 0.$$

We may show in a similar fashion that the integral over Γ'_{-R}, the reflection of Γ'_R in the real axis, tends to zero as R tends to infinity. Thus we may replace the contour C_3 by any simple contour C_4 which coincides with the imaginary axis for large enough |k| and which, since g(x,a,k) is an analytic function of k in S(d), passes to the right of the disc

$\{w : |w| \le d\}$. We pick $r > d$ and choose for C_4 the set

$$\{z \in \mathbf{C} : |z| = r,\ \text{Re}\, z \ge 0\} \cup \{i\sigma : \sigma \in R,\ |\sigma| \ge r\},$$

with orientation in the direction of increasing imaginary part.

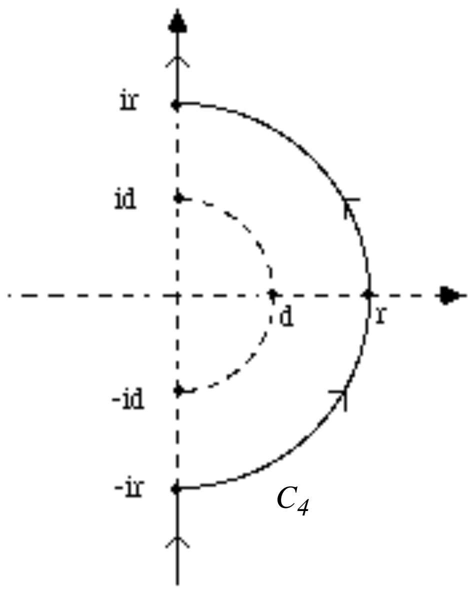

Figure 7

In summary, we have obtained the result that

$$J(x,a,t) = \frac{1}{\pi i} \int_{C_4} g(x,a,k)\, k \exp[k^2 t]\, dk. \tag{11}$$

It is easy to see that the integrals defining the functions

$$J_i(x,a,t) = \frac{1}{\pi i} \int_{C_4} g_i(x,a,k)\, k \exp[k^2 t]\, dk \qquad i \in \{1,2\} \tag{12}$$

converge absolutely, uniformly on compact subsets of $\mathbf{C}^2 \times \{t \in \mathbf{C} : \text{Re}\, t > 0\}$. Thus, the functions J_1 and J_2 are analytic in $\mathbf{C}^2 \times \{t \in \mathbf{C}: \text{Re}\, t > 0\}$ and the theorem has been proved.

□

Recall that we found in Chapter 1 that, for the case of variable coefficients, the function J does not inherit the discontinuities of the function g of that chapter. We can say even more about this in the constant coefficient case, as we see in the following theorem.

Theorem 3

The function J may be continued analytically on to the set $\mathbf{C}^2 \times \{t \in \mathbf{C} : \text{Re } t > 0\}$. Further, if we also denote the continuation by J, we have:

$$\frac{\partial^2 J}{\partial t^2} + \frac{\partial^4 J}{\partial x^4} + \sum_{j=0}^{2} b_j \frac{\partial^{j+1} J}{\partial t \partial x^j} + a_j \frac{\partial^j J}{\partial x^j} = 0 \quad \text{in } \mathbf{C}^2 \times \{t \in \mathbf{C} : \text{Re } t > 0\}. \tag{13}$$

Proof

By property (v) of the functions g_1 and g_2, and by equation (12), it follows easily that equation (13) is satisfied by the functions J_1 and J_2. We prove the theorem by demonstrating that J_1 and J_2 are the same function.

By property (iv) of the functions g_1 and g_2, and by equation (12), we have

$$\frac{\partial^3 J_2}{\partial x^3}(a,a,k) - \frac{\partial^3 J_1}{\partial x^3}(a,a,k) = \frac{1}{\pi i} \int_{C_4} k \exp[k^2 t]\, dk = \frac{1}{\pi i} \int_{-i\infty}^{i\infty} k \exp[k^2 t]\, dk = 0.$$

for all $(a,t) \in \mathbf{C} \times \{t \in \mathbf{C} : \text{Re } t > 0\}$.

Further, by property (ii) of the functions g_1 and g_2, and equation (12), we see that

$$J_1(a,a,k) = J_2(a,a,k),\ \frac{\partial J_1}{\partial x}(a,a,k) = \frac{\partial J_2}{\partial x}(a,a,k) \quad \text{and} \quad \frac{\partial^2 J_1}{\partial x^2}(a,a,k) = \frac{\partial^2 J_2}{\partial x^2}(a,a,k)$$

for all $(a,t) \in \mathbf{C} \times \{t \in \mathbf{C} : \text{Re } t > 0\}$. Since J_1 and J_2 satisfy equation (13), we also get that

$$\frac{\partial^4 J_1}{\partial x^4}(a,a,k) = \frac{\partial^4 J_2}{\partial x^4}(a,a,k)$$

for all $(a,t) \in \mathbf{C} \times \{t \in \mathbf{C} : \text{Re } t > 0\}$. Differentiation of equation (13) with respect to x yields, by induction, that

$$\frac{\partial^n J_1}{\partial x^n}(a,a,k) = \frac{\partial^n J_2}{\partial x^n}(a,a,k) \qquad n = 0, 1, 2, 3, 4, \ldots$$

for all $(a,t) \in \mathbf{C} \times \{t \in \mathbf{C} : \text{Re } t > 0\}$.

Since for, any fixed (a,t), $J_1(x,a,t)$ and $J_2(x,a,t)$ are entire functions of x, it follows that J_1 and J_2 are identical. This completes the proof of the theorem.

□

2.3 Analyticity of the Fundamental Solution of Equation (1)

In this section we see the connection between the semigroup U(t) and the function J of the previous sections. At the end of the section we also remove the restrictions $a_4(x) = 1$ and $a_3(x) = 0$ in equation (1).

The results which we obtain are the analogues of those obtained in Sections 1.7 and 1.8 for the case of variable coefficients with restricted supports. Consequently, the proofs of the results of this section resemble those of Sections 1.7 and 1.8 so closely that we omit them.

As in Section 1.7, we define a mapping on the space of functions $f \in L^2(\mathbf{R}^+)$ with compact support:

$$\mathsf{J}(t)\, f(x) = \int_0^\infty J(x,a,t) f(a)\, da \qquad \text{for } (x,t) \in \mathbf{C} \times \{t \in \mathbf{C} : \text{Re } t > 0\}. \tag{14}$$

We may prove, as in Theorem 14 of Chapter 1, that, for $f_1 \in H_0^2(\mathbf{R}^+)$ and $f_2 \in L^2(\mathbf{R}^+)$ with compact support, and $t > 0$,

$$U(t) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \mathsf{J}'(t) + \mathsf{J}(t)\, L_2 & \mathsf{J}(t) \\ \mathsf{J}''(t) + \mathsf{J}'(t)\, L_2 & \mathsf{J}'(t) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{15}$$

It follows immediately that if f_1 and f_2 are as above and

$$\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = U(t) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{16}$$

then both w and v have analytic continuations on to $\mathbf{C} \times \{t \in \mathbf{C} : \mathrm{Re}\, t > 0\}$.

Now we come to removing the restrictions on a_4 and a_3. This is done exactly as in Section 1.8. We need only check that when the original coefficients a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2 are constants, then the coefficients $\tilde{a}_0$, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_0$, $\tilde{b}_1$, $\tilde{b}_2$ of the transformed equation (109 of Chapter 1) are also constants. But this is easily verified, for in the case of constant coefficients, the transformation Q (see equation (108) in Chapter 1) is nothing more than a scaling of the space variable x by a constant factor, composed with multiplication of the dependent variable w by an exponential function of x.

It is interesting to note that there are equations with variable coefficients which, under the transformation Q, are transformed into equations for which $\tilde{a}_0$, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_0$, $\tilde{b}_1$, $\tilde{b}_2$ are constants. Obviously, the theory of this chapter applies equally well to these equations in the sense that we can deduce the existence of a fundamental solution, which is analytic in t for Re $t > 0$, for them. It is easy to see that such equations can formally be written in the form

$$\frac{\partial^2 p(x)w}{\partial t^2} + \frac{\partial^4 p(x)w}{\partial [q(x)]^4} + \sum_{j=0}^{2} \tilde{b}_j \frac{\partial^{j+1} p(x)w}{\partial t \partial [q(x)]^j} + \tilde{a}_j \frac{\partial^j p(x)w}{\partial [q(x)]^j} = 0,$$

where $\tilde{a}_0$, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_0$, $\tilde{b}_1$, $\tilde{b}_2$ are constants. However, we will say no more about such equations and we return to the case in which a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2 are constants.

We may define a strongly continuous semigroup S(t) by equation (110) of Section 1.8. The infinitesimal generator, B, of S(t) is, of course, still given by equation (111). Further, we may use equation (112) to define a function $K : \mathbf{C}^2 \times \{t \in \mathbf{C} : \mathrm{Re}\, t > 0\} \to \mathbf{C}$, which, for this more general setting, replaces the function J constructed above. Finally, we may still use equation (113) for the definition of the integral operator $\mathsf{K}(t)$, but now we note that it maps members of $L^2(\mathbf{R}^+)$ with compact support into the space of analytic functions on $\mathbf{C} \times \{t \in \mathbf{C} : \mathrm{Re}\, t > 0\}$.

Having made these considerations, we may summarize the results of this chapter in the following theorem, which corresponds to Theorem 16 of Chapter 1.

Theorem 4

If $a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2$ are constants, $a_4 > 0$ and $-2\sqrt{a_4} < b_2 \le 0$, then the following statements hold:

(i) K is analytic on the set $\mathbf{C}^2 \times \{t \in \mathbf{C} : \operatorname{Re} t > 0\}$.

(ii) For $(x,a,t) \in \mathbf{C}^2 \times \{t \in \mathbf{C} : \operatorname{Re} t > 0\}$,

$$\frac{\partial^2 K}{\partial t^2}(x,a,t) + \sum_{j=0}^{4} a_j \frac{\partial^j K}{\partial x^j}(x,a,t) + \sum_{j=0}^{2} b_j \frac{\partial^{j+1} K}{\partial x^j \partial t}(x,a,t) = 0.$$

(iii) Let $f_1 \in H_0^2(\mathbf{R}^+)$ and $f_2 \in L^2(\mathbf{R}^+)$ have compact support. Then for $t > 0$,

$$S(t)\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} K'(t) + K(t)\,L_2 & K(t) \\ K''(t) + K'(t)\,L_2 & K'(t) \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

where $L_2 = b_2\, d^2/dx^2 + b_1\, d/dx + b_0$.

(iv) If $\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = S(t)\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, where f_1 and f_2 are as in (iii), then w and v are both analytic on $\mathbf{C} \times \{t \in \mathbf{C} : \operatorname{Re} t > 0\}$.

□

We end the chapter with an example which illustrates the theory which we have developed.

Example: An Euler-Bernoulli beam with constant physical characteristics.

Littman and Markus [17], in their investigation of the exact boundary controllability of an Euler-Bernoulli beam, considered the solution of the following problem:

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0, \qquad \text{for } (x,t) \in (R^+)^2,$$

$$w(x,0) = w_0(x), \quad v(x,0) = v_0(x), \qquad \text{for } x \geq 0.$$

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0, \qquad \text{for } t \geq 0.$$

In [17], it is shown that, if $(w_0,v_0) \in C_0^\infty(\mathbf{R}^+)^2$, there is a fundamental solution U(x,t,a) such that solutions of the problem are given uniquely† by the expression

$$w(x,t) = \int_0^\infty w_0(a)\, U(x,t,a)\, da + \int_0^t \int_0^\infty v_0(a)\, U(x,\tau,a)\, da\, d\tau.$$

Moreover, the function U is given explicitly:

$$U(x,t,a) = \frac{1}{2\sqrt{2\pi t}} \{\cos[(x-a)^2/4t] + \sin[(x-a)^2/4t] + \cos[(x+a)^2/4t]$$

$$- \sin[(x+a)^2/4t] - 2e^{-2xa/4t} \cos[(x^2-a^2)/4t]\}.$$

Comparing the solution with part (iii) of Theorem 4, we see that the functions U and K are related by the formula

$$K(x,a,t) = \int_0^t U(x,\tau,a)\, d\tau.$$

This formula illustrates part (i) of Theorem 4. Indeed, it shows that there is an analytic continuation of the function K on to the set $\mathbf{C}^2 \times (\mathbf{C} \setminus \{\tau \in \mathbf{R} : \tau \leq 0\})$.

---

†*Uniqueness is proven for the class of continuous, generalized solutions which are bounded by some positive power of x in each strip* $0 \leq t \leq T$.

Chapter 3

Solutions of Euler-Bernoulli Type Equations which Vanish for t ≥ T

3.0 Introduction

In this chapter, given an arbitrarily prescribed $T > 0$, we demonstrate the existence of solutions of equations of Euler-Bernoulli type which satisfy the initial and clamped end conditions (see equations (1), (2) and (3) of Chapter one) and which vanish for $t \geq T$. The method which we use will actually allow us to attack more general equations, the coefficients of which are functions of x and t satisfying for t on compact subsets of $(0,\infty)$ certain Gevrey estimates for their partial derivatives with respect to t:

$$\mathsf{L}w = \frac{\partial^2 w}{\partial t^2}(x,t) + \sum_{i=0}^{4} a_i(x,t)\frac{\partial^i w}{\partial x^i}(x,t) + \sum_{i=0}^{2} b_i(x,t)\frac{\partial^{i+1} w}{\partial x^i \partial t}(x,t) = 0. \tag{1}$$

Our key assumption is that we have at our disposal functions $K_0(x,a,t)$ and $K_1(x,a,t)$, each satisfying for t on compact subsets of $(0,\infty)$ certain Gevrey estimates for their partial derivatives with respect to t, such that if the initial data, w_0 and v_0, are functions of compact support (and in suitable function spaces), then solutions of (1) for $t > 0$ are of the form

$$w(x,t) = \int_0^\infty K_1(x,a,t)\, w_0(a)\, da + \int_0^\infty K_0(x,a,t)\{v_0(a) + L_2(a,0)\, w_0(a)\}\, da, \tag{2}$$

where $L_2(x,t) = b_2(x,t)\, d^2/dx^2 + b_1(x,t)\, d/dx + b_0(x,t)$. We recall that this is the situation in chapters one and two in which solutions are given, for $t > 0$, by equation (2) with

$$K_0(x,a,t) \equiv K(x,a,t), \quad K_1(x,a,t) \equiv \frac{\partial K}{\partial t}(x,a,t). \tag{3}$$

We show in this chapter that for a certain $\mathsf{l} > 0$, depending on the Gevrey estimates of the coefficient functions, we can find functions $\tilde{K}_0$ and $\tilde{K}_1$ defined for $t \geq 0$, $0 \leq x \leq \mathsf{l}$, $0 \leq a < \infty$, such that for $j \in \{0,1\}$,

$$\tilde{K}_j(x,a,t) = K_j(x,a,t) \text{ for } t \leq T/2, \quad \tilde{K}_j(x,a,t) = 0 \text{ for } t \geq T.$$

$\tilde{K}_j(x,a,t)$ satisfies equation (1) as a function of x and t, for $t > 0$.

$$\tilde{K}_j(0,a,t) = \frac{\partial \tilde{K}_j}{\partial x}(0,a,t) = 0. \tag{4}$$

It follows from conditions (4) that, for $t > 0$ and $x \in [0,l]$,

$$\tilde{w}(x,t) = \int_0^\infty \tilde{K}_1(x,a,t)\, w_0(a,t)\, da + \int_0^\infty \tilde{K}_0(x,a,t)\, \{v_0(a,t) + L_2(a,0)\, w_0(a)\}\, da, \tag{5}$$

satisfies equation (1), the clamped end conditions, and takes on the initial data (w_0,v_0) in the same sense that w(x,t) does.

The method of construction of the function $\tilde{w}$ is that described by Littman [16] and used by Littman and Markus in [17]. We start with any function $\psi \in \gamma^2(\mathbf{R})$ satisfying

$$\psi(t) = 1 \quad \text{for} \quad t \leq T/2, \qquad\qquad \psi(t) = 0 \quad \text{for} \quad t \geq T. \tag{6}$$

Hörmander [8] shows how such functions can be explicitly constructed. We now let

$$F(x,t) = \mathsf{L}(w\, \psi)(x,t) = \psi''(t)\, w(x,t) + 2\psi'(t)\, \frac{\partial w}{\partial t}(x,t)$$

This function obviously vanishes for $t \leq T/2$ and for $t \geq T$. If we can now solve the problem

$$\mathsf{L}u = F, \qquad u(0,t) = \frac{\partial u}{\partial x}(0,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^3 u}{\partial x^3}(0,t) = 0, \tag{7}$$

to get a solution which also vanishes for $t \leq T/2$ and for $t \geq T$, it will follow that

$$\tilde{w}(x,t) = \psi(t)\, w(x,t) - u(x,t) \tag{8}$$

will be a solution of equation (1) vanishing for $t \geq T$. Clearly this solution will satisfy the clamped end conditions and the equation $\tilde{w}(x,t) = w(x,t)$ for $t \leq T/2$.

We can in fact treat all solutions w(x,t) given by equation (2) at once. For $j \in \{0,1\}$ we set

$$F_j(x, a, t) = \mathbf{L}(K_j\,\psi)(x, a, t) \tag{9}$$

and then find functions $u_j(x,a,t)$ with support in the strip $T/2 \le t \le T$ satisfying, with 'a' as a parameter,

$$\mathbf{L}u_j = F_j, \qquad u_j(0,a,t) = \frac{\partial u_j}{\partial x}(0,a,t) = \frac{\partial^2 u_j}{\partial x^2} = \frac{\partial^3 u_j}{\partial x^3}(0,a,t) = 0. \tag{10}$$

If this is done, it follows that the functions

$$\tilde{K}_j(x, a, t) = \psi(t)\, K_j(x, a, t) - u_j(x, a, t) \tag{11}$$

would satisfy conditions (4). We see, then, that all we have to do is solve equation (10) to obtain the kernels $\tilde{K}_j$. However, this problem is ordinarily not a well-posed problem and we must rely on the Gevrey smoothness of the functions F_j and of the coefficients of $\mathbf{L}$ in order to solve it.

The chapter is set out as follows: In Section 3.1 we set up the frame work for solving problem (10), in Section 3.2 we study the spaces in which we look for a solution, and finally in Section 3.3 we solve the problem.

We solve (10) by using the Ovcyannikov Theorem which is a generalization of the classical Cauchy-Kowalevski Theorem. Treves [22], [23] has studied applications of this theorem to certain types of problems solvable in Gevrey spaces, however, his work is not directly applicable to our problem. Further, it is desirable for us to use a slight modification of the usual statement of the Ovcyannikov Theorem in order to obtain a solution of equation (10) for $x \in [0,l]$, with as large a value for l as possible. This modified Ovcyannikov Theorem is stated in Section 3.1.

Finally, we remark that problem (10) can sometimes be solved by writing down an explicit series solution and using Gevrey estimates to show the convergence of it. This is how Littman and Markus [17] solve the problem in the case for which equation (1) is the uniform Euler-Bernoulli beam equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0.$$

This approach is easily generalized to handle the case of the equations discussed in Chapter 1. However, when the coefficients of L depend on time, it seems that the easiest method for solving (10) is the one that we use here.

3.1 Statement of the Problem

For fixed $i \in \{0,1\}$, we rewrite problem (10) as follows. We let a column vector y have components $y_1 = u_i$, $y_2 = \frac{\partial u_i}{\partial x}$, $y_3 = \frac{\partial^2 u_i}{\partial x^2}$, $y_4 = \frac{\partial^3 u_i}{\partial x^3}$, and we let $F(x,a) = F_i(x,a,.)\ \vec{e}_4$, $\vec{e}_4$ being the unit vector in the direction of the y_4-axis. Problem (10) may now be written in the form

$$\frac{dy}{dx} = \mathsf{A}(x)\, y + F(x,a), \qquad y(0) = 0. \tag{12}$$

Here $\mathsf{A}(x)$ is formally the operator which maps certain $\mathbf{R}^4$-valued functions of t into certain other $\mathbf{R}^4$-valued functions of t (this will be made more precise below), and is of the form

$$\mathsf{A}(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{a_0(x,.)}{a_4(x,.)} & -\frac{a_1(x,.)}{a_4(x,.)} & -\frac{a_2(x,.)}{a_4(x,.)} & -\frac{a_3(x,.)}{a_4(x,.)} \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{b_0(x,.)}{a_4(x,.)} & \frac{b_1(x,.)}{a_4(x,.)} & \frac{b_2(x,.)}{a_4(x,.)} & 0 \end{bmatrix} \frac{\partial}{\partial t} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{a_4(x,.)} & 0 & 0 & 0 \end{bmatrix} \frac{\partial^2}{\partial t^2}. \tag{13}$$

In the following section, we define a scale of Banach spaces $(E_s, |\ |_s)$ for $s \in [0,1]$, such that:

(I) If $s' \leq s$ then $E_{s'} \supset E_s$ and the natural injection has norm ≤ 1.

(II) For $s' \le s$ and each $x \in [0,\eta]$, $\mathsf{A}(x)$ is a bounded mapping from E_s into $E_{s'}$, and there are constants $M > 0$, $N \ge 0$, both independent of x, s and s', such that the operator norm of this mapping is $\le M(s-s')^{-1} + N$. Further, the mapping is a continuous function of x in the uniform operator topology. (14)

These conditions will allow us to apply the Ovcyannikov Theorem. However, in the usual statement of the Ovcyannikov Theorem, condition (I) is assumed and a condition (II)' is assumed. The condition (II)' is the same as condition (II) except that the estimate of the norm in the mapping is replaced by a quantity of the form

$$M'(s-s')^{-1}. \tag{14)'$$

Clearly, the constant N of (14) can be absorbed into the constant M of (14) to give an estimate of the form (14)' with $M' = M + N$. However, the classical theorem states that the solution of the problem exists in an interval $[0,\delta_0(1-s)]$, where δ_0 is a decreasing function of M'. Specifically,

$$\delta_0 = \min\{\eta, (M'e)^{-1}\}. \qquad (15)'$$

Thus, a direct application of the theorem would give an estimate for δ_0 which would decrease with increasing N. However, a more careful analysis, as in the proof of Theorem 1, shows that if assumption (II) holds, then the constant δ_0 is given by

$$\delta_0 = \min\{\eta, (Me)^{-1}\}. \qquad (15)$$

We prefer to use the modification of the theorem, rather than the original theorem, because, as we shall see, the constants M and N of (14) depend on different coefficients of the differential expression L. For applications it is of use to know which coefficients limit the length of the interval of existence of the solution. We now state the modified Ovcyannikov theorem.

Theorem 1

Suppose that assumptions (I) and (II) hold, that $y_0 \in E_1$ and that $F \in C([0,\eta]; E_1)$. Then the problem

$$\frac{dy}{dx} = A(x)\, y + F(x), \qquad y(0) = y_0$$

has a unique solution belonging to $C^1([0, \delta_0(1-s)]; E_s)$ for each $s \in [0,1]$. The constant δ_0 is given by equation (15).

Proof

We proceed as in the proof of the original theorem and define

$$z_0(x) = y_0 + \int_0^x F(u)\, du,$$

while for $n \geq 0$, we define

$$z_{n+1}(x) = \int_0^x A(u)\, z_n(u)\, du.$$

We put

$$C = |y_0|_1 + \int_0^\eta |F(u)|_1\, du.$$

Thus $|z_0(x)|_s \leq C$. By assumption (II), for all ε satisfying $0 < \varepsilon \leq 1\text{-}s$,

$$|z_1(x)|_s \leq (M\varepsilon^{-1}+N)\ t\ |z_0|_{s+\varepsilon} \leq (M\varepsilon^{-1}+N)\ C\ x.$$

We choose $\varepsilon = 1\text{-}s$ and obtain

$$|z_1(x)| \leq (M(1\text{-}s)^{-1}+N)\ C\ x.$$

Suppose now that we have, for some $k \geq 1$,

$$|z_k(x)|_s \leq (kM(1-s)^{-1}+N)^k\, C\, x^k / k!.$$

It follows that for all ε satisfying $0 < \varepsilon < 1-s$,

$$|z_{k+1}(x)|_s \leq (M\varepsilon^{-1}+N)\ (kM(1-s-\varepsilon)^{-1}+N)^k\, C\, x^{k+1} / [k+1]!.$$

If we choose $\varepsilon = (1-s) / (k+1)$, we obtain

$$|z_{k+1}(x)|_s \leq ([k+1]M(1-s)^{-1}+N)^{k+1}\, C\, x^{k+1} / [k+1]!.$$

Thus by the principle of induction, the formula holds for all k (even for $k = 0$). Now we observe that

$$|z_k(x)|_s \leq C\, [Mx/(1-s)]^k\ (k^k/k!)\ (1+N(1-s)/Mk)^k$$

$$\leq C\, [Mx/(1-s)]^k\, e^k\, e^{N(1-s)/M}.$$

Hence, we can proceed as in [23] and show that the series $z_0 + z_1 + z_2 + \ldots$ converges absolutely, uniformly on the interval $I_s = [0, \delta_0(1-s)]$ to a continuous function y(x) valued in each E_s, and that

$$y(x) = y_0 + \int_0^x F(u)\, du\ + \int_0^x \mathsf{A}(u)\, y(u)\, du.$$

It follows from this that $y \in C^1(I_s; E_s)$ and that y is indeed a solution of the initial value problem. The uniqueness assertion is proved as in [23].

□

We remark that we may replace the system (12) by a more general one, for which

$$\mathsf{A}(x) = A(x,.) + B(x,.)\,\frac{\partial}{\partial t} + \Gamma(x,.)\,\frac{\partial^2}{\partial t^2}\ , \tag{16}$$

where

$$A(x,t) = \begin{bmatrix} \alpha_{11}(x,t) & \alpha_{12}(x,t) & 0 & 0 \\ \alpha_{21}(x,t) & \alpha_{22}(x,t) & \alpha_{23}(x,t) & 0 \\ \alpha_{31}(x,t) & \alpha_{32}(x,t) & \alpha_{33}(x,t) & \alpha_{34}(x,t) \\ \alpha_{41}(x,t) & \alpha_{42}(x,t) & \alpha_{43}(x,t) & \alpha_{44}(x,t) \end{bmatrix},$$

$$B(x,t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \beta_{21}(x,t) & 0 & 0 & 0 \\ \beta_{31}(x,t) & \beta_{32}(x,t) & 0 & 0 \\ \beta_{41}(x,t) & \beta_{42}(x,t) & \beta_{43}(x,t) & 0 \end{bmatrix},$$

$$\Gamma(x,t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_{41}(x,t) & 0 & 0 & 0 \end{bmatrix}. \qquad (17)$$

We define the spaces E_s as

$$E_s = B_{s,0} \times B_{s,1/2} \times B_{s,1} \times B_{s,3/2} \;, \qquad (18)$$

where the spaces $B_{s,\alpha}$ are the subject of the next section.

3.2 The Spaces $B_{s,\alpha}$

Definition: Let K be an interval [a,b] and let θ_0 and θ_1 be two positive constants such that $\theta_1 < \theta_0$. Given $s \in [0,1]$ and $\alpha \in [0,\infty)$, we define $B_{s,\alpha}(K) = B_{s,\alpha}$ to be the space of C^∞ functions f with support in K, satisfying

$$|f|_{s,\alpha} = \sup_{n\geq 0} \max_{t\in K} |f^{(n)}(t)| \, (\theta_0^{1-s}\,\theta_1^{s})^{-(n+\alpha)} \, (n!)^{-2} \, (n+1)^{-2\alpha} < \infty. \qquad (19)$$

It is easily seen that $|\ \ |_{s,\alpha}$ is a norm on $B_{s,\alpha}$ which makes $B_{s,\alpha}$ into a Banach space. Further, for α fixed and s ranging in [0,1], the spaces $B_{s,\alpha}$ form a scale of Banach spaces in the sense that

$$\text{if } s' \leq s, \text{ then } B_{s',\alpha} \supset B_{s,\alpha} \text{ and the natural injection has norm} \leq 1. \tag{20}$$

In the following lemmas, we study the properties of these spaces further. In this study, we will make use of the following simple result:

If $h(x) = (\theta_0/\theta_1)^{-xm}(x+1)^\beta$, where $m > 0$ and $\beta > 0$, then

$$\sup_{x\geq 0} |h(x)| \leq (\theta_0/\theta_1)^m e^{-\beta} \left[\frac{\beta}{m \ln[\theta_0/\theta_1]}\right]^\beta. \tag{21}$$

Lemma 2

(i) If $s' < s$ and $\alpha' < \alpha$, then $B_{s',\alpha'} \supset B_{s,\alpha}$ and the natural injection is continuous. Further, if also $\alpha - \alpha' \leq 1/2$, then there is a constant $C_1 = C_1(\alpha,\theta_0/\theta_1)$ such that the norm of the natural injection no greater than

$$C_1 \, \theta_0^{1/2} \, (s-s')^{-1}. \tag{22}$$

(ii) The m^{th} derivative is a continuous mapping from $B_{s,\alpha}$ into $B_{s',\alpha'}$, provided that

$$s' < s \text{ and } \alpha' < m + \alpha.$$

If we also have $m + \alpha - \alpha' \leq 1/2$, then there is a constant $C_2 = C_2(\alpha,m,\theta_0/\theta_1)$ such that the norm of the mapping is no greater than

$$C_2 \, \theta_0^{1/2} \, (s-s')^{-1}. \tag{23}$$

Proof

We notice that if we allow the case $m = 0$ in (ii), then (ii) yields statement (i). Thus we can prove (i) and (ii) at the same time to get $C_1(\alpha,\theta_0/\theta_1) = C_2(\alpha,0,\theta_0/\theta_1)$.

If $\Phi \in B_{s,\alpha}$, then

$$|\Phi^{(m+n)}(t)|\,[\theta_0^{1-s'}\,\theta_1^{s'}]^{-(n+\alpha')}\,(n+1)^{-2\alpha'}(n!)^{-2}$$

$$= |\Phi^{(m+n)}(t)|\,[\theta_0^{1-s}\,\theta_1^{s}]^{-(m+n+\alpha)}\,(m+n+1)^{-2\alpha}\,((m+n)!)^{-2}$$

$$\cdot\,(\theta_0/\theta_1)^{-n(s-s')}\,(n+1)^{-2\alpha'}\,(m+n+1)^{2\alpha}\,[(m+n)!/n!]^2\,(\theta_0^{1-s'}\theta_1^{s'})^{-\alpha'}\,(\theta_0^{1-s}\,\theta_1^{s})^{\alpha+m}$$

$$\leq |\Phi|_{s,\alpha}\,(\theta_0/\theta_1)^{-n(s-s')}\,(n+1)^{-2\alpha'}\,(m+n+1)^{2\alpha+2m}\,(\theta_0^{1-s'}\,\theta_1^{s'})^{-\alpha'}\,(\theta_0^{1-s}\,\theta_1^{s})^{\alpha+m}$$

$$= |\Phi|_{s,\alpha}\,(\theta_0/\theta_1)^{-n(s-s')}\,(n+1)^{2\alpha+2m-2\alpha'}\,(1+\frac{m}{n+1})^{2\alpha+2m}\,(\theta_0^{1-s'}\,\theta_1^{s'})^{-\alpha'}\,(\theta_0^{1-s}\,\theta_1^{s})^{\alpha+m}$$

$$\leq |\Phi|_{s,\alpha}\,(\theta_0/\theta_1)^{s-s'}\,e^{-2(m+\alpha-\alpha')}\left[\frac{2(m+\alpha-\alpha')}{(s-s')\ln[\theta_0/\theta_1]}\right]^{2(m+\alpha-\alpha')}$$

$$\cdot\,(1+m)^{2\alpha+2m}\,(\theta_0^{1-s'}\,\theta_1^{s'})^{-\alpha'}\,(\theta_0^{1-s}\,\theta_1^{s})^{\alpha+m}.$$

This proves the first parts of each of (i) and (ii). If $m + \alpha - \alpha' = 1/2$, we find that the norm of the mapping is no greater than

$$(1+m)^{2\alpha+2m}\,(\theta_0/\theta_1)^{|1-\alpha-m|}\,(e\,\ln[\theta_0/\theta_1])^{-1}\,\theta_0^{1/2}\,(s-s')^{-1}.$$

But the norm can be no greater than this whenever $m + \alpha - \alpha' < 1/2$, because of (20).

□

Lemma 3

(i) If $\alpha' \geq \alpha$, then $B_{s,\alpha'} \supset B_{s,\alpha}$ and the norm of the natural injection is no greater than $\theta_1^{\alpha-\alpha'}$.

(ii) If $\alpha' \geq \alpha + m$, then the m^{th} derivative is a continuous mapping from $B_{s,\alpha}$ into $B_{s,\alpha'}$ with norm no greater than $\theta_1^{m+\alpha-\alpha'}\,(m+1)^{2m+2\alpha}$.

Proof

As with the previous lemma, there is some redundancy in the statement of Lemma 3: We can prove (i) by proving (ii) and then setting $m = 0$. To prove (ii), we let $\Phi \in B_{s,\alpha}$. Then:

$$|\Phi^{(m+n)}(t)|\,(\theta_0^{1-s}\,\theta_1^{s})^{-(n+\alpha')}\,(n!)^{-2}\,(n+1)^{-2\alpha'}$$

$$= |\Phi^{(n+m)}(t)|\,(\theta_0^{1-s}\,\theta_1^{s})^{-(n+m+\alpha)}\,[(n+m)!]^{-2}\,(n+m+1)^{-2\alpha}$$

$$\cdot\,(\theta_0^{1-s}\,\theta_1^{s})^{m+\alpha-\alpha'}\,[(n+m)!\,/\,n!]^{2}\,(n+m+1)^{2\alpha}\,(n+1)^{-2\alpha'}$$

$$\leq |\Phi|_{s,\alpha}\,\theta_1^{m+\alpha-\alpha'}\,(n+m+1)^{2m+2\alpha}\,(n+1)^{-2\alpha'}$$

$$\leq |\Phi|_{s,\alpha}\,\theta_1^{m+\alpha-\alpha'}\,(n+1)^{2(m+\alpha-\alpha')}\,(m+1)^{2m+2\alpha}$$

$$\leq |\Phi|_{s,\alpha}\,\theta_1^{m+\alpha-\alpha'}\,(m+1)^{2m+2\alpha}.$$

□

It is clear that there should be some kind of relationship between the spaces $B_{s,\alpha}$ and Gevrey class 2 functions. We will be wanting to make use of this relationship later, so now we examine some aspects of it.

Lemma 4

Let $r > 0$, $k \geq 0$ and $f : [0,r] \times \mathbf{R} \to \mathbf{C}$ have support in the strip $[0,r] \times K$. Suppose that the mapping $t \to f(.,t)$ is infinitely differentiable in the topology of $C^k[0,r]$ and that there are constants M and θ such that

$$\left|\frac{\partial^n f}{\partial t^n}(.,t)\right|_{C^k[0,r]} \leq M\theta^n (n!)^2 \qquad \forall\, t \in K \text{ and } n \geq 0. \tag{24}$$

If $\theta < \theta_1$ then the mapping $x \to f(x,.)$ is a member of $C^k([0,r]; B_{s,\alpha})$.

Proof

Since we have the injections (20), it clearly suffices to prove the lemma for $s = 1$. It is clear from the estimate (24) that the mapping $x \to f(x,.)$ is a $B_{s,\alpha}$- valued function, which we denote by $\tilde{f}$. Further, for any x_1 and x_2 in $[0,r]$, $t \in \mathbf{R}$ and $n > 0$,

$$\left|\frac{\partial^n f}{\partial t^n}(x_1,t) - \frac{\partial^n f}{\partial t^n}(x_2,t)\right| \leq 2\,M\theta^n (n!)^2. \tag{25}$$

Given $\varepsilon > 0$, we pick $N > 0$ so that, for $n > N$, we have

$$2\,M\theta^n \theta_1^{-(n+\alpha)} (n+1)^{-2\alpha} < \varepsilon. \tag{26}$$

Next, we use the uniform continuity of $\frac{\partial^n f}{\partial t^n}$ for $n \leq N$ to pick $\delta > 0$ so that for $|x_1 - x_2| < \delta$, we have

$$\left|\frac{\partial^n f}{\partial t^n}(x_1,t) - \frac{\partial^n f}{\partial t^n}(x_2,t)\right| \theta_1^{-(n+\alpha)} (n+1)^{-2\alpha} (n!)^{-2} < \varepsilon \qquad \text{for } n = 0, 1, 2, \ldots, N. \tag{27}$$

It now follows from (25), (26) and (27) that, for $|x_1 - x_2| < \delta$, we have

$$|\tilde{f}(x_1) - \tilde{f}(x_2)|_{1,\alpha} < \varepsilon.$$

This proves the lemma for the case $k = 0$.

We suppose now that we have proved that $\tilde{f}$ is in $C^j([0,r]; B_{s,\alpha})$ for some j satisfying $0 \le j < k$ and that the derivatives are given by

$$\tilde{f}^{(i)}(x) = \frac{\partial^i f}{\partial x^i}(x,.) \qquad \text{for } 0 \le i \le j.$$

Given $\varepsilon > 0$, we again pick N so that (26) is satisfied for $n > N$. Suppose that both x and x+h are in [0,r], with $|h| > 0$. Then we can find $\tilde{x}$ between x and x+h, and depending on n, t and j, such that

$$\frac{1}{h}\left(\frac{\partial^{n+j} f}{\partial t^n \partial x^j}(x+h,t) - \frac{\partial^{n+j} f}{\partial t^n \partial x^j}(x,t)\right) = \frac{\partial^{n+j+1} f}{\partial t^n \partial x^{j+1}}(\tilde{x},t).$$

Thus, if $n > N$,

$$\left|\frac{1}{h}\left(\frac{\partial^{n+j} f}{\partial t^n \partial x^j}(x+h,t) - \frac{\partial^{n+j} f}{\partial t^n \partial x^j}(x,t)\right) - \frac{\partial^{n+j+1} f}{\partial t^n \partial x^{j+1}}(x,t)\right| \theta_1^{-(n+\alpha)} (n+1)^{-2\alpha} (n!)^{-2}$$

$$\le \theta_1^{-(n+\alpha)} (n+1)^{-2\alpha} (n!)^{-2} \left(\left|\frac{\partial^{n+j+1} f}{\partial t^n \partial x^{j+1}}(\tilde{x},t)\right| + \left|\frac{\partial^{n+j+1} f}{\partial t^n \partial x^{j+1}}(x,t)\right|\right)$$

$$< \varepsilon.$$

We next pick $\delta > 0$ so small that if $0 < |h| < \delta$ and $x+h \in [0,r]$ then for $0 \le n \le N$,

$$\left|\frac{1}{h}\left(\frac{\partial^{n+j} f}{\partial t^n \partial x^j}(x+h,t) - \frac{\partial^{n+j} f}{\partial t^n \partial x^j}(x,t)\right) - \frac{\partial^{n+j+1} f}{\partial t^n \partial x^{j+1}}(x,t)\right| \theta_1^{-(n+\alpha)} (n+1)^{-2\alpha} (n!)^{-2} < \varepsilon.$$

It follows that $\tilde{f}$ is j+1 times differentiable and that $\tilde{f}^{(j+1)}(x) = \frac{\partial^{j+1} f}{\partial x^{j+1}}(x,.)$. The proof that

$\tilde{f}^{(j+1)}$ is continuous is the same as the proof that $\tilde{f}$ is continuous. This completes the proof of the lemma by induction.

□

Lemma 5

Suppose $\tilde{f} : [0,r] \to B_{s,\alpha}$ is m times continuously differentiable. We define $f(x,t) = [\tilde{f}(x)](t)$. The function f satisfies the following:

(i) $t \to f(.,t)$ is infinitely differentiable in the topology of $C^m[0,r]$.

(ii) If $\theta = \theta_0^{1-s}\theta_1^s$, then there is a constant M such that

$$\left|\frac{\partial^n f}{\partial t^n}(.,t)\right|_{C^k[0,r]} \le M\,\theta^n\,(n!)^2\,(n+1)^{2\alpha}.$$

Proof

For fixed x, f(x,t) is clearly infinitely differentiable with respect to t. Given $\varepsilon > 0$, we can find $\delta > 0$ so that if x_1 and x_2 are in [0,r] and $|x_1-x_2| < \delta$, then $|\tilde{f}(x_1) - \tilde{f}(x_2)| < \varepsilon$. But this implies that, for $n = 0, 1, 2, 3, \ldots,$

$$\left|\frac{\partial^n f}{\partial t^n}(x_1, t) - \frac{\partial^n f}{\partial t^n}(x_2, t)\right| < \varepsilon\,\theta^n\,(n!)^2\,(n+1)^{2\alpha}.$$

Thus, for each t and n, $\frac{\partial^n f}{\partial t^n}(.,t)$ is a member of C[0,r]. If we set

$$M_j = \max_{0\le x\le r} |\tilde{f}^{(j)}(x)|_{s,\alpha}, \qquad \text{for } j \in \{0,1,2,3,\ldots,m\},$$

then we have

$$\left|\frac{\partial^n f}{\partial t^n}(x, t)\right| \le M_0\,\theta^n\,(n!)^2\,(n+1)^{2\alpha}, \qquad \forall\, n \ge 0.$$

It follows that for $|k| > 0$, there exists $\tilde{t}$ between t and t+k such that

$$\left| \frac{1}{k}\left(\frac{\partial^n f}{\partial t^n}(x, t+k) - \frac{\partial^n f}{\partial t^n}(x, t) \right) - \frac{\partial^{n+1} f}{\partial t^{n+1}}(x, t) \right| = \left| \frac{\partial^{n+1} f}{\partial t^{n+1}}(x, \tilde{t}) - \frac{\partial^{n+1} f}{\partial t^{n+1}}(x, t) \right|$$

$$\leq |t - \tilde{t}|\, M_0\, \theta^{n+2}\, ([n+2]!)^2\, (n+3)^{2\alpha}$$

$$\leq |k|\, M_0\, \theta^{n+2}\, ([n+2]!)^2\, (n+3)^{2\alpha}.$$

This shows that the difference quotients converge in C[0,r]. Hence, we have proved the lemma for the case $m = 0$.

Next, we suppose that for some j satisfying $0 \leq j < m$, we know that:

(a) $\frac{\partial^{n+j} f}{\partial x^j \partial t^n}(x,t)$ exists for each $n \geq 0$, is a continuous function of x and coincides with

$$[\tilde{f}^{(j)}(x)]^{(n)}(t).$$

(b) The difference quotients $\frac{1}{k}\left[\frac{\partial^n f}{\partial t^n}(.,t+k) - \frac{\partial^n f}{\partial t^n}(.,t)\right]$ converge in $C^j[0,r]$ as $k \to 0$.

Given $\varepsilon > 0$, we can find $\delta > 0$ so that if x_1 and x_2 are in [0,r] and $|x_1 - x_2| < \delta$, then

$$|\tilde{f}^{(j+1)}(x_1) - \tilde{f}^{(j+1)}(x_1)|_{s,\alpha} < \varepsilon.$$

Hence $|[\tilde{f}^{(j+1)}(x_1)]^{(n)}(t) - [\tilde{f}^{(j+1)}(x_1)]^{(n)}(t)| < \varepsilon\, \theta^n\, (n!)^2\, (n+1)^{2\alpha}$, for all n, which shows that each of the functions $[\tilde{f}^{(j+1)}(x_1)]^{(n)}(t)$ is a continuous function of x. Moreover, given $\varepsilon > 0$, we can find $\delta > 0$ such that if $0 < |h| < \delta$, and x and x+h are both in [0,r], then

$$\left| \frac{1}{h}\left[\tilde{f}^{(j)}(x+h) - \tilde{f}^{(j)}(x)\right] - \tilde{f}^{(j+1)}(x) \right|_{s,\alpha} < \varepsilon.$$

This implies that

$$\left|\frac{1}{h}\left[\frac{\partial^{n+j}f}{\partial x^j \partial t^n}(x+h,t) - \frac{\partial^{n+j}f}{\partial x^j \partial t^n}(x,t)\right] - [\tilde{f}^{(j+1)}(x)]^{(n)}(t)\right| < \varepsilon\, \theta^n\, (n!)^2\, (n+1)^{2\alpha}.$$

Hence $\dfrac{\partial^{n+j+1}f}{\partial x^{j+1}\partial t^n}$ exists and coincides with $[\tilde{f}^{(j+1)}(x)]^{(n)}(t)$, which is continuous in x. Thus, statement (a) holds with 'j+1' replacing 'j'. Further, for $|k| > 0$, we have

$$\left|\frac{1}{k}\left[\frac{\partial^{n+j+1}f}{\partial x^{j+1}\partial t^n}(x,t+k) - \frac{\partial^{n+j+1}f}{\partial x^{j+1}\partial t^n}(x,t)\right] - \frac{\partial^{n+j+2}f}{\partial x^{j+1}\partial t^{n+1}}(x,t)\right|$$

$$= \left|\frac{1}{k}\left\{[\tilde{f}^{(j+1)}(x)]^{(n)}(t+k) - [\tilde{f}^{(j+1)}(x)]^{(n)}(t)\right\} - [\tilde{f}^{(j+1)}(x)]^{(n+1)}(t)\right|$$

$$= \left|[\tilde{f}^{(j+1)}(x)]^{(n+1)}(\tilde{t}) - [\tilde{f}^{(j+1)}(x)]^{(n+1)}(t)\right|$$

$$\leq |k| M_{j+1}\, \theta^{n+2}\, ([n+2]!)^2\, (n+3)^{2\alpha}.$$

(Here, $\tilde{t}$ is a point between t and t+k). Thus, statement (b) holds with 'j+1' replacing 'j'. By the principle of induction, statements (a) and (b) hold for all j satisfying $0 \leq j \leq m$. This completes the proof of (i).

Part (ii) follows easily:

$$\sum_{j=0}^{m}\left|\frac{\partial^{n+j}f}{\partial x^j\partial t^n}(x,t)\right| = \sum_{j=0}^{m}\left|[\tilde{f}^{(j)}(x)]^{(n)}(t)\right| \leq \left(\sum_{j=0}^{m} M_j\right)\theta^n\, (n!)^2\, (n+1)^{2\alpha}.$$

□

One would expect that the product of a member of $B_{s,\alpha}$ and a function satisfying a suitable Gevrey estimate would also be a member of $B_{s,\alpha}$. The following lemma shows that this is the case, and that such an operation is a bounded mapping.

Lemma 6

Let U be an open subset of **R** containing K, and let $\phi \in C^\infty(U)$ satisfy

$$|\phi^{(n)}(t)| \leq M\,\theta_1^n\,(n!)^2 \qquad \text{for all } t \in K.$$

Then the mapping $f \to \phi f$, which we denote by Φ, is a bounded mapping of $B_{s,\alpha}$ into itself, with operator norm no greater than 3M.

Proof

We first observe that if $f \in B_{s,\alpha}$, then ϕf is a C^∞ function with support in K. Now we compute:

$$|(\phi f)^{(n)}(t)| = \left| \sum_{j=0}^{n} \binom{n}{j} \phi^{(j)}(t)\, f^{(n-j)}(t) \right|$$

$$\leq \sum_{j=0}^{n} \binom{n}{j} M\,\theta_1^j\,(j!)^2\,[\theta_1^s\,\theta_0^{1-s}]^{n-j+\alpha}\,[(n-j)!]^2\,(n+1-j)^{2\alpha}\,\|f\|_{s,\alpha}$$

$$= M\,\|f\|_{s,\alpha}\,(n+1)^{2\alpha}\,[\theta_1^s\,\theta_0^{1-s}]^{n+\alpha}\,(n!)^2 \sum_{j=0}^{n} \frac{j!\,(n-j)!}{n!} \left[\frac{n+1-j}{n+1}\right] [\theta_1/\theta_0]^{(1-s)j}$$

$$\leq 3\,M\,\|f\|_{s,\alpha}\,(n+1)^{2\alpha}\,[\theta_1^s\,\theta_0^{1-s}]^{n+\alpha}\,(n!)^2,$$

because

$$\sum_{j=0}^{n} \frac{j!\,(n-j)!}{n!} \left[\frac{n+1-j}{n+1}\right] [\theta_1/\theta_0]^{(1-s)j} \leq \sum_{j=0}^{n} \frac{j!\,(n-j)!}{n!} \leq 1 + \frac{n-1}{n} + 1 \leq 3.$$

□

Since the coefficients of our differential equation depend on x, the following modification of Lemma 6, which we prove using Lemma 6, is more useful to us.

Lemma 7

Let U be an open subset of **R** containing K, and let $\phi \in C^\infty(U; C^k[0,r])$ satisfy

$$\|\phi^{(n)}(.,t)\|_{C^k} \leq M\,\theta^n\,(n!)^2 \qquad \text{for all } t \in K.$$

Then, if $\theta < \theta_1$, the mapping $f \to \phi(x,.)f$, which we denote by $\Phi(x)$, is a bounded mapping of $B_{s,\alpha}$ into itself, with operator norm no greater than 3M. Further, $x \to \Phi(x)$ is k times continuously differentiable in the uniform operator topology.

Proof

For $x \in [0,r]$, we let $\Phi_0(x), \Phi_1(x), \Phi_2(x), \ldots, \Phi_k(x)$ be the bounded linear operators obtained by applying Lemma 6 to the functions $\phi, \frac{\partial\phi}{\partial x}(x,.), \frac{\partial^2\phi}{\partial x^2}(x,.), \ldots, \frac{\partial^k\phi}{\partial x^k}(x,.)$. We see that the operator norm of each of these operators is no greater than 3M.

Now we can proceed as in the proof of Lemma 4 to show that for any j satisfying $0 \leq j \leq k$, and any $\varepsilon > 0$, one can find $\delta > 0$ so that, if $|x_1-x_2| < \delta$, then

$$\left| \frac{\partial^{j+n}\phi}{\partial x^j \partial t^n}(x_1, t) - \frac{\partial^{j+n}\phi}{\partial x^j \partial t^n}(x_2, t) \right| \leq \varepsilon\, \theta_1^n\, (n!)^2 \qquad \text{for all } n.$$

But, by Lemma 6, this implies that $\|\Phi_j(x_1) - \Phi_j(x_2)\| < 3\,\varepsilon$. This shows that each of the mappings $x \to \Phi_j(x)$ is continuous on [0,r], in the uniform operator topology.

One can also proceed as in the proof of Lemma 4 to show that for any j satisfying $0 \leq j < k$, and any $\varepsilon > 0$, one can find $\delta > 0$ so that if $0 < |h| < \delta$ and both x and x+h are in [0,r], then

$$\left| \frac{1}{h}\left(\frac{\partial^{j+n}\phi}{\partial x^j \partial t^n}(x+h, t) - \frac{\partial^{j+n}\phi}{\partial x^j \partial t^n}(x, t) \right) - \frac{\partial^{j+n+1}\phi}{\partial x^{j+1} \partial t^n}(x, t) \right| \leq \varepsilon\, \theta_1^n\, (n!)^2.$$

But this and Lemma 6 show that $\Phi_j'(x) = \Phi_{j+1}(x)$, where the derivative is taken in the uniform operator topology. This completes the proof of the lemma.

□

Recall that the function K lies in a certain space of functions Y (see the definition of Y preceding Corollary 13 in Chapter 1). We are aiming to modify K without destroying any of its regularity properties. In order to see how to do this, we define spaces similar to Y and see how sets of functions, which satisfy some kind of Gevrey estimate and are valued in such spaces, are related to the spaces $B_{s,\alpha}$.

Definitions: Let R be a compact rectangle in R^2. We denote by $D^{m,n}(\mathsf{R})$ the space of functions $f : \mathsf{R} \to \mathbf{C}$ such that for $0 \le i \le m$ and $0 \le j \le n$, all mixed partial derivatives of order i with respect to the first variable and j with respect to the second variable are equal and continuous. As before, we let it be understood that we define derivatives at the boundary of R as the usual one - sided limits of difference quotients.

We may define a norm $| \ |^{m,n}$ on $D^{m,n}(\mathsf{R})$ by the expression

$$|f|^{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} \max_{(x,a)\in\mathsf{R}} \left| \frac{\partial^{i+j} f}{\partial x^i \partial a^j}(x, a) \right| . \tag{28}$$

It is easily verified that this norm makes $D^{m,n}(\mathsf{R})$ into a Banach space.

Similarly, given a Banach space $(B, \| \ \|)$, we may define $D^{m,n}(\mathsf{R}; B)$ as the space of functions $f : \mathsf{R} \to B$ having the differentiability properties listed above. Again, we may make $D^{m,n}(\mathsf{R}; B)$ into a Banach space, with norm given by the expression

$$\sum_{i=0}^{m} \sum_{j=0}^{n} \max_{(x,a)\in\mathsf{R}} \left\| \frac{\partial^{i+j} f}{\partial x^i \partial a^j}(x, a) \right\| . \tag{29}$$

We will often be considering the space $D^{m,n}(\mathsf{R}; B_{s,\alpha})$, the norm of which we denote by the symbol $| \ |^{m,n}_{s,\alpha}$.

We are now ready to state some relationships between spaces of $D^{m,n}(\mathsf{R})$ - valued functions satisfying a certain Gevrey estimate, and the spaces $D^{m,n}(\mathsf{R}; B_{s,\alpha})$. These results are very similar to the statements of Lemmas 4 and 5, which examine the relationships between spaces of $C^m[0,r]$ - valued functions satisfying a certain Gevrey estimate, and the spaces $C^m([0,r]; B_{s,\alpha})$. We state the relationships in Lemmas 8 and 9 below, the proofs of which are straight - forward modifications of the proofs of Lemmas 4 and 5 respectively.

Lemma 8

Let $f : \mathsf{R} \times \mathbf{R} \to \mathbf{C}$ have support in the strip $\mathsf{R} \times \mathsf{K}$. Suppose that the mapping $t \to f(.,t)$ is infinitely differentiable in the topology of $D^{m,n}(\mathsf{R})$ and that there are constants M and θ such that

$$\left| \frac{\partial^i f}{\partial t^i}(.,.,t) \right|^{m,n} \leq M\theta^i (i!)^2 \qquad \forall\, t \in \mathsf{K} \text{ and } i \geq 0. \tag{30}$$

If $\theta < \theta_1$ then the mapping $(x,a) \to f(x,a, .)$ is a member of $D^{m,n}(\mathsf{R}; B_{s,\alpha})$.

□

Lemma 9

Suppose $\tilde{f} \in D^{m,n}(\mathsf{R}; B_{s,\alpha})$. We define $f(x,a,t) = [\tilde{f}(x,a)](t)$. The function f satisfies the following:

(i) $t \to f(.,.,t)$ is infinitely differentiable in the topology of $D^{m,n}(\mathsf{R})$.

(ii) If $\theta = \theta_0^{1-s}\theta_1^s$, then there is a constant M such that

$$\left| \frac{\partial^i f}{\partial t^i}(.,.,t) \right|^{m,n} \leq M\theta^i (i!)^2 (i+1)^{2\alpha} \qquad \forall\, t \in \mathsf{K} \text{ and } i \geq 0.$$

□

We now investigate how the solution y of the Ovcyannikov Theorem (Theorem 1) depends on a parameter 'a' when $F(x,a) \in D^{0,1}(\mathsf{R}; E_1)$. Here, $\mathsf{R} = [0,\eta] \times [0,\eta']$, where $\eta' > 0$ and η is as in Theorem 1.

Lemma 10

Let $F \in D^{0,1}(\mathsf{R}; E_1)$ and for each $a \in [0,\eta']$, let y(.,a) denote the solution of the problem

$$\frac{dy}{dx} = \mathsf{A}(x)\, y + F(x,a), \qquad y(0) = y_0,$$

which, according to Theorem 1, is a member of $C^1([0, \delta_0(1-s)]; E_s)$. As a function of (x,a), y is a member of $C^1(\mathsf{R}_s; E_s)$, where $\mathsf{R}_s = [0, \delta_0(1-s)] \times [0,\eta']$.

Proof

Recall from the proof of Theorem 1 that we have

$$z_0(x, a) = y_0 + \int_0^x F(u, a)\, du, \tag{31}$$

$$z_{n+1}(x, a) = \int_0^x \mathsf{A}(u)\, z_n(u, a)\, du. \tag{32}$$

But $F \in D^{0,1}(\mathsf{R}; E_1)$. Thus, $z_0 \in C^1(\mathsf{R}; E_1)$ with

$$\frac{\partial z_0}{\partial x}(x, a) = F(x, a), \tag{33}$$

$$\frac{\partial z_0}{\partial a}(x, a) = \int_0^x \frac{\partial F}{\partial a}(u, a)\, du. \tag{34}$$

Inductively, we see that for each $n \geq 1$ and each s satisfying $0 \leq s < 1$, $z_n \in C^1(\mathsf{R}; E_s)$, and that

$$\frac{\partial z_n}{\partial x}(x, a) = \mathsf{A}(x)\, z_{n-1}(x, a), \tag{35}$$

$$\frac{\partial z_n}{\partial a}(x, a) = \int_0^x \mathsf{A}(u) \frac{\partial z_{n-1}}{\partial a}(u, a)\, du. \tag{36}$$

But, from the proof of Theorem 1, we have the estimate

$$|z_n(x, a)|_s \leq \left(\frac{nM}{1-s} + N\right)^n C \frac{x^n}{n!}.$$

It follows from equation (35) that for any ε satisfying $0 < \varepsilon < 1\text{-}s$,

$$\left|\frac{\partial z_n}{\partial x}(x, a)\right|_s \leq (M\varepsilon^{-1}+N) \left(\frac{(n-1)M}{1-s} + N\right)^{n-1} C \frac{x^{n-1}}{(n-1)!}.$$

If we choose $\varepsilon = \frac{1-s}{n}$, we obtain

$$\left|\frac{\partial z_n}{\partial x}(x, a)\right|_s \leq \left(\frac{nM}{1-s} + N\right)^n C \frac{x^{n-1}}{(n-1)!}. \tag{37}$$

If we put $C' = \max_{(x,a)\in \mathsf{R}} \eta \left|\frac{\partial F}{\partial a}(x, a)\right|_1$, it follows from equation (34) that

$$\left|\frac{\partial z_0}{\partial a}(x, a)\right|_s \leq C'.$$

Proceeding with an inductive argument as in the proof of Theorem 1, we can show that for $0 \leq s < 1$,

$$\left|\frac{\partial z_n}{\partial a}(x, a)\right|_s \leq \left(\frac{nM}{1-s} + N\right)^n C' \frac{x^n}{n!}. \tag{38}$$

Estimates (37) and (38) show that the series $y = z_0 + z_1 + z_2 + \ldots$ converges uniformly in $C^1(\mathsf{R}_s; E_s)$. This completes the proof of the lemma.

□

3.3 Solution of the Problem

In this section, we apply the previous results to the solution of the initial value problem (12). We begin by assuming that $\mathcal{A}(x)$ is given by the more general expression (16), and then we consider the case in which $\mathcal{A}(x)$ is given by equation (13) so that we can solve problem (10). Our primary assumptions are as follows:

(i) The entries of A, B, Γ are all infinitely differentiable with respect to t in the topology of C[0,η].

(ii) There exist constants $M > 0, N \geq 0$ and θ with $0 \leq \theta < \theta_1$, such that for all $n \geq 0$,

$$\left|\frac{\partial^n \alpha_{ij}}{\partial t^n}(x, a, t)\right| \leq \begin{cases} M\theta^n (n!)^2 & \text{for } (i, j) \in \{(1, 2), (2, 3), (3, 4)\} \\ N\theta^n (n!)^2 & \text{otherwise} \end{cases}$$

$$\left|\frac{\partial^n \beta_{ij}}{\partial t^n}(x, a, t)\right| \leq \begin{cases} M\theta^n (n!)^2 & \text{for } (i, j) \in \{(2, 1), (3, 2), (4, 3)\} \\ N\theta^n (n!)^2 & \text{otherwise} \end{cases}$$

$$\left|\frac{\partial^n \gamma_{41}}{\partial t^n}(x, a, t)\right| \leq M\theta^n (n!)^2. \tag{31}$$

(iii) Let R be the rectangle $[0,\eta] \times [0,\eta']$. We assume that for $i \in [1,2,3,4\}$, the functions $t \to f_i(.,.,t)$ are infinitely differentiable in the topology of $D^{0,1}(\mathsf{R})$. Also, there is a constant Q such that for all $n \geq 0$,

$$\left|\frac{\partial^n f_i}{\partial t^n}(.,.,t)\right|^{0,1} \leq Q\,\theta^n\,(n!)^2. \tag{32}$$

We let F(x,a) be the column vector with components $f_i(x,a,.)$.

Definition: For $s \in [0,1]$, let E_s denote the Banach space

$$E_s = B_{s,0} \times B_{s,1/2} \times B_{s,1} \times B_{s,3/2}\ , \tag{33}$$

with norm given by $|\ \ |_s$, where $|(y_1,y_2,y_3,y_4)|_s = |y_1|_{s,0} + |y_2|_{s,1/2} + |y_3|_{s,1} + |y_4|_{s,3/2}$.

The spaces $B_{s,\alpha}$ satisfy condition (20), so it is automatic that the spaces E_s satisfy condition (I) of the Ovcyannikov Theorem. We now show that assumptions (i) and (ii) imply that A satisfies condition (II) of our Ovcyannikov Theorem.

Theorem 11

Let $\mathsf{A}(x)$ be given by (16) and let conditions (i) and (ii) be satisfied. Then A satisfies condition (II) for the Ovcyannikov Theorem. Further, if M and N are the constants of (ii) above, and if C_2 is the constant of Lemma 2, then we have the estimate

$$\|\mathsf{A}(x)\|_{b(Es,Es')} \leq 60\ N \max(\theta_1^{-3/2},1) + 6\ M\ C_2(0,2,\theta_0/\theta_1)\ \theta_0^{1/2}\ (s-s')^{-1} \qquad (34)$$

Proof

We consider the operation on $B_{s,(j-1)/2}$ given by

$$f \to \alpha_{ij}(x,.)\ f + \beta_{ij}(x,.)\ f' + \gamma_{ij}(x,.)\ f''.$$

We show that this is, for each fixed x, a bounded mapping from $B_{s,(j-1)/2}$ into $B_{s',(i-1)/2}$ and that the mapping is a continuous function of x in the uniform operator topology. To see this, we consider the separate operations

$$f \to \alpha_{ij}(x,.)\ f\ , \quad f \to \beta_{ij}(x,.)\ f', \quad f \to \gamma_{ij}(x,.)\ f''\ ,$$

which we denote by the symbols $T_{ij}(x)$, $U_{ij}(x)$ and $V_{ij}(x)$ respectively.

(a) The Operation $T_{ij}(x)$: There are three cases to consider:

(i) If $j > i+1$ then $T_{ij}(x)$ is the zero mapping.

(ii) If $j = i+1$ for $i \in \{1,2,3\}$, we know from Lemma 2 that the norm of the natural injection from $B_{s,(j-1)/2}$ into $B_{s',(i-1)/2}$ is no greater than

$$C_2([j-1]/2,0,\theta_0/\theta_1)\ \theta_0^{1/2}\ (s-s')^{-1}.$$

Thus, by Lemma 7, $T_{ij}(x)$ is a bounded mapping from $B_{s,(j-1)/2}$ into $B_{s',(i-1)/2}$ with operator norm no greater than

$$3\ M\ C_2([j-1]/2,0,\theta_0/\theta_1)\ \theta_0^{1/2}\ (s-s')^{-1}.$$

Lemma 7 also shows that $x \rightarrow T_{ij}(x)$ is continuous in the uniform operator topology.

(iii) If $j < i+1$, we know from Lemma 3 that the norm of the natural injection from $B_{s,(j-1)/2}$ into $B_{s',(i-1)/2}$ is no greater than $\theta_1^{(j-i)/2}$. Because of the injection (20) and the results of Lemma 7, we conclude that, in this case, $T_{ij}(x)$ is a bounded mapping from $B_{s,(j-1)/2}$ into $B_{s',(i-1)/2}$ with norm no greater than

$$3\ N\ \max\{\theta_1^{-3/2},1\}$$

and that the mapping is a continuous function of x in the uniform operator topology.

(b) The Operation $U_{ij}(x)$: Again, there are three cases to consider:

(i) If $j > i-1$ then $U_{ij}(x)$ is the zero mapping.

(ii) If $j = i-1$, we know from Lemma 2 that the norm of the mapping $f \rightarrow f'$ from $B_{s,(j-1)/2}$ into $B_{s',(i-1)/2}$ is no greater than

$$C_2([j-1]/2,1,\theta_0/\theta_1)\ \theta_0^{1/2}\ (s-s')^{-1}.$$

We conclude as before that $U_{ij}(x)$ is a continuous function of x in the uniform operator topology of mappings from $B_{s,(j-1)/2}$ into $B_{s',(i-1)/2}$, with operator norm no greater than

$$3\ M\ C_2([j-1]/2,1,\theta_0/\theta_1)\ \theta_0^{1/2}\ (s-s')^{-1}.$$

(iii) If $j < i-1$, we know from Lemma 3 that the norm of the mapping $f \rightarrow f'$ from $B_{s,(j-1)/2}$ into $B_{s',(i-1)/2}$ is no greater than

$$\theta_1^{1+(j-i)/2}\ 2^{1+j} \leq 8\ \max\ \{\theta_1^{-1/2},1\}.$$

Because of the injections (20) and the results of Lemma 7, we conclude that $U_{ij}(x)$ is a continuous function of x in the uniform operator topology of mappings from $B_{s,(j-1)/2}$ into $B_{s',(i-1)/2}$, with operator norm no greater than

$$24\ N \max \{\theta_1^{-1/2}, 1\}.$$

(c) The Operation $V_{ij}(x)$: Here there are only two cases to consider:

(i) If $(i,j) \neq (4,1)$, then this is the zero mapping.

(ii) If $(i,j) = (4,1)$, we get from Lemma 2 that the norm of the mapping $f \rightarrow f''$ from $B_{s,0}$ into $B_{s',3/2}$ is no greater than

$$C_2(0,2,\theta_0/\theta_1)\ \theta_0^{1/2}\ (s-s')^{-1}.$$

We conclude as before that $V_{41}(x)$ is a continuous function of x in the uniform operator topology of mappings from $B_{s,0}$ into $B_{s',3/2}$, with operator norm no greater than

$$3\ M\ C_2(0,2,\theta_0/\theta_1)\ \theta_0^{1/2}\ (s-s')^{-1}.$$

The results of the considerations (a), (b) and (c) above imply the first statement in theorem. To obtain estimate (34), we notice that

$$\|\mathcal{A}(x)\|_{b\ (E_s,E_{s'})} \leq \max_{j\in\{1,2,3,4\}} \sum_{i=1}^{4} \|T_{ij}(x) + U_{ij}(x) + V_{ij}(x)\|_{b\ (B_{s,(j-1)/2}\ ,\ B_{s',(i-1)/2})}.$$

The constant C_2, which is in fact a function of various parameters, is given explicitly in Lemma 2 and it is clear from this that the largest value that C_2 attains in (a), (b) and (c) is $C_2(0,2,\theta_0/\theta_1)$. These facts and a simple computation lead to estimate (34).

□

We are now ready to apply Theorem 1, the Ovcyannikov Theorem, to our problem. First we assume that $\mathcal{A}(x)$ has the general form given in equations (16) and (17).

Theorem 12

Let $\mathsf{A}(x)$ be given by equations (16) and (17), and suppose that this expression satisfies the assumptions (i) and (ii) stated at the beginning of Section 3.3. Further, let $F \in C([0,\eta]; E_1)$. Then the problem

$$\frac{dy}{dx} = \mathsf{A}(x)\, y + F(x), \qquad y(0) = 0$$

has a unique solution belonging to $C^1([0,\delta_0(1-s)]; E_s)$, where the constant δ_0 is given by

$$\delta_0 = \min\{\eta, (M\, C\, \theta_0^{1/2})^{-1}\}. \tag{35}$$

Here C is a constant depending only on θ_0/θ_1 and is given by

$$C(\theta_0/\theta_1) = 6\, e\, C_2(0,2,\theta_0/\theta_1). \tag{36}$$

Proof

By Theorem 11, $\mathsf{A}(x)$ satisfies condition (II) of Theorem 1, so we may apply Theorem 1 to obtain the result stated above.

□

Theorem 13

Let $F(x,a) = (f_1(x,a,.),f_2(x,a,.),f_3(x,a,.),f_4(x,a,.))^t$, where the functions f_1, f_2, f_3 and f_4 satisfy condition (32) in the rectangle $\mathsf{R} = [0,\eta] \times [0,\eta']$, and let $\mathsf{A}(x)$ be as in Theorem 12. Then there is a unique solution, $y(x,a)$, of the problem

$$\frac{dy}{dx} = \mathsf{A}(x)\, y + F(x,a), \qquad y(0) = 0$$

which, for each $s \in [0,1]$, is a member of $C^1(\mathsf{R}_s; E_s)$, where $\mathsf{R}_s = [0, \delta_0(1-s)] \times [0,\eta']$ and δ_0 is given by equation (35).

Proof

By Lemma 8, F is a member of $D^{0,1}(\mathsf{R}; E_s)$ for each $s \in [0,1]$. Thus we may apply Lemma 10 to obtain Theorem 13 as stated.

□

Corollary 14

Suppose that the functions $a_0(x,t)$, $a_1(x,t)$, $a_2(x,t)$, $a_3(x,t)$, $a_4(x,t)$, $b_0(x,t)$, $b_1(x,t)$ and $b_2(x,t)$ are all infinitely differentiable with respect to t in the topology of $C[0,\eta]$. Suppose also that there are constants $M > 0$, $N \geq 0$ and θ satisfying $0 \leq \theta < \theta_1$, such that for $(x,t) \in [0,\eta] \times \mathsf{K}$,

$$\left| \frac{\partial^n}{\partial t^n} [b_2(x,t)/a_4(x,t)] \right| \leq M\theta^n (n!)^2, \tag{37}$$

$$\left| \frac{\partial^n}{\partial t^n} [1/a_4(x,t)] \right| \leq M\theta^n (n!)^2, \tag{38}$$

$$\left| \frac{\partial^n}{\partial t^n} [a_j(x,t)/a_4(x,t)] \right| \leq N\theta^n (n!)^2, \quad \text{for } j \in \{0,1,2,3\}, \tag{39}$$

$$\left| \frac{\partial^n}{\partial t^n} [b_j(x,t)/a_4(x,t)] \right| \leq N\theta^n (n!)^2, \quad \text{for } j \in \{0,1\}. \tag{40}$$

Suppose also that $f(.,.,t)$ is infinitely differentiable with respect to t in the topology of $D^{0,1}(\mathsf{R})$, where $\mathsf{R} = [0,\eta] \times [0,\eta']$, and that there is a constant Q such that

$$\left| \frac{\partial^n f}{\partial t^n}(x,a,t) \right|^{0,1} \leq Q\theta^n (n!)^2, \quad \text{for } n \geq 0,\ (x,a) \in \mathsf{R} \text{ and } t \in \mathsf{K}. \tag{41}$$

If $\mathsf{A}(x)$ is given by equation (13) and if $F(x,a) = (0,0,0,f(x,a,.))^t$, then for each $s \in [0,1]$, the problem

$$\frac{dy}{dx} = \mathsf{A}(x)\, y + F(x,a), \qquad y(0) = 0 \tag{42}$$

has a unique solution, y(x,a), valued in $C^1(R_s; E_s)$, where $R_s = [0, \delta_0(1\text{-}s)] \times [0,\eta']$ and δ_0 is given by equation (35).

□

The following result shows that we indeed get a solution of problem (10) from the solution y of Corollary 14. It also shifts the focus from E_s-valued functions of (x,a) to $D^{m,n}$-valued functions of t.

Theorem 15

Let $z_i(x,a,t)$, $i \in \{1,2,3,4\}$, be the components of the solution [y(x,a)](t) in Corollary 14. Then:

(i) For $i \in \{1,2,3,4\}$, the function $t \to z_i(.,.,t)$ is infinitely differentiable in the topology of $D^{5-i,1}(R_0)$.

(ii) There is a constant $P > 0$ such that, for each $n \geq 0$,

$$\left| \frac{\partial^n z_1}{\partial t^n}(.,.,t) \right|^{4,1} \leq P\,\theta_0^n (n!)^2 (n+1)^3.$$

(iii) $$L\, z_1(x,a,t) = f(x,a,t) \quad \text{for } (x,a,t) \in R_0 \times K,$$

$$z_1(0,a,t) = \frac{\partial z_1}{\partial x}(0,a,t) = \frac{\partial^2 z_1}{\partial x^2}(0,a,t) = \frac{\partial^3 z_1}{\partial x^3}(0,a,t) = 0.$$

Proof

By Lemma 9 and Corollary 14, the functions z_i are infinitely differentiable in the topology of $D^{1,1}(R_0) = C^1(R_0)$, and for each $s \in [0,1]$, there exists a constant M_s such that for (x,a) $\in R_s = [0, \delta_0(1\text{-}s)] \times [0,\eta']$, and $t \in K$,

$$\left| \frac{\partial^n z_i}{\partial t^n}(x,a,t) \right|^{1,1} \leq M_s \left(\theta_0^{1-s}\,\theta_1^{s}\right)^n (n!)^2 (n+1)^{i-1}. \tag{43}$$

Examination of the components of (42) yields the equations, holding for $(x,a,t) \in R_0 \times K$,

$$\frac{\partial z_1}{\partial x} = z_2, \quad \frac{\partial z_2}{\partial x} = z_3, \quad \frac{\partial z_3}{\partial x} = z_4,$$

$$\frac{\partial z_4}{\partial x} = - \sum_{j=0}^{3} \frac{a_j}{a_4} z_{j+1} - \sum_{j=0}^{2} \frac{b_j}{a_4} \frac{\partial z_{j+1}}{\partial t} - \frac{1}{a_4} \frac{\partial^2 z_4}{\partial t^2} + f.$$

It is easy to see that these, along with the equation $y(0) = 0$, establish (iii). Statements (i) and (ii) follow from these equations and inequality (43) with $s = 0$. The constant in (ii) can be taken as $P = 4 M_0$.

□

In the foregoing theory, we have obtained a solution of the problem

$$\frac{dy}{dx} = A(x)\, y + F(x,a), \qquad y(0) = 0,$$

for x in an interval $[0,\delta_0]$, where

$$\delta_0 = \min \{\eta, (M\, C\, \theta_0^{1/2})^{-1}\}$$

(the solution is in $C^1(R_0; E_0)$, where $R_0 = [0, \delta_0] \times [0,\eta']$). Here, M is the constant of Corollary 14 and depends on Gevrey estimates involving only $a_4(x,t)$ and $b_2(x,t)$. Moreover, the constant C depends only on the ratio θ_0/θ_1.

It is of interest to see how large we can make the interval of existence, with given coefficients $a_0(x,t)$, $a_1(x,t)$, $a_2(x,t)$, $a_3(x,t)$, $a_4(x,t)$, $b_0(x,t)$, $b_1(x,t)$, $b_2(x,t)$ and a given function $f(x,t)$. With this in mind, we let θ_2 be the infimum of all constants θ such that the coefficients and f satisfy estimates of the form (37) to (41), and for $\theta > \theta_2$, we let $M(\theta)$ be the smallest constant possible in estimates (37) and (38). It is clear that $M(\theta)$ is defined as a non-increasing function of θ on the interval (θ_2,∞). Let

$$M^* = \inf \{M(\theta)\, \theta^{1/2} : \theta > \theta_1\}, \tag{44}$$

We now claim that we may get a solution of our problem for x in the interval $[0,\mu)$, where

$$\mu = \min \{\eta, (3^6 e M^*)^{-1}\} \qquad (45)$$

To see this, we let $\varepsilon > 0$ and pick $\theta_3 > \theta_2$ such that $M^* > M(\theta_3)(\theta_3)^{1/2} + \varepsilon$. We let $\theta_1 = (1+\varepsilon)\,\theta_3$ and $\theta_0 = \rho\theta_1$, where ρ is a constant > 1. The theory above then implies that we have a solution for x in the interval $[0, \delta_0]$, where

$$\delta_0 = \min \{\eta, [(1+\varepsilon)^{1/2} \rho^{1/2} C\, M(\theta_3)\, (\theta_3)^{1/2}]^{-1}\}, \qquad (46)$$

where $C = C(\rho) = 6\, e\, C_2(0,2,\rho)$, and the constant C_2 is the explicit constant of Lemma 2. In fact, $C_2(0,2,\rho) = 3^4 \rho\, (e \ln \rho)^{-1}$. We choose $\rho = e^{2/3}$, which happens to minimize $\rho^{1/2} C(\rho)$, the minimum being $3^6 e$. Our claim now follows from equation (46), because $\varepsilon > 0$ is arbitrary.

We note that the interval $[0,\mu)$, where μ is given by equation (45), is not necessarily the largest interval of existence, because we have not always chosen the best possible constants in our theory. However, in many important cases estimate (45) is more than sufficient for our purposes. Suppose, for instance, that the functions b_2 and a_4 are independent of t and that estimates of the form (39), (40) and (41) hold for all $\theta > 0$. It follows that the function $M(\theta)$ is just a constant and that $M^* = 0$, the infimum being approached for $\theta \downarrow 0$. Further, if η can be chosen to be arbitrarily large, it then follows that the solution exists for x in the interval $[0,\infty)$. This is the case for the situation examined in Chapter 1.

We now summarize the main results of the chapter. They follow easily from the results proved above and the discussion of Section 3.0.

Theorem 16

Let $T > 0$, $0 < \delta < T$ and let $\mathsf{K} = [T-\delta, T]$ and $\mathsf{R} = [0,\eta] \times [0,\eta']$. We make the following assumptions:

(i) The kernels K_1 and K_0 of equation (2) are infinitely differentiable with respect to t in the topology of $D^{0,1}(\mathsf{R})$ (for t in a neighborhood of K) and there exist constants $\theta > 0$ and $Q \geq 0$ such that

$$\left|\frac{\partial^n K_j}{\partial t^n}(.,.,t)\right|_{0,1} \leq Q\,(n!)^2\,\theta^n \qquad \text{for } t \in \mathsf{K}, \quad j \in \{0,1\} \quad \text{and} \quad n \geq 0.$$

(ii) The functions $a_0(x,t)$, $a_1(x,t)$, $a_2(x,t)$, $a_3(x,t)$, $a_4(x,t)$, $b_0(x,t)$, $b_1(x,t)$ and $b_2(x,t)$ are all infinitely differentiable with respect to t in the topology of $C[0,\eta]$ (for t in a neighborhood of K). Also, there are constants $M > 0$, $N \geq 0$ such that for $(x,t) \in [0,\eta] \times \mathsf{K}$,

$$\left|\frac{\partial^n}{\partial t^n}[b_2(x,t)/a_4(x,t)]\right| \leq M\,\theta^n\,(n!)^2,$$

$$\left|\frac{\partial^n}{\partial t^n}[1/a_4(x,t)]\right| \leq M\,\theta^n\,(n!)^2,$$

$$\left|\frac{\partial^n}{\partial t^n}[a_j(x,t)/a_4(x,t)]\right| \leq N\,\theta^n\,(n!)^2, \quad \text{for } j \in \{0,1,2,3\},$$

$$\left|\frac{\partial^n}{\partial t^n}[b_j(x,t)/a_4(x,t)]\right| \leq N\,\theta^n\,(n!)^2, \quad \text{for } j \in \{0,1\}.$$

Let θ_1 be any constant $> \theta$, and let $\theta_0 = e^{2/3}\,\theta_1$. If we set

$$\delta_0 = \min\{\eta, [3^6\,M\,e\,(\theta_1)^{1/2}]^{-1}\},$$

then for $(x,a) \in \mathsf{R}_0 = [0, \delta_0] \times [0,\eta']$ there exist kernels $\tilde{K}_0(x,a,t)$ and $\tilde{K}_1(x,a,t)$ such that:

(a) If $t \geq T$, then $\tilde{K}_0(x,a,t) = \tilde{K}_1(x,a,t) = 0$.

(b) If $t \leq T - \delta$, then $\tilde{K}_0(x,a,t) = K_0(x,a,t)$ and $\tilde{K}_1(x,a,t) = K_1(x,a,t)$.

(c) For t in a neighborhood of K, the functions $\tilde{K}_0$ and $\tilde{K}_1$ are infinitely differentiable in the topology of $D^{4,1}(\mathsf{R}_0)$, and there is a constant $P > 0$ such that

$$\left| \frac{\partial^n \tilde{K}_j}{\partial t^n}(.,.,t) \right|^{4,1} \le P\,(\theta_0)^n\,(n!)^2, \qquad \text{for } t \in \mathsf{K},\ \ n \ge 0,\ \ j \in \{0,1\}.$$

(d) $$\mathsf{L}\,\tilde{K}_j(x,a,t) = 0 \ \text{ for } (x,a) \in \mathsf{R}_0,\ t > 0 \ \text{ and } \ j \in \{0,1\}.$$

$$\tilde{K}_j(0,a,t) = \frac{\partial \tilde{K}_j}{\partial x}(0,a,t) = 0 \ \text{ for } \ a \in [0,\eta'],\ t > 0,\ \text{ and } \ j \in \{0,1\}.$$

□

The situation of Chapters 1 and 2, for which the coefficients of the partial differential equation are independent of t, deserves special attention. Recall that solutions corresponding to initial data with compact support are given for $t > 0$ by equation (2), where the functions K_0 and K_1 are given by equation (3) for $(x,a,t) \in [0,\infty)^2 \times (0,\infty)$. In this case, we get the following result.

Theorem 17

Suppose that the functions a_0, a_1, a_2, a_3, a_4, b_0, b_1 and b_2 are independent of t and that they satisfy either the assumptions of Theorem 16 in Chapter 1, or the assumptions of Theorem 4 in Chapter 2. Let K be the kernel of the corresponding theorem.

Given constants $T > 0$ and $0 < \delta < T$, there exists a kernel $\tilde{K}$ defined for $(x,a,t) \in [0,\infty)^2 \times (0,\infty)$ such that the following are satisfied:

(a) $t \to \tilde{K}(.,.,t)$ is a member of $\gamma^2((0,\infty),Y,\mathsf{P})$.

(b) For each fixed $a \in [0,\infty)$,

$$\frac{\partial^2 \tilde{K}}{\partial t^2}(x,a,t) + \sum_{j=0}^{4} a_j(x)\,\frac{\partial^j \tilde{K}}{\partial x^j}(x,a,t) + \sum_{j=0}^{2} b_j(x)\,\frac{\partial^{j+1} \tilde{K}}{\partial x^j \partial t}(x,a,t) = 0, \quad \text{for } (x,t) \in (0,\infty)^2.$$

(c) $$\tilde{K}(0,a,t) = \frac{\partial \tilde{K}}{\partial x}(0,a,t) = 0.$$

(d) If $t \ge T$, then $\tilde{K}(x,a,t) = 0$.

(e) If $t \leq T - \delta$, then $\tilde{K}(x,a,t) = K(x,a,t)$.

Proof

We need only observe that Theorem 16 for $K = K_0$ holds with any constants $\theta > 0$, $\eta > 0$ and $\eta' > 0$, and that the constant M does not depend on θ. Instead of constructing $\tilde{K}_1$ as in the previous theorem, we simply let $\tilde{K}_1 = \dfrac{\partial \tilde{K}_0}{\partial t}$.

□

Chapter 4

Applications of the Theory to Exact Boundary Controllability Problems

4.0 Introduction

In this chapter we make use of the theory of Chapters 2 and 3 to solve two control problems involving the simplified SCOLE model mentioned in the Preface. In the first of these, the space shuttle is assumed to be at rest, while the antenna and mast system is initially moving because of some previous maneuver of the shuttle. The control problem

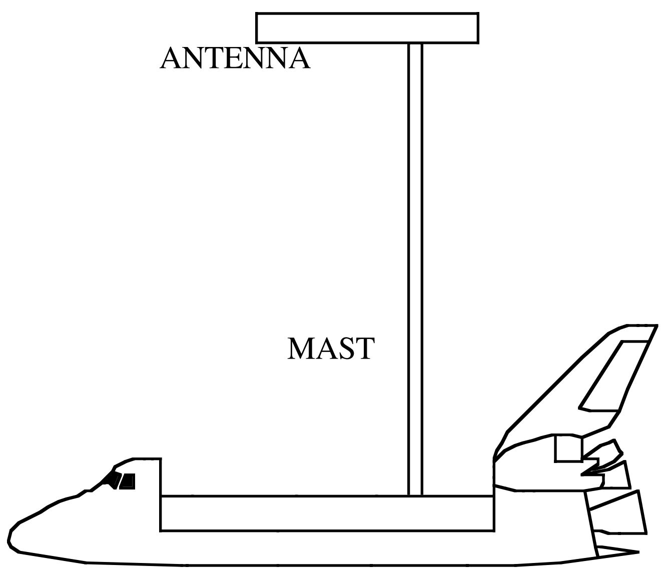

Figure 8

here is to find a force and torque which, when applied only to the antenna, will bring the system to its target rest state. The second control problem occurs for the situation in which the whole system is rotating about its center of mass at an angular velocity which varies with time. In this case, the 'zero state' is no longer a solution of the equations of motion, so we choose another target state. The target state is assumed to be a given solution of the equations of motion (e.g. the 'bent' equilibrium state of the mast for the case of constant angular velocity).

Obviously, the second problem is a generalization of the first. However, the method that we employ (i.e. the use of Transmutation Operators) for the solution of the second problem allows us to obtain most of the results we need from the solution of the first, simpler problem.

We model the system by assuming that the bending of the mast and motion of the antenna occur in a fixed plane. The mast itself is assumed to be an Euler-Bernoulli beam with variable physical characteristics. For the solution of each problem, we find results entirely analogous to the results found by W. Littman and L. Markus who in [17] consider the particular case in which the shuttle is at rest and the mast is modelled as an Euler-Bernoulli beam with constant physical characteristics. The control functions (see Theorems 4 and 13) are continuous at $t = 0$ and are Gevrey regular for $t > 0$. Further, they may be chosen so that the target state is reached after an arbitrarily small time duration. In each case, the response $[w(.,t), \frac{\partial w}{\partial t}(.,t)]$ is a continuous mapping into $H^6(0,d) \times H^4(0,d)$, 'd' being the length of the mast.

We start in Section 4.1 with some details about the model and equations of motion for the first problem. The control problem is explicitly stated there. In Section 4.2, the control problem is solved by applying the results of Chapter 3 (in which it is shown that solutions vanishing for $t > T$ can be obtained from a fundamental solution which vanishes for $t > T$). Much of the theory of this section deals with some regularity results that were not considered in Chapter 1.

In Section 4.3, we discuss the equations of motion for the second control problem. Before we can apply the theory of Chapter 3 to this problem, we must demonstrate the existence of fundamental solutions which are Gevrey regular in the appropriate sense. This does not follow immediately from the theory of Chapter 1, because the partial differential equation that we consider here has certain time dependent coefficients, while the coefficients of the equations considered in Chapter 1 have no time dependence. However, we are very fortunate in this situation, because there exists an invertible mapping (transmutation operator) which maps solutions of our differential equation into solutions of one (the same equation considered in the first control problem) with no time dependent coefficients. Transmutation operators are briefly discussed in Section 4.4, where we also investigate the circumstances under which they preserve Gevrey regularity. Finally, in Section 4.5, we show how to obtain the appropriate fundamental solutions using the transmutation operator and then we apply the theory of Chapter 3 to solve the control problem.

We remark that for practical applications, there is an advantage in solving the control problems by the use of fundamental solutions. This is because once these solutions are known, the required boundary control functions may be quickly calculated by a simple one-dimensional integration. (See the remark after the proof of Theorem 4).

4.1 The Model, Equations of Motion and Control Problem for the Case in which the Shuttle is at Rest

We model the situation described in the Introduction as follows:

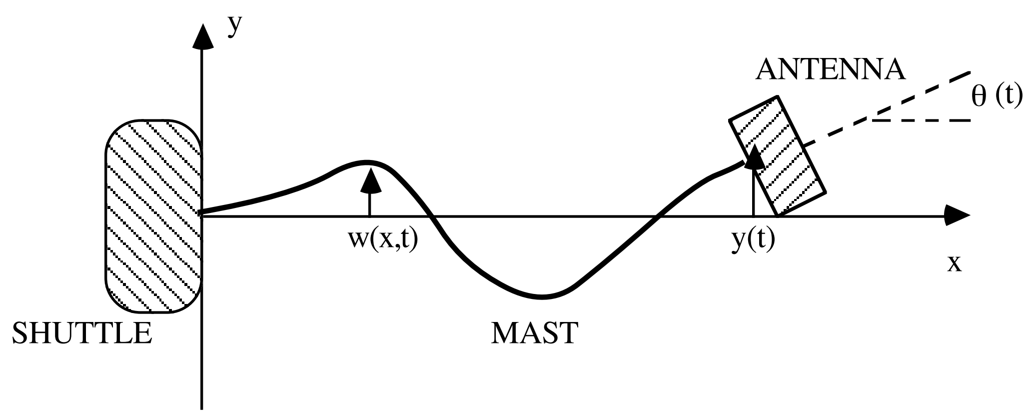

Figure 9

The mast is a beam with physical characteristics which are allowed to vary along its length. We attach a Cartesian coordinate system to the shuttle, which we assume to be so massive, compared with the beam, that the coordinate system can be treated as an inertial reference frame. The directions of all motions are supposed to be in the x - y plane (see Figure 9), while the x - axis coincides with the equilibrium rest state of the beam, which is clamped to the shuttle at $x = 0$. Further, it is assumed that the motion of the beam in the direction of the x - axis is negligible. Let w(x,t) denote the displacement in the direction of the y - axis, at time t, of an element of the beam which has x as its 'x - coordinate'. The beam itself is an Euler - Bernoulli beam (see Rayleigh [19]) of length d and has the equation of motion

$$\rho(x)\frac{\partial^2 w}{\partial t^2}(x,t) \quad + \quad \frac{\partial^2}{\partial x^2}\left[E(x)\ I(x)\ \frac{\partial^2 w}{\partial x^2}(x,t)\right] \ = \ 0. \tag{1}$$

The physical parameters appearing in this equation are:

$\rho(x)$ = mass per unit length of the beam, measured at position x.

E(x) = Young's Modulus of Elasticity, measured at position x.

I(x) = "Moment of inertia of a cross section of the beam. This is measured about an axis parallel with the z - axis, and passing through the neutral curve of the beam (the neutral curve is the curve within the beam along which the filaments of the Euler - Bernoulli beam model do not undergo stretching or compression).

As was mentioned before, the beam is rigidly clamped to the shuttle at $x = 0$. This condition is described mathematically by what we call the 'clamped end conditions':

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0. \tag{2}$$

The antenna is modelled as a rigid body clamped to the other end of the beam (at $x = d$) at its center of mass. We let m denote the mass of the antenna, and J its moment of inertia measured about an axis parallel with the z - axis and passing through the center of mass of the antenna.

For the Euler - Bernoulli beam model, it can be shown that the torque on the antenna, due to the bending of the beam, is equal to $-(E\,I\,\frac{\partial^2 w}{\partial x^2})(d,t)$. If there is also an external (controlling) torque, $f_2(t)$, applied to the antenna, then Newton's law for the angular motion of the antenna is

$$J\,\frac{d^2\theta}{dt^2}(t) = -\,(E\,I\,\frac{\partial^2 w}{\partial x^2})(d,t) + f_2(t). \tag{3}$$

Here θ is the angular deflection of the antenna from the x - axis. In our model, we assume that all displacements are small, so θ is given by

$$\theta(t) = \frac{\partial w}{\partial x}(d,t). \tag{4}$$

The force on the antenna, due to the bending of the beam, is equal to $\frac{\partial}{\partial x}(E\,I\,\frac{\partial^2 w}{\partial x^2})(d,t)$. Thus, taking into account an external (controlling) force $f_1(t)$ applied in the direction of the y - axis, we see that Newton's law of motion for the antenna is

$$m\,\frac{d^2 y}{dt^2}(t) = \frac{\partial}{\partial x}(E\,I\frac{\partial^2 w}{\partial x^2})(d,t) + f_1(t). \tag{5}$$

Here, y(t) denotes the displacement of the antenna in the direction of the y - axis. It is clear that we have

$$y(t) = w(d,t). \tag{6}$$

The state of the mechanical system at any time t is completely described by specifying the pair of functions $(w(.,t)\,,\frac{\partial w}{\partial t}(.,t))$, for if this is known, equations (4) and (6) furnish the values of $\theta(t), \frac{d\theta}{dt}(t)$, $y(t)$ and $\frac{dy}{dt}(t)$.

To complete the description of the evolution of the system, we need only specify the initial conditions

$$w(x,0) = w_0(x), \qquad \frac{\partial w}{\partial t}(x,0) = v_0(x). \tag{7}$$

We now discuss the open - loop boundary control problem for the system. In words, it is as follows: Given an arbitrarily small positive time T and initial data $(w_0(.),v_0(.))$, find a force, $f_1(t)$, and a torque, $f_2(t)$, which, when applied to the antenna, drive the system to rest for times $t \geq T$. Thus, we choose the target state as the stationary equilibrium rest state $w(x,t) = 0, \frac{\partial w}{\partial t}(x,t) = 0$.

Specifically, we must find functions f_1 and f_2 such that the solution w of the following mixed problem, which we call 'Problem A', vanishes for all times $t \geq T$:

Problem A

$$\rho(x)\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left[E(x)\, I(x)\, \frac{\partial^2 w}{\partial x^2}(x,t)\right] = 0, \qquad \text{for } (x,t) \in (0,d) \times (0,\infty). \tag{1}$$

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0, \qquad \text{for } t \geq 0. \tag{2}$$

$$w(x,0) = w_0(x), \qquad \frac{\partial w}{\partial t}(x,0) = v_0(x), \qquad \text{for } 0 \leq x \leq d. \tag{7}$$

$$B_1 w(d,t) = m\,\frac{\partial^2 w}{\partial t^2}(d,t) - \frac{\partial}{\partial x}\left[E\, I\, \frac{\partial^2 w}{\partial x^2}\right](d,t) = f_1(t), \qquad \text{for } t > 0. \tag{8}$$

$$B_2 w(d,t) = J\,\frac{\partial^3 w}{\partial t^2 \partial x}(d,t) + E\, I\, \frac{\partial^2 w}{\partial x^2}(d,t) = f_2(t), \qquad \text{for } t > 0. \tag{9}$$

4.2 Solution of the Control Problem for the Case in which the Shuttle is at Rest

We now use the theory of Chapters 1 and 3 to solve the control problem. For convenience, we set $\alpha(x) = \rho(x)^{-1}$ and $\beta(x) = E(x)\, I(x)$. We assume that α and β are members of $C^4[0,d]$ and we continue them to $C^4[0,\infty)$ functions in such a way that $\alpha(x) = \beta(x) = 1$ for all x sufficiently large, and $\alpha(x)\, \beta(x) > 0$ for all $x \in [0,\infty)$. Equation (1) may, of course, be written in the form

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \alpha(x)\, \beta(x)\, \frac{\partial^4 w}{\partial x^4}(x,t) + 2\alpha(x)\, \beta'(x)\, \frac{\partial^3 w}{\partial x^3}(x,t) + \alpha(x)\, \beta''(x)\, \frac{\partial^2 w}{\partial x^2}(x,t). \tag{10}$$

The coefficients of this equation satisfy assumptions (I), (II) and (III) of Section 1.8. Thus, by the theory of that section, there exists a strongly continuous semigroup S(t) on $\mathsf{H} = H_0^2(\mathbf{R}^+) \times L^2(\mathbf{R}^+)$. The infinitesimal generator of this semigroup is an unbounded operator B, given by:

$$D_B = \{H^4(\mathbf{R}^+) \cap H_0^2(\mathbf{R}^+)\} \times L^2(\mathbf{R}^+),$$

$$B\begin{bmatrix} w \\ v \end{bmatrix}(x) = \begin{bmatrix} v(x) \\ -\alpha(x)\dfrac{\partial^2}{\partial x^2}(\beta(x)\, w(x)) \end{bmatrix}. \tag{11}$$

Further, Theorem 16 of Chapter 1 implies that there exists a kernel K defined on $[0,\infty)^3$ with the properties:

(i) $t \to K(.,.,t)$ is a member of $\gamma^2((0,\infty),Y,\mathsf{P})$.

(ii) For each $a \in [0,\infty)$, K(x,a,t) satisfies equation (1) as a function of (x,t).

(iii) The operator $\mathsf{K}(t)$, defined for $t \geq 0$ by

$$\mathsf{K}(t)\, f(x) = \int_0^\infty K(x, a, t)\, f(a)\, da,$$

satisfies, for $t > 0$,

$$S(t)\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \mathsf{K}'(t) & \mathsf{K}(t) \\ \mathsf{K}''(t) & \mathsf{K}'(t) \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

provided that $f_1 \in H_0^2(\mathbf{R}^+)$ and $f_2 \in L^2(\mathbf{R}^+)$ have compact support.

Given $T > 0$ and $0 < \delta < T$, we let $\tilde{K}$ be the kernel of Theorem 17 in Chapter 3. $\tilde{K}$ satisfies (i) and (ii) above. Moreover, $\tilde{K}(x,a,t)$ vanishes for $t \geq T$ and coincides with K(x,a,t) for $t \leq \delta$. We extend the domain of our initial data (w_0,v_0) so that it has compact support in $[0,\infty)$ and we tentatively set:

$$w(x,t) = \int_0^\infty \tilde{K}(x, a, t)\, v_0(a)\, da + \int_0^\infty \frac{\partial \tilde{K}}{\partial t}(x, a, t)\, w_0(a)\, da. \tag{12}$$

Under suitable regularity assumptions for w_0 and v_0 we will see that equation (12) provides us with a solution of the control problem, in the sense that w(x,t) takes on the initial conditions, satisfies equation (1) and the clamped end conditions, and vanishes for $t \geq T$. The boundary control functions are given by

$$f_1(t) = B_1 w(d,t), \qquad f_2(t) = B_2 w(d,t).$$

The 'suitable regularity assumptions' are required so that the boundary operators can be applied to the function w for all $t \geq 0$, and so that we indeed get a classical solution to Problem A. The following theorem provides us with the necessary information.

Theorem 1

Let S(t) be the strongly continuous semigroup defined by equation (110) of Chapter 1 and let B be the infinitesimal generator of S(t) (see equation (111) of Chapter 1). Suppose that, in addition to satisfying assumptions (I) and (II) of Chapter 1, the coefficients of B are all twice continuously differentiable on $[0,\infty)$. Then the following statements hold:

(i) $D_{B^2} = \{(w,v) \in H^6 \times H^4 \colon w(0) = w'(0) = v(0) = v'(0) = 0,$

$$\sum_{i=0}^{4} a_i(0)\, w^{(i)}(0) + \sum_{i=0}^{2} b_i(0)\, v^{(i)}(0) = 0,$$

$$\sum_{i=0}^{4} a_i'(0)\, w^{(i)}(0) + a_i(0)\, w^{(i+1)}(0) + \sum_{i=0}^{2} b_i'(0)\, v^{(i)}(0) + b_i(0)\, v^{(i+1)}(0) = 0\}.$$

Further, if $\begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \in D_{B^2}$ and $\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = S(t) \begin{bmatrix} w_0 \\ v_0 \end{bmatrix}$, then:

(ii) The mapping $t \to [w(.,t),v(.,t)]$ is a continuous mapping from $[0,\infty)$ into $H^6 \times H^4$ which is continuously differentiable in the norm topology of $H^4 \times H^2$ and twice continuously differentiable in the norm topology of $H^2 \times L^2$.

(iii) $v = \dfrac{\partial w}{\partial t}$.

(iv) $\dfrac{\partial^{i+j}w}{\partial t^i \partial x^j}$ exists and is a continuous function of (x,t) for all i, j satisfying $j+2i \le 5$. Further, the mixed derivatives may be taken in any order.

(v) w satisfies the 'clamped end conditions' and the differential equation

$$\frac{\partial^2 w}{\partial t^2} + \sum_{i=0}^{4} a_i \frac{\partial^i w}{\partial x^i} + \sum_{i=0}^{2} b_i \frac{\partial^{i+1} w}{\partial x^i \partial t} = 0.$$

Proof

For (i), we recall that

$$D_{B^2} = \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \in D_B : \; B \begin{bmatrix} w \\ v \end{bmatrix} \in D_B \right\}.$$

But if $\begin{bmatrix} w \\ v \end{bmatrix} \in D_B = (H^4 \cap H_0^2) \times H_0^2$, then $B \begin{bmatrix} w \\ v \end{bmatrix} \in D_B$ is equivalent to:

$$v \in H^4 \cap H_0^2, \qquad \sum_{i=0}^{4} a_i w^{(i)} + \sum_{i=0}^{2} b_i v^{(i)} \in H_0^2 .$$

Thus, it is easy to see that D_{B^2} contains the given set. For the opposite inclusion, suppose that $\begin{bmatrix} w \\ v \end{bmatrix} \in D_{B^2}$. From this, it is easy to see that $w^{(4)}$ is twice weakly differentiable and that $w^{(5)}$ and $w^{(6)}$ are in L^2. Thus $w \in H^6$. The rest of (i) follows easily.

A substantial part of (ii) will follow from semigroup theory. We first recall that, since S(t) is strongly continuous, the mapping $t \to S(t) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is a continuous mapping from $[0,\infty)$ into $\mathsf{H} = H_0^2 \times L^2$ for any $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathsf{H}$. Further, if $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in D_B$ then the mapping is even continuously differentiable in $H_0^2 \times L^2$ and

$$\frac{d}{dt} S(t) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = B\, S(t) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = S(t)\, B \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} .$$

Now for fixed $\begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \in D_{B^2}$, we let ϕ denote the mapping $t \to S(t)\begin{bmatrix} w_0 \\ v_0 \end{bmatrix}$. Since S(t) carries D_{B^n} into D_{B^n}, we know from (i) that the range of ϕ is contained in $H^6 \times H^4$. Further, ϕ is continuously differentiable in the topology of H and we have

$$\phi'(t) = \frac{d}{dt}\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = B\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = S(t)\, B\begin{bmatrix} w_0 \\ v_0 \end{bmatrix}. \tag{13}$$

The range of ϕ' is contained in $D_B = (H^4 \cap H_0^2) \times H_0^2$. Similarly, ϕ' is continuously differentiable in the topology of H and we have

$$\phi''(t) = \frac{d^2}{dt^2}\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = \frac{d}{dt} B\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = B\frac{d}{dt}\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = B^2\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = S(t)\, B^2\begin{bmatrix} w_0 \\ v_0 \end{bmatrix}. \tag{14}$$

It remains to show that f is continuous in the topology of $H^6 \times H^4$ and continuously differentiable in the topology of $H^4 \times H^2$. For the latter assertion, we observe from equations (13) that the difference quotient $\frac{1}{h}(B\phi(t+h) - B\phi(t))$ converges in $H^2 \times L^2$ to $\phi''(t)$ as $h \to 0$. But

$$B\,\phi(t) = \begin{bmatrix} v(.,t) \\ -\sum_{i=0}^{4} a_i \frac{\partial^i w}{\partial x^i}(.,t) - \sum_{i=0}^{2} b_i \frac{\partial^i v}{\partial x^i}(.,t) \end{bmatrix}.$$

Thus the difference quotient $\frac{1}{h}(v(.,t+h) - v(.,t))$ converges in H^2. Rewriting

$$\sum_{i=0}^{4} a_i \frac{\partial^i f}{\partial x^i} \qquad \text{as} \qquad a_4 \exp\left[-\frac{1}{4}\int \frac{a_3}{a_4}\right] \frac{\partial^4}{\partial x^4}\left\{\exp\left[\frac{1}{4}\int \frac{a_3}{a_4}\right] f\right\} + \mathsf{l}\, f,$$

where l is a second order differential expression with continuous coefficients, shows us immediately that the difference quotient

$$\frac{1}{h}\left\{\frac{\partial^4}{\partial x^4}\exp\left[\frac{1}{4}\int \frac{a_3}{a_4}\right](w(.,t+h) - w(.,t))\right\}$$

converges in L^2 to a continuous L^2-valued function as $h \to 0$. But we know already that the difference quotient $\frac{1}{h}(w(.,t+h) - w(.,t))$ converges in H^2 to a continuous H^2-valued function. So it follows from the interpolation inequality (inequality (36) of Chapter 1) that

$$\frac{1}{h}\left\{\frac{\partial^3}{\partial x^3}\exp\left[\frac{1}{4}\int\frac{a_3}{a_4}\right](w(.,t+h) - w(.,t))\right\}$$

converges in L^2. Thus

$$\frac{1}{h}\left\{\frac{\partial^2}{\partial x^2}\exp\left[\frac{1}{4}\int\frac{a_3}{a_4}\right](w(.,t+h) - w(.,t))\right\}$$

converges in H^2 and we easily see from this that $\frac{1}{h}(w(.,t+h) - w(.,t))$ converges in H^4 to a continuous H^4-valued function. Thus we have proved the assertion that ϕ is continuously differentiable in $H^4 \times H^2$.

To show that ϕ is continuous in the topology of $H^6 \times H^4$, we again use equations (14). Note that

$$B^2\phi = B^2\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = \begin{bmatrix} -\sum_{i=0}^{4} a_i \frac{\partial^i w}{\partial x^i} - \sum_{i=0}^{2} b_i \frac{\partial^i v}{\partial x^i} \\ -\sum_{i=0}^{4} a_i \frac{\partial^i v}{\partial x^i} + \sum_{i=0}^{2} b_i \frac{\partial^i}{\partial x^i}\left\{\sum_{j=0}^{4} a_j \frac{\partial^j w}{\partial x^j} + \sum_{j=0}^{2} b_j \frac{\partial^j v}{\partial x^j}\right\} \end{bmatrix}. \tag{15}$$

The first component of this varies continuously in H_0^2, while the second varies continuously in L^2. Thus $\sum_{i=0}^{4} a_i \frac{\partial^i v}{\partial x^i}$ varies continuously in L^2 and, by the interpolation argument used earlier, $t \to v(.,t)$ is a continuous mapping from $[0,\infty)$ into H^4. Now if we consider again the first component of equation (15), we see that $\sum_{i=0}^{4} a_i \frac{\partial^i w}{\partial x^i}$ varies continuously in H^2. So, once again, we may use the interpolation inequality argument to conclude that $t \to w(.,t)$ is a continuous mapping from $[0,\infty)$ into H^6. Thus, all of (ii) has been proved.

The result (iii) follows even for $\begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \in D_B$, for in H we have

$$\frac{d}{dt}\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = B\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = \begin{bmatrix} v(.,t) \\ -\sum_{i=0}^{4} a_i \frac{\partial^i w}{\partial x^i}(.,t) - \sum_{i=0}^{2} b_i \frac{\partial^i v}{\partial x^i}(.,t) \end{bmatrix}.$$

Statement (iv) follows from (ii) and the Sobolev embedding theorems. Indeed, (ii) implies that the mapping $t \to \begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix}$ is continuous in the topology of $C^{5,1/2}(K) \times C^{3,1/2}(K)$ for any compact set K. Further, the mapping is differentiable in the topology of $C^{3,1/2}(K) \times C^{1,1/2}(K)$ and the first component of it is twice differentiable in the topology of $C^{1,1/2}(K)$.

The function w satisfies more than just the 'clamped end' conditions at $x = 0$, since $\begin{bmatrix} w \\ v \end{bmatrix}$ remains in D_{B^2} (see (i)). To prove the rest of part (v), we merely observe that it is the first component of the equation

$$\frac{d^2}{dt^2}\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix} = B^2\begin{bmatrix} w(.,t) \\ v(.,t) \end{bmatrix}.$$

□

We note that the assumptions which we have already made about our Euler-Bernoulli beam allow us to apply Theorem 1 to obtain results about the solutions when $(w_0,v_0) \in D_{B^2}$. In particular, Theorem 1 shows that in this way we obtain classical solutions of the beam problem.

Before proceeding with the control problem, we state another regularity result which we obtain from the smoothness of the coefficients α and β.

Lemma 2

Let K be any compact subset of $[0,\infty)^2$. Then the kernels K(.,.,t) and $\tilde{K}$(.,.,t) are members of $\gamma^2((0,\infty); D^{6,1}(K))$.

Proof

We know already that the kernels are members of $\gamma^2((0,\infty); D^{4,1}(K))$. We extend this result by observing that they satisfy the equation

$$\frac{\partial^2 k}{\partial t^2} + \alpha \frac{\partial^2}{\partial x^2}\left\{\beta \frac{\partial^2 k}{\partial x^2}\right\} = 0,$$

where α and β are in $C^4[0,\infty)$ with $\alpha\beta > 0$.

□

Remark: We see from Theorem 1 that if $(w_0,v_0) \in D_B2$ with compact support, then the mapping

$$t \to \int_0^\infty w_0(a) \frac{\partial K}{\partial t}(.,a,t)\, da + \int_0^\infty v_0(a)\, K(.,a,t)\, da$$

is continuous in the topology of $H^6(\mathbf{R}^+)$, continuously differentiable in the topology of $H^4(\mathbf{R}^+)$ and twice continuously differentiable in the topology of $H^2(\mathbf{R}^+)$. Lemma 2 implies that the same can be said about the mapping

$$t \to \int_0^\infty w_0(a) \frac{\partial \tilde{K}}{\partial t}(.,a,t)\, da + \int_0^\infty v_0(a)\, \tilde{K}(.,a,t)\, da,$$

if we replace the Sobolev spaces on $\mathbf{R}^+$ with the corresponding spaces on any finite interval of the form $(0,\delta)$.

We now return to the control problem. Our first result is a uniqueness result obtained from energy considerations.

Definition: The *energy*, E of a solution w of Problem A is

$$\mathsf{E}(t) = \frac{1}{2}\int_0^d \rho(x)\left[\frac{\partial w}{\partial t}(x,t)\right]^2 + E(x)\,I(x)\left[\frac{\partial^2 w}{\partial x^2}(x,t)\right]^2 dx + \frac{1}{2}\,m\left[\frac{dy}{dt}(t)\right]^2 + \frac{1}{2}\,J\left[\frac{d\theta}{dt}(t)\right]^2 .$$

This is recognizable from a physical point of view as the total mechanical energy (kinetic + elastic potential) of the system.

Lemma 3

Let w be a solution of Problem A having the regularity properties of Theorem 1 (i.e. the mapping $t \to w(.,t)$ is continuous in the topology of $H^6(0,d)$, continuously differentiable in the topology of $H^4(0,d)$ and twice continuously differentiable in the topology of $H^2(0,d)$). Then:

(i)
$$\frac{d\mathsf{E}}{dt} = f_1(t)\,\frac{dy}{dt} + f_2(t)\,\frac{d\theta}{dt} .$$

(ii) If $\tilde{w}$ is any other such solution of Problem A with the same boundary functions f_1 and f_2, then $w = \tilde{w}$.

Proof

The regularity of w allows us to differentiate E and we obtain (i) by an integration by parts.

The second statement of the lemma is proved by letting $u(x,t) = w(x,t) - \tilde{w}(x,t)$. The function u is thus a solution of Problem A with zero boundary data ($f_1 = f_2 = 0$). Thus, by (i), its energy is $\mathsf{E}(t) = \mathsf{E}(0) = 0$. It is easy to see that this implies that $u(x,t) = 0$.

□

We are now ready to state the boundary controllability result for Problem A.

Theorem 4

Consider the boundary controllability of Problem A. Let $(w_0,v_0) \in H^6(0,d) \times H^4(0,d)$ satisfy the 'clamped end conditions'

$$w_0(0) = w_0'(0) = 0$$

and the 'compatibility conditions'

$$v_0(0) = v_0'(0) = (\beta\, w_0'')''(0) = \{\alpha\, (\beta\, w_0'')''\}'(0) = 0,$$

where $\alpha = 1/\rho$ and $\beta = E\,I$ are members of $C^4[0,d]$ and $\alpha(x) > 0$, $\beta(x) > 0$ for all $x \in [0,d]$.

Then, given $T > 0$, there exist boundary control functions $f_1: [0,\infty) \rightarrow \mathbf{R}$ and $f_2: [0,\infty) \rightarrow \mathbf{R}$, which are continuous on $[0,\infty)$ and are members of $\gamma^2(0,\infty)$, such that Problem A has a solution w with the properties:

(i) For $t \geq 0$, $t \rightarrow w(.,t)$ is continuous in $H^6(0,d)$, continuously differentiable in $H^4(0,d)$ and twice continuously differentiable in $H^2(0,d)$. Moreover, w is unique in the class of solutions of Problem A with these regularity properties.

(ii) $w(.,t) = 0$ for all $t \geq T$.

Proof

First, we extend the domain of the initial data so that the extension is in $H^6(\mathbf{R}^+) \times H^4(\mathbf{R}^+)$ and has compact support. By Theorem 1, (w_0,v_0) is now in D_{B^2}. We let $\tilde{K}$ be the kernel of Theorem 17 in Chapter 3 and set

$$w(x,t) = \int_0^\infty \tilde{K}(x,a,t)\, v_0(a)\, da + \int_0^\infty \frac{\partial \tilde{K}}{\partial t}(x,a,t)\, w_0(a)\, da.$$

According to the remark following Lemma 2, for $t \geq 0$, $t \rightarrow w(.,t)$ is continuous in $H^6(0,\delta)$, continuously differentiable in $H^4(0,\delta)$ and twice continuously differentiable in $H^2(0,\delta)$ for any $\delta > 0$. Also, $w(.,t)$ vanishes for $t \geq T$ because $\tilde{K}(.,.,t)$ does.

We set $f_1(t) = B_1 w(d,t)$ and $f_2(t) = B_2 w(d,t)$. It is clear that f_1 and f_2 are in $\gamma^2(0,\infty)$ because w is in $\gamma^2((0,\infty); C^6[0,\delta])$ for any $\delta > 0$ (see Lemma 2). Moreover, because of the regularity of the mapping $t \rightarrow w(.,t)$ for $t \geq 0$, it is easy to see that the Sobolev embedding theorems imply that f_1 and f_2 are continuous on the interval $[0,\infty)$.

Finally, the uniqueness of w has already been proved in Lemma 3.

□

Remark: For practical applications, it is of interest that we can give the control functions explicitly in terms of the initial data. To see this, let $d' > d$ and let E_1: $H^6(0,d) \rightarrow H^6(\mathbf{R}^+)$ and E_2: $H^4(0,d) \rightarrow H^4(\mathbf{R}^+)$ be any extension operators having ranges which are spaces of functions with support in $[0,d']$. The response w is given by

$$w(x,t) = \int_0^\infty \tilde{K}(x,a,t)\, E_2[v_0(a)]\, da + \int_0^\infty \frac{\partial \tilde{K}}{\partial t}(x,a,t)\, E_1[w_0(a)]\, da,$$

so the boundary control functions are given by

$$f_j(t) = \int_0^\infty B_j \tilde{K}(d,a,t)\, E_2[v_0(a)]\, da + \int_0^\infty B_j \frac{\partial \tilde{K}}{\partial t}(d,a,t)\, E_1[w_0(a)]\, da, \quad \text{for } j \in \{1,2\}. \tag{16}$$

This is perhaps useful for computational purposes, for the four functions $B_1\tilde{K}(d,a,t)$, $B_2\tilde{K}(d,a,t)$, $\frac{\partial}{\partial t} B_1\tilde{K}(d,a,t)$ and $\frac{\partial}{\partial t} B_2\tilde{K}(d,a,t)$, which are functions of only the two variables 'a' and 't', need only be computed once. Thereafter, the control functions can be calculated for any initial data by the integration in equation (16).

4.3 The Equations of Motion and Control Problem For a Rotating Shuttle

We now suppose that the system is rotating about an axis which is parallel with the z-axis and passes through the system's center of mass. We let (-h,-k) be the x - y coordinates of the center of mass and assume that the shuttle is so massive, compared with the mast and antenna, that h and k can be taken as constants. We let $d\phi/dt$ denote the angular velocity of the shuttle and allow it to vary with time.

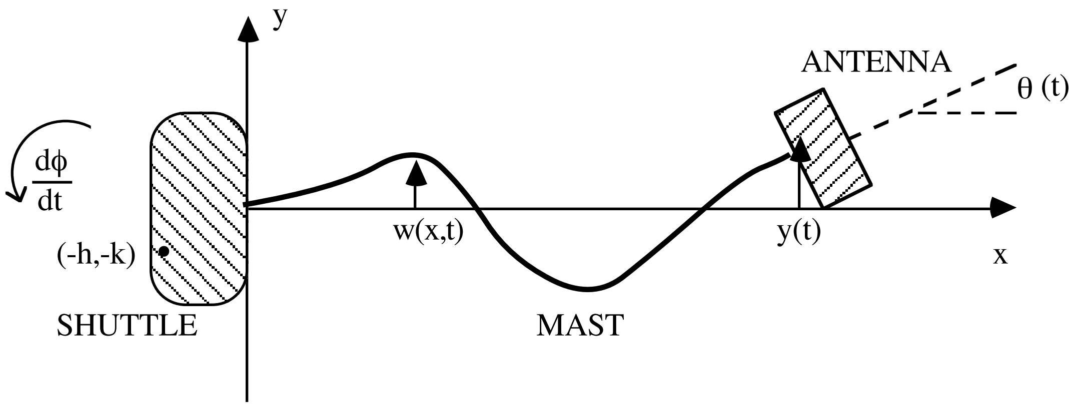

Figure 10

We use the same Cartesian coordinate system attached to the shuttle that we used in the absence of rotation. However, it is no longer an inertial reference frame because of the rotation. Consequently, there are additional force and torque terms in the equations of motion. One can show that the modified equations of motion become (see Littman and Markus [17])

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right] - \rho\,(w+k)\left[\frac{d\phi}{dt}\right]^2 + \rho\,(x+h)\frac{d^2\phi}{dt^2} = 0. \tag{17}$$

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0. \tag{18}$$

$$w(d,t) = y(t), \qquad \frac{\partial w}{\partial x}(d,t) = \theta(t). \tag{19}$$

$$m\frac{d^2 y}{dt^2} = \frac{\partial}{\partial x}\left[EI\frac{\partial^2 w}{\partial x^2}\right](d,t) + m\,(y+k)\left[\frac{d\phi}{dt}\right]^2 - m\,(d+h)\frac{d^2\phi}{dt^2} + f_1(t). \tag{20}$$

$$J\frac{d^2\theta}{dt^2} = -EI\frac{\partial^2 w}{\partial x^2}(d,t) - J\frac{d^2\phi}{dt^2} + f_2(t). \tag{21}$$

$$w(x,0) = w_0(x), \qquad v(x,0) = v_0(x). \tag{22}$$

We choose for the target state a particular solution W of equations (17) - (21). It may be the case that a non-zero force $F_1(t)$ and/or torque $F_2(t)$ are necessary to sustain the solution W, or if desired, the target state may be chosen to be a solution corresponding to no force or torque being applied to the antenna.

In either case, if we denote the solution of equations (17) - (22) by $\tilde{w}$ and set

$$w(x,t) = \tilde{w}(x,t) - W(x,t),$$

we see that the control problem reduces to steering w(x,t) to zero. More precisely, given $T > 0$, we look for functions f_1 and f_2 such that the solution w(x,t) of the following mixed problem, which we call 'Problem B', vanishes for $t \geq T$:

Problem B

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI \frac{\partial^2 w}{\partial x^2}\right] - \rho w \left[\frac{d\phi}{dt}\right]^2 = 0. \tag{23}$$

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0, \qquad \text{for } t \geq 0. \tag{24}$$

$$w(x,0) = w_0(x), \qquad \frac{\partial w}{\partial t}(x,0) = v_0(x), \qquad \text{for } 0 \leq x \leq d. \tag{25}$$

$$B_1 w(d,t) = m \frac{\partial^2 w}{\partial t^2}(d,t) - \frac{\partial}{\partial x}\left[E I \frac{\partial^2 w}{\partial x^2}\right](d,t) - m\, w(d,t)\left[\frac{d\phi}{dt}\right]^2 = f_1(t), \qquad \text{for } t > 0. \tag{26}$$

$$B_2 w(d,t) = J \frac{\partial^3 w}{\partial t^2 \partial x}(d,t) + E I \frac{\partial^2 w}{\partial x^2}(d,t) = f_2(t), \qquad \text{for } t > 0. \tag{27}$$

This system has time-dependent coefficients, so we cannot use the theory of Chapters 1 and 2 to solve the control problem directly. We digress to discuss 'transmutation operators', a tool which will allow us to use, in this time-dependent setting, our results already found for systems which have no time-dependent coefficients.

4.4 Digression: Transmutation Operators

We remark that the only difference between equations (10) and (23) is that to obtain (23) from (10), we replace the differential expression $\frac{\partial^2}{\partial t^2}$ by the differential expression $\frac{\partial^2}{\partial t^2} + q(t)$, where $q(t) = -(d\phi/dt)^2$. It is natural to ask if there exist transformations which will map solutions of equation (10) into solutions of equation (23) and vice versa.

We will see that there is an affirmative answer to this question. In fact, the type of transformation we need has been known of since the 1930's (see Delsarte [5]). The transformations are called *Transmutation Operators,* and have been studied by many researchers. For our purposes it suffices to refer to the work of Lions [14], [15], but for further references, the reader may wish to see the recent report by Trimèche [24] and its bibliography.

We let $\mathsf{U} = (-r,r)$ where $r > 0$ (possibly equal to infinity), and we suppose at first that $q \in C^1(\mathsf{U})$. Let L_1 and L_2 be the differential operators on $C(\mathsf{U})$, with domains $C^2(\mathsf{U})$, given by:

$$\mathsf{L}_1 f = f'', \qquad \mathsf{L}_2 f = f'' + q. \tag{28}$$

We seek an invertible operator B on $C(\mathsf{U})$ with the properties:

$$\mathsf{L}_2 \mathsf{B} = \mathsf{B} \mathsf{L}_1, \tag{29}$$

$$(\mathsf{B} f)(0) = f(0), \qquad (\mathsf{B} f)'(0) = f'(0). \tag{30}$$

The existence of B is guaranteed by the following theorem, which also contains some of the properties of the transmutation operator. To see how such theorems are proved, the reader is referred to Lions [14], [15].

Theorem 5

If $q \in C^1(\mathsf{U})$ then there exists a function $b \in C^2(\mathsf{U} \times \mathsf{U})$ such that the integral operator

$$(\mathsf{B} f)(t) = f(t) + \int_{-t}^{t} b(s,t)\, f(s)\, ds \tag{31}$$

satisfies equations (29) and (30). In fact, b is the unique solution of the problem:

$$\frac{\partial^2 b}{\partial t^2}(s,t) + q(t)\, b(s,t) - \frac{\partial^2 b}{\partial s^2}(s,t) = 0, \qquad \text{for } (s,t) \in \mathsf{U} \times \mathsf{U}, \tag{32}$$

$$b(t,t) = -\frac{1}{2}\int_{0}^{t} q(\tau)\, d\tau, \qquad b(-t,t) = 0, \qquad \text{for } t \in \mathsf{U}. \tag{33}$$

Further, B^{-1} exists and is given by

$$(\mathsf{B}^{-1} f)(t) = f(t) + \int_{-t}^{t} c(s,t)\, f(s)\, ds, \tag{34}$$

where $c \in C^2(\mathsf{U} \times \mathsf{U})$ is the unique solution of the problem:

$$\frac{\partial^2 c}{\partial t^2}(s,t) - q(s)\, c(s,t) - \frac{\partial^2 c}{\partial s^2}(s,t) = 0, \qquad \text{for } (s,t) \in \mathsf{U} \times \mathsf{U},$$

$$c(t,t) = \frac{1}{2}\int_{0}^{t} q(\tau)\, d\tau, \qquad c(-t,t) = 0, \qquad \text{for } t \in \mathsf{U}.$$

□

The case of interest to us is that for which $q \in \gamma^2(\mathsf{U})$. In this case, one would hope that the transmutation operator preserves Gevrey Class 2 regularity. We begin to investigate this possibility with the following lemma.

Lemma 6

Let $q \in \gamma^2(\mathsf{U})$. Then the functions b and c are members of $\gamma^2(\mathsf{U} \times \mathsf{U})$.

Proof

We prove only that $b \in \gamma^2(\mathsf{U} \times \mathsf{U})$, for the proof that $c \in \gamma^2(\mathsf{U} \times \mathsf{U})$ is virtually identical.

First, we change variables as follows:

$$\eta = s + t, \qquad \xi = t - s, \qquad \phi(\eta,\xi) = b(s,t). \tag{35}$$

Under this transformation, the set $\mathsf{U} \times \mathsf{U}$ is mapped on to

$$\mathsf{S} = \{(\eta,\xi) : |\eta| + |\xi| < 2r\}. \tag{36}$$

Since $b \in C^2(\mathsf{U} \times \mathsf{U})$ and satisfies equations (32) and (33), it is apparent that $\phi \in C^2(\mathsf{S})$ and satisfies in S the equations:

$$4\,\frac{\partial^2\phi}{\partial\eta\partial\xi}(\eta,\xi) \;+\; q([\eta+\xi]/2)\,\phi(\eta,\xi) \;=\; 0, \tag{37}$$

$$\phi(0,\xi) = 0, \qquad\qquad \phi(0,\eta) = -\frac{1}{2}\int_0^{\eta/2} q(\tau)\,d\tau. \tag{38}$$

On setting $\psi = \dfrac{\partial\phi}{\partial\eta}$, we see that ϕ and ψ must satisfy the integral equations†

$$\psi(\eta,\xi) = -\frac{1}{4}\,q(\eta/2) \;-\frac{1}{4}\int_0^{\xi} q([\eta+\tau]/2)\,\phi(\eta,\tau)\,d\tau, \tag{39}$$

$$\phi(\eta,\xi) = \int_0^{\eta} \psi(\tau,\xi)\,d\tau. \tag{40}$$

---

†*These integral equations can be used to prove the existence of a solution b of equations (32) and (33).*

Since $q \in C^\infty(\mathsf{U})$, we may differentiate these integral equations and easily find by induction that ϕ and ψ are members of $C^\infty(\mathsf{S})$. We now estimate the derivatives of ϕ and ψ and show that the functions are in fact members of $\gamma^2(\mathsf{S})$.

For this purpose, we let $\delta < r$ and set $\mathsf{S}_\delta = \{(\eta,\xi) : |\eta| + |\xi| \leq 2\delta\}$. For integers $n \geq 0$ and real numbers $\theta > 0$, we consider the following seminorms on functions $f \in C^\infty(\mathsf{S})$:

$$| f |^n_{\delta,\theta} = \max\{\theta^{-n} (n!)^{-2} |D^\alpha f(\eta,\xi)| : (\eta,\xi) \in \mathsf{S}_\delta,\ |\alpha| = n\},$$

$$\| f \|^n_{\delta,\theta} = \max\{ | f |^j_{\delta,\theta} : j \leq n\}.$$

Differentiation of equation (39) yields the equations

$$D^\beta \psi(\eta,\xi) = -\frac{1}{4} D^{\beta - (0,1)}\{q([\eta+\xi]/2)\, \phi(\eta,\xi)\}, \qquad \text{for } \beta = (\beta_1, \beta_2) \text{ with } \beta_2 \neq 0.$$

$$\frac{\partial^n \psi}{\partial \eta^n}(\eta,\xi) = -\frac{1}{4}\frac{d^n}{d\eta^n}\{q(\eta/2)\} - \frac{1}{4}\int_0^\xi \frac{\partial^n}{\partial \eta^n}\{q([\eta+\tau]/2)\, \phi(\eta,\tau)\}\, d\tau.$$

However, since $q \in \gamma^2(\mathsf{U})$, it follows easily that the function $(\eta,\xi) \to q([\eta+\xi]/2)$ is a member of $\gamma^2(\mathsf{S})$, and for any $\theta > 0$, we can find $C_\theta \geq 0$ such that

$$(D^\alpha q([\eta+\xi]/2))\, \theta^{-|\alpha|} (|\alpha|!)^{-2} \leq C_\theta, \quad \text{for all multi-indices } \alpha = (\alpha_1, \alpha_2).$$

We now use a result which we prefer to state as a separate lemma, instead of proving it here. The result is stated and proved as Lemma 7 and follows the proof of this lemma. By Lemma 7, and the last three equations, we obtain the estimates

$$\max\{|D^\beta \psi(\eta,\xi)| : (\eta,\xi) \in \mathsf{S}_\delta\} \leq \frac{9}{4} C_\theta \{(|\beta|-1)!\}^2 \| \phi \|^{|\beta|-1}_{\delta,\theta}\, \theta^{|\beta|-1}, \text{ provided } \beta_2 \neq 0,$$

$$\max\{|D^{(n,0)} \psi(\eta,\xi)| : (\eta,\xi) \in \mathsf{S}_\delta\} \leq \frac{1}{4} C_\theta\, \theta^n (n!)^2 + \frac{9}{2}\delta\, C_\theta (n!)^2 \| \phi \|^n_{\delta,\theta}\, \theta^n.$$

However, equation (40) implies that

$$D^{\alpha}\phi = D^{\alpha-(1,0)}\psi, \quad \text{for } \alpha = (\alpha_1, \alpha_2) \text{ with } \alpha_1 \neq 0,$$

$$\text{and} \quad \frac{\partial^n \phi}{\partial \xi^n}(\eta,\xi) = \int_0^{\eta} \frac{\partial^n \psi}{\partial \xi^n}(\tau,\xi)\, d\tau.$$

From these equations and the two estimates preceding them, we immediately obtain the following estimates (assuming $(\eta,\xi) \in S_\delta$):

$$|D^{\alpha}\phi(\eta,\xi)| \leq \frac{1}{4} C_\theta\, \theta^{n-1}\, [(n-1)!]^2 + \frac{9}{2}\delta\, C_\theta\, [(n-1)!]^2\, \|\phi\|_{\delta,\theta}^{n-1}\, \theta^{n-1}$$

$$+ \frac{9}{4} C_\theta\, [(n-2)!]^2\, \|\phi\|_{\delta,\theta}^{n-2}\, \theta^{n-2}, \quad \text{for } |\alpha| = \alpha_1 + \alpha_2 = n \text{ and } \alpha_1 \neq 0,$$

$$\left| \frac{\partial^n \phi}{\partial \xi^n}(\eta,\xi) \right| \leq \frac{9}{2}\delta\, C_\theta\, [(n-1)!]^2\, \|\phi\|_{\delta,\theta}^{n-1}\, \theta^{n-1}.$$

It follows that

$$|\phi|_{\delta,\theta}^{n} \leq \frac{1}{4n^2\theta} C_\theta + \frac{9\delta}{2n^2\theta} C_\theta \|\phi\|_{\delta,\theta}^{n-1} + \frac{9}{4n^2(n-1)^2\theta^2} C_\theta \|\phi\|_{\delta,\theta}^{n-2}.$$

We choose N so large that for $n > N$ we have $9\delta C_\theta < n^2\theta$ and $9 C_\theta < 2n^2(n-1)^2\theta^2$. Then for $n > N$, we have

$$|\phi|_{\delta,\theta}^{n} \leq \frac{1}{4n^2\theta} C_\theta + \|\phi\|_{\delta,\theta}^{n-1}.$$

If we iterate this inequality, we obtain for $m \geq 0$

$$\|\phi\|_{\delta,\theta}^{n+m} \leq \|\phi\|_{\delta,\theta}^{n-1} + \frac{1}{4\theta} C_\theta \sum_{j=0}^{m} \frac{1}{(n+j)^2}.$$

Hence, $\sup_{m\geq 0} \| \phi \|^{n+m}_{\delta,\theta} \leq \| \phi \|^{n-1}_{\delta,\theta} + \sum_{j=0}^{\infty} \frac{1}{(n+j)^2} < \infty.$

But θ and δ are arbitrary, so this last inequality implies that $\phi \in \gamma^2(S)$. This implies that $b \in \gamma^2(U \times U)$. This completes the proof of the lemma.

□

We now prove Lemma 7, the result of which was used in proving Lemma 6.

Lemma 7

Let the seminorms $|\ |^n_{\delta,\theta}$ and $\|\ \|^n_{\delta,\theta}$ be as in the proof of Lemma 6. Then for functions f and g in $C^\infty(S)$, and integers $k \geq 0$, the following inequality holds:

$$| f g |^k_{\delta,\theta} \leq 9 \| f \|^k_{\delta,\theta} \| g \|^k_{\delta,\theta} .$$

Proof

$$\left| \frac{\partial^{m+n} f g}{\partial\eta^m \partial\xi^n} \right| = \left| \sum_{j=0}^{m} \sum_{i=0}^{n} \binom{m}{j}\binom{n}{i} \frac{\partial^{j+i} f}{\partial\eta^j \partial\xi^i} \frac{\partial^{m+n-i-j} g}{\partial\eta^{m-j} \partial\xi^{n-i}} \right|$$

$$\leq \| f \|^{m+n}_{\delta,\theta} \| g \|^{m+n}_{\delta,\theta} \theta^{m+n} [(m+n)!]^2 \sum_{j=0}^{m} \sum_{i=0}^{n} \binom{m}{j}\binom{n}{i} \frac{[(m+n-i-j)!]^2 [(i+j)!]^2}{[(m+n)!]^2} .$$

Thus the proof has been reduced to showing that

$$\sum_{j=0}^{m} \sum_{i=0}^{n} \binom{m}{j}\binom{n}{i} \frac{[(m+n-i-j)!]^2 [(i+j)!]^2}{[(m+n)!]^2} \leq 9. \tag{41}$$

To see this, we use the standard inequality

$$\binom{m}{j}\binom{n}{i} \leq \binom{m+n}{i+j}.$$

Thus, the left hand side of inequality (41) is no greater than:

$$\sum_{j=0}^{m} \sum_{i=0}^{n} \binom{m}{j}^{-1} \binom{n}{i}^{-1} = \left[\sum_{j=0}^{m} \binom{m}{j}^{-1} \right] \left[\sum_{i=0}^{n} \binom{n}{i}^{-1} \right] < 9.$$

The last inequality here is deduced from the fact that

$$\sum_{s=0}^{p} \binom{p}{s}^{-1} \leq \begin{cases} 1 + (p-1)/p + 1 < 3, & \text{if } p \geq 1 \\ 1, & \text{if } p = 0 \end{cases}$$

Thus, inequality (41) has been established, completing the proof of the lemma.

□

4.5 Solution of the Control Problem for the Rotating System

We write equation (23) in the form

$$\frac{\partial^2 w}{\partial t^2} + q(t)\, w + \alpha \frac{\partial^2}{\partial x^2}\left[\beta \frac{\partial^2 w}{\partial x^2} \right] = 0, \tag{42}$$

where $q(t) = -(d\phi/dt)^2$, and $\alpha = \rho^{-1}$, $\beta = E\,I$ as before. As in the non-rotating system, we assume that α and β have been continued to be in $C^4[0,\infty)$ in such a manner that $\alpha\,\beta > 0$ and, for all x sufficiently large, $\alpha(x) = \beta(x) = 1$. We further assume that $q \in C^1(\mathsf{U})$, while later we restrict attention to the case $q \in \gamma^2(\mathsf{U})$.

We now use the transmutation operator B of Theorem 5 to solve the 'semi-infinite rotating beam problem', which we call **Problem C**, consisting of equation (42) and the equations

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0, \tag{43}$$

$$w(x,0) = w_0(x), \quad \frac{\partial w}{\partial t}(x,0) = v_0(x), \tag{44}$$

for $(x,t) \in [0,\infty) \times \mathsf{U}$.

Our aim is to solve Problem C by applying B to solutions of equation (10). However, this requires the existence of solutions of (10) for negative t values. But we are fortunate in that equation (10) is time-reversible. In fact, we can easily show that we can continue the strongly continuous semigroup S(t), for negative t values, to obtain a strongly continuous group†. Moreover, it is easy to see that for $t \leq 0$,

$$S(t)\begin{bmatrix} w_0 \\ v_0 \end{bmatrix} = S(-t)\begin{bmatrix} w_0 \\ -v_0 \end{bmatrix}. \tag{45}$$

This equation allows us to continue the function K(x,a,t) for negative t values, for if $\begin{bmatrix} w_0 \\ v_0 \end{bmatrix}$ has compact support and $t < 0$, then we obtain from equation (45), and Theorem 16 of Chapter 1:

$$S(t)\begin{bmatrix} w_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \mathsf{K}'(-t) & \mathsf{K}(-t) \\ \mathsf{K}''(-t) & \mathsf{K}'(-t) \end{bmatrix}\begin{bmatrix} w_0 \\ -v_0 \end{bmatrix}$$

Thus, if w(.,t) is the first component of $S(t)\begin{bmatrix} w_0 \\ v_0 \end{bmatrix}$ and if we define for t negative

$$K(x,a,t) = -K(x,a,-t), \tag{46}$$

then we obtain for $t \in \mathbf{R} \setminus \{0\}$ and $\begin{bmatrix} w_0 \\ v_0 \end{bmatrix}$ of compact support,

$$w(x,t) = \int_0^\infty K(x,a,t)\, v_0(a)\, da \;+\; \int_0^\infty \frac{\partial K}{\partial t}(x,a,t)\, w_0(a)\, da. \tag{47}$$

We now use these facts to investigate the solution of Problem C.

Theorem 8

---

† *This follows from the remark after Theorem 4 of Chapter 1 and equation (110) of Chapter 1.*

Let $(w_0,v_0) \in D_B2$ and set w(.,t) equal to the first component of $S(t)\begin{bmatrix} w_0 \\ v_0 \end{bmatrix}$ for $t \in \mathbf{R}$. We define for $t \in \mathsf{U}$,

$$\tilde{w}(.,t) = \mathsf{B}\, w(.,t) = w(.,t) + \int_{-t}^{t} b(s,t)\, w(.,s)\, ds. \tag{48}$$

Then the mapping $\phi(t) = \tilde{w}(.,t)$ satisfies:

(i) ϕ is a continuous mapping from U into $H^6(\mathbf{R}^+)$.

(ii) ϕ is continuously differentiable in the topology of $H^4(\mathbf{R}^+)$.

(iii) ϕ is twice continuously differentiable in the topology of $H^2(\mathbf{R}^+)$.

Moreover, $\tilde{w}$ is a solution of Problem C, and if any other function u is a solution of Problem C and has the regularity listed in (i), (ii), (iii) above, then $u = \tilde{w}$.

Proof

By part (ii) of Theorem 1, the mapping $t \to w(.,t)$ has the regularity of (i), (ii) and (iii) above. Hence (i), (ii) and (iii) follow immediately for the mapping ϕ since $b \in C^2(\mathsf{U} \times \mathsf{U})$. Further, $\tilde{w}(x,0) = w(x,0) = w_0(x)$, and $\frac{\partial \tilde{w}}{\partial t}(x,0) = \frac{\partial w}{\partial t}(x,0) + 2b(0,0)\, w(x,0) = v_0(x)$ because $b(0,0) = 0$. Also, $\tilde{w}$ satisfies the 'clamped end' conditions (equations (43)) because w satisfies these conditions.

The fact that $\tilde{w}$ satisfies equation (42) follows from the properties of the transmutation operator B, for we have:

$$\left[\frac{\partial^2}{\partial t^2} + q(t)\right] \tilde{w}(x,t) = \left[\frac{\partial^2}{\partial t^2} + q(t)\right] \mathsf{B} w\,(x,t) = \mathsf{B}\left[\frac{\partial^2 w}{\partial t^2}\right]$$

$$= \mathsf{B}\left[-\,\alpha\, \frac{\partial^2}{\partial x^2}\, \beta\, \frac{\partial^2 w}{\partial x^2}\right](x,t) = -\,\alpha\, \frac{\partial^2}{\partial x^2}\, \beta\, \frac{\partial^2}{\partial x^2}\, \mathsf{B} w\,(x,t) = -\,\alpha\, \frac{\partial^2}{\partial x^2}\, \beta\, \frac{\partial^2 \tilde{w}}{\partial x^2}(x,t).$$

We now establish the uniqueness of the solution. For this, we define the energy E_0 of a solution W of Problem C as

$$E_0(t) = \frac{1}{2}\int_0^\infty \rho(x)\left[\frac{\partial W}{\partial t}(x,t)\right]^2 + E(x)\,I(x)\left[\frac{\partial^2 W}{\partial x^2}(x,t)\right]^2 dx.$$

If W is a solution of Problem C with the regularity specified in (i), (ii), (iii), then we can differentiate the energy and obtain:

$$E_0'(t) = \int_0^\infty \rho\,\frac{\partial W}{\partial t}\frac{\partial^2 W}{\partial t^2} + E\,I\,\frac{\partial^2 W}{\partial x^2}\frac{\partial^3 W}{\partial t\partial x^2}\,dx$$

$$= -q(t)\int_0^\infty W(x,t)\,\frac{\partial W}{\partial t}(x,t)\,dx$$

$$\leq |q(t)|\,\|W\|_{L^2}\left\|\frac{\partial W}{\partial t}\right\|_{L^2}.$$

But if we now assume that $W(x,0) = 0$, it follows that

$$E_0'(t) \leq |q(t)|\left\|\frac{\partial W}{\partial t}(t)\right\|_{L^2}\int_0^t \left\|\frac{\partial W}{\partial t}(\tau)\right\|_{L^2} d\tau$$

We pick δ so that $U \supset [-\delta,\delta]$ and $C = C(\delta)$ so that $|q(t)| \leq C$ on $[-\delta,\delta]$. We then obtain

$$\sqrt{E_0(t)} \leq C\int_0^t (t-\tau)\sqrt{E_0(\tau)}\,d\tau$$

for $t \in [-\delta,\delta]$, since the function $\tilde{w} - u$ has zero initial energy. This implies that its energy vanishes on $[-\delta,\delta]$. Since $\delta > 0$ is arbitrary, we see that the energy vanishes on U. This implies that $\frac{\partial^2}{\partial x^2}(\tilde{w} - u)$ vanishes for x in $[0,\infty)$. Since $\tilde{w} - u$ satisfies the 'clamped end conditions', it follows that $\tilde{w} - u$ vanishes identically. This completes the proof of the theorem.

□

We now wish to show that the solutions $\tilde{w}$ of the theorem above can be written in the form of equation (2) of Chapter 3. i.e. for (w_0,v_0) of compact support,

$$\tilde{w}(x,t) = \int_0^\infty K_1(x,a,t)\, w_0(a)\, da + \int_0^\infty K_0(x,a,t)\, v_0(a)\, da . \tag{49}$$

If we can establish (49) and show that K_1 and K_0 are of Gevrey class 2, then we can apply the theory of Chapter 3 and construct new kernels which vanish for $t \geq T$.

In view of equations (47) and (48), the obvious choices for the kernels K_1 and K_0 are respectively $\mathsf{B}(\partial K/\partial t)$ and $\mathsf{B}K$. There is no problem in setting $K_0 = \mathsf{B}K$, for K is a continuous function of (x,a,t). However, we must be careful with what we mean by $\mathsf{B}(\partial K/\partial t)$, for $\partial K/\partial t$ may be unbounded for t near the origin. Instead, we write for K_1 what we would formally obtain after an integration by parts of the expression $\mathsf{B}(\partial K/\partial t)$:

Definitions: (i) For $(x,a,t) \in [0,\infty)^2 \times \mathsf{U}$, we define

$$K_0(x,a,t) = K(x,a,t) + \int_{-t}^{t} b(s,t)\, K(x,a,s)\, ds. \tag{50}$$

(ii) For $(x,a,t) \in [0,\infty)^2 \times (\mathsf{U} \setminus \{0\})$, we define

$$K_1(x,a,t) = \frac{\partial K}{\partial t}(x,a,t) + b(t,t)\, K(x,a,t) - b(-t,t)\, K(x,a,-t) - \int_{-t}^{t} \frac{\partial b}{\partial s}(s,t)\, K(x,a,s)\, ds$$

$$= \frac{\partial K}{\partial t}(x,a,t) - \frac{1}{2}\left[\int_0^t q(\tau)\, d\tau\right] K(x,a,t) - \int_{-t}^{t} \frac{\partial b}{\partial s}(s,t)\, K(x,a,s)\, ds. \tag{51}$$

In order to apply the theory of Chapter 3 to these kernels, we must first show that they have the require Gevrey properties and that equation (49) is indeed satisfied. However, there are some difficulties in doing this, because of the unbounded nature of the partial derivatives of K(x,a,t) near $t = 0$. We use the following lemma to overcome these difficulties.

Lemma 9

For integers $n \geq 0$, we define $I_n(x,a,t)$ for (x,a,t) on $[0,\infty)^2 \times \mathbf{R}$ by

$$I_0(x,a,t) = \int_0^t K(x,a,s)\,ds, \quad \text{and for } n \geq 1, \quad I_n(x,a,t) = \int_0^t I_{n-1}(x,a,s)\,ds,$$

i.e. we have for $n \geq 0$,

$$I_n(x,a,t) = \int_0^t \frac{(t-s)^n}{n!} K(x,a,s)\,ds.$$

Let R be any compact rectangle in $[0,\infty)^2$. Then the mapping $t \to I_n(.,.,t)$ is a member of $\gamma^2(\mathbf{R}\backslash\{0\}; D^{4,1}(\mathsf{R}))$ and if $n \geq 2$, it is also a member of $C(\mathbf{R}\,; D^{4,1}(\mathsf{R}))$.

Proof

By equation (46), we need only to examine the behavior of I_n for $t \leq 0$. Because of the way the kernel K was defined in terms of a kernel J (see equation (112) of Chapter 1 for the definition of K, and equation (94) of Chapter 1 for the definition of J), it clearly suffices to prove the lemma for J.

We first prove that the kernels are given by

$$I_n(x,a,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{g(x,a,\sqrt{\lambda}\,)}{\lambda^{n+1}}\, e^{\lambda t}\,d\lambda, \tag{52}$$

where the constant γ is as in equation (94), Chapter 1.
Now by equation (94),

$$\int_0^t J(x,a,s)\,ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} g(x,a,\sqrt{\lambda}\,)\,(e^{\lambda t} - 1)/\lambda\,d\lambda.$$

For $R > 0$, we let C_R denote the contour which consists of the semi-circle which is the intersection of the set $\{\lambda : \text{Re}\,\lambda \geq \gamma\}$ with the circle of radius R, centered at γ. The orientation of C_R is shown in figure 11. By inequality (88), Chapter 1,

$$g(x, a, \sqrt{\lambda}\,) \leq M_3\, |\lambda|^{-3/2}$$

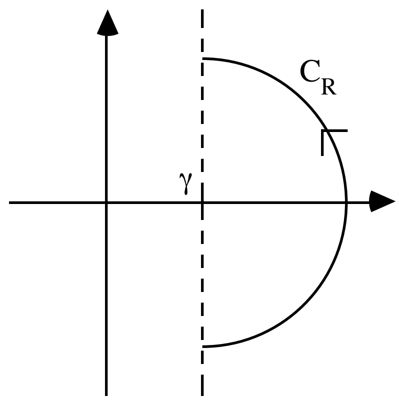

Figure 11

$$g(x, a, \sqrt{\lambda}\,) \leq M_3\, |\lambda|^{-3/2}$$

if $\text{Re}\,\lambda \geq 0$. Thus,

$$\int_{\gamma-i\infty}^{\gamma+i\infty} g(x, a, \sqrt{\lambda}\,)\, \lambda^{-m}\, d\lambda \;=\; \lim_{R\to\infty} \int_{C_R} g(x, a, \sqrt{\lambda}\,)\, \lambda^{-m}\, d\lambda \;=0, \quad \text{for } m \geq 0. \tag{53}$$

Hence equation (52) holds if $n = 0$. We pose the inductive hypothesis that it holds if $n = k$. If this is the case, we use equation (53) and obtain

$$I_{k+1}(x, a, t) = \int_0^t I_k(x, a, s)\, ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} g(x, a, \sqrt{\lambda}\,)\, (e^{\lambda t}-1)/\lambda^{k+2}\, d\lambda$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{g(x,a,\sqrt{\lambda}\,)}{\lambda^{k+2}}\, e^{\lambda t}\, d\lambda.$$

Equation (52) follows by induction.

We omit the proof that $I_n(.,.,t)$ is a member of $\gamma^2(\mathbf{R}\backslash\{0\}; D^{4,1}(\mathsf{R}))$, and point out that this proof is based on equation (52) and is very similar to the last part of the proof of Theorem 11, Chapter 1, in which the Gevrey smoothness of J is established.

To see that $I_n(.,.,t)$ is a member of $C([0,\infty); D^{4,1}(\mathsf{R}))$ for $n \geq 2$, it suffices to prove that the integral in equation (52) converges for $n \geq 2$, absolutely in $D^{4,1}(\mathsf{R})$, uniformly for t on compact subsets of $[0,\infty)$. If we set $\mathsf{R} = [0,d''] \times [0,d']$, then this fact follows immediately from inequality (88) of Chapter 1, which states that

$$\left| \frac{g(.,.,\sqrt{\lambda}\,)}{\lambda^{n+1}}\, e^{\lambda t} \right|_{4,1} (d'',d') \;\leq\; M_3 \frac{e^{\gamma t}}{|\lambda|^n}\,.$$

□

Theorem 10

Let $q \in \gamma^2(\mathsf{U})$. Then the mappings $t \to K_0(.,.,t)$ and $t \to K_1(.,.,t)$ are members of $\gamma^2(\mathsf{U}\backslash\{0\}; D^{6,1}(\mathsf{R}))$, for any compact rectangle R contained in $[0,\infty)^2$. Further, the kernels satisfy, for $i \in \{0,1\}$,

$$K_i(0,a,t) = \frac{\partial K_i}{\partial x}(0,a,t) = 0, \quad \text{for } t \in \mathsf{U}\backslash\{0\}, \tag{54}$$

$$\frac{\partial^2 K_i}{\partial t^2} + \alpha \frac{\partial^2}{\partial x^2}\beta \frac{\partial^2 K_i}{\partial x^2} + q\,K_i = 0, \text{ in } (0,\infty)^2 \times [\mathsf{U}\backslash\{0\}]. \tag{55}$$

If $(w_0,v_0) \in D_{B^2}$ and has compact support, then the solutions of Problem C (as given in Theorem 8) are, for $t \neq 0$, expressible as:

$$\tilde{w}(x,t) = \int_0^\infty K_1(x,a,t)\, w_0(a)\, da \;+\; \int_0^\infty K_0(x,a,t)\, v_0(a)\, da\,. \tag{49}$$

Proof

For the first part of the theorem, we show that K_0 has the properties listed. The proof that K_1 also has these properties is similar.

By Lemma 6, $b \in \gamma^2(\mathsf{U} \times \mathsf{U})$ and thus $b \in C^\infty(\mathsf{U} \times \mathsf{U})$, so we can integrate by parts in equation (50) to obtain:

$$K_0(x,a,t) = K(x,a,t) \;+\; \left[I_0(x,a,s)\, b(s,t) - I_1(x,a,s)\, \frac{\partial b}{\partial s}(s,t) + I_2(x,a,s)\, \frac{\partial^2 b}{\partial s^2}(s,t)\right]_{s=-t}^{s=t}$$

$$- \int_{-t}^{t} \frac{\partial^3 b}{\partial s^3}(s,t)\, I_2(x,a,s)\, ds.$$

Thus, to show that $K_0(.,.,t)$ is in $\gamma^2(\mathsf{U}\backslash\{0\}; D^{4,1}(\mathsf{R}))$, it suffices to prove that

$$M(x,a,t) = \int_{-t}^{t} \frac{\partial^3 b}{\partial s^3}(s,t)\, I_2(x,a,s)\, ds.$$

is in $\gamma^2(\mathsf{U}\backslash\{0\}; D^{4,1}(\mathsf{R}))$. We show that the mapping is in $\gamma^2(\mathsf{U} \cap (0,\infty); D^{4,1}(\mathsf{R}))$, the proof for negative values of t being similar.

Let $[t_0,t_1] \subset \mathsf{U} \cap (0,\infty)$. Then for $t \in \mathsf{U} \cap (0,\infty)$ we have $M(x,a,t) = M_1(x,a,t) + M_2(x,a,t) + M_3(x,a,t)$, where

$$M_1(x,a,t) = \int_{-t}^{-t_0} \frac{\partial^3 b}{\partial s^3}(s,t)\, I_2(x,a,s)\, ds, \qquad M_2(x,a,t) = \int_{-t_0}^{t_0} \frac{\partial^3 b}{\partial s^3}(s,t)\, I_2(x,a,s)\, ds,$$

$$M_3(x,a,t) = \int_{t_0}^{t} \frac{\partial^3 b}{\partial s^3}(s,t)\, I_2(x,a,s)\, ds.$$

Now we apply Lemmas 6 and 9 to obtain for $t \in [t_0,t_1]$

$$\left| \frac{\partial^n M_2}{\partial t^n}(.,.,t) \right|^{4,1} \leq \int_{-t_0}^{t_0} \left| \frac{\partial^{n+3} b}{\partial t^n \partial s^3}(s,t) \right| \left| I_2(.,.,s) \right|^{4,1} ds$$

$$\leq \text{constant} \,.\, \theta^{n+3} [(n+3)!]^2, \quad \text{for any } \theta > 0.$$

Hence $M_2(.,.,t) \in \gamma^2([t_0,t_1]; D^{4,1}(\mathbf{R}))$.

Now we consider M_3 and note that the product $f(.,.,s,t) = \frac{\partial^3 b}{\partial s^3}(s,t)\, I_2(.,.,s)$ is a member of $\gamma^2([t_0,t_1]^2 ; D^{4,1}(\mathbf{R}))$. An easy computation yields

$$\frac{\partial^n M_3}{\partial t^n}(x,a,t) = \sum_{j=0}^{n-1} \left[\frac{d}{dt}\right]^j \frac{\partial^{n-1-j} f}{\partial t^{n-1-j}}(x,a,t,t) + \int_{t_0}^{t} \frac{\partial^n f}{\partial t^n}(x,a,s,t)\, ds$$

$$= \sum_{j=0}^{n-1} \sum_{k=0}^{j} \binom{j}{k} \frac{\partial^{n-1} f}{\partial s^k \partial t^{n-k-1}}(x,a,t,t) + \int_{t_0}^{t} \frac{\partial^n f}{\partial t^n}(x,a,s,t)\, ds.$$

Thus, for all t in $[t_0,t_1]$, and for any $\theta > 0$, we can find C_1 and C_2 so that

$$\left| \frac{\partial^n M_3}{\partial t^n}(.,.,t) \right|^{4,1} \leq C_1 \sum_{j=0}^{n-1} \sum_{k=0}^{j} \binom{j}{k} \theta^{n-1} [(n-1)!]^2 + C_2\, \theta^n [n!]^2$$

$$= C_1\, \theta^{n-1} [(n-1)!]^2 \sum_{j=0}^{n-1} 2^j + C_2\, \theta^n [n!]^2$$

$$\leq 2\, C_1 (2\theta)^{n-1} [(n-1)!]^2 + C_2\, \theta^n [n!]^2.$$

Hence $M_3(.,.,t) \in \gamma^2([t_0,t_1]; D^{4,1}(\mathsf{R}))$. Similarly, we may show that $M_1(.,.,t) \in \gamma^2([t_0,t_1]; D^{4,1}(\mathsf{R}))$. The proofs are almost identical for negative t values and we see that $K_0(.,.,t) \in \gamma^2(\mathsf{U} \cap (0,\infty); D^{4,1}(\mathsf{R}))$.

We now let $\phi \in C_0^\infty(\mathbf{R}^+)$. Clearly $(0,\phi) \in D_{B^2}$. We let u(.,t) be the first component of $S(t)\begin{bmatrix}0\\ \phi\end{bmatrix}$ and we set $\tilde{u}(.,t) = \mathsf{B}u(.,t)$. But

$$u(x,t) = \int_0^\infty \phi(a)\, K(x,a,t)\, da.$$

Thus, we may write

$$\tilde{u}(x,t) = \int_0^\infty \phi(a)\, K(x,a,t)\, da + \int_{-t}^{t} b(s,t) \int_0^\infty \phi(a)\, K(x,a,s)\, da\, ds$$

$$= \int_0^\infty \phi(a)\, K_0(x,a,t)\, da.$$

But $\tilde{u}$ is a solution of Problem C (see Theorem 8). Using this fact and our newly found knowledge of the regularity of K_0 yields the equations

$$\int_0^\infty \phi(a)\, K_0(0,a,t)\, da = \int_0^\infty \phi(a) \frac{\partial K_0}{\partial x}(0,a,t)\, da = 0,$$

$$\int_0^\infty \phi(a) \left[\frac{\partial^2 K_0}{\partial t^2} + q\, K_0 + \alpha \frac{\partial^2}{\partial x^2} \beta \frac{\partial^2 K_0}{\partial x^2}\right](x,a,t)\, da = 0.$$

But, since ϕ is an arbitrary member of $C_0^\infty(\mathbf{R}^+)$, it follows that K_0 satisfies equations (54) and (55).

Now, since K_0 satisfies the partial differential equation (55) and is a member of $\gamma^2(\mathsf{U} \cap (0,\infty); D^{4,1}(\mathsf{R}))$, it follows easily that $K_0 \in \gamma^2(\mathsf{U} \cap (0,\infty); D^{6,1}(\mathsf{R}))$.

We note that all of the results proved above for K_0 may similarly be proved for K_1. Thus, it remains for us to prove equation (49). If $(w_0,v_0) \in D_{B^2}$ with compact support, then we have

$$w(x,t) = \int_0^\infty K(x,a,t)\, v_0(a)\, da + \int_0^\infty \frac{\partial K}{\partial t}(x,a,t)\, w_0(a)\, da.$$

Hence,

$$\tilde{w}(x,t) = \int_0^\infty K(x,a,t)\, v_0(a)\, da + \int_0^\infty \frac{\partial K}{\partial t}(x,a,t)\, w_0(a)\, da$$

$$+ \int_{-t}^{t} b(s,t) \left\{ \int_0^\infty K(x,a,s)\, v_0(a)\, da + \frac{\partial}{\partial s} \int_0^\infty K(x,a,s)\, w_0(a)\, da \right\} ds$$

$$= \int_0^\infty K_0(x,a,t)\, v_0(a)\, da + \int_0^\infty \frac{\partial K}{\partial t}(x,a,t)\, w_0(a)\, da$$

$$+ \left[b(s,t) \int_0^\infty K(x,a,s)\, w_0(a)\, da \right]_{s=-t}^{s=t} - \int_{-t}^{t} \frac{\partial b}{\partial s}(s,t) \int_0^\infty K(x,a,s)\, w_0(a)\, da\, ds$$

$$= \int_0^\infty K_0(x,a,t)\, v_0(a)\, da + \int_0^\infty K_1(x,a,t)\, w_0(a)\, da.$$

This completes the proof of the theorem.

□

We are now ready to use the theory of Chapter 3 to obtain kernels $\tilde{K}_0$ and $\tilde{K}_1$ which vanish for $t \geq T$.

Theorem 11

Let $0 < T < r$ (recall that $U = (-r,r)$), and $\delta > 0$ satisfy $0 < \delta < T$. Then there exist kernels $\tilde{K}_0$ and $\tilde{K}_1$ satisfying ($i \in \{0,1\}$):

(a) $\tilde{K}_i(x,a,t) = K_i(x,a,t)$ for $t \le \delta$.

(b) $\tilde{K}_i(x,a,t) = 0$ for $t \ge T$.

(c) $\tilde{K}_i(0,a,t) = \dfrac{\partial \tilde{K}_i}{\partial x}(0,a,t) = 0$.

(d) $$\frac{\partial^2 \tilde{K}_i}{\partial t^2} + \alpha \frac{\partial^2}{\partial x^2} \beta \frac{\partial^2 \tilde{K}_i}{\partial x^2} + q\,\tilde{K}_i = 0.$$

(e) The mapping $t \to \tilde{K}_i$ is a member of $\gamma^2(U\backslash\{0\}; D^{6,1}(R))$ for any compact rectangle R contained in $[0,\infty)^2$.

Proof

Parts (a) - (d) are direct consequences of Theorem 16 of Chapter 3. The theorem also implies that the mapping in (e) is a member of $\gamma^2(U\backslash\{0\}; D^{6,1}(R))$. However, since each of the kernels satisfies the partial differential equation in (d), it is easily seen that statement (e) holds.

□

Before stating the theorem concerning the controllability of Problem C, we give a uniqueness lemma which shows that the response $\tilde{w}$ to the boundary controls is unique. We define the energy E of a solution of Problem B by the same formula defining the energy for solutions of Problem A, i.e.

$$E(t) = \frac{1}{2}\int_0^d \rho(x)\left[\frac{\partial w}{\partial t}(x,t)\right]^2 + E(x)\,I(x)\left[\frac{\partial^2 w}{\partial x^2}(x,t)\right]^2 dx + \frac{1}{2}\,m\left[\frac{dy}{dt}(t)\right]^2 + \frac{1}{2}\,J\left[\frac{d\theta}{dt}(t)\right]^2.$$

Lemma 12

Let $\tilde{w}$ be a solution of Problem B having the regularity properties: the mapping $t \to \tilde{w}(.,t)$ is continuous in the topology of $H^6(0,d)$, continuously differentiable in the topology of $H^4(0,d)$ and twice continuously differentiable in the topology of $H^2(0,d)$.

Then:

(i) $$\frac{d\mathsf{E}}{dt} = f_1(t)\frac{dy}{dt} + f_2(t)\frac{d\theta}{dt} - q(t)\int_0^d \tilde{w}\frac{\partial \tilde{w}}{\partial t}\,dx.$$

(ii) If W is any other such solution of Problem B with the same boundary functions f_1 and f_2, then $W = \tilde{w}$.

Proof

The regularity of w allows us to differentiate E and we obtain (i) by an integration by parts. The uniqueness assertion may now be proved just as it was proved for Problem C in Theorem 8.

□

We now state the boundary controllability result for Problem B.

Theorem 13

Consider the boundary controllability of Problem B. We let $\mathsf{U} = (-r,r)$ and suppose that $q \in \gamma^2(\mathsf{U})$. Let $(w_0,v_0) \in H^6(0,d) \times H^4(0,d)$ satisfy the 'clamped end conditions'

$$w_0(0) = w_0'(0) = 0$$

and the 'compatibility conditions'

$$v_0(0) = v_0'(0) = (\beta\, w_0'')''(0) = \{\alpha\,(\beta\, w_0'')''\}'(0) = 0,$$

where $\alpha = 1/\rho$ and $\beta = E\,I$ are members of $C^4[0,d]$ and $\alpha(x) > 0$, $\beta(x) > 0$ for all $x \in [0,d]$.

Then, given $T \in (0,r)$, there exist boundary control functions $f_1: [0,r) \to \mathbf{R}$ and $f_2: [0,r) \to \mathbf{R}$, which are continuous on $[0,r)$ and are members of $\gamma^2((0,r))$, such that Problem B has a solution $\tilde{w}$ with the properties:

(i) For $t \in [0,r)$, $t \to \tilde{w}(.,t)$ is continuous in $H^6(0,d)$, continuously differentiable in $H^4(0,d)$ and twice continuously differentiable in $H^2(0,d)$. Moreover, $\tilde{w}$ is unique in the class of solutions of Problem B with these regularity properties.

(ii) $\tilde{w}(.,t) = 0$ for $T \leq t < r$.

Proof

First, we extend the domain of the initial data so that the extension is in $H^6(\mathbf{R}^+) \times H^4(\mathbf{R}^+)$ and has compact support. By Theorem 1, (w_0,v_0) is now in D_B2. We let $\tilde{K}_1$ and $\tilde{K}_0$ be the kernels of Theorem 11 and set

$$\tilde{w}(x,t) = \int_0^\infty \tilde{K}_1(x,a,t)\, w_0(a)\, da + \int_0^\infty \tilde{K}_0(x,a,t)\, v_0(a)\, da .$$

By Theorems 8, 10 and 11, for $t \in \mathrm{U}$, $t \to \tilde{w}(.,t)$ is continuous in $H^6(0,\varepsilon)$, continuously differentiable in $H^4(0,\varepsilon)$ and twice continuously differentiable in $H^2(0,\varepsilon)$ for any $\varepsilon > 0$. Also, $\tilde{w}(.,t)$ vanishes for $t \geq T$ because $\tilde{K}_1(.,.,t)$ and $\tilde{K}_0$ do.

We set $f_1(t) = B_1 w(d,t)$ and $f_2(t) = B_2 w(d,t)$. It is clear that f_1 and f_2 are in $\gamma^2(0,\infty)$ because $\tilde{w}$ is in $\gamma^2((0,r); C^6[0,\varepsilon])$ for any $\varepsilon > 0$ (by Theorem 11, part (e)). Moreover, because of the regularity of the mapping $t \to \tilde{w}(.,t)$ for $t \in [0,r)$, it is easy to see that the Sobolev embedding theorems imply that f_1 and f_2 are continuous on the interval $[0,r)$.

Finally, the uniqueness of $\tilde{w}$ has already been proved in Lemma 12.

□

Chapter 5

Gevrey Semigroups

5.0 Introduction

In this chapter, we develop a theory for the Gevrey regularity of strongly continuous semigroups. Our aim is to try to complement the beautiful theory which already exists and describes necessary and/or sufficient conditions for a semigroup to be differentiable or analytic. The reader may find such results in the work by Pazy [18], which inspires much of the theory of this chapter.

One would expect that Gevrey semigroups should have a behavior somewhat 'between' that of differentiable semigroups and analytic semigroups. This is indeed found to be the case, and we point out the similarities throughout the next section with numerous references to [18].

We begin in Section 5.1 with definitions and theory. The chapter ends (Section 5.2) with a number of examples which demonstrate the application of the theorems to some concrete problems.

5.1 Theory of Gevrey Semigroups

Definition: Let T(t) be a strongly continuous semigroup on a Banach space X and let $\delta > 1$. We say that T(t) is of *Gevrey class* δ *for* $t > t_0$ if T(t) is infinitely differentiable for $t \in (t_0,\infty)$ and, for every compact $K \subset (t_0,\infty)$ and each $\theta > 0$, there exists a constant $C = C(\theta,K)$ such that

$$\|T^{(n)}(t)\| \leq C\, \theta^n\, (n!)^\delta \qquad \text{for all } t \in K \text{ and } n \in \{0, 1, 2, 3, \ldots\}. \tag{1}$$

Remarks: (1) We did not specify in the definition the type of differentiation. We assume that T(t) is infinitely differentiable in the operator norm topology, but this is easily seen to be equivalent to T(t) being infinitely differentiable in the strong topology.

(2) If T(t) is strongly continuous then there exist constants $M \geq 1$ and $\omega \in \mathbf{R}$ such that $\|T(t)\| \leq M e^{\omega t}$. This implies that the estimates (1) for $t \in K$ may be replaced with

equivalent estimates on intervals $[t_1,\infty)$, where $t_1 > t_0$. This is shown in the following theorem.

Theorem 1

Let T(t) be a strongly continuous semigroup satisfying $\|T(t)\| \le M e^{\omega t}$ and suppose that T(t) is infinitely differentiable for $t > t_0$. Then the following statements are equivalent:

(a) T(t) is of Gevrey class δ for $t > t_0$.

(b) For any $t_1 > t_0$ and $\theta > 0$ there exists a constant $C_1 = C_1(t_1,\theta)$ such that

$$\|T^{(n)}(t_1)\| \le C_1\, \theta^n\, (n!)^\delta \qquad \text{for each } n \in \{0, 1, 2, 3, \ldots\}.$$

(c) For any $t_1 > t_0$ and $\theta > 0$ there exists a constant $C_2 = C_2(t_1,\theta)$ such that

$$\|T^{(n)}(t)\| \le C_2\, \theta^n\, (n!)^\delta\, e^{\omega t} \quad \text{for all } t \in [t_1, \infty) \text{ and } n \in \{0, 1, 2, \ldots\}.$$

Proof

It is obvious that (a) implies (b) and that (c) implies (a). To show that (b) implies (c), we let $t_1 > t_0$ and $\theta > 0$ be given, and we choose C_1 as in (b). Now, for any $s_1 \ge 0$, $s_2 > t_0$, we have the identity

$$T^{(n)}(s_1+s_2) = T(s_1)\, T^{(n)}(s_2).$$

Thus, we obtain for $t \ge t_1$

$$\|T^{(n)}(t)\| = \|T(t - t_1)\, T^{(n)}(t_1)\| \le M\, e^{\omega(t-t_1)}\, C_1\, \theta^n\, (n!)^\delta,$$

which yields (c), with $C_2 = M\, C_1 \exp[-\omega t_1]$.

□

While Theorem 1 is of interest with regard to the definition of Gevrey semigroups, it does not help us to identify them. The following theorem is more useful in this respect, for it provides necessary and sufficient conditions on the resolvent of the infinitesimal generator of such semigroups.

Theorem 2

Let T(t) be a strongly continuous semigroup with infinitesimal generator A. Let R_λ denote the resolvent of A and let $\rho(A)$ denote the resolvent set of A. The following statements are equivalent:

(a) T(t) is of Gevrey class δ for $t > t_0$.

(b) For each $t_1 > t_0$ and $b > 0$, there exist constants $a > 0, C_1 \geq 0, C_2 \geq 0$, depending on only b and t_1, such that

$$\rho(A) \supset \Sigma_b = \{ \lambda : \mathrm{Re}\, \lambda \geq a - b\, |\mathrm{Im}\, \lambda|^{1/\delta}\}, \tag{2}$$

$$\|R_\lambda\| \leq C_1 \exp[-t_1 \,\mathrm{Re}\, \lambda] + C_2 \qquad \text{for all } \lambda \in \Sigma_b. \tag{3}$$

Remark: For strongly continuous semigroups, one can always find $\omega \in \mathbf{R}$ such that for $\mathrm{Re}\, \lambda > \omega$, $\|R_\lambda\| \leq C/[(\mathrm{Re}\, \lambda) - \omega]$. Thus, for applications, estimate (3) need only be verified in the subset $\Sigma_b \cap \{\lambda : \mathrm{Re}\, \lambda < \omega + \varepsilon\}$, for some $\varepsilon > 0$.

The proof of Theorem 2 relies on the following lemma. Although the lemma is a simple generalization of a lemma in [18] (Lemma 4.6, page 53), we provide a proof of it because we shall also use elements of this proof to prove Theorem 2. We prove Theorem 2 after proving Lemma 3.

Lemma 3

Suppose T(t) is n times differentiable (in the strong topology) for $t > t_0$. Then, if $t > t_0$,

$$\rho(A) \supset \{\lambda : \lambda^n e^{\lambda t} \in \rho(A^n T(t))\}.$$

Proof of Lemma 3

If $n = 0$, Lemma 3 is proved in [18] (Theorem 2.3, page 45), so we consider only the case $n \geq 1$. As in [18], we define a family of operators $B_\lambda(t)$ on X as:

$$B_\lambda(t)\, x = \int_0^t e^{\lambda(t-s)}\, T(s)\, x\, ds.$$

One easily shows that $B_\lambda'(t) = T(t) + \lambda B_\lambda(t)$ and that (see [18], Lemma 2.2)

$$(\lambda I - A)\, B_\lambda(t)\, x = e^{\lambda t}\, x - T(t)\, x \quad \text{for all } x \in X, \tag{4}$$

$$B_\lambda(t)\, (\lambda I - A)\, x = e^{\lambda t}\, x - T(t)\, x \quad \text{for all } x \in D_A. \tag{5}$$

From now on we assume that $t > t_0$. Then

$$B_\lambda^{(n)}(t) = \lambda^n B_\lambda(t) + \sum_{k=0}^{n-1} \lambda^{n-k-1}\, T^{(k)}(t). \tag{6}$$

Equations (5) and (6) imply that for all $x \in D_A$,

$$B_\lambda^{(n)}(t)\, (\lambda I - A)\, x = [\lambda^n B_\lambda(t) + \sum_{k=0}^{n-1} \lambda^{n-k-1}\, T^{(k)}(t)]\, (\lambda I - A)\, x = \lambda^n e^{\lambda t}\, x - T^{(n)}(t)\, x. \tag{7}$$

We also show that for all $x \in X$,

$$(\lambda I - A)\, B_\lambda^{(n)}(t)\, x = (\lambda I - A)\, [\lambda^n B_\lambda(t) + \sum_{k=0}^{n-1} \lambda^{n-k-1}\, T^{(k)}(t)]\, x = \lambda^n e^{\lambda t} x - T^{(n)}(t)\, x. \tag{8}$$

If $x \in D_A$, then equation (8) follows from equations (4), (5), (6) and (7). But then (8) must hold for all $x \in X$, for each entry in the equation is a bounded operator acting on x, and D_A is dense in X.

Suppose that $\lambda^n e^{\lambda t} \in \rho(A^n T(t))$. Then, on setting $x = (\lambda^n e^{\lambda t} - A^n T(t))^{-1} y$ in equation (8), we obtain

$$y = (\lambda I - A)\, B_\lambda^{(n)}(t)\, (\lambda^n e^{\lambda t} - A^n T(t))^{-1}\, y \qquad \text{for all } y \in X.$$

Equation (7) implies that

$$x = B_\lambda^{(n)}(t)\, (\lambda^n e^{\lambda t} - A^n T(t))^{-1} (\lambda I - A)\, x \quad \text{for all } x \in D_A.$$

The last two equations show that $\lambda I - A$ is invertible. Thus, $\lambda \in \rho(A)$ and the lemma has been proved.

□

Proof of Theorem 2

(a) ⇒ (b): We let $t_1 > t_0$, $b > 0$, set $\theta = e^{-1} [b\, t_1]^{-\delta}$ and choose C so that inequality (1) is satisfied for $t = t_1$. By Lemma 3, we have for $n \geq 0$,

$$\rho(A) \supset \{\lambda : \lambda^n \exp[\lambda t_1] \in \rho(A^n T(t_1))\}$$

$$\supset \{\lambda : |\lambda^n \exp[\lambda t_1]| > C\, \theta^n\, n^{\delta n}\}$$

$$= \{\lambda : t_1 \operatorname{Re} \lambda > \ln C + n \ln[\theta\, n^\delta / |\lambda|]\ \}.$$

For a fixed λ, we pick n to be the largest integer smaller than $(|\lambda|\, [\theta\, e]^{-1})^{1/\delta}$. Thus,

$$n \ln[\theta\, n^{\delta}/|\lambda|] < n \ln e^{-1} = -n < -\left[\frac{|\lambda|}{\theta\, e}\right]^{1/\delta} + 1.$$

Thus, we see that $\rho(A) \supset \Sigma_b = \{ \lambda : \mathrm{Re}\, \lambda \geq a - b\, |\mathrm{Im}\, \lambda|^{1/\delta}\}$, where we have chosen

$$a > \max\{0, [2 + \ln C]/t_1\}.$$

Now we estimate the resolvent on Σ_b. By equation (7) of Lemma 3, with '$R_\lambda x$' in place of 'x', we obtain

$$\lambda^n \exp[\lambda t_1]\, R_\lambda\, x = A^n\, T(t_1)\, R_\lambda\, x + \sum_{k=0}^{n-1} \lambda^{n-k-1}\, T^{(k)}(t_1)\, x + \lambda^n\, B_\lambda(t_1)\, x.$$

Thus,

$$\|R_\lambda\, x\| \leq \frac{C\, \theta^n\, n^{\delta n}}{|\lambda|^n} \exp[-t_1\, \mathrm{Re}\, \lambda]\, \|R_\lambda\, x\| + \frac{\|x\|}{|\lambda|} \sum_{k=0}^{n-1} C\, \frac{\theta^k\, k^{\delta k}}{|\lambda|^k} \exp[-t_1\, \mathrm{Re}\, \lambda]$$

$$+ \left\| \int_0^{t_1} e^{-\lambda s}\, T(s)\, x\, ds \right\| .$$

For a fixed λ, we pick n to be the largest integer smaller than $(|\mathrm{Im}\, \lambda|\, [\theta\, e]^{-1})^{1/\delta}$. Thus,

$$\frac{\theta^n\, n^{\delta n}}{|\mathrm{Im}\, \lambda|^n} \leq e^{-n} \leq \exp[1 - (|\mathrm{Im}\, \lambda|\, [\theta\, e]^{-1})^{1/\delta}].$$

So we obtain

$$\|R_\lambda\, x\| \leq C \exp[-t_1\, \mathrm{Re}\, \lambda + 1 - (|\mathrm{Im}\, \lambda|\, [\theta\, e]^{-1})^{1/\delta}]\, \|R_\lambda\, x\|$$

$$+ \frac{\|x\|}{|\lambda|} \sum_{k=0}^{n-1} C\, e^{-k} \exp[-\, t_1\, \mathrm{Re}\, \lambda] + \int_0^{t_1} |e^{-\lambda s}|\, \|T(s)\, x\|\, ds.$$

But if $\lambda \in \Sigma_b$ then

$$-t_1 \operatorname{Re}\lambda + 1 - (|\operatorname{Im}\lambda|\,[\theta\, e]^{-1})^{1/\delta} = -\, t_1 \operatorname{Re}\lambda + 1 - b\, t_1\, |\operatorname{Im}\lambda|^{1/\delta} < -1 - \ln C.$$

Further, if $\|T(s)\| \le M\, e^{\omega s}$, then we obtain for $\lambda \in \Sigma_b$:

$$\|R_\lambda\, x\| \le e^{-1}\, \|R_\lambda\, x\| + \frac{Ce}{e-1}\, |\lambda|^{-1} \exp[-t_1 \operatorname{Re}\lambda]\, \|x\|$$

$$+ \|x\|\, M\, t_1 \exp[\omega\, t_1] \max\{1, \exp[-t_1 \operatorname{Re}\lambda]\}.$$

This yields estimate (3) since $|\lambda|^{-1}$ is bounded in Σ_b. This completes the proof that statement (a) implies statement (b).

(b) ⇒ (a): Assuming that (b) holds, and given $t_1 > t_0$, $b > 0$, we choose C_1, C_2 and $a > 0$ so that (2) and (3) are satisfied. We suppose at first that $x \in D_{A^2}$. Thus, we can find $\gamma \in \mathbf{R}$ such that for all $t > 0$,

$$T(t)\, x = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} R_\lambda\, x\; e^{\lambda t}\, d\lambda,$$

where the integral converges as an improper Riemann integral in X. Consider, for real numbers R satisfying $a - b\,|R|^{1/\delta} < \gamma$, the contour $\Gamma_R = \{s+iR : a - b\,|R|^{1/\delta} \le s \le \gamma\}$ with orientation in the direction of increasing Re λ. Since, for $x \in D_{A^2}$, we have

$$R_\lambda\, x = \frac{x}{\lambda} + \frac{A\,x}{\lambda^2} + \frac{R_\lambda A^2\, x}{\lambda^2},$$

it follows that we can find constants c_1 and c_2 so that for $\lambda \in \Sigma_b$,

$$\|R_\lambda\, x\| \le \frac{c_1}{|\lambda|} \exp[-t_1 \operatorname{Re}\lambda] + \frac{c_2}{|\lambda|}\,.$$

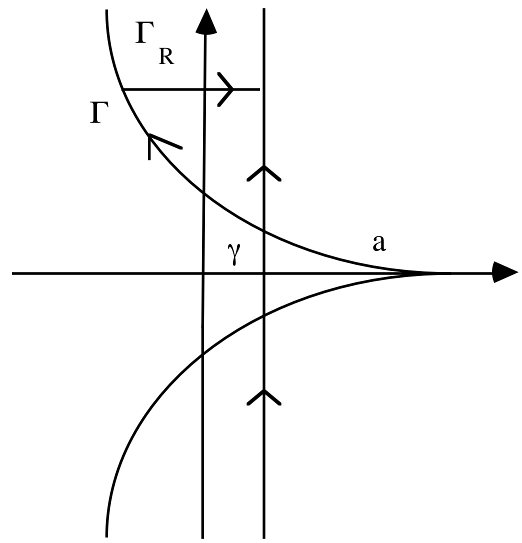

Figure 12

Thus for $t > t_1$, we have

$$\left\| \int_{\Gamma_R} R_\lambda x \ e^{\lambda t} \, d\lambda \right\| \le \int_{a - b|R|^{1/\delta}}^{\gamma} \frac{1}{|R|} (c_1 \exp[(t-t_1)s] + c_2 \exp[t\, s]) \, ds$$

$$\le \frac{1}{|R|} \left[\frac{c_1 \exp[(t - t_1)\gamma]}{t - t_1} + \frac{c_2 \exp[t\gamma]}{t} \right].$$

Clearly $\lim_{R \to \pm\infty} \int_{\Gamma_R} R_\lambda x \ e^{\lambda t} \, d\lambda = 0$ for $t > t_1$ and we may write

$$T(t)\, x = \frac{1}{2\pi i} \int_{\Xi} R_\lambda x \ e^{\lambda t} \, d\lambda \qquad (x \in D_{A^2}, \ t > t_1),$$

where Ξ is the contour consisting of the boundary of Σ_b with orientation in the direction of increasing Im λ.

We now estimate each of the integrals $\int_{\Xi} |\lambda|^n \, \|R_\lambda\| \, |e^{\lambda t}| \, |d\lambda|$. We parametrize Ξ by

$$\lambda = i\,R - b\,|R|^{1/\delta} + a, \qquad\qquad d\lambda = i\,dR \pm \frac{b}{\delta}\,|R|^{-1+1/\delta}\,dR.$$

Let $R_0 > 0$ be such that for $|R| > R_0$ we have

$$-\,b\,|R|^{1/\delta} + a \le 0, \quad |\lambda| \le 2|R|, \quad |d\lambda| \le 2|dR|.$$

Then for $t > t_1$ we obtain

$$\int_{\Xi \cap \{\lambda :\, \mathrm{Im}\,\lambda > R_0\}} |\lambda|^n\, \|R_\lambda\|\, |e^{\lambda t}|\, |d\lambda| \le 2^{n+1}\,(c_1 + c_2)\exp[a(t - t_1)] \int_{R_0}^{\infty} R^n \exp[-(t - t_1)\,b\,R^{1/\delta}]\,dR$$

$$\le 2^{n+1}\,(c_1 + c_2)\exp[a\,(t - t_1)]\,\delta \int_0^{\infty} u^{(n+1)\delta-1}\exp[-(t - t_1)\,b\,u]\,du$$

$$= \left[\frac{2}{\{b(t - t_1)\}^{\delta}}\right]^{n+1} \delta\,(c_1 + c_2)\exp[a(t - t_1)]\;\; \Gamma[(n+1)\delta].$$

But $\Gamma[(n+1)\delta] < [(n+1)\delta]^{(n+1)\delta} = \delta^{(n+1)\delta}\,[n+1]^{(n+1)\delta} < \delta^{(n+1)\delta}\,[e^{n+1}\,(n+1)!]^{\delta}$. Thus, the integral is bounded by

$$\left\{2\left[\frac{e\,\delta}{b(t - t_1)}\right]^{\delta}\right\}^{n+1} \delta\,(c_1 + c_2)\exp[[a(t - t_1)]\;[(n+1)!]^{\delta}$$

We may similarly analyse the corresponding integral over $\Xi \cap \{\lambda : \mathrm{Im}\,\lambda < -R_0\}$ and obtain the same bound for it. Also, if d is the diameter of $\Xi \cap \{\lambda : |\mathrm{Im}\,\lambda| \le R_0\}$, then it is obvious that we can find a constant N such that

$$\int_{\Xi \cap \{\lambda :\, |\mathrm{Im}\,\lambda| \le R_0\}} |\lambda|^n\, \|R_\lambda\|\, |e^{\lambda t}|\, |d\lambda| \le N\, d^n\, e^{at}.$$

We set

$$\theta = \theta(b, t - t_1) = 4\left[\frac{e\,\delta}{b\,(t - t_1)}\right]^{\delta}$$

and conclude from the estimates above that for t in any prescribed compact subset K of (t_1,∞), there exists a constant C_3, depending on only b, K and t_1, such that

$$\int_{\Xi} |\lambda|^n \, \|R_\lambda\| \, |e^{\lambda t}| \, |d\lambda| \le C_3 \, \theta^n \, (n!)^{\delta}. \tag{9}$$

These bounds show that each of the integrals $I_n(t) = \frac{1}{2\pi i}\int_{\Xi} \lambda^n R_\lambda e^{\lambda t}\, d\lambda$ converges

absolutely in the operator norm topology, uniformly for t in compact subsets of (t_0,∞). We can now proceed as in the proof of Theorem 11 in Chapter 1 to show that $I_n'(t) = I_{n+1}(t)$ for $t > t_0$ and $n \ge 0$, where the differentiation is in the sense of the operator norm topology. But for $x \in D_{A^2}$, we know that $I_0(t)\,x = T(t)\,x$. Since D_{A^2} is dense in X, it follows that $I_0(t) = T(t)$ for $t > t_0$.

The above considerations show that we can do the following: Given a compact subset K of (t_0,∞) and $\theta_0 > 0$, we let $\tau = \text{dist}\,(K, t_0)$ and set $t_1 = t_0 + \tau/2$. We choose b so that $\theta(b,\tau/2) < \theta_0$. We can now pick C_3 so that inequality (9) holds for all $t \in K$ and conclude that

$$\|T^{(n)}(t)\| \le C_3\, \theta_0^n\, (n!)^{\delta} \quad \text{for all } t \in K .$$

This completes the proof of the theorem.

□

We now give some sufficient conditions for semigroups to be of Gevrey class. Often, as we shall see in the examples, they are easier to apply than Theorem 2. We remark that all of these conditions are similar to well known conditions for differentiable or analytic semigroups, which may be found in [18].

Theorem 4 (c.f. Theorem 4.9, p 57 of [18])

Let T(t) be a strongly continuous semigroup satisfying $\|T(t)\| \le M e^{\omega t}$. Suppose that, for some $\mu \ge \omega$ and α satisfying $0 < \alpha \le 1$,

$$\limsup_{|\tau| \to \infty} |\tau|^{\alpha} \|R_{\mu+i\tau}\| = C < \infty.$$

Then T(t) is of Gevrey class δ for $t > 0$, for every $\delta > 1/\alpha$.

Proof

R_λ may be written as a Taylor series near $\mu + i\tau$:

$$R_\lambda = \sum_{k=0}^{\infty} (R_{\mu+i\tau})^{k+1} (\mu + i\tau - \lambda)^k. \tag{10}$$

Given $\varepsilon > 0$, we choose τ_0 so that for $|\tau| > \tau_0$ we have $\|R_{\mu+i\tau}\| \le \dfrac{C + \varepsilon/2}{|\tau|^{\alpha}}$. It then follows that the Taylor series converges in the operator norm topology for all λ satisfying $|\mu + i\tau - \lambda| < \dfrac{|\tau|^{\alpha}}{C + \varepsilon}$. Thus, the resolvent exists in the set

$$\{\lambda = \sigma + i\tau : |\mu - \sigma| < \frac{|\tau|^{\alpha}}{C + \varepsilon} \ , |\tau| > \tau_0\}.$$

In this set, we easily find from equation (10) that $\|R_\lambda\| \le \dfrac{(2C + \varepsilon)(C + \varepsilon)}{\varepsilon\, |\tau|^{\alpha}}$.

Further, since $\|T(t)\| \le M e^{\omega t}$, the resolvent also exists in the set $\{\lambda : \mathrm{Re}\, \lambda \ge \mu + \eta\}$, where $\eta > 0$. In this set, $\|R_\lambda\| \le M/(\omega\text{-Re}\, \lambda) \le M/\eta$. Thus, it follows that

$$\rho(A) \supset \{\lambda : \mathrm{Re}\, \lambda \ge \mu + \eta + \frac{\tau_0^{\alpha} - |\mathrm{Im}\, \lambda|^{\alpha}}{C + \varepsilon} \ \}$$

and that $\|R_\lambda\|$ is bounded in Σ. Suppose that $\delta > 1/\alpha$. Given any $b > 0$, it is clear that 'a' can be chosen so that $\Sigma \supset \Sigma_b = \{ \lambda : \text{Re } \lambda \geq a - b\, |\text{Im } \lambda|^{1/\delta}\}$. By Theorem 2, T(t) is of Gevrey class δ for $t > 0$.

□

Theorem 5 (c.f. Theorem 4.11, p 58, and Corollary 5.7, p 68 of [18])

Let T(t) be a strongly continuous semigroup and suppose that there exist constants α, C and ε such that $0 < \alpha \leq 1$, $C > 0$, $\varepsilon > 0$ and

$$\| T(t) - I \| \leq 2 - C\, t^{1-\alpha} \qquad \text{for all } t \in (0,\varepsilon).$$

Then T(t) is of Gevrey class δ for $t > 0$, for every $\delta > 1/\alpha$.

Proof

The proof is just a simple modification of the proof of Theorem 4.11 in [18], so we refer the reader to this source. We remark that the proof here relies on our Theorem 4, while in [18], use is made of the corresponding result for differentiable semigroups (Theorem 4.9, p 57).

□

Remark: If $\alpha = 1$ in the statement of Theorem 5 then the semigroup is analytic. This is a simple consequence of Corollary 5.7, p 68 of [18].

Theorem 6 (c.f. Theorem 5.2 (d), p 61 of [18], for analytic semigroups).

Let T(t) be a differentiable semigroup and let $\delta > 1$. Suppose that

$$\lim_{t \downarrow 0} t^{\delta} \|T'(t)\| = 0.$$

Then T(t) is of Gevrey class δ for $t > 0$.

Proof

Let $\theta > 0$ and $t_1 > 0$. We set $C = (t_1/e)^\delta \theta$ and pick $\varepsilon > 0$ so that $\|T'(t)\| \le C t^{-\delta}$ for $t \in (0,\varepsilon)$. Now, $T^{(n)}(t_1) = (T'(t_1/n))^n$, so if $n > t_1/\varepsilon$, we have

$$\|T^{(n)}(t_1)\| \le [C n^\delta / t_1^\delta]^n \le [C e^\delta / t_1^\delta]^n (n!)^\delta \le \theta^n (n!)^\delta.$$

It follows that we can find a constant C_1 such that

$$\|T^{(n)}(t_1)\| \le C_1 \theta^n (n!)^\delta \qquad \text{for each } n \in \{0, 1, 2, 3, \ldots\}.$$

The result follows, because of Theorem 1.

□

In some of the examples to follow, we use the following simple consequence of the above result:

Corollary 7

Let T(t) be a differentiable semigroup and $\beta \ge 1$. Suppose that there exist constants C and ε such that $\|T'(t)\| \le C t^{-\beta}$ for $0 < t < \varepsilon$. Then T(t) is of Gevrey class δ for $t > 0$, for every $\delta > \beta$.

□

The following result gives a test for Gevrey regularity of semigroups using the behavior of R_λ for only real, positive values of λ. The result itself is similar to a result about analytic semigroups (see Theorem 5.5, p 65, of [18]).

Theorem 8

Let A be the infinitesimal generator of a strongly continuous semigroup T(t). Suppose that there exist constants $C > 0$, $\beta \ge 1$ and $\Lambda \ge 0$ such that

$$\|A (R_\lambda)^{n+1}\| \le C n^{-\beta} \lambda^{\beta-n-1} \quad \text{for } \lambda > n\Lambda, \quad n \in \{0, 1, 2, 3, \ldots\}.$$

Then T(t) is of Gevrey class δ for $t > 0$, for every $\delta > \beta$.

Proof

The proof of Theorem 8 is just a simple modification of the proof of the corresponding result about semigroups, so we refer the reader to [18], (Theorem 5.5). We remark that Theorem 8 is proved by showing that the conditions in the statement of it imply that the conditions in the statement of Corollary 7 hold. Indeed, these two sets of conditions are equivalent.

5.2 Examples of Gevrey Semigroups

Notation: For some of these examples, we use the following spaces:

(i) $\mathcal{l}^2$ is the Hilbert space of 'square summable' sequences on the positive integers, $\mathbf{Z}^+$. The inner product on $\mathcal{l}^2$ is $(\, , \,)_0$, where

$$(\{f_n\}, \{g_n\})_0 = \sum_{n=0}^{\infty} \bar{f}_n \, g_n \, .$$

(ii) For an integer $k \geq 1$, $\mathcal{h}^k$ denotes the Hilbert space of sequences $\{f_n\}$ on $\mathbf{Z}^+$ such that $\{n^k f_n\} \in \mathcal{l}^2$. The inner product here is $(\, , \,)_k$, where

$$(\{f_n\}, \{g_n\})_k = \sum_{n=0}^{\infty} n^{2k} \, \bar{f}_n \, g_n \, .$$

Example 1

Suppose $0 < \alpha \leq 1$, and consider on $\mathcal{l}^2$ the semigroup given by

$$T(t) \{f_n\} = \{f_n \exp[(in - n^{\alpha})t]\}.$$

It is easy to see that T(t) is a strongly continuous semigroup. Moreover, it is easy to see that it is a differentiable semigroup and

$$T'(t) \{f_n\} = \{(in - n^{\alpha}) \, f_n \exp[(in - n^{\alpha})t]\}.$$

Further,

$$\|T'(t)\| \leq \sup_{n\geq 0} 2n\, e^{-n^{\alpha} t} \leq 2\,(e\,\alpha\,t)^{-1/\alpha}.$$

Corollary 7 shows that, for any $\delta > 1/\alpha$, T(t) is of Gevrey class δ for $t > 0$. We may easily compute A, the infinitesimal generator of T(t):

$$D_A = \mathcal{l}^1, \qquad A\,\{f_n\} = \{(in - n^{\alpha})\, f_n \exp[(in - n^{\alpha})t]\}.$$

R_λ, the resolvent of A is given by the expression

$$R_\lambda \{f_n\} = \left\{\frac{f_n}{\lambda - in + n^{\alpha}}\right\},$$

while the spectrum of A is $\sigma(A) = \{in - n^{\alpha} : n \in \mathbf{Z}^+\}$. We note that while the semigroup is uniformly bounded, the resolvent set of A clearly *does not* contain any sector $\{\lambda : |\arg \lambda| < \pi/2 + \varepsilon\}$ for $\varepsilon > 0$, *unless* $\alpha = 1$. Thus, we get an example of a Gevrey semigroup which is *not* analytic.

To further illustrate the theory, we also use Theorem 8 to prove that T(t) is a Gevrey semigroup. Clearly for real λ

$$\|A\,(R_\lambda)^{m+1}\| \leq \sup_{n\geq 0} \frac{(n^2 + n^{2\alpha})^{1/2}}{[(\lambda + n^{\alpha})^2 + n^2]^{(m+1)/2}}$$

$$\leq \sup_{n\geq 0} \frac{\sqrt{2}\; n}{(\lambda + n^{\alpha})^{m+1}} \leq \sqrt{2}\;[(m+1)\alpha]^{-1/\alpha}\; \lambda^{-(m+1)+1/\alpha}.$$

So an application of Theorem 8 yields the same conclusion obtained from Corollary 7.

□

Example 2

Let $\alpha > 1$ and F be the space of functions $f : [0,1] \to \mathbf{C}$ satisfying the conditions:

(i) f is continuous on [0,1].

(ii) f is infinitely differentiable on (1/2,1) and for each $n \geq 0$ $f^{(n)}$ can be continued to be continuous on [1/2,1] with $f^{(n)}(1) = 0$.

(iii) $$p(f) = \sum_{n=0}^{\infty} (n!)^{-\alpha} \max_{x \in [1/2,1]} |f^{(n)}(x)| < \infty.$$

It is easily verified that F is a Banach space with norm given by

$$\| f \| = \max_{x \in [0,1/2]} |f(x)| + p(f).$$

We define a semigroup T(t) on F as follows:

$$[T(t)\, f](x) = \begin{cases} f(x+t) & \text{if } x+t \leq 1 \\ 0 & \text{if } x+t > 1 \end{cases}$$

T(t) is strongly continuous, infinitely differentiable for $t > 1/2$ and is zero for $t > 1$. If $t > 1/2$ then

$$\|T^{(n)}(t)\, f\| = \max_{x \in [0,1/2]} |f^{(n)}(x+t)| \leq \max_{x \in [1/2,1]} |f^{(n)}(x)| \leq (n!)^{\alpha}\, p(f) \leq (n!)^{\alpha} \| f \|.$$

It follows that if δ is any constant satisfying $\delta > \alpha$, then T(t) is of Gevrey class δ for $t > 1/2$.

□

Example 3

Suppose $0 < \alpha \leq 1$, and consider on $\ell^1 \times \ell^2$ the operator A given by $D_A = \ell^2 \times \ell^1$,

$$A\begin{bmatrix}\{f_n\}\\ \{g_n\}\end{bmatrix} = \begin{bmatrix}\{g_n\}\\ \{-n^2 f_n - n^{\alpha} g_n\}\end{bmatrix}.$$

A simple calculation shows that A generates a differentiable semigroup T(t) given by

$$T(t)\begin{bmatrix}\{f_n\}\\ \{g_n\}\end{bmatrix} = \begin{bmatrix}\{f_n \exp[-\mu_n t]\,(\cos[\omega_n t] + \frac{\mu_n}{\omega_n}\sin[\omega_n t])\}\\ \{f_n \exp[-\mu_n t]\,(\frac{-n^2}{\omega_n}\sin[\omega_n t])\}\end{bmatrix}$$

$$+ \begin{bmatrix}\{g_n \exp[-\mu_n t]\,\frac{1}{\omega_n}\sin[\omega_n t]\}\\ \{g_n \exp[-\mu_n t]\,(\cos[\omega_n t] - \frac{\mu_n}{\omega_n}\sin[\omega_n t])\}\end{bmatrix},$$

where $\mu_n = \frac{1}{2} n^{\alpha}$, $\quad \omega_n = n\sqrt{1 - \frac{1}{4} n^{2\alpha - 2}}$.

We remark that the semigroup describes the 'structural damping' of the wave equation in the following system:

$$\frac{\partial^2 w}{\partial t^2} + \tilde{A}^{\alpha/2}\frac{\partial w}{\partial t} + \tilde{A} w = 0,$$

$$w(0,t) = w(\pi,t) = 0, \quad w(x,0) = w_0(x), \; v(x,0) = v_0(x),$$

where $\tilde{A} = -\frac{\partial^2}{\partial x^2}$.

This can be seen by writing the problem as a first order system involving w and $v = \frac{\partial w}{\partial t}$, and then expanding:

$$w(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin[nx], \qquad v(x,t) = \sum_{n=1}^{\infty} g_n(t)\sin[nx].$$

Now,

$$T'(t)\begin{bmatrix}\{f_n\}\\ \{g_n\}\end{bmatrix} = \begin{bmatrix}\{f_n \exp[-\mu_n t]\,(\frac{-n^2}{\omega_n}\sin[\omega_n t])\}\\ \{f_n \exp[-\mu_n t]\,(\frac{n^2\,\mu_n}{\omega_n}\sin[\omega_n t] - n^2\cos[\omega_n t])\}\end{bmatrix}$$

$$+\begin{bmatrix}\{g_n \exp[-\mu_n t]\,(\cos[\omega_n t] - \frac{\mu_n}{\omega_n}\sin[\omega_n t])\}\\ \{g_n \exp[-\mu_n t]\,(\frac{2\mu_n^2 - n^2}{\omega_n}\sin[\omega_n t] - 2\mu_n\cos[\omega_n t])\}\end{bmatrix}.$$

From this, a short calculation reveals the fact that there exists a constant C such that

$$\|T'(t)\| \le C \sup_{n\ge 0}\; n \exp[-\mu_n t] \;\le\; C\, 2^{1/\alpha}(e\,\alpha\, t)^{-1/\alpha}.$$

We conclude, using Corollary 7, that for any $\delta > 1/\alpha$, T(t) is of Gevrey class δ for $t > 0$.

□

Remarks: The above example is a particular case of systems studied by R. Triggiani and S. Chen ([1], [2], [3]), which involve the abstract differential equation

$$\frac{d^2x}{dt^2} + B\frac{dx}{dt} + A\,x = 0$$

in a Hilbert space X. In this, A is supposed to be a strictly positive, self adjoint operator, densely defined, with a compact resolvent. B is assumed to be comparable with A^β in some sense ($0 \le \beta \le 1$). These authors treat the problem as a first order system and show that the solutions are obtained from a strongly continuous semigroup which,

(i) for $1/2 \le \beta \le 1$, is analytic ([2], [3]),

(ii) for $0 < \beta < 1/2$, is differentiable, but not necessarily analytic ([1]),

(iii) for $\beta = 0$, is a group.

W. Littman has conjectured that in case (ii) the semigroup is in a Gevrey δ class, where δ is some function of β. We verify this below for the case $B = 2\rho A^{\beta}$, where ρ is a positive constant satisfying a certain condition. We will see that the conclusion in this case follows immediately from an estimate in [1]. The conjecture is also correct for the more general operators B considered by Triggiani and Chen (see the remark after the example).

Example 4

Let A be as above and let $B = 2\rho A^{\beta}$ $(0 < \beta < 1/2)$. It is shown in [1] that the operator

$$\mathcal{A}_{\rho\beta} = \begin{bmatrix} 0 & I \\ -A & -2\rho A^{\beta} \end{bmatrix}, \quad D_{\mathcal{A}_{\rho\beta}} = D_A \times D_{A^{1/2}}$$

on the Hilbert space $E = D_{A^{1/2}} \times X$, generates a differentiable semigroup of contractions.

The eigenvalues of $\mathcal{A}_{\rho\beta}$ are given by

$$\lambda_n^{+,-} = (-\rho \pm \sqrt{\rho^2 - \mu_n^{1-2\beta}}\,)\,\mu_n^{\beta},$$

where $0 < \mu_1 \le \mu_2 \le \mu_3 \le \ldots \to \infty$ are the eigenvalues of A. It is shown ([1], Corollary 2.3) that if

$$\rho^2 \neq \mu_n^{1-2\beta} \quad \text{for all } n,$$

then the resolvent R_{λ} of $\mathcal{A}_{\rho\beta}$ satisfies:

$$|\tau|^{2\beta}\, \| R_{i\tau} \| \le \text{constant} \quad \text{for all } \tau \in \mathbf{R}.$$

We may immediately conclude from Theorem 4 that the semigroup is of Gevrey class δ for $t > 0$, δ being any constant $> \dfrac{1}{2\beta}$.

□

Remark: The more general case investigated in [1], in which '$2\,\rho\,A^{\beta}$' is replaced in the system above by 'B', also yields a Gevrey semigroup. Here, B is a self adjoint operator such that $D_B = D_{A^\beta}$, and for some constants $0 < \rho_1 < \rho_2 < \infty$,

$$\rho_1\,(A^\beta x,x)_X \le (Bx,x)_X \le \rho_2\,(A^\beta x,x)_X \qquad \text{for all } x \in D_{B^{1/2}} = D_{A^{\beta/2}}.$$

If R_λ denotes the resolvent of the corresponding operator

$$\mathcal{A}_B = \begin{bmatrix} 0 & I \\ -A & -B \end{bmatrix},$$

then we may show that

$$\limsup_{|\tau|\to\infty} |\tau|^{2\beta}\, \|R_{i\tau}\| = C < \infty. \tag{11}$$

Hence we can again conclude from Theorem 4 that the semigroup is of Gevrey class δ for $t > 0$, δ being any constant $> \frac{1}{2\beta}$. The estimate is established as in [1], where a similar estimate is proved (with '$\log|\tau|$' instead of '$|\tau|^{2\beta}$' in the estimate above). In their proof, Triggiani and Chen drop the polynomial dependence of $\|R_{i\tau}\|$ on τ that they had established for the case $B = 2\,\rho\,A^\beta$, and use instead of it a logarithmic estimate, which is all that they need to establish their estimate and obtain the differentiability of the semigroup. However, if the polynomial dependence is retained in their calculations, (11) is obtained.

Bibliography

[1] Chen, S., Triggiani, R., *Differentiable Semigroups Arising from Elastic Systems with Gentle Dissipation : the Case* $0 < \alpha < 1/2$; to appear.

[2] Chen, S., Triggiani, R., *Proof of Two Conjectures by G. Chen and D. L. Russel on Structural Damping for Elastic Systems;* Lecture Notes in Mathematics, 1354, 1987, Springer-Verlag, pp 234-256.

[3] Chen, S., Triggiani, R., *Proof of Extensions of Two Conjectures on Structural Damping for Elastic Systems;* Pacific Journal of Mathematics, vol 136, no. 1, 1989.

[4] Coddington, E. A., Levinson, N., *Theory of Ordinary Differential Equations;* Krieger, 1984.

[5] Delsarte, J., *Sur Certaines Transformations Fonctionelles Relatives aux Équations Linéaires aux Dérivées Partielles du Second Ordre;* C. R. Acad. Sc., vol 206, 1938, p 1780.

[6] Gevrey, M., *Sur la Nature Analytique des Solutions des Équations aux Dérivées Partielles;* Annales Scientifiques de l'École Normale Supérieure, vol 35, 1918, pp 129-190. This is reproduced in: *Oeuvres de Maurice Gevrey;* Centre National de la Recherche Scientifique, Paris, 1970.

[7] Gilbarg, D., Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order,* 2nd Edition, Springer-Verlag, 1983.

[8] Hörmander, L., *Linear Partial Differential Operators;* Academic Press, New York, 1963.

[9] John, F., *Partial Differential Equations,* 4th Edition; Springer-Verlag, 1982.

[10] Lagnese, J. E., Lions, J. L., *Modelling, Analysis and Control of Thin Plates;* 1988.

[11] Lions, J. L., *Contrôlabilité Exacte des Systemes Distribués;* Comptes Rendus, 302, series 1, 1986, pp 471-476.

[12] Lions, J. L., *Exact Controllability, Stabilization and Perturbations for Distributed Systems;* SIAM Review, vol 30, no.1, March 1988.

[13] Lions, J. L., Magenes, E., *Non Homogeneous Boundary Value Problems and Applications,* vol 1; Springer-Verlag, 1972.

[14] Lions, J. L., *Opérateurs de Delsarte et Problèmes Mixtes;* Bull. Soc. Math. France, vol 84, 1956, pp 9-95.

[15] Lions, J. L., *Quelques Applications D'Opérateurs de Transmutation;* Colloque C. N. R. S., Nancy, 1956, pp 125-142.

[16] Littman, W., *Boundary Control Theory for Beams and Plates,* Proceedings, 24th Conference on Decision and Control, Ft. Lauderdale, FL, December, 1985.

[17] Littman, W., Markus, L., *Exact Boundary Controllability of a Hybrid System of Elasticity;* Archive for Rational Mechanics and Analysis, vol 103, no. 3, 1988, pp 193-236.

[18] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations;* Springer-Verlag, 1983.

[19] Rayleigh, J. W. S., *The Theory of Sound,* vol 1; Dover, 1945.

[20] Rudin, W., *Real and Complex Analysis,* 3rd Edition; McGraw-Hill, 1987.

[21] Taylor, L., Balakrishnan, A. V., *The SCOLE Design Challenge,* 4th Annual NASA SCOLE workshop, 1987 (Reprinted from NASA-IEEE conference of 1984), NASA Technical Memo 101503.

[22] Treves, F., *Basic Linear Partial Differential Equations;* Academic Press, 1975.

[23] Treves, F., *Ovcyannikov Theorem and Hyperdifferential Operators;* Notas de Matematica, no. 46, 1968.

[24] Trimèche, K., *Transmutation Operators and Mean-Periodic Functions Associated with Differential Operators;* Mathematical Reports, vol 4, part 1, Harwood Publishers, 1988.

Notation Index

Spaces

Operators

Functions

Norms and Seminorms

Miscellaneous