The smallest symmetric cubic graphs with given type

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Abstract

It is known that arc-transitive group actions on finite cubic (3-valent) graphs fall into seven classes, denoted by 1, 2^1, 2^2, 3, 4^1, 4^2 and 5, where k, k^1 or k^2 indicates that the action is k-arc-regular, and with k^2 indicating that there is no arc-reversing automorphism of order 2 (for $k = 2$ or 4). These classes can be further subdivided into 17 sub-classes, according to the types of arc-transitive subgroups of the full automorphism group of the graph, sometimes called the 'action type' of the graph. In this paper, we complete the determination of the smallest graphs in each of these 17 classes (begun by Conder and Nedela in *J. Algebra* 22 (2009), 722–740).

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1 Introduction

A graph X is *arc-transitive* (or *symmetric*) if its automorphism group acts transitively on the ordered edges of X, and more generally, an arc-transitive group of automorphisms of X is said to be *k-arc-regular* if it acts regularly on the k-arcs of X (namely the walks of length $k+1$ in which every consecutive vertices are distinct). By a classical theorem of Tutte [11, 12], every arc-transitive group of automorphisms of a finite cubic (3-valent) graph X is k-arc-regular for some $k \le 5$, and following some refinement of Tutte's theorem by Djoković & Miller [9] and Conder & Lorimer [6], it is known that such group actions can be partitioned into seven non-empty 'basic' classes, denoted by 1, 2^1, 2^2, 3, 4^1, 4^2 and 5, where k, k^1 or k^2 indicates that the action is k-arc-regular, and k^2 indicates that there is no arc-reversing automorphism of order 2 (for $k = 2$ or 4).

Associated with each of those seven basic types is a finitely-presented group, denoted by G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5, respectively. Each of them is a 'universal' group for actions of the given type, and is an amalgamated free product of the form $V *_A E$, where V and E denote the pre-images of the stabiliser of a vertex v and incident edge $e = \{v, w\}$ of the given graph X, and the intersection $A = V \cap E$ is the pre-image of the stabiliser of the arc (v, w).

For example, G_1 is the modular group $C_3 *_{C_1} C_2$, with presentation $\langle\, h, a \mid h^3 = a^2 = 1 \rangle$, and is isomorphic to $\mathrm{PSL}(2,\mathbb{Z})$. Similarly, $G_2^1 \cong D_3 *_{C_2} V_4 \cong \mathrm{PGL}(2,\mathbb{Z})$.

Djoković & Miller used the corresponding amalgams to exhibit examples of graphs in the five basic classes G_1, G_2^1, G_3, G_4^1 and G_5 in [9], and Conder & Lorimer found the first (but relatively large) examples of graphs in the remaining two classes in [6]. A few years later, Conder & Dobcsányi [5] used computational methods to find smooth finite quotients of the universal groups, in order to completely determine all connected symmetric cubic graphs of order up to 768, thereby completing and extending the 'Foster census' [2] of connected symmetric cubic graphs of order up to 512. (More recently, this author has extended it even further, to all such graphs of order up to 10000; see [4].) In particular, this work led to the discovery of the smallest example of basic type 2^2, namely the graph now known as F448C.

More recently, it was shown by Conder & Nedela [7] that the seven basic classes can be further subdivided into 17 sub-classes, according to the types of arc-regular subgroups of the full automorphism group of the graph. This combination of basic types forms what is now sometimes called the 'action type' of the graph. For example, action type $(4^2, 5)$ indicates that the graph is 5-arc-transitive and admits arc-transitive groups of automorphisms of types 4^2 and 5, but of none of the other five basic types.

In the analysis given in this classification in [7], the smallest known examples of graphs in each of the 17 'action type' classes were identified, and the authors showed that in 14 of those 17 cases, the smallest known was actually the smallest. Obvious cases are the complete graph K_4, the complete bipartite graph $K_{3,3}$ the Petersen graph and Tutte's 8-cage, with action types $(1, 2^1)$, $(1, 2^1, 2^2, 3)$, $(2^1, 3)$ and $(4^1, 4^2, 5)$ respectively, while more interesting cases included F448C, the Sextet graph $S(17)$, the Biggs-Conway graph and Wong's graph, with action types (2^2), (4^1), $(1, 4^1, 4^2, 5)$ and $(4^2, 5)$ respectively.

The cases left open were the action types (4^2), $(4^1, 5)$ and (5). In this paper, we complete that piece of work by showing that the smallest examples in those cases are respectively a 3^{11}-fold cover of Tutte's 8-cage F30, the 5-arc-transitive cubic graph of order 75600 (with automorphism group S_{10}) discovered by the author [3], and a new graph of order 83966400 with the Held simple group He of order 4030387200 as its automorphism group.

We deal with each of these cases in the next three sections, with the help of group theory and a fair amount of computation using Magma [1]. As a result, we can complete the partial table given in [7] by confirming one of the three entries that were left with a question mark, and replacing the other two, to produce Table 1 below. Before proceeding, we give a little further background in Section 2 that will help.

k	Action type	Bipartite?	Smallest example	Unique minimal?
1	(1)	Sometimes	F026	No
2	$(1, 2^1)$	Sometimes	F004 (K_4)	No
2	(2^1)	Sometimes	F084	No
2	(2^2)	Sometimes	F448C	No
3	$(1, 2^1, 2^2, 3)$	Always	F006 ($K_{3,3}$)	No
3	$(2^1, 2^2, 3)$	Always	F020B (GP(10,3))	No
3	$(2^1, 3)$	Never	F010 (Petersen)	No
3	$(2^2, 3)$	Never	F028 (Coxeter)	No
3	(3)	Sometimes	F110	No
4	$(1, 4^1)$	Always	F014 (Heawood)	Yes
4	(4^1)	Sometimes	F102 (S(17))	No
4	(4^2)	Sometimes	3^{11}-fold cover of Tutte's 8-cage	No
5	$(1, 4^1, 4^2, 5)$	Always	Biggs-Conway graph	Yes
5	$(4^1, 4^2, 5)$	Always	F030 (Tutte's 8-cage)	No
5	$(4^1, 5)$	Never	S_{10} graph	No
5	$(4^2, 5)$	Never	F234B (Wong's graph)	No
5	(5)	Sometimes	Held group graph	No

Table 1: The 17 families of finite symmetric cubic graphs, classified by action type

2 Further background

Here we give further details about the universal groups for the seven basic types of arc-transitive action on finite cubic graphs (and also on infinite cubic graphs with finite vertex-stabiliser), which can be found in any of [5, 6, 7].

First, the universal groups themselves are as follows:

G_1 is generated by two elements h and a, subject to the relations $h^3 = a^2 = 1$;

G_2^1 is generated by h, a and p, subject to $h^3 = a^2 = p^2 = 1$, $apa = p$, $php = h^{-1}$;

G_2^2 is generated by h, a and p, subject to $h^3 = p^2 = 1$, $a^2 = p$, $php = h^{-1}$;

G_3 is generated by h, a, p, q, subject to $h^3 = a^2 = p^2 = q^2 = 1$, $apa = q$, $qp = pq$, $ph = hp$, $qhq = h^{-1}$;

G_4^1 is generated by h, a, p, q and r, subject to $h^3 = a^2 = p^2 = q^2 = r^2 = 1$, $apa = p$, $aqa = r$, $h^{-1}ph = q$, $h^{-1}qh = pq$, $rhr = h^{-1}$, $pq = qp$, $pr = rp$, $rq = pqr$;

G_4^2 is generated by h, a, p, q and r, subject to $h^3 = p^2 = q^2 = r^2 = 1$, $a^2 = p$, $a^{-1}qa = r$, $h^{-1}ph = q$, $h^{-1}qh = pq$, $rhr = h^{-1}$, $pq = qp$, $pr = rp$, $rq = pqr$;

G_5 is generated by h, a, p, q, r and s, subject to $h^3 = a^2 = p^2 = q^2 = r^2 = s^2 = 1$, $apa = q$, $ara = s$, $h^{-1}ph = p$, $h^{-1}qh = r$, $h^{-1}rh = pqr$, $shs = h^{-1}$, $pq = qp$, $pr = rp$, $ps = sp$, $qr = rq$, $qs = sq$, $sr = pqrs$.

In each case, let S be the generating set for the universal group U $(= G_k, G_k^1$ or $G_k^2)$. Then the pre-image of the vertex-stabiliser V is generated by $S \setminus \{a\}$, the pre-image of the edge-stabiliser E is generated by $S \setminus \{h\}$, and the pre-image of the arc-stabiliser $A = V \cap E$ is generated by $S \setminus \{h, a\}$. The generator h induces a cyclic rotation (of order 3) about the vertex v chosen for V to stabilise, while the generator a interchanges v with one of its neighbours, and a has order 4 when $U = G_2^2$ or G_4^2, and order 2 otherwise.

Next, every epimorphism from the universal group to a k-arc-regular group of automorphisms of the graph X (with k being the same as for U) is faithful on each of these pre-images, or 'smooth', and so we may abuse notation by denoting the pre-images of V, E and A also by V, E and A. The subgroup A of U has order 1, 2, 4, 8 or 16 when $k = 1$, 2, 3, 4 or 5 respectively, and $|V| = 3|A|$ while $|E| = 2|A|$ in each case. In fact, $V \cong C_3$, S_3, $S_3 \times C_2$, S_4 or $S_4 \times C_2$ (of order 3, 6, 12, 24 or 48) and $A \cong 1$, C_2, V_4, D_4 or $D_4 \times C_2$ (of order 1, 2, 4, 8 or 16) when $k = 1$, 2, 3, 4 or 5 respectively.

Also, importantly, if U is any one of these universal groups, and $\theta\colon U \to G$ is a smooth epimorphism to a group G, then we may construct an arc-transitive cubic graph X upon which G acts (faithfully) as an k-arc-regular group of automorphisms, again with the same value of k. The vertices and edges of X can be taken as the (right) cosets in G of the images of the subgroups V and E, with incidence given by non-empty intersection, and then the action of G on $X = (V, E)$ is given simply by right multiplication.

Finally in this section, we explain some notation that we use later in the paper, about epimorphisms from a universal group U to a finite group G. We will refer to any such G as a *quotient* of U, quite loosely, without always specifying the epimorphism or its kernel K (for which $U/K \cong G$). Also we will say that two such epimorphisms are *equivalent* if one can be obtained from the other by composing it with an inner automorphism of G. (This is the default way in which such epimorphisms are enumerated using the `Homomorphisms` command in Magma.) Note that inequivalent epimorphisms can still have the same kernel K, and hence give the same quotient G, when one can be obtained from the other by composing it with a non-inner automorphism of G.

3 Action type (4^1,5)

The universal group G_5 has three subgroups of index 2 and one subgroup N of index 4, namely the subgroups generated by $\{h, pq, a\}$, $\{h, pq, ap\}$, $\{h, p, aha\}$ and $\{h, pq, aha\}$, respectively. The first two of these are isomorphic to the universal groups G_4^1 and G_4^2, while the third is the pre-image of the stabiliser of a bipartition (when this exists), and the fourth is the intersection of any two of the three subgroups of index 2. Each of the three subgroups of index 2 has abelianisation C_2, and in particular, G_4^1 has a unique subgroup of index 2, namely N.

To obtain a graph with action type $(4^1, 5)$, we need a quotient Q of G_4^1 via a normal subgroup K that is normal in G_5 (so the 4-arc-regular action of $Q = G_4^1/K$ extends to a 5-arc-regular action of G_5/K), but has the property that $G_5 = KG_4^2$ (so the image of G_4^2 in G_5/K does not have index 2, and then G_5/K has no 4-arc-regular subgroup of type 4^2). Note also that this implies that $G_4^1 = G_4^1 \cap G_5 = KG_4^1 \cap KG_4^2 = K(G_4^1 \cap G_4^2) = KN$.

In particular, the quotient $Q = G_4^1/K$ has no subgroup of index 2, for otherwise its pre-image in G_4^1 would have to be N (and contain K), which is impossible since $KN = G_4^1$, rather than a subgroup of index 2 in G_4^1. It follows that the quotient Q is perfect.

Hence we look for epimorphisms (up to equivalence) from the universal group G_4^1 to a finite perfect group Q that can be extended via an automorphism of Q to give an epimorphism from G_5 to a group containing Q as a subgroup of index 2, making the graph 5-arc-transitive.

A Magma computation (using its `SimpleQuotientProcess` function) shows that the smallest simple quotients of G_4^1 in ascending order are $\mathrm{PSL}(2, q)$ for $q = 17, 31, 47, 79, 97, 113$ and 127, followed by A_{10}. In each of first seven of these cases, no such epimorphism from G_4^1 to the simple quotient $Q \cong G_4^1/K$ can be extended in the required way, and so to obtain a perfect quotient of G_4^1 from one of these examples that works, one would need to take at the very least the intersection of the kernel K with its other conjugate in G_5, say L, and then get the quotient $G_4^1/(K \cap L) \cong G_4^1/K \times G_4^1/L \cong \mathrm{PSL}(2, q) \times \mathrm{PSL}(2, q)$, which has order at least $|\mathrm{PSL}(2, 17)|^2 > 10!$. It follows that the smallest suitable perfect quotient of the group G_4^1 is A_{10}.

Moreover, up to equivalence (in A_{10}), there are just two epimorphisms from G_4^1 to A_{10}, but with the same kernel, and in each case, the required automorphism of A_{10} is an outer automorphism, so the resulting graph has automorphism group S_{10}. Similarly up to equivalence in S_{10}, there there is just one epimorphism from G_5 to S_{10}. (In fact there are exactly two conjugacy classes of subgroups of index 10 in G_5, with one giving this graph, and the other giving Tutte's 8-cage.) Hence we have the following:

Theorem 3.1 *The smallest symmetric cubic graph with action type* $(4^1, 5)$ *is a unique one of order* 75600 *with automorphism group* S_{10}*, as found in* [4].

4 Action type (5)

As explained in the previous section, the group G_5 has three subgroups of index 2 and one subgroup N of index 4. Each of the three subgroups of index 2 has abelianisation C_2, and hence in particular, the third subgroup $B = \langle h, p, aha \rangle$ has a unique subgroup of index 2, namely the fourth subgroup $N = G_4^1 \cap G_4^2$.

To obtain a graph with action type (5), we need a quotient $Q = B/K$ via a normal subgroup K that is also normal in G_5, but has the property that $G_5 = KG_4^1$ and $G_5 = KG_4^2$, so that the image of neither G_4^1 nor G_4^2 in G_5/K is a subgroup of index 2, and hence G_5/K has no 4-arc-regular subgroup, and no 1-arc-regular subgroup (by [7, Proposition 2.3]). Note also that this gives $B = B \cap G_5 = KB \cap KG_4^1 = K(B \cap G_4^1) = KN$. In particular, the quotient $Q = B/K$ has no subgroup of index 2, for otherwise its pre-image in B would have to be N (and contain K), which is impossible since $KN = B$, rather than a subgroup of index 2 in B. It follows that the quotient Q is perfect.

Hence we look for perfect quotients of the subgroup B that admit an automorphism giving a 5-arc-transitive graph.

A Magma computation (using its `SimpleQuotientProcess` function) shows that the smallest simple quotient of B is the Mathieu group M_{24}, of order 244823040, but the associated epimorphisms do not extend, and one has to go to $\mathrm{M}_{24} \times M_{24}$ and then $\mathrm{M}_{24} \wr C_2$ (as suggested in [7]) to get an example from that.

The next smallest simple quotient of B is the Held simple group He, of order 4030387200. (Incidentally, the group B also has A_{14} and A_{15} as quotients.) Moreover, two of the six epimorphisms from B to the Held group He (up to equivalence) extend to epimorphisms from G_5 to He, via an inner automorphism of He in each case, while the other four do not. Also the first two have the same kernel, and hence they give the same 5-arc-transitive graph. It follows that there exists a unique symmetric cubic graph with action type (5), having He as its full automorphism group, and with order $4030387200/48 = 83966400$.

As there is no epimorphism from B to a smaller simple group that extends in this way, we have the following.

Theorem 4.1 *The smallest symmetric cubic graph with action type* (5) *is a unique one of order* 83966400*, with automorphism group the Held simple group* He.

5 Action type (4^2)

5.1 Initial remarks

This was the most challenging of the action types to deal with, as will soon become clear. Once again, we note that the group G_5 has three subgroups of index 2 and one subgroup N of index 4, and this time, we use the facts that $N = G_4^1 \cap G_4^2$ is the unique subgroup of index 2 in G_4^2, and that N is perfect.

To obtain a graph with action type (4^2), we need a quotient $Q = N/K$ via a normal subgroup K that is also normal in G_4^2 but not in G_4^1 (and so not in G_5). In particular, as

N is perfect, so is Q. Hence we look for perfect quotients of N admitting an automorphism that gives an arc-transitive graph of type 4^2 rather than one of type 4^1 or 5.

One way to do this is to look for small quotients $Q = N/K$ that do not lead to a group action of type 4^1, and hope that among them, there is one that gives a graph of type 4^2, possibly after taking the larger quotient $N/(K \cap K^c)$ where $c = ap \in G_4^2 \setminus N$. The Mathieu group M_{24} (of order 244823040) is the smallest simple quotient of N satisfying the former condition, but neither of the two epimorphisms from N to M_{24} (up to equivalence) extends to one from G_4^2 to M_{24} or a group containing M_{24} as a subgroup of index 2.

Another way is to look for small quotients $Q = N/K$ where K is normal in G_5, and then try to dig inside K for a normal subgroup L of larger index in N such that L is normal in G_4^2 but not in G_5, in which case G_4^2/L gives a graph of type 4^2. This way works, as we will see, but not all that easily.

Before proceeding, we set some notation that will be helpful. First, for a prime k, the group $\mathrm{PGL}^*(2, k^2)$ is the subgroup of index 2 in $P\Gamma L(2, k^2) \cong \mathrm{Aut}(PSL(2, k^2)$ that is not $PGL(2, k^2)$ or $P\Sigma L(2, k^2)$, the group obtained from $\mathrm{PSL}(2, k^2)$ by adjoining the field automorphism $x \mapsto x^k$. Next, if G is a finite perfect group, then $\mathrm{Cov}(G)$ is the *covering group* (or *universal perfect central extension*) of G, which is the largest perfect group C containing a central subgroup Z for which C/Z is isomorphic to G, and then Z is the *Schur multiplier* of G, which we will denote by $\mathrm{SM}(G)$; see [10, Ch. 11] and [8]. Finally, we will say that a (normal) subgroup of N that is normal in G_4^2 is *admissible*.

5.2 The smallest known example

The smallest quotient $Q = N/K$ of N with K normal in G_5 is $\mathrm{PSL}(2, 9) \cong A_6$, and gives rise to Tutte's 8-cage (of order 30). In this case, the subgroup K contains a normal subgroup L of index 3^{11} that is normal in G_4^2 but not in G_4^1 (or G_5), and hence the quotient G_4^2/L is the automorphism group of a symmetric cubic graph of type 4^2, of order $30 \cdot 3^{11} = 5314410$, which is a 3^{11}-fold cover of Tutte's 8-cage. Its automorphism group is an extension of the elementary abelian group $K/L \cong C_3^8$ by the quotient $(G_4^2/L)/(K/L) \cong G_4^2/K$, which is isomorphic to M_{10}, the point-stabiliser in the Mathieu group M_{11} (and to a subgroup of index 2 in $P\Gamma L(2, 9) \cong \mathrm{Aut}(S_6)$, namely $\mathrm{PGL}^*(2, 9)$).

In fact the author first discovered such a normal subgroup L (of index 3^{11} in K) in 2012, but then re-discovered it in 2015, in answer to a question by Klavdija Kutnar about symmetric cubic graphs of type 4^2 with twice odd order. It can be found as follows.

The subgroup K is free of rank 16, and hence it contains a characteristic subgroup $T = K' \, K^{(3)}$ of index 3^{16} generated by the derived subgroup $K' = [K, K]$ and the cubes of all elements of K, so that K/T is elementary abelian of order 3^{16}. Now the quotient G_4^2/T has other elementary abelian normal subgroups lying inside K/T, including two of order 3^5, and each of these has the form L/T where L has the properties we need. These subgroups can be found with the help of Magma, using either the NormalSubgroups function, or Magma's machinery for finding submodules of kG-modules (where k is a field and G is a finitely-presented group), applied to the 16-dimensional $Z_3G_4^2$-module corresponding to the elementary abelian normal 3-subgroup K/T. The two possibilities for the pre-image of L in G_4^2 are interchanged by conjugation by $a \in G_5 \setminus G_4^2$, and the graphs of type 4^2 obtained from them are isomorphic to each other.

In the rest of this section (and to complete the paper), we explain why there is no smaller cubic graph of type 4^2 than the one found above. A proof of this is not easy, so we give only a skeleton argument, without full details. We find a large number of admissible normal subgroups of N with interesting quotients, but in almost all cases, that subgroup is also normal in G_4^1.

5.3 Step 0: Starting hypotheses

We begin by letting $G = G_4^2/K$ be any quotient of G_4^2 of order at most $720 \cdot 3^{11} = 127545840$ via a normal subgroup K that is not normal in G_5 (and hence gives rise to a graph that has type 4^2 rather than type 5). Then the derived group $D = G' = [G, G]$ of G has index 1 or 2 in G, and is perfect, and $K^a \neq K$. Under these conditions, either K is a subgroup of N and $D = N/K$ has index 2 in G, wth $|D| \leq 360 \cdot 3^{11} = 63772920$ (as for the example in §5.2), or $G_4^2 = NK$ with $G \cong G_4^2/K = NK/K \cong N/(N \cap K)$ being perfect, and $K/(N \cap K) \cong NK/N = G_4^2/N \cong C_2$, so $K/(N \cap K)$ is a central subgroup of order 2 in $G_4^2/(N \cap K)$. (The latter happens when conjugation by $p \in G_4^2 \setminus N$ induces an inner automorphism of $N/(N \cap K) \cong G$, and then $G_4^2/(N \cap K) \cong G \times C_2$.)

We aim to show that K and K^a are the two normal subgroups of index $360 \cdot 3^{11}$ in N described in the previous subsection.

5.4 Step 1: Simple quotients of N

A computation in Magma using the SimpleQuotientProcess and Homomorphisms facilities shows that
(a) N has 69 simple quotients of order at most 127545840, and
(b) the kernel of every epimorphism from N to each one of them is normal in G_4^1, as noted in the observation made earlier about M_{24}, but
(c) for only nine of these simple quotients is the kernel J of at least one such epimorphism admissible.

The nine simple quotients arising in (c) are isomorphic to A_6 of order 360, $\mathrm{PSL}(3,3)$ of order 5616, $\mathrm{PSL}(2,25)$ of order 7800, $\mathrm{PSL}(2,121)$ of order 885720, A_{10} of order 1814400, $\mathrm{PSL}(2,169)$ of order 2413320, $\mathrm{PSU}(3,7)$ of order 5663616, $\mathrm{PSL}(2,361)$ of order 23522760, and J_3 of order 50232960. In just two of those nine cases, the simple quotient S of N has an inner automorphism that makes S also a quotient of G_4^2, namely when S is PSL(3, 3) or PSU(3, 7), and in all nine cases, there is just one admissible kernel – that is, just one normal subgroup J of G_4^2 for which N/J is the given simple quotient S.

In other cases, the element $ap \in G_4^2 \setminus N$ interchanges the kernel J of one epimorphism with another, and the intersection of those two kernels is an admissible subgroup of N, with quotient the direct product of two copies of the simple group $S = N/J$, giving the wreath product $S \wr C_2$ as a quotient of G_4^2. In order for the latter quotient of G_4^2 to have order at most 127545840, we require $|S| < \sqrt{63772920} \simeq 7985.9$, and this gives only three possibilities for $S = N/J$, namely PSL(2, 7), PSL(2, 17) and PSL(2, 23).

But also the direct products of two of the nine simple groups considered two paragraphs above are quotients of N, and this gives three more possibilities worth considering, namely $A_6 \times \mathrm{PSL}(3,3)$, $A_6 \times \mathrm{PSL}(2,25)$, and $\mathrm{PSL}(3,3) \times \mathrm{PSL}(2,25)$. Similarly, the direct product $A_6 \times \mathrm{PSL}(2,7)$ is a small quotient of N, but via a normal subgroup J that is not normal in G_4^2; the core of J in G_4^2 is a subgroup K for which $N/K \cong A_6 \times \mathrm{PSL}(2,7) \times \mathrm{PSL}(2,7)$, of order 10160640, with N/K having index 2 in G_4^2/K. This is the only direct product of a triple of simple quotients of N that is small enough to consider.

The quotients of G_4^2 coming from the simple groups and direct products of simple groups that are quotients of N small enough to be worthy of consideration are summarised in Table 2, with J being a normal subgroup of N, and K being the core of J in G_4^2, and L being either K or the pre-image of a central subgroup of G_4^2/K of order 2.

In all cases in this table, the relevant normal subgroups of N giving the entry in the third column are all normal in G_4^1 as well, because of (b) above, and hence all the associated graphs are 5-arc-transitive. It follows that the (perfect) derived subgroup D of our group G from Step 0 cannot be a simple group or a direct product of simple groups, so D must have at least one cyclic composition factor, and therefore at least one abelian chief factor.

We will consider such possibilities for D by digging below some the possibilities for J or its core in G_4^2 given in Table 2.

5.5 Step 2: Non-trivial soluble normal subgroup S

Here we prove that G has a soluble normal subgroup S contained in D such that D/S is one of the 19 groups given in the fourth column of Table 2.

To do this, we assume G is the smallest possible counter-example, and then let R be the smallest normal subgroup of G contained in $D = G'$ such that D/R is a direct product of simple groups. Also let $G = G_0 \geq G_1 > G_2 > \cdots > G_j > \cdots > G_{k-1} > G_k = \{1\}$ be a chief series of normal subgroups of G, with $G_1 = D$ and $G_j = R$.

Then by choice of G (as the smallest counter-example), the subgroup $M = G_{k-1}$ is insoluble, and so is isomorphic to a direct product $T^m = T \times T \times \cdots \times T$ of (say) m

	N/J	$\|N/J\|$	N/K	$\|N/K\|$	$\|G_4^2/L\|$
1	A_6	360	A_6	360	720
2	$\mathrm{PSL}(3,3)$	5616	$\mathrm{PSL}(3,3)$	5616	5616
3	$\mathrm{PSL}(3,3)$	5616	$\mathrm{PSL}(3,3)$	5616	11232
4	$\mathrm{PSL}(2,25)$	7800	$\mathrm{PSL}(2,25)$	7800	15600
5	$\mathrm{PSL}(2,7)$	168	$\mathrm{PSL}(2,7)\times\mathrm{PSL}(2,7)$	28224	56448
6	$\mathrm{PSL}(2,121)$	885720	$\mathrm{PSL}(2,121)$	885720	1771440
7	A_{10}	1814400	A_{10}	1814400	3628800
8	$A_6\times\mathrm{PSL}(3,3)$	2021760	$A_6\times\mathrm{PSL}(3,3)$	2021760	4043520
9	$\mathrm{PSL}(2,169)$	2413320	$\mathrm{PSL}(2,169)$	2413320	4826640
10	$A_6\times\mathrm{PSL}(2,25)$	2808000	$A_6\times\mathrm{PSL}(2,25)$	2808000	5616000
11	$\mathrm{PSU}(3,7)$	5663616	$\mathrm{PSU}(3,7)$	5663616	5663616
12	$\mathrm{PSU}(3,7)$	5663616	$\mathrm{PSU}(3,7)$	5663616	11327232
13	$\mathrm{PSL}(2,17)$	2448	$\mathrm{PSL}(2,17)\times\mathrm{PSL}(2,17)$	5992704	11985408
14	$\mathrm{PSL}(2,7)\times A_6$	60480	$\mathrm{PSL}(2,7)\times\mathrm{PSL}(2,7)\times A_6$	10160640	20321280
15	$\mathrm{PSL}(2,361)$	23522760	$\mathrm{PSL}(2,361)$	23522760	47045520
16	J_3	50232960	J_3	50232960	50232960
17	$\mathrm{PSL}(2,23)$	2448	$\mathrm{PSL}(2,23)\times\mathrm{PSL}(2,23)$	36869184	73738368
18	$\mathrm{PSL}(3,3)\times\mathrm{PSL}(2,25)$	43804800	$\mathrm{PSL}(3,3)\times\mathrm{PSL}(2,25)$	43804800	87609600
19	J_3	50232960	J_3	50232960	100465920

Table 2: Admissible quotients of N that are direct products of simple groups

copies of some non-abelian simple group T. Also $|G/M| \geq |G/R| \geq 720$, and hence $|M| \leq \lfloor 127545840/720 \rfloor = 3^{11} = 177147$, while on the other hand, $|T| \geq 60$, so $m \leq 2$.

Next, conjugation of M by elements of D gives a homomorphism from D to $\mathrm{Aut}(M)$ with kernel $C_D(M)$, and the latter subgroup intersects M trivially (since $Z(M) = \{1\}$), so the normal subgroup $MC_D(M)$ of D is a direct product $M \times C_D(M)$. Furthermore, $\mathrm{Aut}(M) \cong \mathrm{Aut}(T^m) \cong \mathrm{Aut}(T) \wr S_m$. The latter contains T^m as a normal subgroup with quotient $\mathrm{Out}(T) \wr S_m$, which is soluble by Schreier's conjecture (now known to be true, as a consequence of the Classification of Finite Simple Groups) and the fact that $m \leq 2$. Hence $D/(MC_D(M))$ is isomorphic to a subgroup of $\mathrm{Out}(T) \wr S_m$, and then since D is perfect, it follows that $D/(MC_D(M))$ is trivial, so $D = M \times C_D(M)$. This, however, implies that $C_D(M)$ is isomorphic to D/M, which has D/R as a quotient, and so D has a quotient isomorphic to the direct product of $M \cong T^m$ and D/R, contradicting the choice of R.

This proves the claim. Equivalently, it shows that smallest normal subgroup S of G contained in $D = G'$ such that D/S is a direct product of simple groups is soluble. We now proceed to consider possibilities for S.

5.6 Step 3: Eliminating some possibilities for D/S

By what we saw in Step 1, the soluble normal subgroup S of G is non-trivial, and therefore $|G/S| \leq 127545840/2 = 63772920$. This eliminates from consideration the possibilities for D/S given in rows 17 to 19 of Table 2.

Next, if S is non-trivial but so small that $\mathrm{Aut}(S)$ is soluble or its order is not divisible by the order of any simple quotient of D, then $D/C_D(S)$ is trivial, so S is central in D, and it follows that S is isomorphic to a quotient of the Schur multiplier of D/S.

Before taking this further, we note that the Schur multipliers of A_6, $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(3,7)$ and J_3 are C_6, C_1, C_1 and C_3, respectively, while those of A_{10} and the groups $\mathrm{PSL}(2,q)$ for $q \in \{7, 17, 25, 121, 169, 361\}$ are all C_2. Hence $\mathrm{SM}(\mathrm{PSL}(2,q) \times \mathrm{PSL}(2,q)) \cong C_2 \times C_2$ for $q \in \{7, 17, 23\}$, while $\mathrm{SM}(A_6 \times PSL(3,3))$, $\mathrm{SM}(A_6 \times \mathrm{PSL}(2,25))$ and $\mathrm{SM}(PSL(2,7) \times PSL(2,7) \times A_6)$ are C_6, $C_6 \times C_2$ and $C_6 \times C_2 \times C_2$, respectively.

We now find that D/S cannot be $PSU(3,7)$ or J_3, because their Schur multipliers are trivial and C_3 (and if $D/S \cong \mathrm{J}_3$ then $|S| \leq \lfloor 127545840/|\mathrm{J}_3| \rfloor = 2$). This eliminates eliminates the 11th, 12th and 16th entries in Table 2 from consideration.

Similarly, if $|D/S|$ is so large that S is central, and we know that $\mathrm{SM}(D/S) \cong C_2$, then $|S| = 2$, so the centre of $D = N/K$ has order 2, and hence also the centre of $N/(K^a)$ has order 2, and then $N/(K \cap K^a)$ is an extension of its centre, of order 2 or 4, by a quotient isomorphic to D/SL. But then the centre of $N/(K \cap K^a)$ has order 2 as well, and so $K \cap K^a = K$, which makes K normal in G_5, contradiction. Accordingly, D/S cannot be $\mathrm{PSL}(2,121)$, A_{10}, $\mathrm{PSL}(2,169)$, $\mathrm{PSL}(2,17) \times \mathrm{PSL}(2,17)$ or $\mathrm{PSL}(2,361)$, and this eliminates the 6th, 7th, 9th, 13th and 15th entries in Table 2 from consideration.

Finally, suppose that D/S is $\mathrm{PSL}(2,7) \times \mathrm{PSL}(2,7) \times A_6$, of order 10160640. Then D is a quotient of $\mathrm{Cov}(D/S)$ by a subgroup of its centre $\mathrm{SM}(D/S) \cong C_2 \times C_2 \times C_6$, of index between 2 and $\lfloor 63772920/10160640 \rfloor = 6$. There are 22 such quotients, but a Magma computation shows that only 8 of them are quotients of N, and in each case, the corresponding kernel is normal in G_4^1, so none of them gives a graph of type 4^2.

As a result, we conclude that D/S is isomorphic to A_6, or $\mathrm{PSL}(3,3)$, or $\mathrm{PSL}(2,25)$, or $\mathrm{PSL}(2,7) \times \mathrm{PSL}(2,7)$, or $A_6 \times \mathrm{PSL}(3,3)$, or $A_6 \times \mathrm{PSL}(2,25)$.

5.7 Step 4: The remaining cases

We now deal with the possibilities from rows 1, 2, 3, 4, 5, 8 and 10 of Table 2.

Row 1: Suppose $D/S \cong A_6$. Then $|S| \leq 3^{11} = 177147$.

If S is central in D, then it is a proper quotient of $\mathrm{SM}(A_6) \cong C_6$, and hence must be C_2, C_3 or C_6. In the first and third of those cases, however, that quotient is a group (of order 720 or 2160 respectively) that is not a quotient of N: in both cases it has just one involution, making an epimorphism from N impossible. Hence $S \cong C_3$, and $D = N/K$

is an extension of C_3 by A_6. In this case, however, K is normal in G_5 and G is the automorphism group of the 5-arc-transitive triple cover of Tutte's 8-cage, contradiction. Thus S is non-central.

Next, suppose S is an elementary abelian k-group for some prime k, upon which $D/S \cong A_6$ acts non-trivially. Then we can apply MAGMA's module machinery (specifically the `GModuleH` and `PullBackH` functions that are used in the `LowIndexNormalSubgroups` function) to the corresponding action of G_4^2 on the pre-image K $(= L)$ of S, and find the following, without placing any restriction on $|S|$:

(a) if $k = 2$ then $|S| = 2^{16}$, and

(b) if $k = 3$ then $|S| = 3^6$, 3^7, 3^{15} or 3^{16} (one module each), or 3^{11} (two modules), and

(c) if $k = 5$ or 7 then $|S| = k^{16}$.

Now case (c) and the sub-cases of (b) with $|S| = 3^{15}$ or 3^{16} can be eliminated immediately as they would make $|S|$ too large. Moreover, as the prime divisors of $|G_4^1/K| = 720$ are 2, 3 and 5, all cases with $k \geq 7$ can be eliminated too, because the sub-module dimensions would be the same as for characteristic zero and hence for characteristic 7, giving $|S| = k^{16}$. Also case (a) and the sub-cases of (b) where $|S| = 3^6$ or 3^7 can be eliminated, because the normal subgroup K of G_4^2 is normal in G_5. This leaves only the possibility $|S| = 3^{11}$, and for that one, the two modules for the action of G_4^2 on K produce the normal subgroups of G_4^2 found in §5.2.

Finally, suppose S is neither central nor elementary abelian. Then S must contain a characteristic subgroup T of index 3, 2^{16}, 3^6 or 3^7 that is normal in D, with D/T being an extension of C_3 by A_6, or one of the quotients of N arising from case (a) or (b) above.

If $|D/T| = 3 \cdot 360 = 1080$ then $|T| \leq 3^{10} = 59049$, and digging further (using Schur multipliers and module actions) reveals just three further small possibilities, with $|T| = 2^{12}$, 2^{13} and 2^{14}, but in all three cases the corresponding subgroup K is again normal in G_5.

In the other three sub-cases, $|T|$ is at most 2, 243 or 81, respectively, and then digging further gives just one further small possibility, with $(|D/T|, |T|) = (2^{16} \cdot 360, 2)$, but again in this case the corresponding subgroup K is normal in G_5. (The Schur multiplier method also gives a potential example with $(|D/T|, |T|) = (3^6 \cdot 360, 3)$, but that is the same as the case where S is elementary abelian of order 3^7, from case (b) above, and this can be taken no further, given the upper bound on $|G|$.)

In summary, we find the following possibilities for S when $N/K \cong A_6$, under the extra condition that $|S| \leq 3^{11}$, but in advance of considering whether or not K is normal in G_5:

- S is cyclic of order 3, or elementary abelian of order 2^{16}, 3^6 or 3^7, and in each of these cases the kernel K is unique and gives a unique 5-arc-transitive graph;
- S is elementary abelian of order 3^{11}, associated with two possibilities for K, both of which give the graph of type 4^2 found earlier;
- S is a group of order 2^{17}, $3 \cdot 2^{12}$, $3 \cdot 2^{13}$ or $3 \cdot 2^{14}$, and in each of these cases K is unique

and gives a unique 5-arc-transitive graph.

In particular, we get no further examples than the two found already.

Rows 2 and 3: Suppose $D/S \cong \mathrm{PSL}(3,3)$. Then $|S| \leq \lfloor 127545840/5616 \rfloor = 22711$.

In this case S cannot be central in D, because $\mathrm{SM}(\mathrm{PSL}(3,3))$ is trivial.

Next, if S is an elementary abelian k-group (for some k), on which $D/S \cong \mathrm{PSL}(3,3)$ acts non-trivially, then application of MAGMA's module machinery shows that if $k = 2$ then $|S| = 2^{12}$ or 2^{13} (with S splitting as a direct product $C_2 \times (C_2)^{12}$ in the latter case), while if $k = 3$ then $|S| = 3^7$ or 3^8, but again the corresponding subgroup K is normal in G_5. There are no possibilities for $k \geq 5$ (given the upper bound on $|G|$), and also digging further inside the perfect group N also gives no more examples. (The Schur multiplier method gives potential examples with $|S| = 2 \cdot 2^{12}$ and $3 \cdot 3^7$, but these are the same as the cases where S is elementary abelian with order 2^{13} and 3^8, respectively.)

The resulting possibilities for G, however, depend on whether $G/S = D/S \cong \mathrm{PSL}(3,3)$, or $|G\!:\!D| = 2$ and $G/S \cong \mathrm{PSL}(3,3) \times C_2$, and on the effect of the C_2 factor on S in the latter case. Just two possibilities arise when $G/S = D/S \cong \mathrm{PSL}(3,3)$, namely with S elementary abelian of order 3^7 or 3^8, respectively. Similarly, when $G/S \cong \mathrm{PSL}(3,3) \times C_2$ then there are just two possibilities where S is an elementary abelian 2-group (of order 2^{12} or 2^{13}). On the other hand, when $G/S \cong \mathrm{PSL}(3,3) \times C_2$ there are two possibilities for K for which S is elementary abelian of order 3^7, and another two for which S is elementary abelian of order 3^8 (with $Z(G)$ being trivial and cyclic of order 6 for those last two). Furthermore, in all of these eight cases, the kernel K is unique and gives a unique 5-arc-transitive graph.

Hence this case can be eliminated.

Row 4: Suppose $D/S \cong \mathrm{PSL}(2,25)$. Then $|S| \leq \lfloor 63772920/7800 \rfloor = 8176$.

In this case S cannot be central, because $\mathrm{Cov}(\mathrm{PSL}(2,25)) \cong \mathrm{SL}(2,25)$, which has a unique involution and therefore cannot be a quotient of N. But also S cannot be an elementary abelian k-group, because in that case the module approach shows that $|S| \geq 5^6$ when $k = 5$, while $|S| \geq k^{25}$ for every other small prime k, and in both cases, $|S|$ is too large. Obviously digging deeper does not help, and so this case can be eliminated as well.

Row 5: Suppose $D/S \cong \mathrm{PSL}(2,7) \times \mathrm{PSL}(2,7)$. Then $|S| \leq \lfloor 63772920/(168^2) \rfloor = 2259$.

If S is central in D, then D is a quotient of $\mathrm{Cov}(D/S) \cong \mathrm{SL}(2,7) \times \mathrm{SL}(2,7)$. The latter, however, has only one quotient of order greater than $|D/S| = 168^2$ that is a quotient of N, namely one of order $2 \cdot 168^2$ obtained by factoring out the subgroup generated by the product of the central involutions of the two copies of $\mathrm{SL}(2,7)$, and the corresponding kernel is normal in G_4^1, so this does not give a graph of type 4^2.

On the other hand, if S is elementary abelian but non-central, then the same approach as taken for rows 1 to 4 (using MAGMA's module machinery) shows there is no possibility: without imposing the bound $|S| \leq 2259$, the smallest S would have order at least 3^{14} or 7^6 or k^{16} for some small prime $k \notin \{3,7\}$. (Note: Here the index $|G_4^2\!:\!L|$ of the pre-image L of S

in N is too large for the standard approach to work, but the normal subgroup L is generated by conjugates of the element $w = ahahahah^{-1}ah^{-1}ah^{-1}ah^{-1}ahah^{-1}ah^{-1}ahahah^{-1}ah$, so we can add the relation $w^k = 1$ to the defining relations for the group G_4^2 to make it work.)

Finally, if S is abelian-by-abelian, it must have an abelian subgroup T of index 2 that is normal in D, with D/T being the quotient of N of order $2|D/S|$ found two paragraphs above, but then the 'module' approach shows that $|T/S|$ is again at least 3^{14} or 7^6 or k^{16} for some small prime $k \notin \{3, 7\}$, and so this case can be eliminated.

Row 8: Suppose $D/S \cong A_6 \times \mathrm{PSL}(3, 3)$. Then $|S| \leq \lfloor 63772920/(360 \cdot 5616) \rfloor = 31$.

If S is central in D, then it must be a non-trivial quotient of $\mathrm{SM}(A_6 \times \mathrm{PSL}(3, 3)) \cong C_6$, so $S \cong C_2$ or C_3. In this case, we may consider the kernel R of the projection from D to A_6, which is an extension of S by $\mathrm{PSL}(3, 3)$. Now because $\mathrm{SM}(\mathrm{PSL}(3, 3))$ is trivial, R is the direct product of S with a subgroup L isomorphic to $\mathrm{PSL}(3, 3)$, and as L is characteristic R, it is normal in D and hence can be factored out, giving a central extension of A_6 as a quotient of D and hence of N. By what we found above for row 1, however, this gives $S \cong C_3$ and so the pre-image of L in N is normal in G_5, and then it follows that the kernel of the epimorphism from N to D is normal in G_5 as well, contradiction.

Thus S is non-central, and hence must be isomorphic to either $(C_2)^4$ or $(C_3)^3$, the only groups of order at most 31 with the property that the order of their automorphism group is divisible by $|A_6| = 360$ or $|\mathrm{PSL}(3, 3)| = 5616$. We consider these two sub-cases separately.

If $S \cong (C_2)^4$, then S is central in the kernel R of the epimorphism from D to A_6, and then since $\mathrm{SM}(\mathrm{PSL}(3, 3))$ is trivial, R cannot be perfect. Indeed R must contain a perfect characteristic subgroup L isomorphic to $\mathrm{PSL}(3, 3)$ with quotient R/L of order 16, and so N has a quotient $Q = D/L$ of order $16 \cdot 360 = 5760$, isomorphic to an extension of $(C_2)^4$ by A_6. Cohomology computations in Magma show that if M is the A_6-module corresponding to the normal subgroup $(C_2)^4$, then $H^2(A_6, M)$ is trivial, so the extension splits, and therefore $Q = D/L$ is a unique group of order 5760, namely the non-central split extension $(C_2)^4 \rtimes A_6$. The latter group, however, is not a quotient of N, contradiction.

Similarly, if $S \cong (C_3)^3$, then S is central in the kernel R of the epimorphism from D to $\mathrm{PSL}(3, 3)$, and then since $\mathrm{SM}(A_6) \cong C_6$, we find that R cannot be perfect. Indeed R must contain a perfect characteristic subgroup L isomorphic to A_6 or a central C_3-cover of A_6, with quotient R/L of order 27 or 9, and it follows that D has a quotient $Q = D/L$ of order $27 \cdot 5616$ or $9 \cdot 5616$, isomorphic to an extension of $(C_3)^3$ or $(C_3)^2$ by $\mathrm{PSL}(3, 3)$. Now Q must be perfect (being a non-trivial quotient of D), and as $\mathrm{PSL}(3, 3)$ has trivial Schur multiplier and no faithful action on a group of order 3^2, we find that Q is a non-central extension of $(C_3)^3$ by $\mathrm{PSL}(3, 3)$. Again, cohomology computations in Magma show that this extension splits, and therefore $D/L = Q$ is a non-central split extension $(C_3)^3 \rtimes \mathrm{PSL}(3, 3)$, but this is not a quotient of N, contradiction.

Hence this case is eliminated too.

Row 10: Suppose $D/S \cong A_6 \times \mathrm{PSL}(2,25)$. Then $|S| \le \lfloor 63772920/(360 \cdot 7800) \rfloor = 22$.

If S is central in D, then D must be a quotient of $\mathrm{Cov}(A_6 \times \mathrm{PSL}(2,25)$ by a proper subgroup L of its centre (which is isomorphic to $\mathrm{SM}(A_6 \times \mathrm{PSL}(2,25)) \cong C_6 \times C_2$). Now although $\mathrm{Cov}(A_6 \times \mathrm{PSL}(2,25))$ has nine such subgroups, the quotients by only three of them are quotients of N, and in those three cases (with quotient of $\mathrm{SM}(A_6 \times \mathrm{PSL}(2,25))$ by L being isomorphic to C_2, C_3 and C_6, respectively), the corresponding kernel in N is normalised by $a \in G_5 \setminus N$, so all three of them give 5-arc-transitive graphs, contradiction.

Thus S is non-central, and so must be isomorphic to $(C_2)^4$, the only group of order at most 22 whose automorphism group has order divisible by $|A_6| = 360$ or $|\mathrm{PSL}(3,3)| = 5616$. In this case S is central in the kernel R of the epimorphism from D to A_6, and then since $\mathrm{SM}(\mathrm{PSL}(2,25)) \cong C_2$, we find that R cannot be perfect. Indeed R must contain a perfect characteristic subgroup L isomorphic to $\mathrm{PSL}(2,25)$ or a central C_2-cover of $\mathrm{PSL}(2,25)$, with quotient R/L of order 16 or 8, and it follows that D has a quotient $Q = D/L$ of order $16 \cdot 360$ or $8 \cdot 360$, isomorphic to an extension of $(C_2)^4$ or $(C_2)^3$ by A_6. Now Q is perfect (as a non-trivial quotient of D), and since A_6 has Schur multiplier C_6 and no faithful action on a group of order 2^3, we find that Q must be a non-central extension of $(C_2)^4$ by A_6. But this extension splits and therefore $D/L = Q$ is not a quotient of N, by the same argument as in the case for Row 8, contradiction.

Hence this final case can be eliminated too, leaving only the case arising from row 1.

5.8 Conclusion

Thus we obtain the following:

Theorem 5.1 *The smallest symmetric cubic graph with type* (4^2)*, and hence the smallest with action type* (4^2)*, is a unique one of order* 5314410*. This graph is a* 3^{11}*-fold cover of Tutte's* 8*-cage, with automorphism group an extension of an elementary abelian group of order* 3^{11} *by* M_{10} (*the point-stabiliser in the Mathieu group* M_{11}*, also known as* $\mathrm{PGL}^*(2,9)$).

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