Calculus for Teachers: Vision and Considerations of Mathematicians

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Acknowledging the significant contribution of mathematicians to the mathematical education of teachers, we explore the views of mathematicians on an envisioned Calculus course for prospective teachers. We analyzed the semi-structured interviews with 24 mathematicians, using the EDW (Essence-Doing-Worth) framework (Hoffmann & Even, 2018, 2019); and subsequently, we adapted the framework by extending and refining the existing themes. The findings of our study indicate that the mathematicians believe the primary purpose of a Calculus course for teachers is to communicate the nature of mathematics as a discipline. By providing a variety of examples that could shape and expand the teachers' understanding of mathematics, the majority of the mathematicians participated in the study emphasized the value of mathematical investigation in an envisioned Calculus course for teachers, as well as connections within and beyond the subject.

Keywords: advanced mathematical knowledge, mathematicians as educators, calculus, teacher education

Calculus is one of the standard mathematics courses offered to a wide range of student populations in most universities. Special Calculus courses are usually designed for engineering, life sciences, or business majors. While prospective secondary mathematics teachers are often required to take one or two Calculus courses at university, a special Calculus course designed for teachers is rarely available. Research on the mathematical development of prospective teachers has shown that learning Calculus at the undergraduate level enriches prospective teachers' understanding of the mathematical concepts introduced in school (e.g., Keene et al., 2014; Wasserman & Weber, 2017). These research findings make us wonder how a Calculus course for teachers would look like. In this paper, we report on mathematicians' views of a Calculus course designed specifically for prospective teachers.

Theoretical Underpinnings

Research Background

The relevance and contribution of university-level mathematics to the education of secondary school mathematics teachers have been debated for decades. Nevertheless, a broad consensus reached among researchers is that mathematics teachers at the secondary level need to have insight into advanced mathematics (e.g., Dörfler & McLone 1986; Ferrini-Mundy & Findell, 2001; Murray et al., 2017; Winsløw & Grønbæk, 2014).

Zazkis and Leikin (2010) described advanced mathematical knowledge (AMK) as knowledge of the subject matter acquired in mathematics courses taken as part of a degree from a university or college. Mathematics as a scientific discipline taught at university has an axiomatic-deductive structure and focuses on the rigorous establishment of theory concerning definitions, theorems, and proofs (Klein, 2016). This approach is often adopted in mathematics courses for prospective teachers to help them gain advanced mathematical knowledge (AMK) and develop advanced mathematical thinking (AMT). Tall (1991) examined differences between elementary and advanced mathematical thinking as transitions from *describing* to *defining*, from *convincing* to *proving* based on abstract entities. These transitions are often considered challenging by prospective teachers. In fact, research on secondary mathematics teachers' conceptions of the role and usage of AMK in their teaching practice, has shown that while some teachers acknowledged its importance, others are unaware of the connections between advanced and secondary mathematics, and are often dismissive of their upper-level training (e.g., Goulding, Hatch, & Rodd, 2003; Even, 2011; Zazkis & Leikin, 2010).

Dreher Lindmeier, Heinze, and Niemand (2018) pointed out that the gap between school mathematics and the advanced mathematics taught at university is often too wide, and as a result, it is difficult for prospective teachers to make connections between the two. A few studies have sought possible connections related to mathematics content and disciplinary practices. Concerning mathematics content, Jukic and Brückler (2014) showed that Calculus tasks that connect embodied conceptions and symbolic manipulations promote flexibility in mathematical thinking of prospective teachers. Concerning disciplinary practices, Wasserman, Fukawa-Connelly, Villanueva, Mejia-Ramos, and Weber's (2017) research explored how the study of proofs in real analysis could be used to enhance the teachers' ability to engage in quality instructional practices at the secondary school level. Other studies shed light on how abstract algebra might support teachers to unpack particular secondary mathematics topics and how they might shape pedagogy in the secondary classroom (e.g., Christy & Sparks, 2015; McCrory et al., 2012; Murray et al., 2017). However, the gap between advanced and school mathematics is still evident. To support prospective teachers in making possible connections, researchers need to further explore the relationship between advanced and school mathematics.

Leikin, Zazkis, and Meller (2018) noted that as mathematicians teach mathematics to prospective teachers, they "act as teacher educators de facto, without explicitly identifying themselves in this role" (p. 452). As such, understanding mathematicians' views is relevant in an effort to facilitate change in undergraduate mathematics teaching that supports prospective teachers. Moreover, Bass (2005) and Hodge et al. (2010) claimed that mathematicians' knowledge, practices, and habits of mind are essential for maintaining the mathematical balance and integrity of the educational process.

Research on mathematicians' views regarding advanced mathematics studies of teachers has not received much attention. The existing empirical research literature is mainly based on interviews with teachers. Only few studies sought the relevance and contribution of academic mathematics studies to secondary school mathematics teaching, taking into account the views of mathematicians in curriculum planning and course design (e.g., Goos, 2013; Hoffmann & Even, 2018; Leikin, Zazkis & Meller, 2018). Nevertheless, many questions remain. For instance, which connections between advanced and school mathematics are important, and how they might lead to the improvement of teachers' practice (Murray, Baldinger, Wasserman, Broderick, & White, 2017).

Theoretical Framing

Ziegler and Loos (2014) identified two dimensions of mathematical knowledge that appear to be critical for teaching: one dimension is knowledge of specific topics, concepts, and procedures, and the other is a more general epistemological knowledge about what mathematics is and what doing mathematics entails. Focusing on the latter dimension, Hoffmann and Even (2018; 2019) identified three aspects of mathematics that the research mathematicians wanted teachers to acquire in their study: (1) the Essence of mathematics, (2) Doing mathematics, and (3) the Worth of mathematics. Each of these aspects includes several sub-themes, a total of nine sub-themes can be found on the left-hand side of Figure 1. These aspects form the Essence-Doing-Worth conceptual framework (henceforth referred to as the EDW framework) intended to serve in studying the relevance and contribution of academic mathematics courses to the teaching of mathematics in secondary schools, and in analyzing teachers' views on what mathematics is.

As we are concerned with potential affordances of a Calculus course to the education of secondary school mathematics teachers, the EDW framework is highly relevant to our study (we additionally expanded this framework, as described in the next section). In particular, we are interested in what mathematicians wish to teach in a Calculus course for prospective teachers. In this paper, we address the following research questions: *How do mathematicians envision a Calculus course designed specifically for teachers? What particular features do they consider important in such a course?*

Method

Participants and Data Collection

24 mathematicians from 10 research universities participated in our study. All the participants were Mathematics Faculty members, representing a variety of specializations within mathematics. All participants have taught a Calculus course in the past or at the time of the data collection.

The mathematicians participated in individual semi-structured interviews aimed to gain insight into how they envisioned a Calculus course designed for teachers. The following guiding questions were posed to the interviewees:

1. What would you like teachers to know and experience about the mathematics taught in university?
2. If you were to design a Calculus course for teachers, how would you adapt an existing Calculus course?

These questions were followed by prompts for expansion and elaboration, as necessary. All the interviews were audio-recorded and transcribed. Additional written artifacts generated by the interviewees were collected for qualitative analysis.

Data Analysis

The interview transcripts were analyzed using iterative and comparative data analysis with the assistance of Nvivo 12. The nine themes from Hoffmann and Even's (2019) conceptual framework were set up as the initial thematic codes in a hierarchical structure (see left-hand side of Figure 1).

In the first round of the data analysis, relevant quotes in the transcripts were coded by theme. As the analysis proceeded, several new themes emerged: *joy, developing creativity, human endeavor,* and *investigating through technology.* In the second round, the connections between initial and emerged themes were explored and identified. The two initial themes *wide and varied* and *lively and developing* were combined for a new theme *human endeavour*, given that two themes were frequently mentioned together. In addition, a third sub-theme *the worth of mathematics as human activity* was added to the main theme *the worth of mathematics*.

In the third round of the data analysis, the quotes assigned to the frequently mentioned themes *investigating* and *using intuition and formalism* were re-examined due to multiple meanings given by the participants in relation to these themes. As a result, we refined the theme *investigating* to three sub-themes: 1) investigating for learning, 2) investigating for teaching, and 3) investigating through technology. Similarly, the theme *using intuition and formalism* was also refined to three sub-themes: 1) use/confront intuition, 2) formalization, and 3) mathematical language and notation. (In the next round of our data analysis we will focus on refining additional recurring themes *thinking and understanding* and *the practical worth of mathematics*)

The right-hand side of Figure 1 shows an overview of our adaption of the EDW conceptual framework. The combined, refined, and new themes are shaded in grey. The numbers in parentheses at the end of each theme indicate the number of the mathematicians, out of 24, who have discussed the corresponding theme during the interviews.

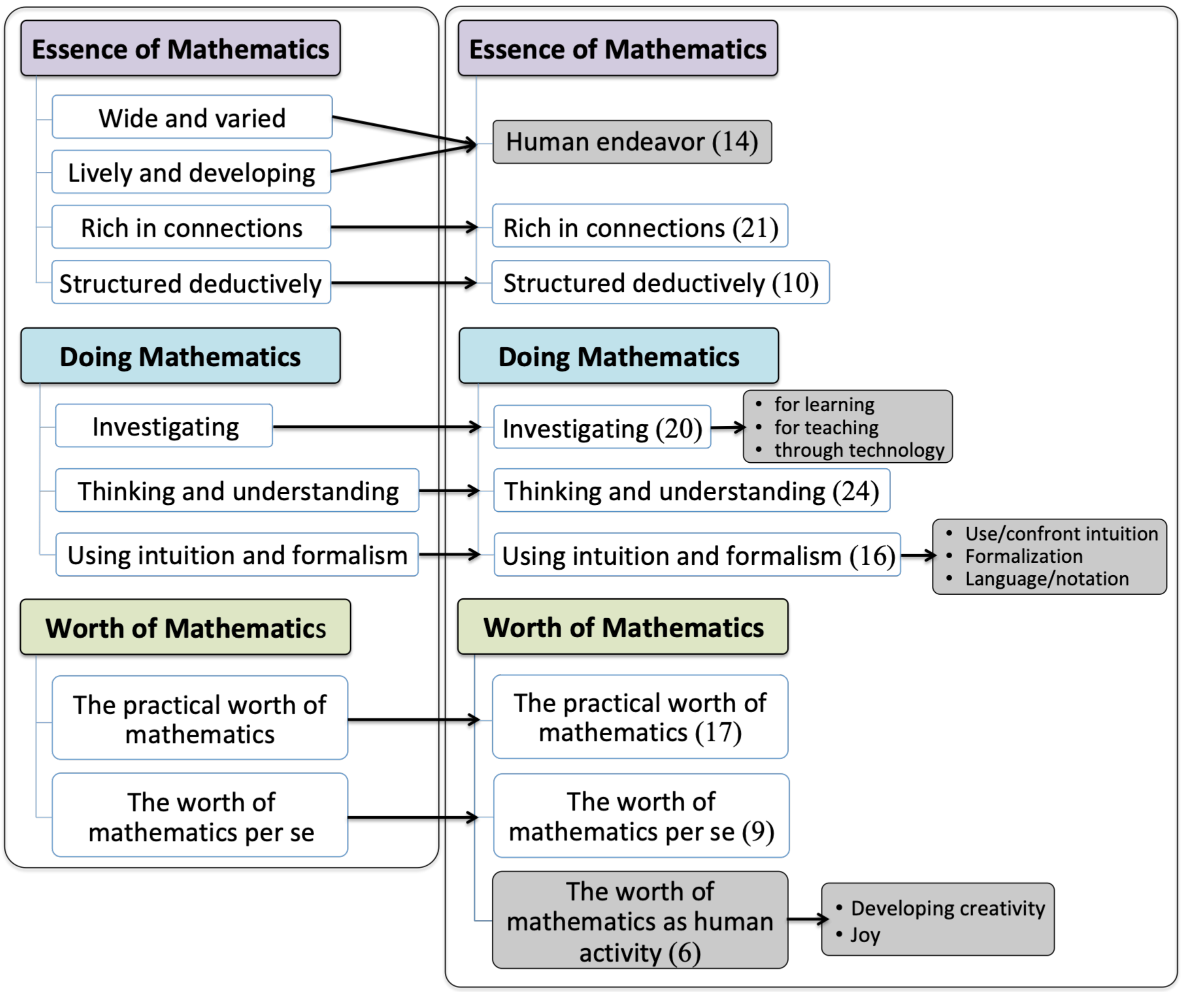

Figure 1. Hoffmann and Even's EDW conceptual framework (on the left) and our adaption (on the right).

Findings

As shown in Figure 1, the mathematicians' responses speak to all the initial themes in Hoffmann and Even's (2019) conceptual framework; however, some themes were mentioned more frequently than others. For example, all 24 participating mathematicians explained how *doing mathematics* enriches the thinking and understanding of the subject. Taking into account the needs of prospective teachers, 20 mathematicians particularly emphasized the important role of investigating in the sense of *doing mathematics*. Additionally, a vast majority of the mathematicians discussed connections between mathematics branches as well as connections to other disciplines.

We present the results of our analysis by focusing on two themes: 1) the essence of Mathematics/Calculus – rich in connections, and 2) doing Mathematics/Calculus – using intuition and formalism. These two themes were chosen based on 1) the limited literature on possible connections between advanced and secondary mathematics, and how these might lead to the improvement of teachers' practice, and 2) an understanding of the key concepts in Calculus, such as derivatives, requires both intuition and formal reasoning. In what follows, we elaborate on these themes. We refer to the participating mathematician as Mi, where i = 1…24.

On the Essence of Calculus – Rich in Connections

Three participants mentioned that regardless of which Calculus course they teach, approximately 70 percent of the course contents overlap (M11), and this is precisely the core – "the most valuable component of a Calculus course" (M9) that they would like to teach prospective teachers. M1 believed that rather than adding another "flavor" to Calculus, generic Calculus would benefit prospective teachers the most. He explained:

> When Calculus is a flavor, there are two things that happen. One is level. You can have more or less proof, more or less theorems, more or less explanation compared to the technical stuff. That is a decision that can be made in a particular course. The other is so-called "word problems in situations that are considered." I am not convinced that the "real" problems that are posted are valuable. Instead of maximizing a function, call that function a revenue function, so suddenly, it is Calculus for business. I think everyone can learn generic Calculus. […] You need to learn the essence.

One critical aspect of the essence of Calculus lies in its richness in connection. As M11 put it, "Calculus is less like a bucket of techniques that you carry around with you but more like a reticulation of ideas that are all at some level connected. To learn mathematics is really to learn about the connections." In the same spirit, M7 argued that the purpose of a Calculus course for teachers, if appropriately tailored, would be to help teachers see the connections between concepts and value these connections.

M6 suggested cutting off about a third of the topics in a regular Calculus course to open up space so that prospective teachers could have time to dig deeper into the essentials – "make it less of a race and more of exploration." M10 also suggested focusing on a handful of central ideas in Calculus rather than cover many topics at a very shallow depth. With interest in making connections between seemingly unrelated concepts, M19 envisioned his Calculus course for teachers with far less content but more time allocated in class for connecting concepts:

> Upon considering the choice of what it is that we ought to teach high school teachers, I think we should pay attention to how deliberate should we be in connecting what we teach high school teachers to what they are going to teach in high school. Giving the teachers really interesting mathematics for an almost purely appreciative goal is maybe insufficient. […] It would be great if teachers think of Calculus as a part of a coherent network of ideas that constitutes mathematics. And I think one way to encourage that view of mathematics is to teach it that way to the high school teachers.

"*Rich in connections*" was also interpreted by the mathematicians in the sense that a mathematical concept can be approached and explained using multiple models and multiple representations. More importantly, developing the flexibility of switching between models and representations should be the goal of such a Calculus course (M4). Taking "π" as an example, M15 argued that while the most usual definition of π is the ratio of the circumference of a circle to its diameter, π could be defined in various ways:

> For example, you can define π analytically: you can define $\frac{\pi}{2}$ to be the place where cosine is 0, having defined cosine as a power series [$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$]. Then say, well, let's define π as twice the smallest positive zero of the power series. And then you can derive all the other properties. […] I think this is closer to where they [teachers] are living because they have known π like since Grade 6... Now that is advanced mathematical knowledge that high school teachers need.

In a Calculus course for teachers, M15 would further challenge teachers by asking why πr^2 is the area when $2\pi r$ is the circumference. "That needs some explanation. And that actually does come up in the first year of Calculus."

On Doing Calculus – Using Intuition and Formalism

Mathematics is not a spectator sport (Phillips, 2005). "Roll up your sleeves and do it" (M10) because "mathematics embodies structure and one has to spend time doing it to understand the structure" (M9). In discussing what it means to "do Calculus", the mathematicians placed value on building and using intuition to understand concepts in Calculus. For example, to introduce the idea of what a limit is, M14 imagined teachers using a computer to plot curves and experiment with the derivatives of those curves:

> I would have them pick a point on the curve. I would have it set up so that they could zoom in repeatedly over and over and over again until eventually they see it looks like a line. […] Where that intuition comes from? To me the intuition comes from making a physical change and seeing what the physical outcome is.

While building and using intuition is an important aspect of doing Calculus, the mathematicians also pointed out situations in which intuition needs to be formalized. For example, M11, M19, and M23 used a typical question on the rate of change to show the necessity of formalization: If a car has traveled 90km in one hour on a road with an 80km/h speed limit, did the car break the speed limit? Intuitively, the answer is obvious. Justifying the intuition, however, requires a careful examination of the global and local behavior of a function, and an understanding of average versus instantaneous speed. The transformation from intuition to formalization is often challenging, yet this is where the fundamental concepts in Calculus come into play (M23).

In addition to formalizing intuition, the mathematicians also discussed situations in which intuition and mathematical formalization need to be reconciled. M11 stated that,

> It is dangerous to believe your intuition is always right. But you should trust your intuition because you want to believe by reasoning through you will have a deeper understanding. […] It is intuition, math, intuition, math… that kind of pattern.

Furthermore, M11 provided two related problems that he would work with students while attending to their intuition:

1. If a full bucket of water is poured into a small cup, then the cup is going to flow over. If this bucket of water is poured into the Lake of Ontario, have the beaches been flooded?

2. One makes a metal piece of track around a basketball tight. If the metal track is increased by a meter, then there will be a gap between the basketball and the track. Now replace the basketball with the earth, would this one-meter make any difference?

In explaining how intuition may support or hinder reasoning, M11 stated:

> Your intuition is a key guide. And in most of the time in a Calculus course, intuition is really something to be developed. But you also want to use the mathematics to verify whether your intuition is correct. And sometimes a simple mathematical tool, the idea of relative sizes, in this case, can actually show you where you have misconceptions.

In M11's view, it is crucial for teachers to learn to confront intuition with mathematical tools and reasoning. This resonates with M14, "I would put much more focus on knowing when to use what tool to evaluate whether you have a sensible answer."

In line with M11 and M14, M18 shared her experience in teaching derivatives, which highlights the role that counter-examples play in challenging one's intuition. As M18 explained, in a first-year Calculus course, many students could get an intuitive understanding that if the derivative is positive at a point then the function is monotonically increasing on the interval around that point. This intuitive understanding can be challenged by exemplifying oscillating functions and can be further corrected as follows: if the derivative is positive on an *interval*, then the function is monotonically increasing on that interval, by the Mean Value Theorem. As stated by M18, "There are these weird functions that have oscillations, which is not obvious. […] However, counter-examples break their [students'] intuition and allow them to appreciate why a theorem is needed."

Conclusion

Acknowledging the significant contribution of mathematicians to the mathematical education of teachers, we set to address how mathematicians envision a Calculus course designed specifically for teachers and which features they consider important in such a course. Utilizing the EDW framework (Hoffmann & Even, 2018, 2019), we analyzed interviews with 24 mathematicians, as a result of which the framework was extended and refined.

Our findings suggest that mathematicians believe the primary purpose of a Calculus course for teachers, if such a course were to be designed, is to communicate the nature of mathematics and to provide prospective teachers with opportunities for mathematical investigations. This resonates with Ziegler and Loos' (2014) discussion on the importance of knowing and understanding the nature of mathematics and what constitutes doing mathematics. This is also in line with Burton's (1998) argument that mathematicians often see connections set within a global image of mathematics as an important part of knowing mathematics. With particular attention to Calculus, the mathematicians provided a variety of examples that can shape and expand the teachers' understanding of mathematics, and in turn, contribute to the teaching of mathematics in school.

The contribution of our findings is twofold: a) we expanded prior research on the role of mathematicians in teacher education by focusing on an envisioned Calculus course for teachers; and b) we extended and refined a theoretical framework which we believe would be useful in future research on how advanced mathematical knowledge may serve secondary mathematics teachers. To conclude, the mathematicians' broad visions of how to teach Calculus to prospective teachers deserve attention among mathematics educators and researchers. The findings of our study call for further research that focuses on mathematics courses designed specifically for teachers and their potential contributions to teacher development.

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