

# Two infinite families of chiral polytopes of type $\{4,4,4\}$ with solvable automorphism groups

Marston D.E. Conder<sup>a</sup>, Yan-Quan Feng<sup>b</sup>, Dong-Dong Hou<sup>b</sup>

<sup>a</sup>Department of Mathematics, University of Auckland, PB 92019, Auckland 1142, New Zealand

<sup>b</sup>Department of Mathematics, Beijing Jiaotong University, Beijing, 100044, P.R. China

## Abstract

We construct two infinite families of locally toroidal chiral polytopes of type  $\{4,4,4\}$ , with  $1024m^2$  and  $2048m^2$  automorphisms for every positive integer  $m$ , respectively. The automorphism groups of these polytopes are solvable groups, and when  $m$  is a power of 2, they provide examples with automorphism groups of order  $2^n$  where  $n$  can be any integer greater than 9. (On the other hand, no chiral polytopes of type  $[4,4,4]$  exist for  $n \leq 9$ .) In particular, our two families give a partial answer to a problem proposed by Schulte and Weiss in [Problems on polytopes, their groups, and realizations, *Periodica Math. Hungarica* 53 (2006), 231-255].

**Keywords:** Chiral 4-polytope, locally toroidal, solvable group, 2-groups.

## 1 Introduction

Abstract polytopes are combinatorial structures with properties that generalise those of classical polytopes. In many ways they are more fascinating than convex polytopes and tessellations. Highly symmetric examples of abstract polytopes include not only classical regular polytopes such as the well known Platonic solids, and more exotic structures such as the 120-cell and 600-cell, but also regular maps on surfaces (such as Klein's quartic); see [7, Chapter 8] for example.

Roughly speaking, an abstract polytope  $\mathcal{P}$  is a partially ordered set endowed with a rank function, satisfying certain conditions that arise naturally from a geometric setting. Such objects were proposed by Grünbaum in the 1970s, and their definition (initially as 'incidence polytopes') and theory were developed by Danzer and Schulte.

An *automorphism* of an abstract polytope  $\mathcal{P}$  is an order-preserving permutation of its elements, and every automorphism of  $\mathcal{P}$  is uniquely determined by its effect on any maximal chain in  $\mathcal{P}$  (which is known as a 'flag' in  $\mathcal{P}$ ). The most symmetric examples are regular, with all flags lying in a single orbit. The comprehensive book written by

Peter McMullen and Egon Schulte [14] is nowadays seen as the principal reference on this subject.

An interesting class of examples which are not quite regular are the chiral polytopes. For these, the automorphism group has two orbits on flags, with any two flags that differ in a single element lying in different orbits. Chirality is a fascinating phenomenon that does not have a counterpart in the classical theory of traditional convex polytopes. The study of chiral abstract polytopes was pioneered by Schulte and Weiss (see [18, 19] for example), but it has been something of a challenge to find and construct finite examples.

For quite some time, the only known finite examples of chiral polytopes had ranks 3 and 4. In rank 3, these are given by the irreflexible (chiral) maps on closed compact surfaces (see Coxeter and Moser [7]). Some infinite examples of chiral polytopes of rank 5 were constructed by Schulte and Weiss in [20], and then some finite examples of rank 5 were constructed just over ten years ago by Conder, Hubbard and Pisanski [5].

Many small examples of chiral polytopes are now known. These include all chiral polytopes with at most 4000 flags, and all that are constructible from an almost simple group  $\Gamma$  of order less than 900000. These have been assembled in collections, as in [3, 10], for example. In early 2009 Conder and Devillers devised a construction for chiral polytopes whose facets are simplices, and used this to construct examples of finite chiral polytopes of ranks 6, 7 and 8 [unpublished].

At about the same time, Pellicer devised a quite different method for constructing finite chiral polytopes with given regular facets, and used this construction to prove the existence of finite chiral polytopes of every rank  $d \geq 3$ ; see [16]. A few years later, Cunningham and Pellicer proved that every finite chiral  $d$ -polytope with regular facets is itself the facet of a chiral  $(d + 1)$ -polytope; see [9].

The work of Conder and Devillers was later taken up by Conder, Hubbard, O'Reilly Regueiro and Pellicer [4], to prove that all but finitely many alternating groups  $A_n$  and symmetric groups  $S_n$  are the automorphism group of a chiral 4-polytope of type  $\{3, 3, k\}$  for some  $k$  (dependent on  $n$ ). This has recently been extended to every rank greater than 4 by the same authors as [4].

Also Conder and Zhang in [6] introduced a new covering method that allows the construction of some infinite families of chiral polytopes, with each member of a family having the same rank as the original, but with the size of the members of the family growing linearly with one (or more) of the parameters making up its 'type' (Schläfli symbol). They have used this method to construct several new infinite families of chiral polytopes of ranks 3, 4, 5 and 6. Furthermore, Zhang constructed in her PhD thesis [23] a number of chiral polytopes of types  $\{4, 4, 4\}$ ,  $\{4, 4, 4, 4\}$  and  $\{4, 4, 4, 4, 4\}$ , with automorphism groups of orders  $2^{10}$ ,  $2^{11}$ ,  $2^{12}$ , and  $2^{15}$ ,  $2^{16}$ ,  $\dots$ ,  $2^{22}$ , and  $2^{18}$ ,  $2^{19}$ , respectively.

Now let  $\mathcal{P}$  be a regular or chiral 4-polytope. We say that  $\mathcal{P}$  is *locally toroidal* if its facets and its vertex-figures are maps on the 2-sphere or on the torus, and either its facets or its vertex-figures (or both) are toroidal – which means they have type  $\{3, 6\}$ ,  $\{4, 4\}$  or  $\{6, 3\}$ . Up to duality, rank 4 polytopes that are locally toroidal are of type  $\{4, 4, 3\}$ ,  $\{4, 4, 4\}$ ,  $\{6, 3, 3\}$ ,  $\{6, 3, 4\}$ ,  $\{6, 3, 5\}$ ,  $\{6, 3, 6\}$  or  $\{3, 6, 3\}$ .

Schulte and Weiss [19] developed a construction that starts with a 3-dimensional regular hyperbolic honeycomb and a faithful representation of its symmetry group as a group

of complex Möbius transformations (generated by the inversions in four circles that cut one another at the same angles as the corresponding reflection planes in hyperbolic space), and then derived chiral 4-polytopes by applying modular reduction techniques to the corresponding matrix group (see Monson and Schulte [15]). They then used the simple group  $\mathrm{PSL}(2, p)$  with  $p$  an odd prime to construct infinite families of such polytopes.

Some years later, Breda, Jones and Schulte [2] developed a method of ‘mixing’ a chiral  $d$ -polytope with a regular  $d$ -polytope to produce a larger example of a chiral polytope of the same rank  $d$ . They used this to construct such polytopes with automorphism group  $\mathrm{PSL}(2, p) \times \Omega$ , where  $\Omega$  is the rotation group of a finite regular locally-toroidal 4-polytope. For example,  $\Omega$  could be  $A_6 \times C_2$  or  $A_5$ , when the corresponding chiral polytope has type  $\{4, 4, 3\}$  or  $\{6, 3, 3\}$ , respectively.

One can see that almost all of the examples mentioned above involve non-abelian simple groups. On the other hand, there appear to be few known examples of chiral polytopes with solvable automorphism groups, apart from some of small order, and families of rank 3 polytopes arising from chiral maps on the torus (of type  $\{3, 6\}$ ,  $\{4, 4\}$  or  $\{6, 3\}$ ). This was the main motivation for the research leading to this paper. It was also motivated in part by a problem posed by Schulte and Weiss [21, Problem 30], namely the following:

**Problem 1.1** *Characterize the groups of orders  $2^n$  or  $2^n p$ , with  $n$  a positive integer and  $p$  an odd prime, which are automorphism groups of regular or chiral polytopes.*

Here we construct two infinite families of locally toroidal chiral 4-polytopes of type  $\{4, 4, 4\}$ , with solvable automorphism groups. Each family contains one example with  $1024m^2$  or  $2048m^2$  automorphisms, respectively, for every integer  $m \geq 1$ . In particular, if we let  $m$  be an arbitrary power of 2, say  $2^k$  (with  $k \geq 0$ ), then the automorphism group has order  $2^{10+2k}$  or  $2^{11+2k}$ , which can be expressed as  $2^n$  for an arbitrary integer  $n \geq 10$ .

This extends some earlier work by the second and third authors [11, 12], which showed that all conceivable ranks and types can be achieved for *regular* polytopes with automorphism group of 2-power order. It also extends both the work by Zhang [23] mentioned above, and a construction by Cunningham [8] of infinite families of tight chiral 3-polytopes of type  $\{k_1, k_2\}$  with automorphism group of order  $2k_1k_2$  (considered here for the special cases where  $k_1$  and  $k_2$  are powers of 2).

## 2 Additional background

In this section we give some further background that may be helpful for the rest of the paper.

### 2.1 Abstract polytopes: definition, structure and properties

An abstract polytope of rank  $n$  is a partially ordered set  $\mathcal{P}$  endowed with a strictly monotone rank function with range  $\{-1, 0, \dots, n\}$ , which satisfies four conditions, to be given shortly.

The elements of  $\mathcal{P}$  are called *faces* of  $\mathcal{P}$ , and the faces of rank  $n - 1$  are called the *facets* of  $\mathcal{P}$ . More generally, the elements of  $\mathcal{P}$  of rank  $j$  are called  $j$ -faces, and a typical  $j$ -face is denoted by  $F_j$ . Two faces  $F$  and  $G$  of  $\mathcal{P}$  are said to be *incident* with each other if  $F \leq G$  or  $F \geq G$  in  $\mathcal{P}$ . A *chain* of  $\mathcal{P}$  is a totally ordered subset of  $\mathcal{P}$ , and is said to have *length*  $i$  if it contains exactly  $i + 1$  faces. The maximal chains in  $\mathcal{P}$  (with length  $n + 1$ ) are called the *flags* of  $\mathcal{P}$ . Two flags are said to be  $j$ -*adjacent* if they differ in just one face of rank  $j$ , or simply *adjacent* (to each other) if they are  $j$ -adjacent for some  $j$ .

If  $F$  and  $G$  are faces of  $\mathcal{P}$  with  $F \leq G$ , then the set  $\{H \in \mathcal{P} \mid F \leq H \leq G\}$  is called a *section* of  $\mathcal{P}$ , and is denoted by  $G/F$ . Such a section has rank  $m - k - 1$ , where  $m$  and  $k$  are the ranks of  $G$  and  $F$  respectively. A section of rank  $d$  is called a  $d$ -section. Moreover, if  $F_{n-1}$  is any facet, then the section  $F_{n-1}/F_{-1}$  is also called a facet of  $\mathcal{P}$ , while if  $F_0$  is any vertex, then the section  $F_n/F_0 = \{G \in \mathcal{P} \mid F_0 \leq G\}$  is the *vertex-figure* of  $\mathcal{P}$  at  $F_0$ .

We can now give the four conditions that are required of  $\mathcal{P}$  to make it an abstract polytope. These are listed as (P1) to (P4) below:

- (P1)  $\mathcal{P}$  contains a least face and a greatest face, denoted by  $F_{-1}$  and  $F_n$ , respectively.
- (P2) Each flag of  $\mathcal{P}$  has length  $n + 1$  (so has exactly  $n + 2$  faces, including  $F_{-1}$  and  $F_n$ ).
- (P3)  $\mathcal{P}$  is *strong flag-connected*, which means that any two flags  $\Phi$  and  $\Psi$  of  $\mathcal{P}$  can be joined by a sequence of successively adjacent flags  $\Phi = \Phi_0, \Phi_1, \dots, \Phi_k = \Psi$ , each of which contains  $\Phi \cap \Psi$ .
- (P4) The rank 1 sections of  $\mathcal{P}$  have a certain homogeneity property known as the *diamond condition*, namely as follows: if  $F$  and  $G$  are incident faces of  $\mathcal{P}$ , of ranks  $i - 1$  and  $i + 1$ , respectively, where  $0 \leq i \leq n - 1$ , then there exist precisely *two*  $i$ -faces  $H$  in  $\mathcal{P}$  such that  $F < H < G$ .

An easy case of the diamond condition occurs for polytopes of rank 3 (or polyhedra): if  $V$  is a vertex of a 2-face  $F$ , then there are two edges that are incident with both  $V$  and  $F$ .

Next, every 2-section  $G/F$  of  $\mathcal{P}$  is isomorphic to the face lattice of a polygon. Now if it happens that the number of sides of every such polygon depends only on the rank of  $G$ , and not on  $F$  or  $G$  itself, then we say that the polytope  $\mathcal{P}$  is *equivelar*. In this case, if  $k_i$  is the number of edges of every 2-section between an  $(i - 2)$ -face and an  $(i + 1)$ -face of  $\mathcal{P}$ , for  $1 \leq i \leq n$ , then the expression  $\{k_1, k_2, \dots, k_{n-1}\}$  is called the Schläfli type of  $\mathcal{P}$ . (For example, if  $\mathcal{P}$  has rank 3, then  $k_1$  and  $k_2$  are the valency of each vertex and the number of edges of each 2-face, respectively.)

## 2.2 Automorphisms of polytopes

An *automorphism* of an abstract polytope  $\mathcal{P}$  is an order-preserving permutation of its elements. In particular, every automorphism preserves the set of faces of any given rank. Under permutation composition, the automorphisms of  $\mathcal{P}$  form a group, called the automorphism group of  $\mathcal{P}$ , and denoted by  $\text{Aut}(\mathcal{P})$  or sometimes more simply as  $\Gamma(\mathcal{P})$ . Also it is easy to use the diamond condition and strong flag-connectedness to prove that if an

automorphism fixes a flag of  $\mathcal{P}$ , then it fixes every flag of  $\mathcal{P}$  and hence every element of  $\mathcal{P}$ . It follows that  $\Gamma(\mathcal{P})$  acts fixed-point-freely, and hence semi-regularly, on the flags of  $\mathcal{P}$ .

A polytope  $\mathcal{P}$  is said to be *regular* if its automorphism group  $\Gamma(\mathcal{P})$  acts transitively (and hence regularly) on the set of flags of  $\mathcal{P}$ . In this case, the number of automorphisms of  $\mathcal{P}$  is as large as possible, and equal to the number of flags of  $\mathcal{P}$ . In particular,  $\mathcal{P}$  is equivelar, and the stabiliser in  $\Gamma(\mathcal{P})$  of every 2-section of  $\mathcal{P}$  induces the full dihedral group on the corresponding polygon. Moreover, for a given flag  $\Phi$  and for every  $i \in \{0, 1, \dots, n-1\}$ , the polytope  $\mathcal{P}$  has a unique automorphism  $\rho_i$  that takes  $\Phi$  to the unique flag  $(\Phi)^i$  that differs from  $\Phi$  in precisely its  $i$ -face, and then the automorphisms  $\rho_0, \rho_1, \dots, \rho_{n-1}$  generate  $\Gamma(\mathcal{P})$  and satisfy the defining relations for the string Coxeter group  $[k_1, k_2, \dots, k_{n-1}]$ , namely

$$\langle \rho_0, \rho_1, \dots, \rho_{n-1} \mid \rho_i^2 = 1 \text{ for } 0 \leq i \leq n-1, (\rho_i \rho_{i+1})^{k_{i+1}} = 1 \text{ for } 0 \leq i \leq n-2, (\rho_i \rho_j)^2 = 1 \text{ for } 0 \leq i < j-1 < n-1 \rangle,$$

where the  $k_i$  are as given in the previous subsection for the Schläfli type of  $\mathcal{P}$ . The generators  $\rho_i$  also satisfy a certain ‘intersection condition’, which follows from the diamond and strong flag-connectedness conditions. These and many more properties of regular polytopes may be found in [14].

We now turn to chiral polytopes, for which some good references are [18, 5, 17].

A polytope  $\mathcal{P}$  is said to be *chiral* if its automorphism group  $\Gamma(\mathcal{P})$  has two orbits on flags, with every two adjacent flags lying in different orbits. (Another way of viewing this definition is to consider  $\mathcal{P}$  as admitting no ‘reflecting’ automorphism that interchanges a flag with an adjacent flag.) Here the number of flags of  $\mathcal{P}$  is  $2|\Gamma(\mathcal{P})|$ , and  $\Gamma(\mathcal{P})$  acts regularly on each of two orbits. Again  $\mathcal{P}$  is equivelar, with the stabiliser in  $\Gamma(\mathcal{P})$  of every 2-section of  $\mathcal{P}$  inducing at least the full cyclic group on the corresponding polygon.

Next, let  $\Phi$  be any flag of  $\mathcal{P}$ , denote by  $F_i$  the  $i$ -face of  $\Phi$ , for  $0 \leq i \leq n$ . Then for  $1 \leq j \leq n-1$ , the chiral polytope  $\mathcal{P}$  admits an automorphism  $\sigma_j$  that takes  $\Phi$  to the flag  $(\Phi)^{j,j-1}$  which differs from  $\Phi$  in only its  $(j-1)$ - and  $j$ -faces  $F_{j-1}$  and  $F_j$ , and so fixes each  $F_i$  with  $i \notin \{j-1, j\}$ , and cyclically permutes consecutive  $j$ - and  $(j-1)$ -faces in the 2-section  $F_{j+1}/F_{j-2}$ . This automorphism  $\sigma_j$  is the analogue of the abstract rotation  $\rho_{j-1}\rho_j$  in the regular case, for each  $j$ . Now the automorphisms  $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$  generate  $\Gamma(\mathcal{P})$ , and if  $\mathcal{P}$  has Schläfli type  $\{k_1, k_2, \dots, k_{n-1}\}$ , then they satisfy the defining relations for the orientation-preserving subgroup of (index 2 in) the string Coxeter group  $[k_1, k_2, \dots, k_{n-1}]$ . Also they satisfy a ‘chiral’ form of the intersection condition, which is a variant of the one mentioned earlier for regular polytopes.

Chiral polytopes occur in pairs (or *enantiomorphic* forms), such that each member of the pair is the ‘mirror image’ of the other. Suppose one of them is  $\mathcal{P}$ , and has Schläfli type  $\{k_1, k_2, \dots, k_{n-1}\}$ . Then  $\Gamma(\mathcal{P})$  is isomorphic to the quotient of the orientation-preserving subgroup  $\Lambda^\circ$  of the string Coxeter group  $\Lambda = [k_1, k_2, \dots, k_{n-1}]$  via some normal subgroup  $K$ . By chirality,  $K$  is not normal in the full Coxeter group  $\Lambda$ , but is conjugated by any orientation-reversing element  $c \in \Lambda$  to another normal subgroup  $K^c$  which is the kernel of an epimorphism from  $\Lambda^\circ$  to the automorphism group  $\Gamma(\mathcal{P}^c)$  of the mirror image  $\mathcal{P}^c$  of  $\mathcal{P}$ .

The automorphism groups of  $\mathcal{P}$  and  $\mathcal{P}^c$  are isomorphic to each other, but their canonical generating sets satisfy different defining relations. In fact, replacing the elements  $\sigma_1$  and

$\sigma_2$  in the canonical generating tuple  $(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1})$  by  $\sigma_1^{-1}$  and  $\sigma_1^2\sigma_2$  gives a set of generators for  $\Gamma(\mathcal{P})$  that satisfy the same defining relations as a canonical generating tuple for  $\Gamma(\mathcal{P}^c)$ , but chirality ensures that there is no automorphism of  $\Gamma(\mathcal{P})$  that takes  $(\sigma_1, \sigma_2)$  to  $(\sigma_1^{-1}, \sigma_1^2\sigma_2)$  and fixes all the other  $\sigma_j$ .

Conversely, any finite group  $G$  that is generated by  $n - 1$  elements  $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$  which satisfy both the defining relations for  $\Lambda^\circ$  and the chiral form of the intersection condition is the ‘rotation subgroup’ of an abstract  $n$ -polytope  $\mathcal{P}$  that is either regular or chiral. Indeed,  $\mathcal{P}$  is regular if and only if  $G$  admits a group automorphism  $\rho$  of order 2 that takes  $(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1})$  to  $(\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \dots, \sigma_{n-1})$ , and hence behaves in the analogous way to conjugation by the reflection  $\rho_0$  (when this exists).

We now focus our attention on the rank 4 case. Here the generators  $\sigma_1, \sigma_2, \sigma_3$  for  $\Gamma(\mathcal{P})$  satisfy the canonical relations  $\sigma_1^{k_1} = \sigma_2^{k_2} = \sigma_3^{k_3} = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1$ , and the chiral form of the intersection condition can be abbreviated to

$$\langle \sigma_1 \rangle \cap \langle \sigma_2, \sigma_3 \rangle = \{1\} = \langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_3 \rangle \quad \text{and} \quad \langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle = \langle \sigma_2 \rangle.$$

The following proposition is useful for the groups we will deal with in the proof of our main theorem. It is called the *quotient criterion* for chiral 4-polytopes.

**Proposition 2.1** [2, Lemma 3.2] *Let  $G$  be a group generated by elements  $\sigma_1, \sigma_2, \sigma_3$  such that  $(\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1$ , and let  $\theta : G \rightarrow H$  be a group homomorphism taking  $\sigma_j \mapsto \lambda_j$  for  $1 \leq j \leq 3$ , such that the restriction of  $\theta$  to either  $\langle \sigma_1, \sigma_2 \rangle$  or  $\langle \sigma_2, \sigma_3 \rangle$  is injective. If  $(\lambda_1, \lambda_2, \lambda_3)$  is a canonical generating triple for  $H$  as the automorphism group of some chiral 4-polytope, then the triple  $(\sigma_1, \sigma_2, \sigma_3)$  satisfies the chiral form of the intersection condition for  $G$ .*

## 2.3 Group theory

We use standard notation for group theory, as in [22] for example. In this subsection we briefly describe some of the specific aspects of group theory that we need.

Let  $G$  be any group. We define the *commutator*  $[x, y]$  of elements  $x$  and  $y$  of  $G$  by  $[x, y] = x^{-1}y^{-1}xy$ , and then define the *derived subgroup* (or commutator subgroup) of  $G$  as the subgroup  $G'$  of  $G$  generated by all such commutators. Then for any non-negative integer  $n$ , we define the  $n$ th derived group of  $G$  by setting

$$G^{(0)} = G, \quad G^{(1)} = G', \quad \text{and} \quad G^{(n)} = (G^{(n-1)})' \quad \text{when } n \geq 1.$$

A group  $G$  is called *solvable* if  $G^{(n)} = 1$  for some  $n$ . (This terminology comes from Galois theory, because a polynomial over a field  $\mathbb{F}$  is solvable by radicals if and only if its Galois group over  $\mathbb{F}$  is a solvable group.) Every abelian group and every finite  $p$ -group is solvable, but every non-abelian simple group is not solvable. In fact, the smallest non-abelian simple group  $A_5$  is also the smallest non-solvable group.

We also need the following, which are elementary and so we give them without proof.

**Proposition 2.2** *If  $N$  is a normal subgroup of a group  $G$ , such that both  $N$  and  $G/N$  are solvable, then so is  $G$ .*

**Proposition 2.3** *Let  $G$  be the free abelian group  $\mathbb{Z} \oplus \mathbb{Z}$  of rank 2, generated by two elements  $x$  and  $y$  subject to the single defining relation  $[x, y] = 1$ . Then for every positive integer  $m$ , the subgroup  $G_m = \langle x^m, y^m \rangle$  is characteristic in  $G$ , with index  $|G : G_m| = m^2$ .*

Finally, we will use some Reidemeister-Schreier theory, which produces a defining presentation for a subgroup  $H$  of finite index in a finitely-presented group  $G$ . An easily readable reference for this is [13, Chapter IV], but in practice we use its implementation as the `Rewrite` command in the MAGMA computation system [1]. We also found the groups that we use in the next section with the help of MAGMA in constructing and analysing some small examples.

### 3 Main results

**Theorem 3.1** *For every positive integer  $m \geq 1$ , there exist chiral 4-polytopes  $\mathcal{P}_m$  and  $\mathcal{Q}_m$  of type  $\{4, 4, 4\}$  with solvable automorphism groups of order  $1024m^2$  and  $2048m^2$ , respectively.*

**Proof.** We begin by defining  $\mathcal{U}$  as the finitely-presented group

$$\langle a, b, c \mid a^4 = b^4 = c^4 = (ab)^2 = (bc)^2 = (abc)^2 = (a^2b^2)^4 = a^2c^2b^2(ac)^2 = [a, c^{-1}]b^2 = 1 \rangle.$$

This group  $\mathcal{U}$  has two normal subgroups of index 1024 and 2048, namely the subgroups generated by  $\{(ac^{-1})^4, (c^{-1}a)^4\}$  and  $\{(bc^{-1})^4, (c^{-1}b)^4\}$ , respectively. The quotients of  $\mathcal{U}$  by each of these give the initial members of our two infinite families.

Case (1): Take  $N$  as the subgroup of  $\mathcal{U}$  generated by  $x = (ac^{-1})^4$  and  $y = (c^{-1}a)^4$ .

A short computation with MAGMA shows that  $N$  is normal in  $\mathcal{U}$ , with index 1024. In fact, the defining relations for  $\mathcal{U}$  can be used to show that

$$\begin{aligned} a^{-1}xa = y, & \quad b^{-1}xb = y & \quad \text{and} & \quad c^{-1}xc = y, \\ a^{-1}ya = x, & \quad b^{-1}yb = x^{-1} & \quad \text{and} & \quad c^{-1}yc = x^{-1}. \end{aligned}$$

The first, third and fifth of these are easy to prove by hand, while the second and fourth can be verified in a number of ways, and the sixth follows from the other five. One way to prove the second and fourth is by hand, which we leave as a challenging exercise for the interested reader. Another is by a partial enumeration of cosets of the identity subgroup in  $\mathcal{U}$ . For example, if this is done using the `ToddCoxeter` command in MAGMA, allowing the definition of just 8000 cosets, then multiplication by each of the words  $b^{-1}xby^{-1}$  and  $a^{-1}yax^{-1}$  is found to fix the trivial coset, and therefore  $b^{-1}xby^{-1} = 1 = a^{-1}yax^{-1}$ .

It follows that conjugation by  $a$ ,  $b$  and  $c$  induce the three permutations  $(x, y)(x^{-1}, y^{-1})$ ,  $(x, y, x^{-1}, y^{-1})$  and  $(x, y, x^{-1}, y^{-1})$  on the set  $\{x, y, x^{-1}, y^{-1}\}$ , and then  $(ac^{-1})^2$  and  $(c^{-1}a)^2$  centralise both  $x$  and  $y$ , so  $x$  and  $y$  centralise each other.

Also MAGMA's `Rewrite` command gives a defining presentation for  $N$ , with  $[x, y] = 1$  as a single defining relation. Hence the normal subgroup  $N$  is free abelian of rank 2.

The quotient  $\mathcal{U}/N$  is isomorphic to the automorphism group of the chiral 4-polytope of type  $\{4, 4, 4\}$  with 1024 automorphisms listed at [3].

Now for any positive integer  $m$ , let  $N_m$  be the subgroup generated by  $x^m = (ac^{-1})^{4m}$  and  $y^m = (c^{-1}a)^{4m}$ . By Proposition 2.3, we know that  $N_m$  is characteristic in  $N$  and hence normal in  $\mathcal{U}$ , with index  $|\mathcal{U} : N_m| = |\mathcal{U} : N||N : N_m| = 1024m^2$ . Moreover, in the quotient  $G_m = \mathcal{U}/N_m$ , the subgroup  $N/N_m$  is abelian and normal, with quotient  $(\mathcal{U}/N_m)/(N/N_m) \cong \mathcal{U}/N$  being a 2-group, and so  $G_m$  is solvable, by Proposition 2.2.

Next, we use Proposition 2.1 to prove that the triple  $(\bar{a}, \bar{b}, \bar{c})$  of images of  $a, b, c$  in  $G_m$  satisfies the chiral form of the intersection condition. To do this, we observe that the group with presentation  $\langle u, v \mid u^4 = v^4 = (uv)^2 = (u^2v^2)^4 = 1 \rangle$  has order  $2^7 = 128$ , as does its image in the group  $G_1 = \mathcal{U}/N_1$  under the epimorphism taking  $(u, v) \mapsto (aN_1, bN_1)$ . These claims are easily verifiable using MAGMA. Then since  $G_1 = \mathcal{U}/N_1$  is a quotient of  $G_m = \mathcal{U}/N_m$ , the subgroup generated by  $\bar{a}$  and  $\bar{b}$  in  $G_m = \mathcal{U}/N_m$  must have order 128 as well, and hence the restriction to  $\langle \bar{a}, \bar{b} \rangle$  of the natural homomorphism from  $G_m$  to  $G_1$  is injective, as required.

Accordingly,  $G_m$  is the rotation group of a chiral or regular 4-polytope  $\mathcal{P}_m$  of type  $\{4, 4, 4\}$ . Assume for the moment that  $\mathcal{P}_m$  is regular. Then there exists an automorphism  $\rho$  of  $G_m$  taking  $(\bar{a}, \bar{b}, \bar{c})$  to  $(\bar{a}^{-1}, \bar{a}^2\bar{b}, \bar{c})$ , and it follows from the relation  $1 = \bar{a}^2\bar{c}^2\bar{b}^2(\bar{a}\bar{c})^2 = \bar{a}^2\bar{c}^2\bar{b}^2(\bar{a}\bar{c})^2$  that also  $1 = (\bar{a}^2\bar{c}^2\bar{b}^2(\bar{a}\bar{c})^2)^\rho = \bar{a}^{-2}\bar{c}^2(\bar{a}^2\bar{b})^2(\bar{a}^{-1}\bar{c})^2$  in  $G_m$ . But the image of this element in  $G_1$  has order 2, and as this is also the image of  $a^{-2}c^2(a^2b)^2(a^{-1}c)^2$  in  $G_1$ , it follows that  $(\bar{a}^2\bar{c}^2\bar{b}^2(\bar{a}\bar{c})^2)^\rho = \bar{a}^{-2}\bar{c}^2(\bar{a}^2\bar{b})^2(\bar{a}^{-1}\bar{c})^2$  is non-trivial in  $G_m$ , a contradiction.

Thus  $\mathcal{P}_m$  is chiral, with automorphism group  $G_m$  of order  $1024m^2$ .

Case (2): Take  $K$  as the subgroup of  $\mathcal{U}$  generated by  $z = (bc^{-1})^4$  and  $w = (c^{-1}b)^4$ .

Another computation with MAGMA shows that  $K$  is normal in  $\mathcal{U}$ , with index 2048, and moreover, the `Rewrite` command tells us that  $K$  is free abelian of rank 2. In this case, the defining relations for  $\mathcal{U}$  give

$$\begin{aligned} a^{-1}za = z^{-1}, \quad b^{-1}zb = w \quad & \text{and} \quad c^{-1}zc = w, \\ a^{-1}wa = w, \quad b^{-1}wb = z^{-1} \quad & \text{and} \quad c^{-1}wc = z^{-1}. \end{aligned}$$

The quotient  $\mathcal{U}/K$  is the automorphism group of the chiral 4-polytope of type  $\{4, 4, 4\}$  with 2048 automorphisms found by Zhang in [23].

Now for any positive integer  $m$ , let  $K_m$  be the subgroup generated by  $z^m = (bc^{-1})^{4m}$  and  $w^m = (c^{-1}b)^{4m}$ . Using Proposition 2.3, we find that  $K_m$  is characteristic in  $K$  and hence normal in  $\mathcal{U}$ , with index  $|\mathcal{U} : K_m| = |\mathcal{U} : K||K : K_m| = 2048m^2$ . Also the quotient  $H_m = \mathcal{U}/K_m$  is solvable, again by Proposition 2.2.

Next, the image of the subgroup generated by  $a$  and  $b$  in  $H_1 = \mathcal{U}/K$  has order 128, so just as in Case (1) above, we can apply Proposition 2.1 and find that the triple  $(\bar{a}, \bar{b}, \bar{c})$  of images of  $a, b, c$  in  $H_m = \mathcal{U}/K_m$  satisfies the chiral form of the intersection condition.

Thus  $H_m$  is the rotation group of a chiral or regular 4-polytope  $\mathcal{Q}_m$  of type  $\{4, 4, 4\}$ .

Moreover, the same argument as used in Case (1) shows that  $\mathcal{Q}_m$  is chiral, because the image in  $H_1$  of the element  $a^{-2}c^2(a^2b)^2(a^{-1}c)^2$  has order 2, and hence the image of that element in  $H_m$  is non-trivial.

Thus  $\mathcal{Q}_m$  is chiral, with automorphism group  $H_m$  of order  $2048m^2$ . □

As a special case we have the following Corollary, which is an immediate consequence of Theorem 3.1 when  $m$  is taken as a power of 2.



**Corollary 3.2** *There exists a chiral 4-polytope of type  $\{4, 4, 4\}$  with automorphism group of order  $2^n$ , for every integer  $n \geq 10$ .*

On the other hand, an inspection of the lists at [3] shows that there exists no such chiral polytope with automorphism group of order  $2^n$ , where  $n \leq 9$ .

## Acknowledgements

The first author acknowledges the hospitality of Beijing Jiaotong University, and partial support from the N.Z. Marsden Fund (project UOA1626). The second and the third authors acknowledge the partial support from the National Natural Science Foundation of China (11731002) and the 111 Project of China (B16002). Together we acknowledge the help of the MAGMA system [1] in constructing examples that help lead us to a proof of our main theorem.

## References

- [1] W. Bosma, J. Cannon, C. Playoust, The Magma Algebra System I: the user language, *J. Symbolic Comput.* 24 (1997) 235-265.
- [2] A. Breda D’Azevedo, G.A. Jones, E. Schulte, Constructions of chiral polytopes of small rank, *Canad. J. Math.* 63 (2011) 1254-1283.
- [3] M.D.E. Conder, Chiral polytopes with up to 4000 flags, <https://www.math.auckland.ac.nz/~conder/ChiralPolytopesWithUpTo4000Flags-ByOrder.txt>.
- [4] M.D.E. Conder, I. Hubard, E. O’Reilly Regueiro, D. Pellicer, Construction of chiral 4-polytopes with an alternating or symmetric group as automorphism group, *J. Algebraic Combin.* 42 (2015) 225-244.
- [5] M.D.E. Conder, I. Hubard, T. Pisanski, Constructions for chiral polytopes, *J. London Math. Soc.* 77 (2008) 115-129.
- [6] M.D.E. Conder, W.-J. Zhang, Abelian covers of chiral polytopes, *J. Algebra* 478 (2017) 437-457.
- [7] H.S.M. Coxeter, W.O.J. Moser, *Generators and Relations for Discrete Groups*, Springer-Verlag, 1972.
- [8] G. Cunningham, Tight chiral polyhedra, *Combinatorica* 38 (2018) 115-142.
- [9] G. Cunningham, D. Pellicer, Chiral extensions of chiral polytopes, *Discrete Math.* 330 (2014) 51-60.
- [10] M. Hartley, I. Hubard, D. Leemans, An Atlas of Chiral Polytopes for Small Almost Simple Groups, <http://homepages.ulb.ac.be/~dleemans/CHIRAL/index.html>.
- [11] D.-D. Hou, Y.-Q. Feng, D. Leemans, Existence of regular 3-polytopes of order  $2^n$ , *J. Group Theory* 22 (2019) 579-616.

- [12] D.-D. Hou, Y.-Q. Feng, D. Leemans, On regular polytopes of 2-powers, *Discrete Comput. Geom.* 64 (2020) 339-346.
- [13] D.L. Johnson, *Topics in the Theory of Group Presentations*, Cambridge University-Press, Cambridge, 1980.
- [14] P. McMullen, E. Schulte, *Abstract Regular Polytopes*, Cambridge University Press, Cambridge, 2002.
- [15] B. Monson, E. Schulte, Modular reduction in abstract polytopes, *Canad. Math. Bull.* 52 (2009) 435-450.
- [16] D. Pellicer, A construction of higher rank chiral polytopes, *Discrete Math.* 310 (2010) 1222-1237.
- [17] D. Pellicer, Developments and open problems on chiral polytopes, *Ars Math. Contemp* 5 (2012) 333-354.
- [18] E. Schulte, A.I. Weiss, Chiral polytopes, *Applied Geometry and Discrete Mathematics* 4 (1991) 493-516.
- [19] E. Schulte, A.I. Weiss, Chirality and projective linear groups, *Discrete Math.* 131 (1994) 221-261.
- [20] E. Schulte, A.I. Weiss, Free extensions of chiral polytopes, *Canad. J. Math.* 47 (1995) 641-654.
- [21] E. Schulte, A.I. Weiss, Problems on polytopes, their groups, and realizations, *Periodica Math. Hungarica* 53 (2006) 231-255.
- [22] M.Y. Xu, *Introduction to Group Theory I*, Science Publishing House, Beijing, 1999.
- [23] W.-J. Zhang, *Constructions For Chiral Polytopes*, PhD Thesis, University of Auckland, 2016.

**E-mail addresses for authors:**

Marston Conder <m.conder@auckland.ac.nz>  
 Yan-Quan Feng <yqfeng@bjtu.edu.cn>  
 Dong-Dong Hou <16118416@bjtu.edu.cn>